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Abstract in English

Fixed point theory has a long history of being used in nonlinear differential equations, in order to prove existence, uniqueness, or other qualitative properties of solutions. However, using the contraction mapping principle for stability and asymptotic stability of solutions is of more recent appearance. Lyapunov's direct method has been very effective in establishing stability results for a wide variety of general nonlinear systems without solving the systems themselves. Nevertheless, the application of this method to problems of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms. Applying Lyapunov techniques can be challenging, and the Banach fixed point method has been shown to yield less restrictive criteria for stability of delayed FDEs. The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise. In this Thesis, we will extend a contraction mapping stability result that gives mean square asymptotic stability of a nonlinear stochastic differential equations with variables delays.

Keywords: Fixed points theorem; Contraction; Asymptotic stability in mean square; Neutral stochastic differential equations; Variable delays.

Abstract in French

La théorie du point fixe a une longue histoire d'utilisation dans les équations différentielles non linéaires, afin de prouver l'existence, unicité ou d'autres propriétés qualitatives des solutions. Cependant, l'utilisation du principe de contraction pour la stabilité et la stabilité asymptotique des solutions est d'apparition plus récente. La méthode directe de Lyapunov a été très efficace pour établir des résultats de stabilité pour une grande variété de systèmes non linéaires sans résoudre les systèmes eux-mêmes. Néanmoins, cette méthode a rencontré de sérieux obstacles et il existe encore un tas de problèmes qui résistent à cette méthode. La classe d'équations différentielles fonctionnelles à retard fait partie du nombre de problèmes qui ont résisté à la méthode directe de Lyapounov. En général, l'insuffisance de la méthode de Lyapounov se manifeste lorsque les fonctions utilisées dans les équations ne sont pas bornées en temps, si le délai n'est pas borné ou si sa dérivée n'est pas petite. Dans cette thèse on montre aussi que la technique de point fixe reste applicable pour démonter la stabilité asymptotique en moyenne quadratique pour des équations différentielles stochastiques non linéaires avec plusieurs retards non bornés.

Mots clés: Points fixe; Contraction; Stabilité asymptotique en moyenne quadratique; Equations differentielles stochastiques de type neutres; Retard.

Abstract in Arabic

تتمتع نظرية النقطة الثابتة بتاريخ طويل من الاستخدام في المعادلات التفاضلية غير الخطية، من أجل إثبات وجود أو تفرد أو غيرها من الخصائص النوعية للحلول. ومع ذلك، فإن استخدام مبدأ الانكماش من أجل الاستقرار والاستقرار غير المقارب للحلول هو مظهر أكثر حداثة.كانت طريقة Lyapunov المباشرة فعالة جدًا في تحقيق نتائج الاستقرار لمجموعة واسعة من الأنظمة غير الخطية دون حل الأنظمة نفسها. ومع ذلك، واجهت هذه الطريقة عقبات خطيرة ولا يزال هناك الكثير من المشاكل التي تقاوم هذه الطريقة. فئة المعادلات الوظيفية التفاضلية مع التأخير هي واحدة من عدد من المشاكل التي قاومت طريقة Lyapunov المباشرة. بشكل عام ، يتضح عدم كفاية طريقة Lyapunov ، عندما لا تكون الدوال المستخدمة في المعادلات محددة في الوقت المناسب إذا لم يكن التأخير محددًا أو إذا كان مشتقه غير صغير. في هذه الرسالة ، نبيّن أيضًا أن تقنية النقطة الثابتة تظل قابلة للتطبيق لإظهار الثبات المقارب كوسيلة تربيعية للمعادلات التفاضلية العشوائية غير الخطية مع العديد من التأخيرات غير المحدودة.

Dedication

To my beloved parents.

The reason of what I become today.

Thanks for your great support and care.

To my sisters, my brothers, and all my friends.

They have been my inspiration, and my soul mates.

Thank you for your everlasting love and warm encouragement throughout my research.

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Acronyms

Abbreviation	**Meaning**
ODEs	Ordinary Differential Equations
FDEs	Functional Differential Equations
SFDEs	Stochastic Functional Differential Equations
DDEs	Delay Differential Equations
SDEs	Stochastic Differential Equations
SDDEs	Stochastic Delay Differential Equations
NDDEs	Neutral Delay Differential Equations

List of Symbols

Here we state some conventions regarding mathematical notation that we will use in this thesis.

$a.s., a.e.$	almost surely, almost everywhere.
$r.v.$	random variable.
$\mathbb{R}^n$	Euclidean space of $n-$ dimensions.
$\mathbb{R}^+ = [0, \infty)$	set of positive real numbers.
$\mathbb{N}$	set of natural numbers.
$\frac{d}{dt}$	first derivative with respect to t.
$(a, b), [a, b]$	open, closed interval from a to b.
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space.
$C([a, b], \mathbb{R})$	space of continuous functions mapping from the interval $[a, b]$ to $\mathbb{R}$.
$\mathbb{P}, \mathbb{E}, Var$	probability, expected value, variance.
ϕ	empty set.
$w(t)$	brownian motion.

$\mathbb{E}[x \mid \mathcal{G}]$	conditional expectation of x with respect to $\mathcal{G}$.
$\mathcal{B}$	the Borel $-\ \sigma$ algebra on $\mathbb{R}$.
$\mathbb{R}^{n\times m}$	the space of real $n \times m$- matrices.
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2.
χ_A	the indicator function of a set A, i.e. $\chi_A(x) = 1$ if $x \in A$ or otherwise 0.
A^T	the transpose of a vector or matrix A.

$C([-\tau, 0]; \mathbb{R}^n)$	the space of all continuous $\mathbb{R}^n$ – valued functions φ defined on $[-\tau, 0]$ with a norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} \lvert\varphi(\theta)\rvert$.
$f : A \to B$	the mapping f from A to B.
A^c	the complement of A in Ω, $i.e.$ $A^c = \Omega - A$.
$\sigma(C)$	the σ – algebra generated by C.
$L^p(\Omega; \mathbb{R}^n)$	the family of $\mathbb{R}^n$ – valued random variables x with $\mathbb{E}\lvert x\rvert^p < \infty$.

$\dot{w}$	the derivative of the Brownian motion.
$\dot{V}$	the derivative of the Lyapunov function.
$\lvert x\rvert$	the Euclidean norm of a vector x.
$C(S_1, S_2)$	the set of all continuous functions $\phi : S_1 \to S_2$ with the supremum norm.
$\sum_{i=0}^{n}$	Summation from index $i = 0$ to $i = n$.

Other notations will be explained when they first appear.

Introduction

In nature, physics, society, engineering, and so on we always meet two kinds of functions with respect to time: one is deterministic and another is random. Stochastic differential equations (SDEs) arise in mathematical models of physical systems which possess inherent noise and uncertainly. Such models have been used with great success in a variety of application area, biology, finance, mechanics, and so forth. Up to now, it is an important branch of stochastic analysis.

Currently, the study of analysis and synthesis of stochastic time delay systems, described by stochastic delayed differential equations (SDDE for short), is a popular topic in the field of control theory [Cong, 2013]. Delays in the dynamics can represent memory or inertia in the financial system [Øksendal and Sulem, 2000]. Because, It is now well known that the existence of delays in a dynamical system has been the source of oscillation, instability and poor system performances, the study on time delay systems stability and control has important theoretical and practical values. Furthermore, real systems depend on not only present and past states but also involve derivatives with delays. As a result, these systems are often built in the form of neutral differential equations.

So far, these topics have received a lot of attention and there are so many references about them. For instance, this type of equation is very important in several models, including population biology [Mao, 1997], and financial mathematics [Anh and Inoue, 2005], population ecology [Kuang, 1993], and other engineering systems [Kolmanovskii and Myshkis,1992], [Hobson and Rogers, 1998], and [Bouchaud and Cont., 1998], [Itô and Nisio, 1964] wrote the first paper in this field in 1964. For neutral stochastic delay differential equations, we refer to [60, 73, 81].

Stability plays an important role in the theory of dynamical systems and control. It characterizes the property of an unperturbed trajectory that all perturbed trajectories starting nearby stay nearby: small perturbations cause only small changes in the system behavior. The most important concept of stability has been introduced by the Russian mathematician Lyapunov in 1892. Based on his famous work a general "Lyapunov theory" has been developed to investigate the stability behavior of general dynamical systems. In [77] La Selle and Lefschetz refer to Lyapunov's paper (French translation) "*Probleme general de la stabilite du mouvement*" which proposed two different methods for determining the stability of deterministic systems. The first is known as the "First method" which requires the existence of a known explicit solution. Unfortunately this method is restrictive since for most differential equations (deterministic or stochastic) an explicit solution can be determined for only a very few cases e.g. linear SDEs driven by Brownian motion. For the vast majority of them, this is not possible. On the other hand, the " *Second method or direct Lyapunov method*" for determining the stability of a system, is more applicable since it does not require the knowledge of the explicit solution and that's why in recent years it has exhibited great power in applications specifically in engineering sciences and to mechanical and structural systems that have non-linear behavior [see Ariaratnam and Xie [7]]. With the direct Lyapunov method we can get a lot of useful qualitative information about the behavior of the solution, without solving the equation. This includes asymptotic behavior and sensitivity of the solutions to small changes in the initial conditions. This information can be found from the coefficients in the differential equation.

Recently, many researchers have studied the stability of stochastic differential equations using Lyapunov functions and obtained interesting results, for example, Liao [75], Mao [80], Caraballo et al. [28] , [29], Yang [99] amongst others. However, there are also several difficulties in the applications of the corresponding theories to specific problems. Lyapunov's direct method usually requires pointwise conditions, the stability result we offer asks conditions of an averaging nature. There are no general rules for constructing Lyapunov functions. The constructions are merely based on a researchers' experience and some particular techniques. It is therefore necessary to seek some new methods to deal with the stability in order to overcome those difficulties.

To this end, Burton and other authors have applied the fixed point theory to investigate the stability of deterministic systems during the last years, and have obtained some more applicable conclusions which can be found, for example, in the monograph [20] and the works ([5] , [17]-[19],[21],[22] , [30],[34], [88],[100],[101] , [104],[105]). In addition, there are some papers where the fixed point theory is used to investigate the stability of stochastic (delayed) differential equations (see for instance [68] , [69] , [89] , [90]). More precisely, in ([69] , [70] , [71]) the authors used the fixed point theory to study the exponential stability of mild solutions for stochastic partial differential equations with bounded delays and with infinite delays. In [89] , [90] the fixed point theory is used to discuss the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and infinite delays, and in [72] Luo used the fixed point theory to study the exponential stability of stochastic Volterra-Levin equations. Motivated by the works mentioned above, in this thesis we study the mean square asymptotic stability of a class of neutral stochastic differential systems with variable delays by using a contraction mapping principle, and the obtained stability criteria are easily checked. In our result, the delays can be unbounded and the coefficients in the equations can change their sign.

The main aim of this thesis is to examine stability properties of the solutions to stochastic differential equations (SDDEs). These equations have complicated characteristics; therefore, it is better to start with discussing the properties of stochastic differential equations (SDEs) and delay differential equations (DDEs), which are the special case of SDDEs. Once the behavior of DDEs and SDEs is understood, it is easier to follow the fundamental properties of SDDEs. For this reason we start with building notation and terminology on DDEs and SDEs. Moreover, the necessity for their existence, the general definition of SDE, conditions to have a unique solution and the properties of that solution will be discussed. Also, we give some examples to make these concepts clear. For detailed information and proofs one can see [Evans [36], Lamberton [76] and, Mao [81], Øksendal [85] , Friedman [38]].

This thesis consists of four chapters

Chapter 1 : This chapter contains various theories and results from probability theory as well as stochastic calculus that are required in later chapters.

Chapitre 2 : This chapter falls into two parts. The first, is devoted to DDEs analysis, we present some basic preliminaries and we discuss the existence and uniqueness theorem for the solution and properties of them, while the second concerns stability theory. The former provides the appropriate mathematical tools that are needed to understand the concepts that will be developed in this thesis for the study of the stability of stochastic delay differential equations (SDDEs). In the last stage of this chapter, we compare results from a certain application of fixed point theory with a certain common Lyapunov functional.

Chapter 3 : We will present some general theory of stochastic delayed FDEs, starting with the essential definitions, discussions of some basic differences with respect to SDEs, as well as foundational theoretical results. Moreover, we touch upon stability definitions and give some stability results, which will be sufficient for a working in the topics of this thesis. In the last section, with the help of Itô's formula, solution processes of some SDDEs are derived to see the effect of the delay terms in the equations. To understand how the solution process is obtained if it exists, some examples are given.

Chapter 4 : This chapiter exposes results published in [26] and relates to study the mean square asymptotic stability of the zero solution for a system of nonlinear stochastic neutral differential equations with variable delays by using a contraction mapping principle,

$$d\left[x_i(t)-\sum_{j=1}^{n} q_{ij}(t)x_j(t-\tau_j(t))\right] = \left[\sum_{j=1}^{n} a_{ij}(t)x_j(t)+\sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t))\right.$$
$$\left.+\sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\delta_j(t)))\right] dt+\sum_{j=1}^{n} \sigma_{ij}(x_j(t))dw_j(t), t\geq t_0,$$

for $i=1,2,3,...,n$, with an assumed intial condition

$$x_i(s)=\varphi_i(s) \text{ for } s\in[m(t_0),t_0], \text{for each } t_0\geq 0.$$

A mean square asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Guo and al. [41].

Chapter 5 : This chapter collects the other works published in renowned international journals of high quality, namely:

- Mean square asymptotic stability in nonlinear stochastic neutral Volterra-Levin equations with Poisson jumps and variable delays;
- Stability analysis of neutral stochastic differential equations with Poisson jumps and variable delays;
- Existence of solutions and stability for impulsive neutral stochastic functional differential equations;
- Stability results for neutral differential equations by Krasnoselskii fixed point theorem.

CHAPTER 1

Stochastic Caculus- Itô's formula

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This chapter surveys the theory of stochastic differential equations (SDEs). First, we introduce some notions in stochastic calculus such as stochastic Integrals, Itô integrals, and related results. Then, definition of SDEs and of solutions are presented. The basic theorems in this chapter are stated without proof. For proofs of the theorems and details of the definitions, the reader is referred to [Karatzas and Shreve, 1991], [Evans, 2006], and [ØKsendal, 2013], [Applebaum, 2009],[Friedman, 1975], [Mao, 1997]. The presentation of Itô integrals are mostly based on those references mentioned above.

1.1 On Stochastic Integrals

In the first chapter, we explain the need for and importance of SDEs and give a short literature review.

Motivation

Modeling of physical systems by ordinary differential equations (ODEs) ignores stochastic effects. Addition of random elements into the differential equations leads to what is called stochastic differential equations (SDEs), and the term stochastic is called noise [Klebaner, 2005]. The necessity of this inclusion is due to the fact that almost every natural phenomenon in this world is influenced by environmental noise. For example, consider the simple population growth model

$$\frac{dx(t)}{dt} = a(t)x(t), \tag{1.1}$$

with initial value $x(0) = x_0$, where $x(t)$ is the size of the population at time t and $a(t)$ is the relative rate of growth. It might happen that $a(t)$ is not completely known, but subject to some random environmental effects. In other words,

$$a(t) = r(t) + \sigma(t)\text{“}noise\text{”},$$

so equation (1.1) becomes

$$\frac{dx(t)}{dt} = r(t)x(t) + \eta_t\sigma(t)x(t), \tag{1.2}$$

where η_t is a white noise process. It is then very hard to treat this equation mathematically. One cannot give any reasonable definition to the product of the stochastic process $\sigma(t)x(t)$ and the noise η_t. The inclusion of the noise term in differential equations may lead to a fundamentally different methods of analysis. Certainly, a reasonable mathematical interpretation of the noise term is a white noise, which is formally regarded as the derivative of a Brownian motion $w(t)$, i.e. $\dot{w}(t) = dw(t)\backslash dt$. In integral form, Eq. (1.2) is written as

$$\begin{aligned} x(t) &= x_0 + \int_0^t r(s)x(s)ds + \int_0^t \sigma(s)x(s)\eta_s ds \\ &= x_0 + \int_0^t r(s)x(s)ds + \int_0^t \sigma(s)x(s)\dot{w}(s)ds \\ &= x_0 + \int_0^t r(s)x(s)ds + \int_0^t \sigma(s)x(s)dw(s). \end{aligned}$$

The questions are: What is the mathematical interpretation for the "noise" term and what is the integration $\int_0^t \sigma(s)x(s)dw(s)$?

Since the simplest or say, the most basic continuous stochastic perturbation, intuitively will have the below four properties, the modeling of the general continuous stochastic perturbation by a stochastic integral with respect to this basic Brownian motion $(w(t), t \geq 0)$ is quite natural. However, the Brownian motion also has some stronge property: Even though it is continuous in t, it is nowhere differentiable in t. So we cannot define the stochastic integral $\int_0^t \sigma(s)x(s)dw(s)$ as the Riemann-Stieltjes integral. That is why K. Itô in 1949 invented a completely new way to define this stochastic integral.

1.1.1 Stochastic Integrals

As mentioned above, the Brownian motion $(w(t), t \geq 0)$ is nowhere differentiable and it is not bounded variation on any bounded interval. Moreover, the consequences of unbounded variation property make this integral

$$\int_\alpha^\beta f(t)dw\,(t)\,,$$

cannot even be interpreted as the Riemann Stieltjes integral for each sample path. However, we can define the integral for a large class of stochastic processes by making use of the stochastic nature of Brownian motion. This integral was first defined by K. Itô in 1949 and is now known as Itô stochastic integral. Our aim in this section is to introduce the Itô stochastic integral and discuss its properties for later applications. We shall now start to define the stochastic integral step by step.

Approximation of functions by step functions

Let $(w(t), t \geq 0)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing family of $\sigma-$fields, i.e., $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ if $t_1 < t_2$, such that $\mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}(w(s), 0 \leq s \leq t)$ is in $\mathcal{F}_t$, and

$$\mathcal{F}(w(\lambda + t) - w(t), \lambda \geq 0) \text{ is independant of } \mathcal{F}_t,$$

for all $t \geq 0$. One can take, for instance, $\mathcal{F}_t = \sigma(w(s), 0 \leq s \leq t)$.

Definition 1.1 [38] Let $0 \leq \alpha < \beta < \infty$. A stochastic process $f(t)$ defined for $\alpha \leq t \leq \beta$ is called a nonanticipative function with respect to $\mathcal{F}_t$ if:
i) $f(t)$ is a separable process;
ii) $f(t)$ is a measurable process, i.e., the function $(t,\omega) \to f(t,\omega)$ from $[\alpha,\beta] \times \Omega$ into $\mathbb{R}$ is measurable;
iii) for each $t \in [\alpha,\beta]$, $f(t)$ is $\mathcal{F}_t$ -measurable;
When (iii) holds we say that $f(t)$ is adapted to $\mathcal{F}_t$. We denote by $\mathbb{L}_w^p[\alpha,\beta]$ $(1 \leq p \leq \infty)$ the class of all nonanticipative functions $f(t)$ satisfying:

$$\mathbb{L}_w^p[\alpha,\beta] = \left\{ \begin{array}{l} f \text{ nonanticipative functions: } \mathbb{P}\left[\int_\alpha^\beta |f(t)|^p \, dt < \infty\right] = 1, \ 1 \leq p < \infty, \\ \mathbb{P}\left[\underset{\alpha\leq t\leq\beta}{ess\ sup} |f(t)| < \infty\right] = 1 \text{ if } p = \infty. \end{array} \right\}$$

We denote by $\mathbb{M}_w^p[\alpha,\beta]$ the subset of $\mathbb{L}_w^p[\alpha,\beta]$ consisting of all functions f

with

$$\mathbb{M}_w^p[\alpha,\beta] = \left\{ \begin{array}{l} f \in \mathbb{L}_w^p[\alpha,\beta] : \mathbb{E}\int_\alpha^\beta |f(t)|^p\,dt < \infty,\ 1 \le p < \infty, \\ \mathbb{E}\left[\underset{\alpha\le t\le\beta}{ess\ sup} |f(t)| \right] < \infty \ if\ p = \infty. \end{array} \right\}$$

Let us first introduce the concept of step (or simple) processes.

Definition 1.2 [38] *A stochastic process $f(t)$ defined on $[\alpha,\beta]$ is called a step function if there exists a partition $\alpha = t_0 < t_1 < ... < t_r = \beta$ of $[\alpha,\beta]$ such that*

$$f(t) = f(t_i)\ if\ t_i \le t < t_{i+1},\ 0 \le i \le r-1.$$

The next Lemma is given in literatures without details of the proof.

Lemma 1.1 [38] *Let $f \in \mathbb{L}_w^2[\alpha,\beta]$. Then:*

i) there exists a sequence of continuous functions g_n in $\mathbb{L}_w^2[\alpha,\beta]$ such that

$$\lim_{n\to\infty} \int_\alpha^\beta |f(t) - g_n(t)|^2\,dt = 0 \quad a.s.;$$

ii) there exists a sequence of step functions f_n in $\mathbb{L}_w^2[\alpha,\beta]$ such that

$$\lim_{n\to\infty} \int_\alpha^\beta |f(t) - f_n(t)|^2\,dt = 0 \quad a.s.$$

Lemma 1.2 [38] *Let $f \in \mathbb{M}_w^2[\alpha,\beta]$. Then:*

i) there exists a sequence of continuous functions k_n in $\mathbb{M}_w^2[\alpha,\beta]$ such that

$$\lim_{n\to\infty} \mathbb{E}\int_\alpha^\beta |f(t) - k_n(t)|^2\,dt = 0;$$

ii) there exists a sequence of bounded step functions l_n in $\mathbb{M}_w^2[\alpha,\beta]$ such that

$$\lim_{n\to\infty} \mathbb{E}\int_\alpha^\beta |f(t) - l_n(t)|^2\,dt = 0.$$

Definition of the stochastic integral

We now give the definition of the Itô integral.

Definition 1.3 [38] *Let $f(t)$ be a step function in $\mathbb{L}_w^2[\alpha,\beta]$,* say $f(t)=f(t_i)$ if $t_i \leq t < t_{i+1}$, $0 \leq i \leq r-1$ where $\alpha = t_0 < t_1 < ... < t_r = \beta$. *The random variable*

$$\sum_{k=0}^{r-1} f(t_k)\left[w(t_{k+1}) - w(t_k)\right],$$

is denoted by

$$\int_\alpha^\beta f(t)dw(t),$$

and is called the stochastic integral of f with respect to the Brownian motion w, it is also called the Itô integral.

Clearly, the stochastic integral $\int_\alpha^\beta f(t)dw(t)$ is $\mathcal{F}_\beta$-measurable.

Theorem 1.1 [38] *Let f, f_n be in $\mathbb{L}_w^2[\alpha,\beta]$ and suppose that:*

$$\lim_{n\to\infty} \int_\alpha^\beta |f_n(t) - f(t)|^2 dt \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Then:

$$\int_\alpha^\beta f_n(t)dw(t) \xrightarrow{\mathbb{P}} \int_\alpha^\beta f(t)dw(t) \text{ as } n \to \infty.$$

Where $\xrightarrow{\mathbb{P}}$ refers that the converge is in probability.

Lemma 1.3 [38] ***Linearity:*** *let f, g be two step functions in $\mathbb{L}_w^2[\alpha,\beta]$ and λ_1, λ_2 be real numbers. Then $\lambda_1 f + \lambda_2 g$ is in $\mathbb{L}_w^2[\alpha,\beta]$ and*

$$\int_\alpha^\beta [\lambda_1 f(t) + \lambda_2 g(t)]\, dw(t) = \lambda_1 \int_\alpha^\beta f(t)\,dw(t) + \lambda_2 \int_\alpha^\beta g(t)\,dw(t).$$

Lemma 1.4 [38] *If f is a step function $\mathbb{M}_w^2[\alpha,\beta]$, then*

$$i)\ \mathbb{E}\left[\int_\alpha^\beta f(t)\,dw(t)\right] = 0,$$
$$ii)\ \mathbb{E}\left[\int_\alpha^\beta f(t)\,dw(t)\right]^2 = \mathbb{E}\int_\alpha^\beta |f^2(t)|\,dt.$$

Proof. *i) Since $f(t_k)$ is $\mathcal{F}_{t_k}$-measurable whereas $w(t_{k+1})-w(t_k)$ is independent of $\mathcal{F}_{t_k}$,*

$$\begin{aligned}\mathbb{E}\left[\int_\alpha^\beta f(t)\,dw(t)\right] &= \sum_{k=0}^{r-1}\mathbb{E}\left[f(t_k)\left(w(t_{k+1})-w(t_k)\right)\right]\\ &= \sum_{k=0}^{r-1}\mathbb{E}\left(f(t_k)\right)\mathbb{E}\left(w(t_{k+1})-w(t_k)\right)=0.\end{aligned}$$

Moreover, note that $w(t_{k+1})-w(t_k)$ is independent of $f(t_i)\,f(t_k)\,(w(t_{i+1})-w(t_{i+1}))$ if $i<k$. Thus

$$\begin{aligned}&\mathbb{E}\left[\int_\alpha^\beta f(t)\,dw(t)\right]^2\\ &= \sum_{0\le i,k\le r-1}\mathbb{E}\left[f(t_i)\,f(t_k)\left(w(t_{k+1})-w(t_k)\right)\left(w(t_{i+1})-w(t_{i+1})\right)\right]\\ &= \sum_{k=0}^{r-1}\mathbb{E}\left[\left(f^2(t_k)\left(w(t_{k+1})-w(t_k)\right)\right)^2\right]\\ &= \sum_{k=0}^{r-1}\mathbb{E}\left(f^2(t_k)\,\mathbb{E}\left(w(t_{k+1})-w(t_k)\right)\right)^2\\ &= \sum_{k=0}^{r-1}\mathbb{E}f^2(t_k)\left(t_{k+1}-t_k\right)=\mathbb{E}\int_\alpha^\beta \left|f^2(t)\right|dt. \quad \blacksquare\end{aligned}$$

Lemma 1.2 extend to any functions from $\mathbb{L}_w^2[\alpha,\beta]$:

Theorem 1.2 [38] *If $f\in\mathbb{L}_w^2[\alpha,\beta]$ and f is continuous, then, for any sequence Π_n of partitions $\alpha=t_{n,0}<t_{n,1}<\ldots<t_{n,m_n}=\beta$ of $[\alpha,\beta]$ with mesh $|\Pi_n|=\max\left(t_{n,k}-t_{n,k-1}\right)$ converging to 0, then*

$$\int_\alpha^\beta f(t)\,dw(t)\xrightarrow{\mathbb{P}}\sum_{k=0}^{m_n-1}f(t_{n,k})\left[w(t_{n,k+1})-w(t_{n,k})\right]\ \text{ as } n\to\infty. \tag{1.3}$$

Proof. Introduce the step function g_n:

$$f(t_{n,k})=g_n(t)\text{ if } t_{n,k}\le t<t_{n,k+1},\, 0\le k\le m_n-1.$$

For a.a. ω, $g_n(t)\to f(t)$ uniformly in $t\in[a,b)$ as $n\to\infty$. Hence:

$$\int_\alpha^\beta \left|g_n(t)-f(t)\right|^2dt\to 0,\ a.s.$$

By Theorem 1.1, we then have:

$$\int_{\alpha}^{\beta} g_n(t)dw\,(t) \xrightarrow{\mathbb{P}} \int_{\alpha}^{\beta} f(t)dw\,(t)\,.$$

Since

$$\int_{\alpha}^{\beta} g_n(t)dw\,(t) = \sum_{k=0}^{m_n-1} f\left(t_{n,k}\right)\left[w\left(t_{n,k+1}\right) - w\left(t_{n,k}\right)\right],$$

then assertion (1.3) follows. ■

Example 1.1 To show the existence of $\int_0^T w(t)dw(t)$, from Theorem 1.2 we need to show that the Wiener process $(w(t), t \geq 0)$ belongs to $\mathbb{L}_w^2[0,T]$. Since for all T,

$$\mathbb{E}\left(\int_0^T |w(t)|^2\, dt\right) = \int_0^T \mathbb{E}\left(\left|w^2(t)\right|\right) dt) = \int_0^T tdt < \infty.$$

Thus $w(t)$ belongs to $\mathbb{L}_w^2[0,T]$. Also, we have seen that $w(t)$ satisfies all asumptions of theorem. Hence the existence of the Itô integral $\int_0^T w(t)dw(t)$, is justified.

$\int_0^T w(s)dw(s) = \lim\limits_{n\to\infty}\sum\limits_{k=0}^{n} w(t_k)\left[w\left(t_{k+1}\right) - w\left(t_k\right)\right]$

$= \lim\limits_{n\to\infty}\dfrac{1}{2}\left[\sum\limits_{k=0}^{n}\left[w^2(t_k) - w^2\left(t_{k+1}\right)\right] - \sum\limits_{k=0}^{n}\left[w(t_k) - w\left(t_{k+1}\right)\right]^2\right]$

$= \dfrac{1}{2}w^2\left(T\right) - \dfrac{1}{2}\lim\limits_{n\to\infty}\sum\limits_{k=0}^{n}\left[w(t_k) - w\left(t_{k+1}\right)\right]^2$

$= \dfrac{1}{2}w^2\left(T\right) - \dfrac{1}{2}T.$

The last convergence follows from the fact that the quadratic variation of Weiner process is T. Therfore we conclude that

$$\int_0^T w(s)dw(s) = \frac{1}{2}w^2\left(T\right) - \frac{1}{2}T.$$

In the case of a deterministic integral

$$\int_0^T w(s)dw(s) = \frac{1}{2}w^2(T),$$

whereas the Itô integral differs by the term $-\frac{1}{2}T$. This example shows that the rules of differentiation (in particular the chain rule) and integration need to be re-formulated in the stochastic calculus.

Lemma 1.5 [38] *If f is a function in $\mathbb{L}_w^2[\alpha,\beta]$, then, for any $\varepsilon > 0$, $N > 0$,*

$$\mathbb{P}\left\{\left[\left|\int_\alpha^\beta f(t)\,dw(t)\right| > \varepsilon\right]\right\} \leq \mathbb{P}\left\{\int_\alpha^\beta f^2(t)\,dt > N\right\} + \frac{N}{\varepsilon^2}.$$

The next Lemma improves Theorem 1.1.

Lemma 1.6 [38] *Let $f \in \mathbb{M}_w^2[\alpha,\beta]$. Then*

i) $\mathbb{E}\left[\int_\alpha^\beta f(t)\,dw(t) \mid \mathcal{F}_\alpha\right] = 0,$
ii) $\mathbb{E}\left\{\left|\int_\alpha^\beta f(t)\,dw(t)\right|^2 \mid \mathcal{F}_\alpha\right\} = \mathbb{E}\left\{\int_\alpha^\beta f^2(t)\,dt \mid \mathcal{F}_\alpha\right\}$
$= \int_\alpha^\beta \mathbb{E}\left[f^2(t) \mid \mathcal{F}_\alpha\right] dt.$

Properties of the Stochastic Integral The basic properties of the Itô integral are summarized in the following Theorem:

Theorem 1.3 [38] *The following properties hold for any $f, g \in \mathbb{L}_w^2[0,T]$, any $a, b \in \mathbb{R}$, and any $0 \leq s < T$:*

1) ***Linearity:***

$$\int_0^T (af(t) + bg(t))\,dw(t) = a\int_0^T f(t)\,dw(t) + b\int_0^T g(t)\,dw(t),$$

2) ***Isometry:***

$$\mathbb{E}\left(\left|\int_0^T f(t)\,dw(t)\right|^2\right) = \mathbb{E}\left(\int_0^T |f(t)|^2\,dt\right),$$

3) ***Martingale Property:***

$$\mathbb{E}\left[\int_0^t f(t)\,dw(t) \mid \mathcal{F}_{t'}\right] = \int_0^{t'} f(t)\,dw(t),\ 0 \leq t' < t < T,$$

in particular, $\mathbb{E}\left[\int_0^T f(t)\,dw(t)\right] = 0,$

4) If $0 < t_1 < t_2$*, then* $\int_0^{t_2} f(t)\,dw(t) = \int_0^{t_1} f(t)\,dw(t) + \int_{t_1}^{t_2} f(t)\,dw(t)$,

5) For $f \in \mathbb{L}_w^2[0,T]$*, define* $\xi(s) = \int_0^s f(t)\,dw(t) = \int_0^T f(t)\,\chi_{[0,T]}dw(t)$. *Then there exists a version of the process* ξ *such that* $s \to \xi(s)$ *is continuous almost surely.*

Proof. See Friedman [38] . ■

Lemma 1.7 [38] *Let* $f, g \in \mathbb{L}_w^2[\alpha,\beta]$ *and assume that* $f(t) = g(t)$ *for all* $\alpha \leq t \leq \beta, w \in \Omega_0$*. Then*

$$\int_\alpha^\beta f(t)\,dw(t) = \int_\alpha^\beta g(t)\,dw(t) \ \text{ for a.a. } w \in \Omega_0.$$

Definition 1.4 [38] *Let* $f \in \mathbb{L}_w^2[0,T]$ *and consider the integral*

$$I(t) = \int_0^T f(s)\,dw(s)\,,\ 0 \leq t \leq T.$$

By definition, $\int_0^0 f(s)\,dw(s) = 0$. We refer to $I(t)$ as indefinite integrale of f. Notice that $I(t)$ is $\mathcal{F}_t$ mesurable.

Theorem 1.4 [38]

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t f(s)\,dw(s)\right|^2\right] &\leq 4\mathbb{E}\left|\int_0^T f(t)\,dw(t)\right|^2 \\ &= 4\mathbb{E}\int_0^T \left|f^2(t)\right| dt.\end{aligned}$$

The fact that the indefinite integrals $\int_0^t f(s)\,dw(s)$ *is a continuous martingale.*

1.2 Stochastic Differential Equations

Following are some fundamental and necessary concepts in the theory of stochastic differential equations, which are needed later on in the proof of the existence and uniqueness theorem. Most of the definitions and results from this section are based on the books [80] and [38] .

After we have understood the stochastic integral $\int_0^t f(s)dw(s)$ we can study the following general stochastic differential equation (SDE):

$$x(t) = x_0 + \int_0^t b(s, x(s))ds + \int_0^t \sigma(s, x(s))dw(s), t \geq 0,$$

or equivalently, we write

$$\begin{aligned} dx(t) &= b(t, x(t))dt + \sigma(t, x(t))dw(t), t \geq 0, &(1.4)\\ x(t) &= x_0 \quad a.s. &(1.5) \end{aligned}$$

1.2.1 Existence and uniqueness

The existence and uniqueness theorem is so fundamental in science that it is sometimes called " *the principle of determinism. The idea is that if we know the initial conditions, then we can predict the future states of the equation"*.

If $\sigma = (\sigma_{ij})$ is a matrix, we write $|\sigma|^2 = \sum_{i,j} |\sigma_{ij}|^2$. Let $b(t,x) = (b_1(t,x), ..., b_n(t,x)), \sigma(t,x) = (\sigma_{ij}(t,x))_{i,j=1}^n$ and suppose the functions $\sigma_{ij}(t,x)$, $b_i(t,x)$ are mesurable in $(t,x) \in \mathbb{R}^n \times [0,T]$. If $(x(t), 0 \leq t \leq T)$ is a stochastic process such that

$$\begin{aligned} dx(t) &= b(t, x(t))dt + \sigma(t, x(t))dw(t), &(1.6)\\ x(t) &= x_0 \quad a.s., &(1.7) \end{aligned}$$

then we say that $x(t)$ satisfies the system of stochastic differential equations (1.6) and the initial condition (1.7). Note that it is implicity assumed that $b(t, x(t)) \in \mathbb{L}_w^1 [0,T]$ and $\sigma(t, x(t)) \in \mathbb{L}_w^2 [0,T]$.

Theorem 1.5 [80] *Suppose* $b(t,x), \sigma(t,x)$ *are mesurable in* $(t,x) \in [0,T] \times \mathbb{R}^n$ *and*

$$|b(t,x) - b(t,y)| \leq K|x-y|, \quad |\sigma(t,x) - \sigma(y,t)| \leq K|x-y|,$$
$$|b(t,x)| \leq K'(1+|x|), \quad |\sigma(t,x)| \leq K'(1+|x|),$$

where K, K' *are constants. Let* x_0 *be any* n*-dimentional random vector independent of* $\mathcal{F}(w(t), 0 \leq t \leq T)$*, such that* $\mathbb{E}|x_0|^2 < \infty$*. Then there exists a unique solution of (1.6), (1.7) in* $\mathbb{M}_w^2[0,T]$*.*

The proof is based on the Picard-Lindelof method of successive approximations. The detailed proof of the theorem can be found in [Friedman,1975] and [Mao, 1994].

- The assertion of uniqueness means that if $x_1(t), x_2(t)$ are two solutions of (1.6),(1.7) and if they belong to $\mathbb{M}_w^2[0,T]$, then

$$\mathbb{P}\{x_1(t) = x_2(t) \text{ for all } 0 \leq t \leq T\} = 1.$$

- If the coefficient functions b and σ are in the form of

$$\begin{aligned} b(t,x) &:= a(t) + e(t)x, \\ \sigma(t,x) &:= c(t) + d(t)x, \end{aligned}$$

where $a, e, c, d \in C(\mathbb{R}^+, \mathbb{R})$, then we say that equation (1.6) defines a linear SDE.

- If $a = c = 0$ for $0 \leq t \leq T$, then the linear SDE is called homogeneous.

Corollary 1.1 [80] *Under the assumptions of Theorem* 1.5*,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |x(t)|^2\right] \leq C^*\left(1 + \mathbb{E}|x_0|^2\right),$$

where C^* *is a constant depending only on* K', T*.*

Weak and strong solutions

There are two types of solutions and two types of uniqueness to stochastic differential equations referred as weak and strong. Weak uniqueness is also referred to as uniqueness in law. Strong uniqueness is referred as pathwise uniqueness. In this section we give precise definitions to each of the terms and show existence for the solution to Equation (1.6).

Definition 1.5 [65] *We will say that (1.6) admits a strong solution if whenever* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *is a stochastic basis on which an* n*-dimensional* $\mathcal{F}$ *- Brownian motion* w *is defined, we find an* $\mathcal{F}$*- adapted, continuous process* $x : \Omega \to \mathbb{R}$ *such that*

1) $\mathbb{P}(x(0) = x_0) = 1$;

2) $\mathbb{P}\left(\int_0^T |b(s, x(s))|\, ds + \int_0^T |\sigma(s, x(s))|^2\, ds < \infty\right) = 1$;

3) The integrated version of (1.6) holds true, i.e. almost surely

$$x(t) = x_0 + \int_0^t b(t, x(t))dt + \int_0^t \sigma(t, x(t))dw(t), \forall t \in [0, T].$$

Note that here we require that we can find such a solution on every stochastic basis carrying a Wiener process and that it is a solution with respect to that given Wiener process. For a weak solution, we make the stochastic basis a part of the solution thus, we cannot allow random initial data, as we are not given a priori a probability space. It is, however, possible to generalize this definition by prescribing a certain initial distribution. We will not go into details at this moment.

Definition 1.6 [65] *We say that there exists a weak solution to the equation (1.6) if there exist a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$*, and an* $(\mathcal{F}_t)$*- Wiener* w *and an* $(\mathcal{F}_t) -$ *adapted process* x *defined in it such that* x *solves equation*

(1.6), (p.s). We denote this solution by the triplet

$$\left(x, \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right), w\right).$$

We now define weak and strong uniqueness.

Definition 1.7 (**Strong Uniqueness,** [65]) *We say that strong uniqueness holds for equation (1.6) if whenever x and y are (strong) solutions defined on the same stochastic basis and with respect to the same Brownian motion with $x(0) = y(0)$, then*

$$\mathbb{P}(x(t) = y(t) : \forall t \in [0, T]) = 1.$$

Definition 1.8 (**Weak Uniqueness,** [65]) *We say that uniqueness in law (or weak uniqueness) holds for equation (1.6) if whenever $\left(x, \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right), w\right)$ and $\left(y, \left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \left(\widetilde{\mathcal{F}_t}\right)_{t\geq 0}, \widetilde{\mathbb{P}}\right), \widetilde{w}\right)$ are two weak solutions starting at the same $\eta \in \mathbb{R}$, the laws of x and y as $C([0,T])$ -valued random variables are the same. That is to say, $\forall B \in \mathcal{B}$, we have*

$$\mathbb{P}\left\{\omega \mid x_t(\omega) \in \mathcal{B}\right\} = \widetilde{\mathbb{P}}\left\{\omega \mid y_t(\omega) \in \mathcal{B}\right\}.$$

Note: The two notions of uniqueness are not equivalent but strong uniqueness implies weak uniqueness.

The solution of a stochastic differential system as a Markov process

We shall assume:

(A) The n-vector $b(t,x)$ and the $n \times n$ matrix $\sigma(t,x)$ are measurable functions for $(t,x) \in [0,\infty) \times \mathbb{R}^n$ and, for any $T > 0$,

$$|b(t,x)| \leq K(1+|x|), |b(t,x) - b(t,y)| \leq K|x-y|,$$

$$|\sigma(t,x)| \leq K(1+|x|), |\sigma(t,x) - \sigma(y,t)| \leq K|x-y|,$$

if $0 \leq t \leq T, x \in \mathbb{R}^n, y \in \mathbb{R}^n$ where K is a constant depending on T.

By Theorem 1.5 there exists a unique solution in $\mathbb{M}_w^2[0, \infty)$ of

$$\begin{aligned} d\xi(t) &= b(t, \xi(t))dt + \sigma(t, \xi(t))dw(t), && (1.8) \\ \xi(0) &= \xi_0 \quad a.s., && (1.9) \end{aligned}$$

provided ξ_0 is independent of $\mathcal{F}(w(\lambda), \lambda \geq 0)$ and $\mathbb{E}|\xi_0|^2 < \infty$. Similarly, for any $s \geq 0$ there exists a unique solution in $\mathbb{M}_w^2[s, \infty)$ of (1.8) and

$$\xi(s) = \xi_s,$$

provided ξ_s is independent of $\mathcal{F}(w(\lambda + s) - w(s), \lambda \geq 0)$ and $\mathbb{E}|\xi_s|^2 < \infty$.

If $\xi_s = x \ a.s.$, where x is a point in $\mathbb{R}^n$, then we denote the solution of (1.8), (1.9) by $\xi_{x,s}(t)$.

For any Borel set A in $\mathbb{R}^n$ and for any $t \geq s$, let

$$p(s, x, t, A) = \mathbb{P}\left(\xi_{x,s}(t) \in A\right).$$

Theorem 1.6 [38] *Let (A) hold and let ξ_0 be independent of $\mathcal{F}(w(t), t \geq 0)$, $\mathbb{E}|\xi_0|^2 < \infty$. Denote by F_t the $\sigma-$field spanned by ξ_0 and $w(s), 0 \leq s \leq t$. Then the unique solution $\xi(t)$ of (1.8), (1.9) satisfies*

$$\mathbb{P}(\xi(t) \in A \mid \mathcal{F}_s) = \mathbb{P}(\xi(t) \in A \mid \xi(s)) = \ p(s, \xi(s), t, A), \ a.s.$$

for all $t > s$ and for any Borel set A. Further, $p(s, x, t, A)$ is a transition probability function.

1.3 Itô's Formula

Example 1.1 illustrates that the basic definition of Itô integrals is not very useful when we try to evaluate a given integral. This is similar to the situation for ordinary Riemann integrals, where we do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in the explicit calculations. The Itô formula is, as we will show by examples, very useful for evaluating Itô integrals.

For a function $f(x, y)$ of the variables x and y it is not at all hard to justify that the equation below is correct to first order terms.

$$df = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy.$$

However, what if we have a function f which depends not only on a real variable t, but also on a stochastic process such as Brownian motion. Suppose that $f = f(t, w(t))$, where $w(t)$ denotes Brownian motion. One is tempted to write as before that

$$df = \frac{\delta f}{\delta t}dt + \frac{\delta f}{\delta w}dw.$$

However, in this case we would be badly mistaken. To see that this is so, we expand df using Taylor's formula; this time keep the terms involving the second derivatives of f

$$df = \frac{\delta f}{\delta t}dt + \frac{\delta f}{\delta w}dw + \frac{1}{2}\frac{\delta^2 f}{\delta^2 t}(dt)^2 + \frac{\delta^2 f}{\delta t \delta w}dtdw + \frac{1}{2}\frac{\delta^2 f}{\delta^2 w}(dw)^2 + \text{higher order terms}.$$

Let us define formally a multiplication table: $dw_i dw_j = 0$ $(i \neq j)$,$dw_i dw_i = dt$, $dtdt = 0$, $dtdw_i = 0$, and write

$$df = \left(\frac{\delta f}{\delta t} + \frac{1}{2}\frac{\delta^2 f}{\delta^2 w}\right)dt + \frac{\delta f}{\delta w}dw. \tag{1.10}$$

Equation (1.10) is called Itô's lemma, and gives us the correct expression for calculating differentials of composite functions which depend on Brownian processes.

Definition 1.9 [38] *Let* $(\xi(t), 0 \leq t \leq T)$ *be a process such that for any* $0 \leq t_1 < t_2 \leq T$

$$\xi(t_2) - \xi(t_1) = \int_{t_1}^{t_2} a(s)ds + \int_{t_1}^{t_2} b(s)dw(s),$$

where $a \in \mathbb{L}_w^1[0,T]$, $b \in \mathbb{L}_w^2[0,T]$. *Then we say that* $\xi(t)$ *has stochastic differential* $d\xi$, *on* $[0,T]$, *given by*

$$d\xi(t) = a(t)dt + b(t)dw(t).$$

Definition 1.10 [38] *Let* $d\xi(t) = a(t)dt + b(t)dw(t)$ *and let* $f(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ *be a continuous function with continuous derivatives* f_t, f_x, *and* f_{xx}. *Then, the process* $f(\xi(t),t)$ *has a stochastic differential, given by*

$$\begin{aligned} df(\xi(t),t) &= [f_t(\xi(t),t) + f_x(\xi(t),t)a(t) + \frac{1}{2}f_{xx}(\xi(t),t)b^2(t)]dt \\ &\quad + f_x(\xi(t),t)b(t)dw\,(t)\,. \end{aligned}$$

This is called Itô's formula. Notice that if $w\,(t)$ were continuously differentiable in t, then by the standard calculus formula for total derivatives the terme $\frac{1}{2}f_{xx}b^2dt$ would not appear.

Now, let us discuss some examples to understand solution strategy better.

Example 1.2 Determine an expression for $\int_0^t \sin(w)\,dw$, that does not involve Itô integrals.

Since Version (1.10) of Itô's formula tells us that

$$f\,(w(t)) - f\,(w(0)) = \int_0^t f'\,(w\,(s))\,dw\,(s) + \frac{1}{2}\int_0^t f''\,(w\,(s))\,ds,$$

if we choose $f'(x) = \sin(x)$ so that $f(x) = -\cos(x)$ and $f''(x) = \cos(x)$, then

$$-\cos(w\,(t)) + \cos(w\,(0)) = \int_0^t \sin(w\,(s))dw\,(s) + \frac{1}{2}\int_0^t \cos(w\,(s))ds.$$

The fact that $w\,(0) = 0$ implies

$$\int_0^t \sin(w\,(s))dw\,(s) = 1 - \cos(w\,(t)) - \frac{1}{2}\int_0^t \cos(w\,(s))ds.$$

Let us continuous to work on our first Example 1.2.

Example 1.3 Let $\xi(t) = w(t)$ and $f(\xi(t), t) = \xi^m(t)$. Then $d\xi = dw$ and thus $a = 0, b = 1$.

Hence Itô's formula gives

$$d(w^m) = mw^{m-1}dw + \frac{1}{2}m(m-1)w^{m-2}dt.$$

In particular the case $m = 2$ reads

$$d\left(w^2(t)\right) = 2w(t)\, dw(t) + 1dt.$$

This integrated is the identity

$$\int_s^r w(t)\, dw(t) = \frac{w^2(r) - w^2(s)}{2} - \frac{(r-s)}{2}.$$

Let us consider this integral with the assumption $s = 0, r = T$ and use $w(0) = 0$, we get

$$\int_0^T w(t)\, dw(t) = \frac{1}{2}w^2(T) - T,$$

as in Example 1.1.

We see in this exemple that Itô's formula is not only useful for evaluating the Itô integrals but, more importantly, solution of SDEs can be obtained by it. Let us consider an example and corresponding solution to clarify the solution technique.

Example 1.4 (**Geometric Brownian Motion**) *Assume that $S(t)$ denote the stock price at time $t \geq 0$ which changes randomly. The dynamics of the price of the stock is given as:*

$$dS(t) = \mu S(t)dt + \sigma S(t)dw(t), \tag{1.11}$$

where, the both μ and σ are positive constants we want to find the SDE for the process g related to S as follows: $g(t) = \phi(t, S) = \ln(S(t))$.

The partial derivatives are: $\frac{\delta\phi}{\delta t}(S,t)=0$, $\frac{\delta\phi}{\delta S}(S,t)=2S$, and $\frac{\delta^2\phi}{\delta S^2}(S,t)=0$. Therefore, according to Itô we get,

$$\begin{aligned} dg(t) &= \left(\frac{\delta\phi}{\delta t}(S,t)+\frac{\delta\phi}{\delta S}(S,t)\,\mu S(t)+\frac{1}{2}\frac{\delta^2\phi}{\delta S^2}(S,t)\,\sigma^2 S^2(t)\right)dt \\ &\quad +\frac{\delta\phi}{\delta S}(S,t)\,\sigma S(t)dw\,(t)\,. \end{aligned}$$

Hence

$$dg(t)=(\mu-\frac{1}{2}\sigma^2)dt+\sigma dw\,(t)\,. \tag{1.12}$$

Since the right hand side of (1.12) is independent of $g(t)$, we are able to compute the stochastic integral:

$$g(t)=g\,(0)+\int_0^t(\mu-\frac{1}{2}\sigma^2)dt+\int_0^t\sigma dw,$$

$$g(t)=g\,(0)+(\mu-\frac{1}{2}\sigma^2)t+\sigma w(t).$$

Since $g(t)=\ln(S(t))$ we have found a solution $S(t)$ for (1.12):

$$\ln(S(t))=\ln(S\,(0))+(\mu-\frac{1}{2}\sigma^2)t+\sigma w(t),$$

and so

$$S(t)=S\,(0)\exp\left[(\mu-\frac{1}{2}\sigma^2)t+\sigma w(t)\right],$$

where $(w(t),t\geq 0)$ is a standard Brownian motion.

Remark 1.1 Since S has a Markovian property, it is also possible to write the solution as follows:

$$S(t)=S\,(u)\exp\left[(\mu-\frac{1}{2}\sigma^2)\,(t-u)+\sigma\,(w(t)-w(u))\right],\ t\geq u.$$

CHAPTER 2

Retarded functional differential equations with applications

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This chapter provides background material necessary for the rest of the thesis. Some preliminaries and basic definitions are given for delay differential equations. Strictly speaking a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process. Like ordinary differential equations, delay differential equations have several features which make their analysis more complicated. The survey of the theory related to delay differential equations can be found e.g. in books, [9] , [10] , [14] , [16] , [35] , [37] , [39] , [42] , [48] , [49] , [56] − [62] , [93].

2.1 Basic concepts of delay differential equations

Motivation

The questions have been asked by many researchers "Why study this subject?" Why study differential equations with time delays when so much is known about equations without delays, and they are so much easier? The answer is because so many of the processes, both natural and manmade, in biology, medicine, chemistry, physics, engineering, economics, etc., involve time delays. Like it or not, time delays occur so often, in almost every situation, that to ignore them is to ignore reality. To clarify more, we give a biological system in which the present rate of change of some unknown function depends upon past values of the same function.

Real exemple of delay differential equation

To have a better understanding and reading of this section, we will focus on a simple real example. The goal is to help the reader to understand the most relevant aspects of delay differential equations. The following is an example presented in [9]. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t = \alpha\rho$. Then the dynamics of N can be described by the linear delay differential equation

$$\frac{d}{dt}N(t) = -\mu N(t) - rN(t-h), \tag{2.1}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N. Equation (2.1) describes the changes in N.

With deeper study and understanding of population dynamics, people started to consider introducing state-dependent delay into population models, as was pointed out in Arino et al. [9] :

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population.

Mathematical point of view:

To determine a solution past time t_0, we need to prescribe the value of $N(t_0-h)$. Suppose we have the initial value $N(t_0-h)$. Once we advance, say to $N(\varepsilon)$, with $t_0<\varepsilon<t_0+h$ small, notice that to calculate the derivative at $t=\varepsilon$ so that we can advance the next step, we need to know

$$\frac{d}{dt}N(\varepsilon)=-\mu N(\varepsilon)-rN(\varepsilon-h),$$

where $\varepsilon-h\in(t_0-h,t_0)$. In this manner, we realize that we need to know the values of $N(.)$ on the whole interval $[t_0-h,t_0]$. If we do not specify these values, we obtain an unsatisfactory notion of uniqueness, as the following example

$$x'(t)=-\frac{\pi}{2}x(t-1),x(0)=\frac{1}{\sqrt{2}}.$$

Here

$$x_1(t)=\sin\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right] \text{ and } x_2(t)=\cos\left[\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right],$$

are both solutions to the above equation at $t_0=0$. But if we specify the initial behavior on the interval $[-1,0]$, we obtain that only one solution exists to each delay differential equations, by the existence-uniqueness result in Theorem 3.1 that we give below.

Clearly, to begin with, an initial value problem requires more information than an analogous problem for a system without delays. For an ordinary differential system, a unique solution is determined by an initial point in Euclidean space at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - h, t_0]$. Each such initial function determines a unique solution to the delay differential equation. If we require that initial functions be continuous, then the space of solutions has the same dimensionality as $C([t_0 - h, t_0], \mathbb{R})$.

In the next section, with the previous discussion as a guide, let us now define the DDEs problem for a given initial function.

2.1.1 A general initial value problem

Suppose $\tau > 0$ is a given real number $\tau > 0$, denote $C([a, b], \mathbb{R}^n)$, the Banach space of continuous functions mapping the interval $[a, b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. We will denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ as $|x|$ from now on in order to avoid confusion with another norm we shall use. If $[a, b] = [-\tau, 0]$, we let C=$C\left([-\tau, 0], \mathbb{R}^n\right)$ and designate the norm of an element φ in C by

$$\|\varphi\|_\tau := \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|.$$

Let $\sigma \in \mathbb{R}, A > 0$ and $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$, be defined by

$$x_t = x(t + \theta) \text{ for } -\tau \leq \theta \leq 0.$$

Definition 2.1 [49] *If Ω is a subset of $\mathbb{R}\times C$, Let $f : \mathbb{R}\times C \to \mathbb{R}^n$ is a given function and represents the right-hand derivative, we say that the relation*

$$\begin{cases} x'(t) = f(t, x_t), t \geq \sigma, \\ \qquad and\ x_\sigma = \varphi, \end{cases} \tag{2.2}$$

is a retarded functional differential equation on and will denote this equation by DDEs.

Definition 2.2 [49] *A function x is said to be a solution of (2.2) if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, and x satisfies (2.2) for $t \in [\sigma, \sigma + A]$. In such a case we say that x is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ for a given $\sigma \in \mathbb{R}$ and a given $\varphi \in C$ we say that $x = x(\sigma, \varphi)$, is a solution of (2.2) with initial value at σ or simply a solution of (2.2) through (σ, φ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (2.2) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.*

Equation (2.2) is a very general type of equation and includes ordinary differential equations ($\tau = 0$). Although the structure of these equations is similar to ordinary differential equations, the crucial difference is that a delay differential equation (or a system of equations) is an infinite dimensional problem and the corresponding phase space is a functional space usually the space of continuous functions is considered.

Remark 2.1 The quantitie $\tau \geq 0$, is called the delay. The delay may be constant, function $\tau(t)$ of t (time-dependent delay), or function $\tau(t, x(t))$ (state-dependent delay).

Definition 2.3 *Equation (2.2) is called*:

i*) linear if $f(t, \varphi) = L(t, \varphi)$, where L is linear in φ.*

ii) nonhomogeneous if $f(t, \varphi) = L(t, \varphi) + h(t)$, where $h(t) \neq 0$, it is called *homogeneous if* $h = 0$.

iii) autonomous if $f(t, \varphi) = g(\varphi)$, *where* g *does not depend on* t.

Equation (2.2) is a very general type of equation and includes differential-difference equations. To be more explicit we give some classes of equations that can be expressed by (2.2), we have equations with a fixed delay (the simplest possible case) such as

$$x'(t) = f(t, x(t), x(t-\tau)),$$

or nonlinear nonautonomous differential equations with multiple time varying delays on the same state x

$$x'(t) = f(t, x(t), x(t-\tau_1(t)), ..., x(t-\tau_p(t))),$$

with $0 \le \tau_i(t) \le \tau$ for all $i = 1, ..., p$. We also have integrodifferential equations with a distributed delay

$$x'(t) = \int_{-\tau}^{0} g(t, x(t+\theta))\, d\theta,$$

where we see how in the integration process we need to know the values of x in $[t-\tau, t]$ for each t where the vector field is defined.

2.1.2 Existence and uniqueness theory

The existence and uniqueness theory for delay equations can be derived from the more general theory of functional differential equations. Since we intend to consider only equations of the form (2.2) we will not make use of the full generality available. Nevertheless, the more general theory leads to a presentation that is simpler and also benefits from an analogy with similar results in the theory of ordinary differential equations.

We now state the basic theory of DDEs.

Lemma 2.1 [49] *Let $\sigma \in \mathbb{R}$ and $\varphi \in C$ be given and f be continuous on the product $\mathbb{R} \times C$. Then, finding a solution of equation (2.2) through (σ, φ) is equivalent to solving the integral equation:*

$$x(t) = \varphi(\sigma) + \int_{\sigma}^{t} f(s, x_s)\, ds \text{ for } t \geq \sigma, \text{ and } x_\sigma = \varphi.$$

Lemma 2.2 [49] *If $x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n)$, then, x_t is a continuous function of t for $t \in [\sigma - \tau, \sigma + A]$.*

Proof. Since x is continuous on $[\sigma - \tau, \sigma + A]$, it is uniformly continuous and thus $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x(t) - x(s)| < \varepsilon$ if $|t - s| < \delta$. Consequently for t, s in $[\sigma, \sigma + A]$, $|t - s| < \delta$, we have $|x(t + \theta) - x(s + \theta)| < \varepsilon, \forall \theta \in [-\tau, 0]$. ■

The existence and uniqueness of the solutions of DDEs are given by the following Theorems.

Theorem 2.1 ***(Local existence,*** [49]*) Suppose $\overline{\Omega}$ is an open subset in $R \times C$ and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous. For any $(\sigma, \varphi) \in \overline{\Omega}$, there exists a solution of equation (2.2) through (σ, φ).*

Definition 2.4 **(** ***Lipschitzian,*** [49] **).** *We say $f(t, \varphi)$ is Lipschitz in φ in a compact set K of $\mathbb{R} \times C$ if there is a constant $k > 0$ such that, for any $(t, \varphi_i) \in K, i = 1, 2,$*

$$|f(t, \varphi_1) - f(t, \varphi_2)| < k\, |\varphi_1 - \varphi_2|\,.$$

Theorem 2.2 ***(Existence and uniqueness,*** [49] *) Suppose $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$, $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, and $f(t, \varphi)$ is Lipschitzian in φ in each compact set in $\overline{\Omega}$. If $(t_0, \varphi) \in \overline{\Omega}$, then there is a unique solution of Eq. (2.2) through (t_0, φ).*

Proposition 2.1 *(**Global existence,** [9]). If f is at most affine i.e. $f(t,\varphi) \leq a + b\,|\varphi|\,,$ with $a, b > 0$, then there exists a global solution of the equation (2.2) i.e. $\forall\varphi$, the solution $x(\sigma,\varphi)$ is defined on $[A,\infty[$.*

In the following we also require continuous dependence of solutions on initial conditions, for which the following theorem gives a result analogous to that for ordinary differential equations.

Theorem 2.3 *(**Continuous dependence,** [9]). Suppose x is a solution through (t_0,φ) of the equation (2.2) and that it is unique on $[t_0-\tau,\beta]$. If $\{(t_n,\varphi_n)\} \subset \mathbb{R}\times C$ is a sequence such that $(t_n,\varphi_n) \to (t_0,\varphi)$ as $n\to\infty$, then for all sufficiently large n every solution x_n through φ_n exists on $[t_n-\tau,\beta]$, and $x_n \to x$ uniformly on $[t_0-\tau,\beta]$.*

2.1.3 Neutral delay differential equations

Now are ready to give the definition of an other class of delay differential equations so-called the Neutral delay diffferential equations (NDDEs).

Definition 2.5 [49] *Suppose that $\mathbb{R}\times C$ is open with elements (t,φ). A function $D : \overline{\Omega} \to \mathbb{R}^n$ is said to be atomic at β on $\overline{\Omega}$ if D is continuous together with its first and second Fréchet derivatives with respect to φ; and D_φ, the derivative with respect to φ, is atomic at β on $\overline{\Omega}$.*

Definition 2.6 [49] *Suppose that $\overline{\Omega} \subseteq \mathbb{R}\times C$ is open, $f : \overline{\Omega} \to \mathbb{R}^n$, $D : \overline{\Omega} \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. The equation*

$$\frac{dD}{dt}(t,x_t) = f\,(t,x_t)\,, \tag{2.3}$$

is called the neutral delay differential equation NDDE (D,f).

If the delayed argument occurs in the highest order derivative of the state we call it neutral functional differential equation.

The following equations are some examples of neutral differential equations

Example 2.1 [49] *If $\tau > 0$, B is an $n \times n$ constant matrix, $D(\varphi) = \varphi(0) - B\varphi(-\tau)$, and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDDE,*

$$\frac{d}{dt}[x(t) - Bx(t-\tau)] = f(t, x_t).$$

Example 2.2 [49] *If $\tau > 0$, x is a scalar, $D(\varphi) = \varphi(0) - \sin(-\tau)$, and $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous, then the pair (D, f) defines an NDDE,*

$$\frac{d}{dt}[x(t) - \sin x(t-\tau)] = f(t, x_t). \tag{2.4}$$

Remark 2.2 *Note that when x is continuous differentiable, (2.4) is equivalent to*

$$x'(t) - (\cos x(t-\tau))\, x'(t-\tau) = f(t, x_t).$$

Definition 2.7 [49] *A function x is said to be a solution of (2.3) on $[\sigma - \tau, \sigma + A]$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that*

$$x \in C([\sigma - \tau, \sigma + A], \mathbb{R}^n), \ \ (t, x_t) \in \overline{\Omega}, \ t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (2.3) on $[\sigma, \sigma + A]$. For a given $t_0 \in \mathbb{R}, \varphi \in C$, and $(\sigma, \varphi) \in \overline{\Omega}$, we say $x(t, \sigma, \varphi)$ is a solution of equation (2.3) with initial value φ at σ or simply a solution through (σ, φ) if there is an $A > 0$ such that $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$; we say $x(t, \sigma, \varphi)$ is a solution of (2.3) on $[\sigma - \tau, \infty)$, if for every $A > 0$, $x(t, \sigma, \varphi)$ is a solution of equation (2.3) on $[\sigma - \tau, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.

Theorem 2.4 ***(Existence,*** [49]***)*** *If $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$ and $(t_0, \varphi) \in \overline{\Omega}$, then there exists a solution of the NDDE (L, f) through (t_0, φ).*

Theorem 2.5 ***(Existence and Uniqueness,***[49]***)*** *If $\overline{\Omega}$ is an open set in $\mathbb{R} \times C$ and $f : \overline{\Omega} \to \mathbb{R}^n$ as Lipschitz in on compact sets of $\overline{\Omega}$, then, for any $(t_0, \varphi) \in \overline{\Omega}$, there exists a unique solution of the NDDE (D, f) through (σ, φ).*

2.1.4 Method of steps

It is known that the exact solution of delay differential equations can be found just in some special cases. There is no unified approach to solve the delayed differential equations, even in the linear case. The theory of ordinary differential equations gives various methods to obtain analytical solution (e.g. the variation of constants method, the separation of variables method and others). But these methods are inapplicable dealing with delay differential equations. Hence qualitative and numerical analysis of these equations gather great importance. The method of steps was first proposed by Bellman and Cooke [14]. This approach, furnishes a method of finding explicit solutions. The desired solution is found on successive intervals by solving ordinary differential equations without delays in each interval. As an illustration to this approach, consider the DDE:

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau)), t \geq t_0 \\ x(t) = \varphi_0(t), \ t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.5}$$

For such equations the solution is constructed step by step as follows:

Given that a function $\varphi_0(t)$ continuous on $[t_0 - \tau, t_0]$, therefore one can obtain the solution in the next step interval $[t_0, t_0 + \tau]$ by solving the following ordinary differential equation:

$$x'(t) = f(t, x(t), \varphi_0(t-\tau)) = g_0(t, x(t), \ \text{for } t_0 \leq t \leq t_0 + \tau.$$

Under suitable hypotheses on g_0, existence and uniqueness of a solution of this equation (hence a solution of (2.5)) on $[t_0 - \tau, t_0]$ can be established. Denoting this solution by $\varphi_1(t)$ and restricting equation (2.5) to the interval $[t_0 + \tau, t_0 + 2\tau]$, we find the ordinary differential equations

$$x'(t) = f(t, x(t), \varphi_1(t - \tau)) = g_1(t, x(t) \text{ for } t_0 + \tau \leq t \leq t_0 + 2\tau,$$

with the initial condition $x(t_0+\tau) = \varphi_1(t_0+\tau)$, for which we can again establish existence and uniqueness of a solution φ_2. Thus we have now extended the solution x to the interval $[t_0 + \tau, t_0 + 2\tau]$, and we now have a formula for $x(t)$ when $t \in [t_0 - \tau, t_0 + 2\tau]$.

In general, by assuming that $\varphi_{k-1}(t), \forall(k = 1, 2, ...)$ is defined on the interval $[t_0 + (k-2)\tau, t_0 + (k-1)\tau]$, then, one can find the solution $\varphi_k(t)$ to the equation:

$$x'(t) = f(t, x(t), \varphi_{k-1}(t - \tau)), \text{ for } t_0 + (k-1)\tau \leq t \leq t_0 + k\tau,$$

with the initial condition $x(t_0 + (k-1)\tau) = \varphi_{k-1}(t_0 + (k-1)\tau)$. We can continue this process indefinitely, showing that the uniquely defined $x(t)$ exists on $[t_0 - \tau, \infty)$.

Remark 2.3 *The method of steps can be extended to differential equations with other types of delays, such as multiple delays, variable delay and even state dependent delay or for neutral systems. The difficulty is to locate the primary discontinuities.*

Definition 2.8 [37] *If the solution of a DDE and its derivatives of order μ are continuous at some point in the time interval, but the derivative of order $\mu + 1$ is not, then such a point is called a primary discontinuity of the given problem.*

Theorem 2.6 [37] *The points $\xi_\mu := \mu\tau$ the primary discontinuities of problem (2.5). More precisely, $x^{(\mu)}$ is continuous at ξ_μ but $x^{(\mu+1)}$ is, in general, not, even if the functions φ and f have continuous derivatives of all orders.*

Proof. See [37]. Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t = 0$, the first derivative $x'(t)$ has a primary discontinuity, since the integrable equation

$$x'(t) = f(t, x(t), \varphi(t-\tau)), t \in [0, \tau],$$

may satisfy the condition $x(0) = \varphi(0)$, but it is unlikely to satisfy the additional condition $x'(0^+) = \varphi'(0^-)$. Only for special choices of the initial function $\varphi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\varphi'(0^-) = f(0, \varphi(0), \varphi(-\tau))$. ■

Example 2.3 *We illustrale this method by using the special cases of equation (2.5), the following is an example presented in* [106]. *Let*

$$\begin{aligned} x'(t) &= ax(t-\tau), t \in [0, +\infty), \\ x(t) &= 1, t \in [-\tau, 0], \end{aligned}$$

where a is positive constant. Using the method of steps, it is easy to see that the solution $x(t)$ is a piecewise polynomial. On each subinterval $[i\tau, (i+1)\tau]$, $x(t)$ is an $(i+1)$-th. order polynomial, i.e.,

$$x(t) = \sum_{j=1}^{i+1} \frac{a^j}{j!}(t-(j-1)\tau)^j, i \in \mathbb{N}.$$

It is also clear that integer multiples of τ are primary discontinuities for this particular problem.

As a generalization of (2.5), we consider

$$x'(t) = f(t, x(t), x(t - \tau(t)), x'(t - \tau(t))), t \in [0, \overline{a}],$$

where $t- \tau(t)$ is a strictly increasing function and

$$0 < \tau(t) \leq t,\ \overline{a} = \inf_{t \geq 0} \{t - \tau(t)\}.$$

Remark 2.4 *The method of steps can be extended to delay differential equations with additive noise term (see chapitre 3).*

2.2 Stability of delay differential equations

In 1982, Lyapunov introduced the concept of stability of a dynamic system. To make the stochastic stability theory more understandable (in chapitre 3 and 4) let us recall a few basic facts on the theory of stability of deterministic systems described by delay differential equations. For a greater amount of results, we refer [9],[20] , [37] , [39] , [42] , [43] , [56] $-$ [59] as main sources.

Motivation

A fundamental problem in the theory of differential equations is to study the motion of the system using the vector field that induces the differential equations. Qualitative analysis involves questions of the type: Do the solutions go to infinity, or do they remain bounded within a certain region? What conditions must a vector field satisfy in order for the solutions to remain within a given region? Do nearby solutions act similarly to a particular solution of interest?

These are questions of qualitative type, in contrast with analytic methods which tend to search for a formula to express each solution of a differential equation. As our vector fields get more complicated, such as when one goes from ordinary differential equations to functional differential equations, analytic methods go out the window, since solving the equations becomes even more impossible. Thus qualitative methods take the leading role.

This section presents generalization of the Lyapunov method to DDEs. The stability notions are defined, and Lyapunov-Krasvoskii and Lyapunov-Razumikhin stability theorems are stated. First, we will recall sufficient stability results for our purposes, though there are many small variations of similar results for each different type of stability behavior we define, such as for stability, asymptotic stability,..., etc. We will concentrate on asymptotic stability results, since that is the kind of stability that the results developed ahead will treat.

Definition 2.9 *A point $x(t) = x_e$ in the state space is said to be an equilibrium point of the autonomous system $x' = f(x)$ if and only if it has the property that whenever the state of the system starts at x_e, it remains at x_e for all future time.*

Let each $t_0 \in J$, $D \subset \mathbb{R}^n$, $\varphi \in C([t_0 - \tau, t_0], D)$ induce an initial value problem

$$\begin{aligned} x'(t) &= f(t, x_t) \text{ for } t \geq t_0 \\ x_{t_0} &= \varphi. \end{aligned} \tag{2.6}$$

We have $f : J \times C([t_0 - \tau, t_0], D) \to \mathbb{R}^n$ with $J \subset \mathbb{R}$ an infinite interval of the form $[a, \infty)$, $a \geq 0$, we can assume $J = \mathbb{R}^+ = [0, \infty)$ for simplicity. We suppose that f is continuous and is supposed to satisfy all the conditions which guarantee a solution.

For stability analysis, we assume that $0 \in D$, which implies that $0 \in C([t_0 - \tau, t_0], D)$ and that $f(t, 0) = 0$ for all $t \in J$. Thus 0 is an equilibrium solution.

Remark 2.5 *Of course, just as we did for ODEs, we can study the translation of a nonzero equilibrium solution $t \to z(t)$ of a FDE $y^{'}(t) = g(t, y_t)$ by defining the change of variable $x(t) = y(t) - z(t)$ and obtaining a new vector field $f(t, x_t)$ with a zero equilibrium. Thus, studying the stability of the trivial solution $z(t) = 0$ is sufficient.*

In the following, we state the definitions of the stability notions of DDEs.

Definition 2.10 [49] *The zero solution of (2.6) is said to be*

1. **Stable,** *if for every $\varepsilon > 0$ and $t_0 \in J$, there exists a $\delta(t_0, \varepsilon) > 0$ such that if*

$$[\varphi \in C([t_0 - \tau, t_0], D) \text{ with } \|\varphi\|_\tau < \delta \text{ and } t \geq t_0] \Rightarrow |x(t, t_0, \varphi)| < \varepsilon,$$

2. **Unstable,** *if it is not stable,*

3. **Uniformly Stable,** *if δ of (1) is independent of t_0,*

4. **Asymptotically stable,** if *it is stable and if $\forall t_1 \geq t_0$, $\exists \eta > 0$ such that if*

$$[\varphi \in C([t_1 - \tau, t_1], D) \textit{ with } \|\varphi\|_\tau < \eta \textit{ and } t \geq t_1] \Rightarrow |x(t, t_1, \varphi)| \to 0 \textit{ as } t \to \infty,$$

5. **Uniformly asymptotically stable,** *if it is uniformly stable and if there exist $\eta > 0$ and for $\gamma > 0, \exists T > 0$ such that*

$$[t_1 \geq t_0, \varphi \in C([t_1 - \tau, t_1], D) \text{ with } \|\varphi\|_\tau < \eta \text{ and } t \geq t_1 + T] \Rightarrow |x(t, t_1, \varphi)| < \gamma,$$

6. **Exponentially stable,** *if there exist positive constants δ, k and λ such that*

$$|x(t, t_0, \varphi)| \leq k \|\varphi\|_\tau e^{-\lambda(t-t_0)}, \textit{ whenever } \|\varphi\|_\tau < \delta; \textit{ and } t \geq t_0;$$

7. **Globally exponentially stable,** *if (6) holds with an arbitrary large constant δ.*

2.2.1 Lyapunov's direct method

Just as in the case of ODEs, it is possible to derive sufficient conditions for the stability of DDEs of the form of (2.6) through the use of auxiliary functions. It is possible to follow the approach used in the ODE system case directly, however, the infinite dimensional character of the functional differential equation system (2.6) makes the issue here more complicated. The evolution of the system from time t is now dependent not on t and $x(t)$, as is the case with ODEs, but on t and the function x_t. Consequently, it should be expected that the Lyapunov function will now in fact be a functional, of the form $V(t, x_t)$. It is this functional character that makes this approach more difficult.

For DDEs, there are two main direct Lyapunov methods for stability analysis: Krasovskii (1956) method of Lyapunov functionals [56, 57] and Razumikhin (1956) method of Lyapunov functions [96] . we will recall them later.

Lyapunov–Krasovskii approach

The use of Lyapunov Krasovskii-Lyapunov functionals is a natural generalizations of the direct Lyapunov method of ODEs (1980), since a proper state for DDEs is x_t. It consists in selecting functionals, i.e. (functions of the functional state x_t) of the form $V(t, x_t)$, that are positive definite and decreasing along the trajectories of system (2.6). The Lyapunov -Krasovskii theorem is stated below.

Let $V : \mathbb{R} \times C\left[-\tau, 0\right] \to \mathbb{R}$ be a continuous functional, and let $x\left(t, \sigma, \varphi\right)$ be the solution of (2.6) at time $\varrho \geq t$ with the initial condition $x_t = \varphi$. We define the right upper derivative $\dot{V}(t,\varphi)$ along (2.6) as follows:

$$\dot{V}(t,\varphi) = \lim_{\Delta t \to 0^+} \sup \frac{1}{\Delta t}\left[V\left(t+\Delta t, x_{t+\Delta t}(t,\varphi)\right) - V(t,\varphi)\right].$$

Intuitively, a non-positive $\dot{V}(t,\varphi)$ indicates that x_t does not grow with t, meaning that the system under consideration is stable.

Theorem 2.7 *(**Lyapunov–Krasovskii Theorem**, [57]). Suppose $f : \mathbb{R} \times C[-\tau, 0] \to \mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in $C[-\tau,0]$) into bounded sets of $\mathbb{R}^n$ and that $u, v, \overline{w} : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous non decreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. The trivial solution of (2.6) is uniformly stable if there exists a continuous functional $V : \mathbb{R} \times C[-\tau, 0] \to \mathbb{R}^+$, which is positive-definite, i.e.*

$$u(|\varphi(0)|) \leq V(t,\varphi) \leq v(\|\varphi\|_c),$$

such that its derivative along (2.6) is non-positive in the sense that

$$\dot{V}(t,\varphi) \leq -\overline{w}(|\varphi(0)|).$$

If $\overline{w}(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. If in addition $\lim\limits_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

The main idea behind the statement of this theorem is to determine a positive definite functional V, such that its derivative with respect to time along the trajectories of the system (2.6) is negative definite. The main problem within the application of this theorem is the design functional and then to provide some conditions that guarantee its positive definiteness and the negative definiteness of its derivative.

Lyapunov-Razumikhin approach

In this approach, the goal is to consider a classical Lyapunov function $V(t, x(t))$, as the one employed for the delay free case (i.e. for ordinary differential equations). The main idea of the Lyapunov-Krasovskii theorem is that it is not necessary to ensure the negative definiteness of $\dot{V}(t, x(t))$ along all the trajectories of the system. Indeed, it is sufficient to ensure its negative definiteness only for the solutions that tend to escape the neighborhood of $V(t, x(t)) \leq c$ of the equilibrium. The difficulties associated with the utility of functionals can, however, often be avoided by means of an approach due to another Russian mathematician, Boris Sergeevich Razumikhin. In his (1956) work [96], Razumikhin introduced the idea of considering a functional of the form

$$\overline{V}(t, x_t) = \sup_{-\tau \leq \theta \leq 0} V(t + \theta, x(t + \theta)),$$

and then established conditions on the function $V(t, x)$ that would ensure that the functional $V(t, x_t)$ would be decreasing along the system trajectories. In this way, we study the properties of the functional indirectly, focusing directly on the function $V(t, x)$ which is generally significantly easier to study. To give a precise formulation of Razumikhin method, we consider a differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ and define the derivative of V along the solution $x(t)$ of (2.6) as

$$\dot{V}\left(t, x\left(t\right)\right) = \frac{d}{dt} V\left(t, x\left(t\right)\right) = \frac{\partial V\left(t, x\left(t\right)\right)}{\partial t} + \frac{\partial V\left(t, x\left(t\right)\right)}{\partial x} f\left(t, x_t\right).$$

This idea is formalized in the following theorem

Theorem 2.8 *(**Lyapunov-Razumikhin Theorem**, [96]). Let $x = 0$ be a solution of (2.6). Suppose that $f : \mathbb{R} \times C\left([-\tau, 0]\right) \to \mathbb{R}^n$ in (2.6) takes $\mathbb{R}\times$ (bounded sets in $C\left([-\tau, 0]\right)$) into bounded sets of $\mathbb{R}^n$, and that $u, v, \overline{w}$*

$: \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous non-decreasing functions, with* $u(s), v(s) > 0$ *for all* $s > 0$, $u(0) = v(0) = 0$, v *is strictly increasing. Suppose further that there exists a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$, *which is positive-definite, i.e.*
i) $u(|x|) \le V(t,x) \le v(|x|), \forall t \in \mathbb{R}^+, \forall x \in \mathbb{R}^n$, *such that the derivative of* V *along the solution* $x(t)$ *of (2.6) satisfies,*
ii) $\dot{V}(t, x(t)) \le -\overline{w}(|x(t)|)$ *if* $V(t+\theta, x(t+\theta)) < V(t, x(t)), \forall \theta \in [-\tau, 0]$,
where $x(t)$ *is any trajectory of (2.6). Then the solution* $x = 0$ *is uniformly stable.*

Proof. For arbitrary fixed $\epsilon > 0$, let us fix δ in the range $0 < \delta < v^{-1}(u(\epsilon))$, which may be done by virtue of the assumed properties of the functions u and v. Now take any function $\varphi \in C$ such that $\|\varphi\| < \delta$. Then, according to property (i), we will have, for any t_0,

$$V(t_0 + \theta, \varphi(\theta)) < v(\varphi(\theta)) \le v(\delta),$$

for all $\theta \in [-\tau, 0]$. Let us now let x be the solution of (2.6) with initial data $x_{t_0} = \varphi$. What this then tells us is that $V(t_0 + \theta, x(t_0 + \theta)) \le v(\delta)$ for all $\theta \in [-\tau, 0]$.
Suppose that r is the earliest time after t_0 such that $V(r, x(r)) = \nu(\delta)$. As a consequence of the above, we now know that we must have $V(r + \theta, x(r + \theta)) < v(\delta)$ for all $\theta \in [-\tau, 0]$. But then the condition within property (ii) of the theorem must be satisfied, which means that we have $\dot{V}(r, x(r)) \le 0$. Therefore, the continuity of $V(., x(.))$ implies that

$$V(t, x(t)) \le v(\delta) < u(\epsilon),$$

for all $t \ge t_0 - \tau$. But condition (i) tells us that if $|x(t| \ge \epsilon$, then $V(t, x(t)) \ge u(\epsilon)$, and so we would have a contradiction. Consequently, we conclude that, in fact, $|x(t| < \epsilon$ for all $t \ge t_0 - \tau$.

Therefore, this shows that, given any $\epsilon > 0$, we can choose a $\delta > 0$ independently of t_0 such that whenever $\|x_{t_0}\| < \delta$, we must have $|x(t)| < \epsilon$ for all $t \geq t_0 - \tau$. This is precisely the definition of uniform stability, and so the theorem is proved. ■

Theorem 2.9 *(**Razumikhin Theorem for Uniform Asymptotic Stability**, [96]) Suppose that all the assumptions of Theorem 2.8 are satisfied and also $\overline{w}(s) > 0$ for $s > 0$. Suppose that, in addition, there exists a continuous non-decreasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $q(s) > s$ for all $s > 0$ such that (ii) can be strengthened to*

$$\dot{V}(t, x(t)) \leq -\overline{w}(|x(t)|) \text{ if } V(t+\theta, x(t+\theta)) < q(V(t, x(t))),$$

for $\theta \in [-\tau, 0]$ where $x(t)$ is any trajectory of (2.6). Then the solution $x = 0$ is uniformly asymptotically stable. If in addition $u(s) \to \infty$ as $s \to \infty$, then the solution $x = 0$ is globally uniformly asymptotically stable.

Lyapunov was interested in stability of mechanical systems, and Lyapunov functions generalize the total energy function in mechanical or electrical systems. That is why it is very often that in applications in these areas, a good candidate for a Lyapunov function tends to be the sum of the kinetic plus potential energy, or the Hamiltonian. When these energy functions fail to act as Lyapunov functions (or are not convenient enough, they might give us stability, but not asymptotic stability, for instance), a certain amount of ingenuity and experience is required to find a suitable function.

The following example uses the previous result. It is taken from Y. Kuang [60] .

Example 2.4 [60] *Consider the linear delay differential equation*

$$x'(t) = -a(t)x(t) - b(t)x(t - \tau(t)), \tag{2.7}$$

where $a, b \in C(\mathbb{R}^+, \mathbb{R})$ *and* $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ *are bounded continuous functions,* $a(t) > 0$, $\tau(t) > 0$, $\tau'(t) < 1$.

If $b(t) = 0$, then (2.7) becomes an ordinary differential equation; a trivial Lyapunov function is $V_1(x(t)) = x^2(t)\backslash 2$, or equivalently, if viewed as a functional, then $V_1(\phi) = \phi^2(0)\backslash 2$. Its derivative a long the solution of (2.7) is

$$\begin{aligned}\dot{V}_1(x(t)) &= x(t)\left[-a(t)x(t) - b(t)x(t-\tau(t))\right] \\ &= -a(t)x^2(t) - b(t)x(t)x(t-\tau(t)).\end{aligned}$$

We cannot determine the sign of $\dot{V}_1$, since we do not know the ratio of $x(t)\backslash x(t-\tau(t))$. In order to find a Lyapunov functional V, we want to generate a term like $-x^2(t-\tau(t))$ in the $\dot{V}$ for the equation (2.7). Let us apply the Theorem of Krasovskii to this example with

$$V(\phi) = \frac{1}{2}\phi^2(0) + \alpha\int_{-\tau(t)}^{0}\phi^2(s)\,ds,$$

where $\alpha > 0$ is a constant, or equivalently,

$$V(x_t) = V(t, x_t) = \frac{1}{2}x^2(t) + \alpha\int_{-\tau(t)}^{0}x^2(t+\theta)\,d\theta.$$

We have

$$\begin{aligned}\dot{V}(x_t) &= (-a(t) - \alpha)x^2(t) - b(t)x(t)x(t-\tau(t)) \\ &\quad -\alpha\left(1-\tau'(t)\right)x(t-\tau(t)),\end{aligned}$$

since

$$\int_{-\tau(t)}^{0}x^2(t+\theta)\,d\theta = \int_{t-\tau(t)}^{t}x^2(\theta)\,d\theta \text{ and } \frac{d}{dt}\int_{-\tau}^{0}x(t+\theta)\,d\theta = x(t) - x(t-\tau).$$

Clearly, if

$$b^2(t) < 4(a(t)-\alpha)\left(1-\tau'(t)\right)\alpha, \tag{2.8}$$

then $\dot{V}(x_t) < 0$. Let $\tau(t) < \tau$, where τ is a positive constant; $u(s) = s^2 \backslash 2$, $\upsilon(s) = ((1\backslash 2) + \alpha\tau)s^2$. Then,

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq \upsilon(\|\phi\|).$$

If $\alpha > 0$ satisfies (2.8), then there may be a positive constant $\epsilon > 0$ such that

$$\dot{V}(x_t) \leq -\epsilon x^2(t).$$

Thus, we may take $\overline{w}(s) = -\epsilon s^2$. By Theorem 2.7, we know $x = 0$ is uniformly asymptotically stable. Indeed, since (2.7) is linear, we see that all solutions of (2.7) tend to $x = 0$ if (2.8) is true for some positive constant α.

When a, b and τ are constants, (2.8) reduces to

$$b^2 < 4(a - \alpha) \leq a^2,$$

which implies that if $|b| < a$, then $x = 0$ is globally asymptotically stable; i.e., $\lim_{t \to +\infty} x_t(\phi) = 0$ for $\phi \in C$. Note that the length of delay τ is not restricted.

Now, we will study very standard delay differential equation, it was proposed by Burton (see [22]). In his paper, the author compared results from a certain application of fixed point theory with a certain common Lyapunov functional. Burton proved that under one set of assumptions we obtain bettre results using contraction mapping, while better results are obtained using Lyapunov theory under a different set of assumptions.

Example 2.5 [22] *Consider the linear delay differential equation*

$$x'(t) = b(t)x(t - \tau), \tag{2.9}$$

the equation (2.9) *can be written as the form*

$$x'(t) = -b(t + \tau)x(t) + \frac{d}{dt}\int_{t-\tau}^{t} b(s + \tau)x(s)ds, \tag{2.10}$$

where $\tau > 0$ *is a constant,* $b : [0, \infty) \to \mathbb{R}$ *is a bounded and continuous function.*

Although we can treat solutions with any initial time, we will always look at a solution $x(t) := x(t, 0, \varphi)$ where $\varphi : [-\tau, 0] \to \mathbb{R}$ is a given continuous initial function and $x(t, 0, \varphi) = \varphi(t)$ on $[-\tau, 0]$. Burton [22] obtained the following results as follows:

Theorem 2.10 [22] *If there is $\delta > 0$ with*

$$b(t+\tau) \geq \delta, \text{ for all } t \geq 0, \tag{2.11}$$

an $\epsilon > 0$ with

$$b(t+\tau)\int_{t-\tau}^{t} b(s+\tau)ds - 2 + \tau \leq -\epsilon, \text{ for all } t \geq 0, \tag{2.12}$$

and if there is a $\gamma > 0$ with

$$\gamma[b(t) + b(t+\tau)] \leq (\epsilon\backslash 2)b(t+\tau), \text{ for all } t \geq 0, \tag{2.13}$$

then the zero solution of (2.9) is uniformly asymptotically stable.

Proof. Rewrite (2.9) in the following equivalent form

$$\left(x(t)) - \int_{t-\tau}^{t} b(s+\tau)x(s)ds\right)' = -b(t+\tau)x(t).$$

By constructing the following the Lyapunov functional $V(t, x_t) = V_1(t, x_t) + V_2(t, x_t)$, we select a first Lyapunov functional as

$$V_1(t, x_t) = \left(x(t) - \int_{t-\tau}^{t} b(s+\tau)x(s)ds\right)^2 + \int_{-\tau}^{0}\int_{t+s}^{t} b(u+\tau)x^2(u)duds,$$

so the derivative along a solution of (2.9) satisfies $\dot{V}_1(t, x_t) \leq -\epsilon\delta x^2$. Now, we need to select a second Lyapunov functional and add them together to make a positive definite Lyapunov functional. Define

$$V_2(t, x_t) = \gamma\left[x^2(t) + \int_{t-\tau}^{t} b(s+\tau)x^2(s)ds\right],$$

so the derivative along a solution of (2.9) satisfies $\dot{V}_2(t, x_t) \leq (\epsilon\backslash 2)\, b(t+\tau)x^2$. Then we have $\dot{V}(t, x_t) \leq -(\epsilon\backslash 2)\, \delta x^2$. We can now find wedges with

$$u(|x(t)|) \leq V(t, x_t) \leq \upsilon(\|x_t\|),$$

and, since (2.11) and (2.12) imply that $x(t)$ is bounded, conclude that the zero solution is uniformly asymptotically stable. ■

Example 2.6 [22] *Let* $b(t) = 1.1 + \sin t$ *in (2.9), we have*

$$x^{'}(t) = -(1.1 + \sin t)\, x(t-\tau). \tag{2.14}$$

Theorem 2.10 holds if there is an $\varepsilon > 0$ *such that*

$$2.1\,(1.1\tau + 2\sin(\tau\backslash 2)) - 2 + \tau < -\varepsilon. \tag{2.15}$$

Using a rough estimate (taking $\sin(\tau\backslash 2) = \tau\backslash 2$ *), we have that* $\tau < 0.37$*. Therefore, the zero solution of (2.9) is asymptotically stable if* $\tau < 0.37$*.*

2.2.2 Fixed point methods

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the “Banach Contraction Principle” (BCP) which is one of the most important results of analysis and considered as the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with contractive condition on the map which is easy to test in this setting. The BCP has been generalized in many different directions. In fact, there is vast amount of literature dealing with extensions / generalizations of this remarkable theorem.

Definition 2.11 Let $\mathcal{H}$ *be a mapping in the set* M*. We call fixed point of* H *any point* x *satisfying* $H(x) = x$*. If there exists such* x*, we say that* $\mathcal{H}$ *has a fixed point, which is equivalent to saying that the equation* $\mathcal{H}(x) - x = 0$ *has a null solution.*

Definition 2.12 *Let* (X, d) *be a complete metric space. The operator* $\mathcal{H} : X \to X$ *is called the contraction operator, if there exists a constant* $0 < \alpha < 1$ *such that*

$$\forall x, y \in X, d(\mathcal{H}(x), \mathcal{H}(y)) \leq \alpha d(x, y).$$

To prove our main result we will use a classical contraction mapping principle. We recall it below for the readers convenience.

Theorem 2.11 [92] ***(Banach's fixed point theorem)*** *(see* ***Smart****,* 1974*)* *Let* $\mathcal{H}$ *be a contraction operator on a complete metric space* X*, then there exists a unique point* $x^* \in X$ *such that* $\mathcal{H}(x^*) = x^*$*.*

Hence, to solve a problem using a fixed point approach we have to identify (see Chen [30]):

(a) a set S consisting of points which would be acceptable solutions;

(b) a mapping $P : S \to S$ with the property that a fixed point solves the problem;

(c) a fixed point theorem stating that this mapping on the set S will have a fixed point.

The following steps represent the way in which we can establish stability of the zero solution of a delay differential equation by applying fixed point theory.

Step 1. An examination of the differential equation reveals that for a given initial time σ there is an initial interval we denote it to be E_σ and we require an initial function $\phi : E_\sigma \to \mathbb{R}^n$. We then must determine a set S of functions $\varphi : E_\sigma \cup [\sigma, \infty) \to \mathbb{R}^n$ with $\varphi(t) = \phi(t)$ on E_σ which could serve as acceptable functions. Usually, this means that we would ask some other conditions on φ, for example, the boundedness, and sometimes we require that $\varphi(t) \to 0$ as $t \to \infty$.

Step 2. Next, invert the differential equation and define a mapping P from S to S.

Step 3. Finally, we select a fixed point theorem which will show that the mapping P has a fixed point in S.

Notice that the process of application of a fixed point method relies on three principles: an elementary variation of constants formula, a complete metric space and the contraction mapping principle. Moreover, in one step, a fixed point argument yields existence, uniqueness and stability. Hence, our major problem, when using fixed point theory to deal with stability analysis, is to define a suitable Banach space and a suitable mapping.

Let us continue to work on our first example.

In the following, some results are presented to illustrate the application of a fixed point method. We return to study the equation (2.9), where $b : \mathbb{R}^+ \to \mathbb{R}$ is bounded and continuous, and τ is positive constant.

$$x'(t) = -b(t)\,x(t-\tau).$$

Denote $x(t) = x(t, 0, \varphi) \in \mathbb{R}$ where $\varphi : [-\tau, 0] \to \mathbb{R}$ is a given continuous initial condition and $x(t, 0, \varphi) = \varphi(t)$ on $t \in [-\tau, 0]$. It is then known that there is a unique continuous solution $x(t)$ satisfying (2.9) for $t > 0$ and with $x(t) = \varphi(t)$ on $[-\tau, 0]$.

For this, we build a mapping that inverts (2.9), We now re-write this equation (2.9) in an equivalent form, or, on the other hand, rewrite the equation (2.9) as integral equation suitable for the Banach Theorem.

$$\int_0^t \left(x\left(u\right) e^{\int_0^u b(s+\tau)ds}\right)' du = \int_0^t e^{-\int_0^u b(s+\tau)ds} \frac{d}{du}\left(\int_{u-\tau}^u b(s+\tau)x(s)ds\right) du. \tag{2.16}$$

As a consequence, we arrive at

$$\begin{aligned} x\left(t\right) &= \varphi\left(0\right) e^{-\int_0^t b(s+\tau)ds} + \int_{t-\tau}^t b(u+\tau)x(u)du - e^{-\int_0^t b(s+\tau)ds}\int_{t-\tau}^t b(u+\tau)x(u)du \\ &\quad - \int_0^t b(s+\tau)e^{-\int_s^t b(u+\tau)ds}\int_{s-\tau}^s b(u+\tau)x(u)du\ ds. \end{aligned} \tag{2.17}$$

Let C be the space of all continuous functions from $\mathbb{R} \to \mathbb{R}$ and with this intial function φ, set

$$S := \left\{x \in C\left(\mathbb{R}\right) : x\left(t\right) = \varphi\left(t\right) \text{ if } t \leq 0, x\left(t\right) \to 0, \text{ as } t \to \infty\right\}.$$

S is a complete metric space under the metric

$$\rho\left(x, y\right) := \sup_{t \geq 0} \left|x\left(t\right) - y\left(t\right)\right|.$$

Theorem 2.12 [22] *Suppose there exists a constant $0 < \alpha < 1$ such that,*

$$\int_{t-\tau}^t \left|b(s+\tau)\right| ds + \int_0^t \left|b(s+\tau)\right| e^{-\int_s^t b(u+\tau)du} \int_{s-\tau}^s \left|b(u+\tau)\right| du\ ds \leq \alpha, \tag{2.18}$$

for all $t \geq 0$. As we are interested in asymptotic stability we will need

$$\int_0^t \left|b(s+\tau)\right| ds \to \infty \text{ as } t \to \infty. \tag{2.19}$$

Then for every continuous initial function $\varphi : [-\tau, \infty) \to \mathbb{R}$, the solution $x(t) = x(t, 0, \varphi)$ of (2.9) is bounded and tends to zero as $t \to \infty$. Then every solution $x(t, 0, \varphi)$ of (2.9) with small continuous initial function $\varphi(t)$, is bounded and approaches zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.

Proof. Let $(S, \|.\|)$ be the Banach space of bounded and continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}$ with the supremum norm. Let $(\mathbb{B}, \|.\|)$ be the complete metric space with supremum norm consisting of functions $x \in \mathbb{B}$ such that $x(t) = \varphi(t)$ on $[-\tau, 0]$ and $x(t) \to 0$ as $t \to \infty$. In order to apply the Banach fixed point theorem, we need to prove that P maps S to S. Use (2.17) to define the operator $P : S$ to S by

$$\begin{aligned}
(Px)(t) &= \varphi(t) \text{ on } [-\tau, 0], \\
(Px)(t) &= \varphi(0)\, e^{-\int_0^t b(s+\tau)ds} + \int_{t-\tau}^{t} b(u+\tau)x(u)du \\
&\quad -e^{-\int_0^t b(s+\tau)ds} \int_{t-\tau}^{t} b(u+\tau)x(u)du \\
&\quad -\int_0^t b(s+\tau)e^{-\int_s^t b(u+\tau)ds} \int_{s-\tau}^{s} b(u+\tau)x(u)du\; ds \text{ if } t \geq 0.
\end{aligned} \tag{2.20}$$

Clearly, $Px : \mathbb{R} \to \mathbb{R}$ is continuous, and by definition $(Px)(0) = \varphi(0)$, and from (2.18) it follows that P is bounded. Also, P is a contraction by (2.18). We can show that the last term tends to zero by using the classical proof that the convolution of an L_1 -function with a function tending to zero, does also tend to zero. Here are the details. Let $x \in \mathbb{B}$ be fixed and let $0 < T < t$. Denote the supremum of $|x|$ by $\|.\|$ and the supremum of $|x|$ on $[T, +\infty)$ by $\|.\|_{[T,+\infty)}$. Consider (2.18) and (2.19). We have

$$\begin{aligned}
&\int_0^t |b(s+\tau)|\, e^{-\int_s^t b(u+\tau)du} \int_{s-\tau}^{s} |b(u+\tau)x(u)|\, duds \\
\leq\ &\int_0^T |b(s+\tau)|\, e^{-\int_s^T b(u+\tau)ds} \int_{s-\tau}^{s} |b(u+\tau)|\, duds\, \|x\|\, e^{-\int_T^t b(u+\tau)du} \\
&+ \int_T^t |b(s+\tau)|\, e^{-\int_s^t b(u+\tau)ds} \int_{s-\tau}^{s} |b(u+\tau)|\, duds\, \|x\|_{[T-\tau,+\infty)} \\
\leq\ &\alpha\, \|x\|\, e^{-\int_T^t b(u+\tau)du} + \alpha\, \|x\|_{[T-\tau,+\infty)}\,.
\end{aligned}$$

For a given $\varepsilon > 0$ take T so large that $\alpha \|x\|_{[T-\tau,+\infty)} < \varepsilon\backslash 2$. For that fixed T, take t^* so large that $\alpha \|x\| e^{-\int_T^t b(u+\tau)du}$ for all $t > t^*$. Then have that last term smaller than ϵ for all $t > t^*$. By the contraction mapping theorem there exists a fixed point $x \in \mathbb{B}$, which solves (2.9), for each $\varphi \in C(\delta_0)$, and by the definition of $\mathbb{B}$ we have that $x(t) = x(t, t_0, \varphi) \to 0$ as $t \to \infty$. ∎

Example 2.7 [22] *Consider the differential equation*

$$x^{'}(t) = -(1 + 2\sin t)\, x(t - \tau), \tag{2.21}$$

where $0 < \tau < 1$. *The zero solution of (2.21) is asymptotically stable when* $(\tau + 4\sin(\tau\backslash 2)))\,(2 + 2e^2) < 1$ *this is approximately* $0 \leq \tau < 0.02$.

Since 1+ $2\sin t$ changes sign for $t \geq 0$, Theorem 2.10 is not applicable to Example 2.7. Consider Example 2.6 by using Theorem 2.12, we obtain that the zero solution of (2.14) is asymptotically stable if $2(1.1\tau + 2\sin(\tau\backslash 2)) < 1$. This is approximated by $0 < \tau < 0.2$, compared to $\tau < 0.37$ by using Luypunov's direct method.

Example 2.8 [22] *In (2.9), let* $b(t) = 1.1 + \sin t$. *The conditions of Theorem 2.12 are satisfied if* $(\tau + 4\sin(\tau\backslash 2)))\,(2 + 2e^2) < 1$. *This is approximated by* $0 < \tau < 0.2$.

Remark 2.6 *From the above discussion, we find it is very difficult to find a way to interpret a relation between the fixed point condition (2.18) and the Lyapunov condition (2.12). Sometimes one is better than the other. Sometimes the fixed point condition gives results when the Lyapunov condition can not. Nonetheless, the success of the contraction mapping method for delayed differential equations in* [20, 30, 88, 100, 104, 105] *was, in the author's opinion, sufficient justification to begin the study of this method in more complex systems.*

CHAPTER 3

Existence of solutions and stability for stochastic delay differential equations

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In this chapter we will concentrate on stochastic functional differential equations (SFDEs). In SFDEs the drift, diffusion coefficients depend on the past of the solution, while in SDEs the coefficients just depend on the present. Obviously an SDE belongs to the class of SFDEs with zero time delay but stability theory of SFDEs and SDEs has significant differences. Also SFDEs contain an important class of stochastic delay equations (SDDEs), which (in the Brownian motion case) have found a number of applications to e.g. mathematical finance (see e.g. Kazmerchuka, Swishchukb, and Wu [64]). Stochastic functional differential equations appear in many different contexts including the study of materials with memory (viscoelastic materials), mathematical demography and population dynamics (see Mohammed [78]).

3.1 Basic facts on stochastic stability

Stability in the deterministic or stochastic case means insensitivity of the system to changes i.e. whether small changes in the initial condition lead to small changes (stability) or to large changes (instability) in the solution of the system. Starting from a small vicinity of the trivial solution of the SDE we will investigate constraints on the parameters which ensures that the solution of the SDE converges to the trivial solution. In a stable system, trajectories which start very close to the origin remain close to the origin after a long time has passed and for unstable systems they may have moved a large distance away.

When we try to carry over the principles of the Lyapunov stability theory for deterministic systems to stochastic ones, we face the following problems:

- What is a suitable definition of stochastic stability?
- What conditions should a stochastic Lyapunov function satisfy?
- With what should the inequality $\dot{V}(t, x) \leq 0$ be replaced in order to get stability assertions?

In this part we shall answer these questions one by one. First, we will describe the celebrated Lyapunov theorem and we will give an introduction to the stochastic stability theory of SDEs.

3.1.1 Stability of stochastic differential equations

Consider the n-dimensional stochastic differential equation of Itô type

$$dx(t) = f(t, x(t))dt + g(t, x(t))dw(t) \text{ on } t \geq t_0, \tag{3.1}$$

where $f(t, x)$ is a function in $\mathbb{R}^n$ defined in $[t_0, +\infty) \times \mathbb{R}^n$, and $g(t, x)$ is a $n \times m$ is a matrix, f, g are locally Lipschitz functions in x and $w(t)$ is an m -dimensional Wiener process. We assume

$$f(t, 0) = g(t, 0) = 0, \text{ for all } t \geq t_0,$$

which imply that system (3.1) admits a trivial equilibrium $x = 0$.

Throughout this section we shall assume that the assumptions of the existence and uniqueness Theorem 1.10 are fulfilled (see chapiter 1). Hence, for any given initial value $x(t_0) = x_0 \in \mathbb{R}^n$, equation (3.1) has a unique global solution that is denoted by $x(t, t_0, x_0)$.

Firstly, let us introduce a few necessary notations and definitions, for which we can refer to $[47, 59, 80]$, let $\mathcal{K}$ denote the family of all continuous nondecreasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$. And for $h > 0$, let $S_h = \{x \in \mathbb{R}^n : |x| < h\}$.

Definition 3.1 [80] *A continuous function $V(t, x)$ defined on $[t_0, +\infty) \times S_h$ is said to be positive-definite (in the sense of Lyapunov) if $V(t, 0) = 0$ and, for some $\mu \in K$,*

$$V(t, x) \geq \mu(|x|) \textit{ for all } (t, x) \in [t_0, +\infty) \times S_h.$$

A function V is said to be negative-definite if $-V$ is positive-definite (negative definite and negative semidefinie are defined similarly with the inequatilies reversed).

A continuous nonnegative function $V(t, x)$ is said to be decrescent (i.e. to have an arbitrarily small upper bound) if for some $\mu \in \mathcal{K}$,

$$V(t, x) \leq \mu(|x|) \textit{ for all } (t, x) \in [t_0, +\infty) \times S_h.$$

A function $V(t,x)$ defined on $[t_0,+\infty)\times\mathbb{R}^n$ is said to be radially unbounded if

$$\lim_{|x|\to\infty}\inf_{t\geq t_0} V(t,x)=\infty.$$

Denote by $C^{1,2}(\mathbb{R}^+\times S_h;\mathbb{R}^+)$ the family of all nonnegative function $V(t,x)$ defined on $\mathbb{R}^+\times S_h$ such that they are continuously once differentiable in t and twice in x. Define the differential operator L associated with equation (3.1) by

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{n} f_i(t,x)\,.\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n}\left[g^T(t,x)\,g(t,x)\right]_{ij}\frac{\partial^2}{\partial x_i x_j}.$$

The acts of L on a function $V(t,x)\in C^{1,2}(\mathbb{R}^+\times S_h;\mathbb{R}^+)$ is

$$L(V)(t,x)=\frac{\partial V}{\partial t}+f^T.\frac{\partial V}{\partial x}+\frac{1}{2}Trace\left[g^T.\frac{\partial^2 V}{\partial x^2}.g\right]. \tag{3.2}$$

By Itô's formula, if $x\in S_h$, then

$$dV(t,x(t))=L(V)(t,x(t))dt+\frac{\partial V}{\partial x}g(t,x(t))dw(t) \text{ on } t\geq t_0,$$

and this explains why the differential operator L is defined as above. We shall see that the inequality $\dot{V}(t,x)\leq 0$ will be replaced by $LV(t,x)\leq 0$ in order to get the stochastic stability assertions.

Now we present the following theorems which give conditions for the stability of the trivial solution of the stochastic system in terms of Lyapunov function [80].

Theorem 3.1 [80] *i) If there exists a positive-definite function $V(t,x)\in C^{1,2}(\mathbb{R}^+\times S_h;\mathbb{R}^+)$ such that $L(V)(t,x)\leq 0$ for all $(t,x)\in[t_0,+\infty)\times S_h$, then the trivial solution of the system (3.1) is stochastically stable.*

ii) If there exists a positive-definite decrescent function $V(t,x)\in C^{1,2}(\mathbb{R}^+\times S_h;\mathbb{R}^+)$ such that $L(V)(t,x)$ is negative-definite, then the trivial solution of the system (3.1) is stochastically asymptotically stable.

Stability theory for SDEs appears to be much richer than for deterministe systems. We will mainly focus on the three most important types of stochastic stability, these being the following:

- **stability in pobability**
- **moment stability**
- **almost sure stability**

There also exists other types of stochastic stability but these may be too weak from the point of view of practical significance (e.g. asymptotic convergence in probability see Kozin [63] Example 2.1 pp. 96).

We quote the following definitions from Mao [80].

Definition 3.2 [80] *i) The trivial solution of equation (3.1) is said to be stochastically stable or stable in probability if for every pair of* $\varepsilon \in (0,1)$ *and* $r > 0$*, there exists a* $\delta = \delta(\varepsilon, r, t_0) > 0$ *such that*

$$\mathbb{P}\left\{|x(t;t_0,x_0)| < r \text{ for all } t \geq t_0\right\} \geq 1 - \varepsilon,$$

whenever $|x_0| < \delta$*. Otherwise, it is said to be stochastically unstable.*
ii) The trivial solution of equation (3.1) is said to be stochastically asymptotically stable if it is stochastically stable and moreover, for every $\varepsilon \in (0,1)$*, there exists a* $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ *such that*

$$\mathbb{P}\left\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\right\} \geq 1 - \varepsilon,$$

whenever $|x_0| < \delta$*.*
iii) *The trivial solution is said to be stochastically asymptotically stable in the large if it is stochastically stable and, moreover, for all* $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}\left\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\right\} = 1.$$

Definition 3.3 [80] *The trivial solution of (3.1) is said to be almost surely exponentially stable if*

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| < 0 \quad a.s., \tag{3.3}$$

for all $x_0 \in \mathbb{R}^n$.

The quantity in the left hand side of (3.3) is called the sample Lyapunov exponent.

Definition 3.4 [80]. *Assume that* $p > 0$. *The trivial solution of (3.1) is said to be pth moment exponentially stable if there is a pair of constants* $\lambda > 0$ *and* $C > 0$ *such that*

$$\mathbb{E}\,|x(t)|^p \leq C\,|x_0|^p \exp\left(-\lambda\,(t-t_0)\right), \quad \textit{for all } t \geqslant t_0, \tag{3.4}$$

for all $x_0 \in \mathbb{R}^n$.

When $p = 2$, it is usually said to be exponentially stable in mean square. It follows from (3.4) that

$$\frac{1}{t}\left(\log \mathbb{E}\,|x(t)|^p\right) \leq \frac{1}{t}\left(\log C\,|x_0|^p\right) + \frac{1}{t}\log\left(\exp\left(-\lambda\,(t-t_0)\right)\right),$$

for all $t \geqslant t_0$. Hence

$$\limsup_{t\to\infty} \frac{1}{t}\left(\log \mathbb{E}\,|x(t)|^p\right) < -\lambda.$$

So the trivial solution of (3.1) is pth moment exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t}\left(\log \mathbb{E}\,|x(t)|^p\right) < 0 \quad a.s., \tag{3.5}$$

for all $x_0 \in \mathbb{R}^n$. The quantity in the left hand side of (3.5) is called the pth moment Lyapunov exponent of the solution.

The difference between almost sure exponential stability and pth moment exponential stability is that almost sure exponential stability implies that almost all sample paths of the solution will tend to the the equilibrium position exponentially fast, while the pth moment exponential stability implies that the pth moment of the solution will tend to zero exponentially fast.

It should also be pointed out that when $g\,(t, x) = 0$, these definitions reduce to the corresponding deterministic ones.

3.1.2 Stability of stochastic functional differential equations

In this section, we first give the general formulation of stochastic delay differential equations. Second, the existence and uniqueness theorem for SDDEs will be discussed. Before giving this theorem, we introduce some definitions to understand the conditions clearly. Stochastic delay differential equations (SDDEs) have become widespread in the last 30 years. Phenomena of time delay occur in many different areas of the real world. Stochastic delay differential equations are their mathematical reflection. A description without time delay is nowadays not to think of. In physics it is the time of transportation of particles or information from one system to another. In financial mathematics it is the time to react on developments in financial markets. In econometrics time delay corresponds to the reaction time of the client to behave in a certain way. A first survey on the theory of SDDEs is presented in Mohammed [78] and Mao [79] , [80]. We shall consider SDDEs of the kind

$$dx(t) = f(t, x_t)dt + g(t, x_t)dw(t) \text{ on } t \geq t_0. \tag{3.6}$$

Let $f : [t_0, T] \times C \to \mathbb{R}^n$, $g : [t_0, T] \times C \to \mathbb{R}^{n \times m}$, $0 < t_0 < T < \infty$. We impose the initial data:

$$\begin{aligned} x_{t_0} = \varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\} \text{ is an } \mathcal{F}_{t_0} - \text{measurable}, \\ C([-\tau, 0], \mathbb{R}^n) - \text{valued random variable such that } \mathbb{E}\|\varphi\|^2 < \infty. \end{aligned} \tag{3.7}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $w(t) = (w_1(t), w_2(t), ..., w_m(t))^T, t \geq 0$, be an m-dimensional standard Brownian motion on that probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We let $x_t \in C$, be defined by $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is the past history of the state, where $\tau > 0$ denotes the length of memory. We exclude the case $\tau = \infty$ which has been treated in Riedle [95].

• In the case $\tau = 0$ the system (3.6) is a stochastic ordinary differential equation. This means that SDEs are actually a kind of SDDE.

• SDDEs are actually a kind of stochastic functional differential equations (SFDEs). Stochastic delay differential equation, stochastic differential equation with delay, stochastic differential equation with memory all have the same meaning.

• If the function g does not depend on x, we say that the above equation has an additive noise.

• If the function f and g do not depend on t explicitly, we call the above SDDE as autonomous SDDE.

Definition 3.5 [79] *An $\mathbb{R}^n$-valued stochastic process $x(t)$ on $t_0 - \tau \leq t \leq T$ is called a solution to equation* (3.6) *with initial data* (3.7) *if it has the following properties:*

i) *it is continuous and $\{x_t\}_{t_0\leq t\leq T}$ is $\mathcal{F}_t$-adapted;*

ii) $\{f(t, x_t)\} \in \mathbb{L}^1_w([t_0, T], \mathbb{R}^n)$ *and* $\{g(t, x_t)\} \in \mathbb{L}^2_w([t_0, T]; \mathbb{R}^{n\times m})$;

iii) $x_{t_0} = \varphi$ *and, for every* $t_0 \leq t \leq T$,

$$x(t) = x(0) + \int_0^t f(s, x_s)\, ds + \int_0^t g(s, x_s)\, ds, a.s.$$

A solution solution $x(t)$ is said to be unique if any other solution $\overline{x}(t)$ is indistinguishable from it, that is

$$\mathbb{P}\left\{x(t) = \overline{x}(t) \text{ for all } t_0 - \tau \leq t \leq T\right\} = 1.$$

Let us now begin to establish the theory of the existence and uniqueness of the solution.

Theorem 3.2 [79] *Assume that there exist two positive constants $\overline{K}$ and K such that*

i) (uniform Lipschitz condition) for all ζ, $\phi \in C\left([-\tau, 0], \mathbb{R}^n\right)$ and $t \in [t_0, T]$,

$$\left|f\left(t, \zeta\right) - f\left(t, \phi\right)\right|^2 \vee \left|g\left(t, \zeta\right) - g\left(t, \phi\right)\right|^2 \leq \overline{K}||\zeta - \phi||^2; \tag{3.8}$$

ii) (linear growth condition) for all $(t, \zeta) \in [t_0, T] \times C([-\tau, 0], R^n)$

$$\left|f(t, \zeta)\right|^2 \vee \left|g(t, \zeta)\right|^2 \leq K\left(1 + ||\zeta||^2\right). \tag{3.9}$$

Then there exists a unique solution $x\left(t\right)$ to equation (3.6) with initial data (3.7). Moreover, the solution belongs to $\mathbb{M}^2_w\left([t_0 - \tau, T], \mathbb{R}^n\right)$.

We need a lemma in order to prove this theorem.

Lemma 3.1 [79] *Let the linear growth condition (3.9) hold. If $x(t)$ is a solution to equation (3.6) with initial data (3.7), then*

$$\mathbb{E}\left(\sup_{t_0 - \tau \leq t \leq T} |x(t)|^2\right) \leq (1 + 4\mathbb{E}||\varphi||^2)e^{3K(T-t_0)(T-t_0+4)}.$$

In particular, $x(t)$ belongs to $\mathbb{M}^2_w([t_0 - \tau, T], R^n)$.

The proof can be shown by using Picard-Lindelof iterative method. The detailed proof of the theorem can be found in [see, Mao, 1994].

Firstly, let us to introduce some notations for the study of stability stochastic functional differential equations. Denote with $\mathcal{H}$ the space $\mathcal{F}_0-$adapted random variables φ, with $\varphi(s) \in \mathrm{R}^n$ for $s \leq 0$, and

$$\|\varphi\| = \sup_{s\leq 0} |\varphi(s)|, \quad \|\varphi\|_1^2 = \sup_{s\leq 0} \mathbb{E} |\varphi(s)|^2 .$$

Let $V : [0, \infty) \times \mathcal{H} \to \mathrm{R}$ be a functional defined for $t \geq 0$ and $\varphi \in \mathcal{H}$. Reduce the arbitrary functional $V(t, \varphi), t \geq 0,\ \varphi \in \mathcal{H}$ to the form

$$V(t, \varphi) = V(t, \varphi(0), \varphi(s)), s < 0,$$

and define the function

$$V_\varphi(t, x) = V(t, \varphi) = V(t, x_t) = V(t, x, x(t+s)), s < 0,\ x_t = \varphi,\ x = \varphi(0).$$

Let $\mathcal{D}$ be the class of functionals $V(t, \varphi)$ for which the functions $V_\varphi(t, x)$ has continuous partial derivatives with respect to x of order two, and bounded derivative for all $t \geq 0$.

For all $V \in \mathcal{D}$, the differential operator L associated with equation (3.10), where the initial conditions $x_0 = \varphi \in \mathcal{H}$ by

$$L(V) = \frac{\partial V_\varphi}{\partial t} + f^T . \frac{\partial V_\varphi}{\partial x} + \frac{1}{2} Tr \left[g^T . \frac{\partial^2 V_\varphi}{\partial x^2} . g \right].$$

Now we introduce the following definitions of stability for stochastic differential delay equations [see, Kolmanovskki and Nosov, 1986].

Definition 3.6 [59] i) *the trivial solution of SFDEs (3.6) is said to be mean square stable if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any initial process $\varphi(\theta)$, the inequalities*

$$\sup_{\tau\leq\theta\leq 0} \mathbb{E} |\varphi(s)|^2 \leq \delta(\varepsilon),$$

imply that $\mathbb{E}\left|(t, t_0, \varphi)\right|^2 < \varepsilon$ *for* $t \geq 0$ *and exponentially mean square stable if, for any positive constants* c_1 *and* c_2,

$$\mathbb{E}\left|(t, t_0, \varphi)\right|^2 \leq c_1 \sup_{\tau \leq \theta \leq 0} \mathbb{E}\left|\varphi(s)\right|^2 \exp\left(-c_2 t\right), t \geq 0,$$

ii) *The trivial solution of SFDE (3.6) is said to be asymptotically mean square stable if it is mean square stable and for all functions satisfying (3.6) we have*

$$\lim_{t \to \infty} \mathbb{E}\left|x(t, t_0, \varphi)\right|^2 = 0.$$

iii) *The trivial solution of SFDE (3.6) is said to be stochastically stable if for each* $\varepsilon_1 > 0$ *and* $\varepsilon_2 > 0$, *there exists a* $\delta > 0$ *such that for* $t > 0$ *the solution* $x(t, t_0, \varphi)$ *satisfies the inequality*

$$\mathbb{P}\left\{\sup_{t \geq 0} \left|x(t, t_0, \varphi)\right| \leq \varepsilon_1\right\} \geq 1 - \varepsilon_2,$$

provided that $\mathbb{P}\{\|\varphi\| \leq \delta\} = 1$.

The following theorem gives conditions for stability of equilibruim states of as SFDE [see, Kolmanovskki and Nosov, 1986].

Theorem 3.3 [59] *: i) Suppose that there exists a functional* $V(t, \varphi) \in \mathcal{D}$ *such that*

$$c_1 \mathbb{E}\left(\left|x(t)\right|^2\right) \leq \mathbb{E}\left(V(t, x_t)\right) \leq c_2 \left\|x_t\right\|_1^2,$$

and

$$\mathbb{E}\left(LV(t, x_t)\right) \leq -c_3 \mathbb{E}\left(\left|x(t)\right|^2\right),$$

with $c_i > 0, i = 1, 2$, *where* x_t *is the solution of system (3.6) verifying the initial condition* $x_0 = \varphi$. *Then the trivial solution of system (3.6) is asymptotically mean square stable.*

ii) Suppose that there exists a functional $V(t, \varphi) \in \mathcal{D}$ *such that for all functions*

$$c_1 \left|\varphi(0)\right|^2 \leq V(t, \varphi) \leq c_2 \left\|\varphi\right\|^2,$$

and

$$LV(t, x_t) \leq 0,$$

with $c_i > 0, i = 1, 2$, *where* x_t *is the solution of system (3.6) verifying the initial condition* $x_0 = \varphi$, *for all functions* $\varphi \in \mathcal{H}$ *such that* $\mathbb{P}\{\|\varphi\| \leq \delta\} = 1$ *where* $\delta > 0$ *is sufficiently small. Then the trivial solution of system (3.6) is stochastically stable.*

It should also be pointed out that when $g(t, x_t) = 0$, these definitions reduce to the corresponding deterministic delay differential equations (2.6) (see chapitre 2).

Now, we analyze one example to illustrate the applicability of Lyapunov function for stochastic delay differential equations.

Using this idea we will prove stability conditions for the equations *(3.13)* by contradiction. We will consider the solution of the appropriate equation with a deterministic initial function *(3.7)* satisfying

$$\sup_{-\tau \leq \theta \leq 0} |x(\theta)| < \delta_1, \tag{3.10}$$

and assume that the solution is not stable. This, in turn, means that there must exist some moment of time $t = T > \tau$ which is a first exit time of the solution from the stability domain with radius $\rho > \delta_1$ about the origin. From this it follows that, except for a subset of probability zero, trajectories satisfy

$$|x(T-\tau)| < x(T) = \rho, \tag{3.11}$$

so

$$\mathbb{E}\left\{|x(T-\tau)|^2\right\} < \mathbb{E}|x(T)|^2 = \rho^2, \tag{3.12}$$

Example 3.1 [82] *Consider the scalar linear differential delay equation damped with stochastic term*

$$dx(t) = [-ax(t) - bx(t-\tau)]\,dt + \sigma dw(t),\ t \geq 0, \tag{3.13}$$

where $\tau > 0$ *is a constant delay, and* $a, b \in \mathbb{R}$.

Theorem 3.4 *If*

$$a < -|b| - \frac{\sigma^2}{2}\frac{1}{|x(0)|^2}, \tag{3.14}$$

then the solution $x(t)$ *of (3.13) is mean square stable, where*

$$a < -|b|. \tag{3.15}$$

Proof. To prove condition (3.14) we pick a Lyapunov function $V(x) = |x|^2$. By Itô differential rule the stochastic differential of $V(x(t))$ is

$$dV(x(t)) = \left[2|x(t)|(ax(t) + bx(t-\tau)) + \sigma^2\right]dt + 2|x(t)|\sigma dw(t),$$

Integrating this relation from zero to t, taking the mathematical expectation of both parts, using the properties of the stochastic integral, and then differentiating with respect to t, we obtain

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\{V(x(t))\} &< \mathbb{E}\left\{2a|x(t)|^2 + 2b|x(t)|x(t-\tau) + \sigma^2\right\} \\ &< \mathbb{E}\left\{2a|x(t)|^2 + 2|b||x(t)||x(t-\tau)| + \sigma^2\right\}. \end{aligned} \tag{3.16}$$

Now assume that $x(t)$ is not mean square stable, which implies that there is some time $t = T$ such that (3.12) holds. From (3.16) and (3.12) then we obtain for $t = T$,

$$\frac{d}{dt}\mathbb{E}\{V(x(t))\} \leq 2(a + |b|)\mathbb{E}|x(t)|^2 + \sigma^2. \tag{3.17}$$

Solving the differential inequality (3.17) gives

$$\mathbb{E}\{V(x(t))\} < \left[\mathbb{E}\{V(x(0))\} + \frac{\sigma^2}{2(a+|b|)}\right]e^{2(a+|b|)t} - \frac{\sigma^2}{2(a+|b|)}. \tag{3.18}$$

Note that if $a + |b| < 0$ holds, then $e^{2(a+|b|)t} < 1$. Furthermore, assume that

$$\mathbb{E}\{V(x(0))\} + \frac{\sigma^2}{2(a+|b|)} > 0. \tag{3.19}$$

Then (3.18) and (3.19) together imply that

$$\mathbb{E}\{V(x(t))\} < \mathbb{E}\{V(x(0))\}, \tag{3.20}$$

and since $V(x) = |x|^2$, (3.20) becomes

$$\mathbb{E}\left\{|x(t)|^2\right\} < |x(0)|^2,$$

Thus we conclude that

$$\mathbb{E}\left\{|x(T)|^2\right\} < |x(0)|^2 < \delta_1^2, \tag{3.21}$$

which is clearly in contradiction with (3.12). Thus, with condition (3.19) and $a + |b| < 0$, the assumption that there exists a first exit time T from the stability domain is not valid. Consequently, we have proved mean square stability of $x(t)$ in the sense of Definition 3.5 with $\varepsilon = \delta = \delta_1^2$. Rewriting (3.19) gives the final form (3.14), which completes the proof. ∎

3.2 Solving stochastic delay differentials

The procedure for finding the solution again depends on Itô formula, but it is a little bit different than the way used in SDE because of the delay effect that we insert in the equation. We need to proceed step by step in the intervals with equal step-size starting from the initial point. Now, let us examine the following examples to understand the solution technique better and see the difference.

Example 3.2 [1] *Let us consider an example of SDDE such that the delay effects only the drift term and diffusion term is a constant real number :*

$$\begin{aligned} dx(t) &= x(t-\tau)dt + \beta dw(t); t \geq 0; \\ x(t) &= \varphi(t); t \in [-\tau, 0]. \end{aligned}$$

We first check the conditions for the existence and uniqueness theorem and then solve this SDDE with the given initial condition. Note that, $f(t,x,y) = y(t-\tau)$, $g(t,x,y) = \beta$, then

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &= |y_1 - y_2| \leq |x_1 - x_2| + |y_1 - y_2|, \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &= |\beta - \beta| \leq |x_1 - x_2| + |y_1 - y_2|, \end{aligned}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in \mathbb{R}^+$. Moroever,

$$\begin{aligned} |f(t,x,y)|^2 &= |y|^2 \leq 1 + |y|^2, \\ |g(t,x,y)|^2 &= |\beta|^2 \leq |\beta|^2 \left(1 + |x|^2\right). \end{aligned}$$

As a result, given SDDE in the example has a unique strong solution and that solution $x(t)$ satisfies

$$\mathbb{E}\left(\sup_{-\tau \leq t \leq T} |x(t)|^2\right) < \infty, \text{ for all } T > 0.$$

After we show that the solution is unique, we are ready to solve it by Itô formula.

Define $\varphi(t) =: \varphi_1(t)$.

For $t \in [0,\tau] : t-\tau \in [-\tau, 0]$, which implies that $x(t-\tau) = \varphi_1(t-\tau)$ and our SDDE actually defines the following SDE:

$$dx(t) = \varphi_1(t-\tau)dt + \beta dw(t).$$

Applying Itô formula for $H(x) = x$:

$$\begin{aligned} x(t) &= \varphi_1(0) + \int_0^t \varphi_1(s_1 - \tau) ds_1 + \int_0^t \beta dw(s_1) \\ &= \varphi_1(0) + \int_0^t \varphi_1(s_1 - \tau) ds + \beta w(t) \\ &= \varphi_2(t) \,. \end{aligned}$$

For $t \in [\tau, 2\tau]$: $t - \tau \in [0, \tau]$. Hence, $x(t-\tau) = \varphi_2(t-\tau)$ and the equation becomes:

$$dx(t) = \varphi_2(t-\tau)dt + \beta dw(t).$$

Applying Itô formula for $H(x) = x$ again.

$$\begin{aligned} x(t) &= \varphi_2(\tau) + \int_\tau^t \varphi_2(s_2 - \tau) ds_2 + \int_\tau^t \beta dw(s_2) \\ &= \varphi_2(\tau) + \int_\tau^t \left[\varphi_1(0) + \int_0^{s_2 - \tau} \varphi_1(s_1 - \tau) ds_1 + \beta w(s_2 - \tau) \right] ds_2 + \int_\tau^t \beta dw(s_2) \\ &= \varphi_2(\tau) + \varphi_1(0)(t - \tau) + \int_\tau^t \int_0^{s_2 - \tau} \varphi_1(s_1 - \tau) ds_1 ds_2 \\ &\quad + \beta \int_\tau^t w(s_2 - \tau) ds_2 + \beta \left(w(t) - w(\tau) \right) \\ &: = \varphi_3(t). \end{aligned}$$

For $t \in [2\tau, 3\tau] : t - \tau \in [\tau, 2\tau]$. Thus, $x(t-\tau) = \varphi_3(t-\tau)$ and the equation turns to

$$dx(t) = \varphi_3(t-\tau)dt + \beta dw(t).$$

Applying Itô formula again:

$$
\begin{aligned}
x(t) &= \varphi_3(2\tau) + \int_{2\tau}^{t} \varphi_3(s_3-\tau)ds_3 + \int_{2\tau}^{t} \beta dw(s_3) \\
&= \varphi_3(2\tau) + \int_{2\tau}^{t} \Big[\varphi_2(\tau) + \varphi_1(0)(s_3-2\tau) + \int_{\tau}^{s_3-\tau}\int_{0}^{s_2-\tau} \varphi_1(s_1-\tau)ds_1 ds_2 \\
&\quad + \int_{\tau}^{s_3-\tau} \beta w(s_2-\tau)ds_2 + \beta\left(w(s_3-\tau)-w(\tau)\right)\Big] ds_3 + \beta\left(w(t)-w(2\tau)\right) \\
&= \varphi_3(2\tau) + \varphi_2(\tau)(t-2\tau) + \int_{2\tau}^{t} \varphi_1(0)(s_3-2\tau)ds_3 \\
&\quad + \int_{2\tau}^{t}\int_{\tau}^{s_3-\tau}\int_{0}^{s_2-\tau} \varphi_1(s_1-\tau)ds_1 ds_2 ds_3 \\
&\quad + \int_{2\tau}^{t}\int_{\tau}^{s_3-\tau} \beta w(s_2-\tau)ds_2 ds_3 + \int_{2\tau}^{t} \beta\left(w(s_3-\tau)ds_3 - w(\tau)\right) + \beta\left(w(t)-w(2\tau)\right) \\
&:= \varphi_4(t).
\end{aligned}
$$

Notice that in this example, the calculations for $t \in [2\tau, 3\tau]$ quickly became larger, and this is why the method of steps may not give us much qualitative information about the solution, it might give an explicit formula, but in general no essential properties of the solution are revealed. We can repeat this procedure over the intervals $[i\tau, (i+1)\tau]$, for $i = 1, 2, 3, ...$, and construct the solution recursively for this SDDE. The recurrence relation for the solution $\varphi_n(t)$ for $t \in [(n-2)\tau, (n-1)\tau]$ can be written as follow:

$$
\varphi_n(t) = \begin{cases} \varphi_{n-1}(t)((n-2)\tau) + \int_{(n-2)\tau}^{t} \varphi_{n-1}(s-\tau)ds + \beta\left(w(t) - w((n-2)\tau)\right), n = 2, 3, ..., \\ \varphi_1(t), \ \ n = 1, \end{cases} .
$$

Example 3.3 *Consider the following SDDE:*

$$
dx(t) = x(t-1)\,dt + x(t-2)dw(t), t \in [0, 1],
$$

with initial condition:

$$
x(t) = \varphi_0(t) = t, \text{ for } -2 \leq t \leq 0.
$$

Hence to find the solution, consider the first time step interval $[0,1]$, i.e., consider:

$$dx(t) = \varphi_0(t-1)dt + \varphi_0(t-2)dw(t),$$

which is reduced to:

$$dx(t) = (t-1)dt + (t-2)dw(t),$$

and to solve this SDE by using the Itô formula to $H(t, w(t))$, let $x(t) = H(t, w(t))$, then, we get:

$$\begin{aligned} H(t, w(t)) &= H(0,0) + \int_0^t \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^2}(s, w(s)) + \frac{\partial H}{\partial s}(s, w(s))\right] ds \\ &\quad + \int_0^t \frac{\partial H}{\partial x}(s, w(s))dw(s), \end{aligned}$$

such that:

$$\begin{aligned} \frac{\partial^2 H}{\partial x^2}(s, w(s)) + \frac{\partial H}{\partial s}(s, w(s)) &= s-1, \\ \frac{\partial H}{\partial x}(s, w(s)) &= s-2, \\ H(0,0) &= x_0. \end{aligned}$$

Hence $\dfrac{\partial H}{\partial x}(s, w(s)) = s-2$, wich implies that:

$$H(s,x) = sx - 2x + k(s). \tag{3.22}$$

Therefore, substituting in the partial differential equation:

$$\frac{1}{2}\frac{\partial^2 H}{\partial x^2}(s, w(s)) + \frac{\partial H}{\partial s}(s, w(s)) = s-1,$$

which implies $k'(s) = s-1-x$, and therefore $k(s) = \dfrac{s^2}{2} - s - 2x + c$. Now, substituting in Eq. (3.22), yields to:

$$H(t,x) = \frac{1}{2}t^2 + x_0 - t - 2x,$$

and the candidate solution to the SDE is:

$$x(t) = H(t, w(t)) = x_0 + \frac{1}{2}t^2 - t - 2w(t).$$

In this example, it is so difficult to find the solution for the next time interval, unless when a new method is proposed or numerical methods are used to solve for further time intervals [see, Al-Kubeisy, 2004].

CHAPTER 4

Mean square asymptotic stability for stochastic neutral differential equations by the contraction mapping principle

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The goal of this chapter is to present a very recent work published in [26] , namely, Mimia Benhadri, Tomás Caraballo & Halim Zeghdoudi. Stability results for neutral stochastic functional differential equations via fixed point methods. International Journal of Control,To link to this article: https: //doi.org / 10.1080/00207179.2018. 1530431.

In this chapter we prove some results on the mean square asymptotic stability of a class of neutral stochastic differential systems with variable delays by using a contraction mapping principle. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved. The assumption does not require neither boundedness or differentiability of the delay functions, nor do they ask for a fixed sign on the coefficient functions. In particular, the results improve some previous ones proved by [41] ,(Guo et al., 2017). Finally, an example is exhibited to illustrate the effectiveness of the proposed results.

Keywords. Fixed points theory; Asymptotic stability in mean square; Neutral stochastic differential equations; Variable delays.

4.1 Statement of the problem and preliminaries

In this section, we consider the following class of neutral stochastic differential systems with variable delays,

$$\begin{aligned} d\left[x_i(t) - \sum_{j=1}^{n} q_{ij}(t)x_j(t-\tau_j(t))\right] &= \left[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t))\right. \\ &\left. + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\delta_j(t)))\right] dt + \sum_{j=1}^{n} \sigma_{ij}(x_j(t))dw_j(t), t \geq t_0, \end{aligned} \tag{4.1}$$

for $i = 1, 2, 3, ..., n$, which can be written in a vector–matrix form as follows:

$$\begin{aligned} d[x(t) - Q(t)x(t-\tau(t))] &= [A(t)x(t) + B(t)f(x(t)) + C(t)g(x(t-\delta(t))] dt \\ &+ \sigma(x(t))dw(t), t \geq t_0, \end{aligned} \tag{4.2}$$

where $x(t) = [x_1(t), x_2(t), .., x_n(t)]^T \in \mathbb{R}^n$, and $a_{ij}, b_{ij}, c_{ij}, q_{ij} \in C(\mathbb{R}^+, \mathbb{R})$, are continuous functions, $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $Q(t) = (q_{ij}(t))_{n\times n}$, are real matrices and $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times n}$ is the diffusion coefficient matrix, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))]^T \in \mathbb{R}^n$, $g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), ..., g_n(x_n(t))]^T \in \mathbb{R}^n$, and $\tau_j, \delta_j, j = 1, ..., n$, which are the variable delays, are continuous functions satisfying appropriate conditions described below.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space and let $w(t) = [w_1(t), w_2(t), ..., w_n(t)]^T$ be an $n-$dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $w(t)$ (i.e $\mathcal{F}_t$ is the completion of $\sigma\{w(s) : 0 \leq s \leq t\}$). Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|.\|$.

Denote by $x(t) = x(t; s, \varphi) = (x_1(t; s, \varphi_1), ..., x_n(t; s, \varphi_2))^T \in \mathbb{R}^n$ the solution to (4.1) with a deterministic intial condition

$$x_i(s) = \varphi_i(s) \text{ for } s \in [m(t_0), t_0], \text{ for each } t_0 \geq 0, \tag{4.3}$$

where

$$\begin{aligned} m_j(t_0) &= \min\{\inf\{t - \tau_j(t), t \geq t_0\}, \inf\{t - \delta_j(t), t \geq t_0\}\}, \\ m(t_0) &= \min\{m_j(t_0), 1 \leq j \leq n\}, \end{aligned} \tag{4.4}$$

and $\varphi_i(\cdot) \in C([m(t_0), t_0], \mathbb{R})$, and $s \to \varphi(s) = (\varphi_1(s), ..., \varphi_n(s))^T \in \mathbb{R}^n$ belongs to the space $C([m(t_0), t_0], \mathbb{R}^n)$, with the norm defined by $\|\varphi\| = \sum_{i=1}^{n} \sup_{m(t_0)\leq s\leq t_0} |\varphi_i(s)|$. Finally, $\mathbb{E}$ will denote expectation. Before proceeding, we firstly introduce some assumptions to be imposed later on

(A1) The delay functions $\tau_j, \delta_j \in C(\mathbb{R}^+, \mathbb{R}^+)$, with $t - \delta_j(t) \to \infty$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$ for $j = 1, 2, ..., n$.

(A2) there exist nonnegative constants α_j such that for all $x, y \in \mathbb{R}$,

$$|f_j(x) - f_j(y)| \leq \alpha_j |x - y|, \ j = 1, 2, ..., n. \tag{4.5}$$

(A3) there exist nonnegative constants β_j such that for all $x, y \in \mathbb{R}$,

$$|g_j(x) - g_j(y)| \leq \beta_j |x - y|, \ j = 1, 2, ..., n. \tag{4.6}$$

(A4) there exist nonnegative constants L_{ij} such that for all $x, y \in \mathbb{R}$,

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \leq L_{ij} |x - y|, \ i, j = 1, 2, ..., n. \tag{4.7}$$

Throughout this paper, we always assume that

$$f_j(0) = g_j(0) = \sigma_{ik}(0) = 0, \text{ for } i, j, k = 1, 2, \ldots, n \tag{4.8}$$

thereby, problem (4.1) admits the trivial equilibrium $x = 0$.

Very recently, Guo et al. published in (2017) related results on the solutions of a particular case of (4.1). More precisely, the following result was established.

Theorem 4.1 [41] *Suppose that assumptions (A1)–(A4) hold and that there exist positive scalars a_i such that, for all $t \geq 0$,*

$$\sum_{i=1}^{n} \left\{ \left[\sum_{j=1}^{n} \left(|q_{ij}(t)| + \int_0^t e^{-a_i(t-s)} |\overline{a_{ij}}(s)| \, ds \right.\right.\right.$$
$$+ \int_0^t e^{-a_i(t-s)} a_i |q_{ij}(s)| \, ds + \int_0^t e^{-a_i(t-s)} |b_{ij}(s)| \, \alpha_j ds$$
$$\left.\left.\left. + \int_0^t e^{-a_i(t-s)} |c_{ij}(s)| \, \beta_j ds \right) \right]^2 + \frac{2}{a_i} \sum_{j=1}^{n} L_{ij}^2 \right\} \leq \gamma < \frac{1}{2}, \tag{4.9}$$

where $\overline{a_{ij}}(t) = a_{ij}(t) (i \neq j)$, $\overline{a_{ii}}(t) = a_{ii}(t) + a_i$. Then, for any $\varphi \in C([m(0), 0], \mathbb{R}^n)$, there exists a unique global solution $x(t, 0, \varphi)$. Moreover, the zero solution is mean-square asymptotically stable.

Our objective here is to generalize Theorem A to the general case of equation (4.1) by proving a necessary and sufficient condition for the asymptotic stability of the zero solution. We also provide an example to illustrate our results.

For each $t_0 \geq 0$ and $\varphi \in C\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}^n\right)$ fixed, we define $X^{l_i}_{\varphi_{i,t_0}}$ as the following space of stochastic processes

$$\begin{aligned}
&X^{l_i}_{\varphi_{i,t_0}} = \{x_i(t,\omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R} \text{ which is} \\
&\mathcal{F}_t - \text{adapted and almost surely continuous} \\
&\text{in } t \text{ for fixed } \omega \text{,such that } x_i(t,.) = \varphi_i\left(t\right) \\
&\text{for } t \in \left[m\left(t_0\right), t_0\right], \|x_i\|_{X^{l_i}_{\varphi_{i,t_0}}} \leq l_i \text{ for } t \geq t_0 \text{ and } \mathbb{E}\left|x_i(t)\right|^2 \to 0 \text{ as } \ t \to \infty\},
\end{aligned}$$

where $\|x_i(t,\omega)\|_{X^{l_i}_{\varphi_{i,t_0}}} = \left(\mathbb{E}\left(\sup_{t\geq m(t_0)} \left|x_i(t)\right|^2\right)\right)^{1\backslash 2}$.

Now, we denote $X^l_{\varphi,t_0} = X^{l_1}_{\varphi_{1,t_0}} \times X^{l_2}_{\varphi_{2,t_0}} ... \times X^{l_n}_{\varphi_{n,t_0}}$, which can be rewritten as

$$\begin{aligned}
X^l_{\varphi,t_0} = \{x(t,\omega) : [m(t_0), \infty) \times \Omega \to \mathbb{R}^n \text{ which is} \\
\mathcal{F}_t - \text{adapted and almost surely continuous} \\
\text{in } t \text{ for fixed } \omega \text{, such that } x(t,.) = \varphi\left(t\right) \ \text{ for } t \in \left[m\left(t_0\right), t_0\right], \\
\|x\|_X \leq l \text{ for } t \geq t_0 \text{ and } \mathbb{E}\sum_{i=1}^n \left|x_i(t)\right|^2 \to 0 \text{ as } \ t \to \infty\},
\end{aligned}$$

where $\|x\|_X := \left\{\sum_{i=1}^n \mathbb{E}\left(\sup_{t\geq m(t_0)} \left|x_i(t)\right|^2\right)\right\}^{\frac{1}{2}}$. It is easy to check that X^l_{φ,t_0} is a complete metric space with metric induced by the norm $\|\cdot\|_X$.

When no confusion is possible we will not write X^l_{φ,t_0},$X^{l_i}_{\varphi_{i,t_0}}$ but X^l_{φ},$X^{l_i}_{\varphi_i}$ respectively, and we will also omit the random parameter ω, where l, l_i are positive numbers such that $l^2 = \sum_{i=1}^n l_i^2$.

Let us know recall the definitions of stability that will be used in the next section.

Definition 4.1 *The zero solution of the system (4.1) is said to be :*

i) stable if for any $\varepsilon > 0$ *and* $t_0 \geq 0$, *there exists a* $\delta = \delta(\varepsilon, t_0) > 0$ *such that* $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ *and* $\|\varphi\| < \delta$ *imply* $\mathbb{E}\sum_{i=1}^{n} |x_i(t, t_0, \varphi_i)|^2 < \varepsilon$ *for* $t \geq t_0$.

ii) asymptotically stable if the zero solution is stable and for any $\varepsilon > 0$ *and* $t_0 \geq 0$, *there exists a* $\delta = \delta(\varepsilon, t_0) > 0$ *such that* $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ *and* $\|\varphi\| < \delta$ *imply* $\mathbb{E}\sum_{i=1}^{n} |x_i(t, t_0, \varphi_i)|^2 \to 0$ *as* $t \to \infty$.

4.2 Stability of the zero solution

Our purpose here is to extend the work carried out in (Guo et al., 2017) by providing a necessary and sufficient condition for asymptotic stability of the zero solution of equation (4.1). Zhang (2004, 2005) was the first to establish necessary and sufficient condition for the stability of solutions of functional differential equation by the fixed point theory. The necessity of condition (4.12) below for the main stability result was first established in (Zhang, 2004). It is well known that studying the stability of an equation using a fixed point technique involves the construction of a suitable fixed point mapping. This can be an arduous task. Thus, to construct our mapping $\mathcal{P}$, we begin by transforming (4.1) into a more tractable, but equivalent, equation, which we will then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After that, we use a suitable complete metric space X_{φ}^{l} defined above, which may depend on the initial condition φ. Using the contraction mapping principle, we obtain a fixed point for this mapping and hence a solution for (4.1), which in addition is mean square asymptotically stable.

Now, we can state our main result.

Theorem 4.2 [26] *Suppose that assumptions (A1)–(A4) hold, and there exist continuous functions* $a_i : [t_0, \infty) \to \mathbb{R}$ *such that for* $t \geq t_0$

$$\liminf_{t\to\infty} \int_{t_0}^{t} a_i(s)\, ds > -\infty, \quad i = 1, ..., n, \tag{4.10}$$

$$\begin{aligned}
\sum_{i=1}^{n} \Bigg\{ \Bigg[\sum_{j=1}^{n} \Bigg(& |q_{ij}(t)| + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} |\overline{a_{ij}}(s)|\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} |a_i(s)|\, |q_{ij}(s)|\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} |b_{ij}(s)|\, \alpha_j ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} |c_{ij}(s)|\, \beta_j ds \Bigg) \Bigg]^2 \\
& + 4 \sum_{j=1}^{n} \int_{t_0}^{t} L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi} \Bigg\} \leq \gamma < \frac{1}{4},
\end{aligned} \tag{4.11}$$

where $\overline{a_{ij}}(t) = a_{ij}(t) (i \neq j)$, $\overline{a_{ii}}(t) = a_{ii}(t) + a_i(t)$. *Then for any* $\varphi \in C([m(t_0), t_0], \mathbb{R}^n)$ *there exists a unique global solution* $x(t, t_0, \varphi)$*. Moreover, the zero solution is mean-square asymptotically stable if and only if*

$$\int_{t_0}^{t} a_i(s)\, ds \to \infty \text{ as } t \to \infty. \tag{4.12}$$

Proof: Set

$$M_i = \sup_{t \geq t_0} \left\{ e^{-\int_{t_0}^t a_i(s)ds} \right\}, \tag{4.13}$$

which is well defined thanks to (4.10). Suppose also that (4.12) holds.

We now re-write equation (4.1) in an equivalent form. For this end, we use the variation of parameter formula and rewrite the equation in an integral from which we derive a contracting mapping.

We rewrite (4.1) as

$$\begin{aligned} & d\left[x_i(t) - \sum_{j=1}^{n} q_{ij}(t)x_j(t-\tau_j(t))\right] \\ = & \left[-a_i(t)x_i(t) + \sum_{j=1}^{n}\overline{a_{ij}}(t)x_j(t)\right. \\ & \left. + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t-\delta_j(t)))\right] dt \\ & + \sum_{j=1}^{n}\sigma_{ij}(x_j(t))dw_j(t), t \geq t_0, \end{aligned} \tag{4.14}$$

with the intial condition $x_i(t) = \varphi_i(t)$ for $t \in [m(t_0), t_0]$.

Multiplying both sides of (4.14) by $e^{\int_0^t a_i(\xi)d\xi}$ and integrating from t_0 to t,

$$\int_{t_0}^t \left[e^{\int_0^s a_i(\xi)d\xi}x_i(s)\right]' ds = \int_{t_0}^t e^{\int_0^s a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)x_j(s-\tau_j(s))\right)\right.$$
$$\left. + \sum_{j=1}^{n}\overline{a_{ij}}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\delta_j(s)))\right\} ds$$
$$+ \int_{t_0}^t e^{\int_0^s a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(x_j(s))dw_j(s).$$

As a consequence, we arrive at

$$e^{\int_0^t a_i(\xi)d\xi}x_i(t) - e^{\int_0^{t_0} a_i(\xi)d\xi}x_i(t_0) = \int_{t_0}^t e^{\int_0^s a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)x_j(s-\tau_j(s))\right)\right.$$
$$\left. + \sum_{j=1}^{n}\overline{a_{ij}}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\delta_j(s)))\right\} ds$$
$$+ \int_{t_0}^t e^{\int_0^s a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(x_j(s))dw_j(s).$$

Dividing both sides of the above equation by $e^{\int_0^t a_i(\xi)d\xi}$, we obtain

$$x_i(t) = e^{-\int_{t_0}^t a_i(\xi)d\xi}x_i(t_0) + \int_{t_0}^t e^{-\int_s^t a_i(\xi)d\xi}\left\{d\left(\sum_{j=1}^{n} q_{ij}(s)x_j(s-\tau_j(s))\right)\right.$$
$$\left. + \sum_{j=1}^{n}\overline{a_{ij}}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\delta_j(s)))\right\} ds$$
$$+ \int_{t_0}^t e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(x_j(s))dw_j(s).$$

Performing now an integration by parts, we have for $t \geq t_0$, $i = 1, 2, ..., n$,

$$
\begin{aligned}
& x_i(t) \\
= & \left[\varphi_i(t_0) - \left(\sum_{j=1}^{n} q_{ij}(t_0)\varphi_j(t_0 - \tau_j(t_0))\right)\right] e^{-\int_{t_0}^{t} a_i(\xi)d\xi} \\
& + \left(\sum_{j=1}^{n} q_{ij}(t)x_j(t - \tau_j(t))\right) - \int_{t_0}^{t} a_i(s)\, e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} q_{ij}(s)x_j(s - \tau_j(s))\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \overline{a_{ij}}(s)x_j(s)\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s - \delta_j(s)))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(x_j(s))dw_j(s) .
\end{aligned} \tag{4.15}
$$

Use (4.15) to define the operator $\mathcal{P} : X_\varphi^l \to X_\varphi^l$ by

$$(\mathcal{P}x)(t) := [(\mathcal{P}_1 x_1)(t), (\mathcal{P}_2 x_2)(t), ..., (\mathcal{P}_n x_n)(t)]^T \in X_\varphi^l,$$

where $\mathcal{P}_i : X_{\varphi_i}^{l_i} \to X_{\varphi_i}^{l_i}$ by $(\mathcal{P}_i x_i)(t) = \varphi_i(t)$ for $t \in [m(t_0), t_0]$ and for $t \geq t_0$, where $\mathcal{P}_i(x_i) : [m(t_0), +\infty) \to \mathbb{R}$ $(i = 1, 2, ..., n)$ is defined as follows:

$$
\begin{aligned}
& (\mathcal{P}_i x_i)(t) \\
= & \left[\varphi_i(t_0) - \left(\sum_{j=1}^{n} q_{ij}(t_0)\varphi_j(t_0 - \tau_j(t_0))\right)\right] e^{-\int_{t_0}^{t} a_i(\xi)d\xi} \\
& + \left(\sum_{j=1}^{n} q_{ij}(t)x_j(t - \tau_j(t))\right) - \int_{t_0}^{t} a_i(s)\, e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} q_{ij}(s)x_j(s - \tau_j(s))\right) ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \overline{a_{ij}}(s)x_j(s)\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s))\, ds \\
& + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s - \delta_j(s)))\, ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(x_j(s))dw_j(s) \\
= & \sum_{m=1}^{7} Q_{im}(t) ,
\end{aligned} \tag{4.16}
$$

where,

$$Q_{i1}(t)=\left[\varphi_i(t_0)-\left(\sum_{j=1}^{n}q_{ij}(t_0)\varphi_j(t_0-\tau_j(t_0))\right)\right]e^{-\int_{t_0}^{t}a_i(\xi)d\xi},$$

$$Q_{i2}(t)=\sum_{j=1}^{n}q_{ij}(t)x_j(t-\tau_j(t)),$$

$$Q_{i3}(t)=\int_{t_0}^{t}a_i(s)\,e^{-\int_s^t a_i(\xi)d\xi}\left(\sum_{j=1}^{n}q_{ij}(s)x_j(s-\tau_j(s))\right)ds,$$

$$Q_{i4}(t)=\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\overline{a_{ij}}(s)x_j(s)\,ds,$$

$$Q_{i5}(t)=\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}b_{ij}(s)f_j(x_j(s))\,ds,$$

$$Q_{i6}(t)=\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}c_{ij}(s)g_j(x_j(s-\delta_j(s)))\,ds,$$

$$Q_{i7}(t)=\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\sigma_{ij}(x_j(s))dw_j(s).$$

Now we split the rest of our proof into three steps.

First step: We prove that $\mathcal{P}\left(X_{\varphi}^{l}\right)\subset X_{\varphi}^{l}$. First we show the mean square continuity of $\mathcal{P}$ on $[t_0,\infty)$. For $x_i\in X_{\varphi_i}^{l_i}$, it is necessary to show that $\mathcal{P}_i(x_i)\in X_{\varphi_i}^{l_i}$. It is clear that $\mathcal{P}_i$ is continuous on $[m(t_0),t_0]$. For fixed time $t\geq t_0$, each $i\in\{1,2,3,...,n\}$, $x_i\in X_{\varphi_i}^{l_i}$, and $|r|$ be sufficiently small, we then have

$$\mathbb{E}\left|(\mathcal{P}_i(x_i))(t+r)-(\mathcal{P}_i(x_i))(t)\right|^2\leq 7\sum_{m=1}^{7}\mathbb{E}\left|Q_{im}(t+r)-Q_{im}(t)\right|^2. \quad (4.17)$$

We must prove the mean square continuity of $\mathcal{P}_i$ on $[t_0,\infty[$.

It is easy to obtain that

$$\mathbb{E}\left|Q_{im}(t+r)-Q_{im}(t)\right|^2\to 0,\ \text{as } r\to 0,\ i=1,2,...,6.$$

As for the last term,

$\mathbb{E}\left|Q_{i7}(t+r)-Q_{i7}(t)\right|^2$

$$
\begin{aligned}
&= \mathbb{E}\left| \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left(e^{-\int_t^{t+r} a_i(\xi)d\xi} - 1 \right) \sum_{j=1}^{n} \sigma_{ij}(x_j(s))dw_j(s) \right. \\
&\left. + \int_t^{t+r} e^{-\int_s^{t+r} a_i(\xi)d\xi} \sum_{j=1}^{n} \sigma_{ij}(x_j(s))dw_j(s) \right|^2 \\
&\leq 2\mathbb{E}\left| \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \left(e^{-\int_t^{t+r} a_i(\xi)d\xi} - 1 \right) \sigma_{ij}(x_j(s))dw_j(s) \right|^2 \\
&+ 2\mathbb{E}\left| \sum_{j=1}^{n} \int_t^{t+r} e^{-\int_s^{t+r} a_i(\xi)d\xi} \sigma_{ij}(x_j(s))dw_j(s) \right|^2 \\
&\leq 2\mathbb{E}\left(\sum_{j=1}^{n} \int_{t_0}^{t} e^{-2\int_s^t a_i(\xi)d\xi} \left(e^{-\int_t^{t+r} a_i(\xi)d\xi} - 1 \right)^2 \sigma_{ij}^2(x_j(s))ds \right) \\
&+ 2\mathbb{E}\left(\sum_{j=1}^{n} \int_t^{t+r} e^{-2\int_s^{t+r} a_i(\xi)d\xi} \sigma_{ij}^2(x_j(s))ds \right) \to 0,
\end{aligned}
$$

as $\varepsilon \to \infty$. Thus, $\mathcal{P}_i$ ($i = 1, 2, ..., n$) is mean square continuous on $[t_0, \infty)$. Then $\mathcal{P}$ is indeed mean square continuous on $[t_0, \infty)$.

Next, we verify that $\|\mathcal{P}(x)\|_X \leq l$. Let φ be a small bounded initial function with $\|\varphi\| < \delta$, where we choose $\delta > 0$, $(\delta < l)$ such that

$$
2\delta \sum_{i=1}^{n} \left(1 + \sum_{j=1}^{n} |q_{ij}(t_0)| \right)^2 M_i^2 \leq l^2 (1 - 4\gamma). \tag{4.18}
$$

Let $x \in X_\varphi^l$, then $\|x\|_X \leq l$. Since f, g, σ, satisfy a Lipschitz condition, it follows from (4.16), condition (4.11) and L^p-Doob inequality that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \sup_{t \geq m(t_0)} |(\mathcal{P}_i x_i)(t)|^2 \right] \\
&\leq 2\sum_{i=1}^{n} \left[|\varphi_i(t_0)| + \left(\sum_{j=1}^{n} |q_{ij}(t_0)| \left| \varphi_j(t_0 - \tau_j(t_0)) \right| \right) \right]^2 e^{-2\int_{t_0}^{t} a_i(\xi)d\xi} \\
&+ 4\sum_{i=1}^{n} \left\{ \mathbb{E}\sup_{t \geq t_0} \left[\left(\sum_{j=1}^{n} |q_{ij}(t)| \left| x_j(t - \tau_j(t)) \right| \right) \right. \right. \\
&+ \int_{t_0}^{t} |a_i(s)| e^{-\int_s^t a_i(\xi)d\xi} \left(\sum_{j=1}^{n} |q_{ij}(s)| \left| x_j(s - \tau_j(s)) \right| \right) ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} |\overline{a_{ij}}(s)| \left| x_j(s) \right| ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi} \sum_{j=1}^{n} |b_{ij}(s)| \left| f_j(x_j(s)) \right| ds
\end{aligned}
$$

$$+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|c_{ij}(s)|\,|g_j\left(x_j(s-\delta_j\left(s\right))\right)|\,ds\Bigg]^2$$

$$+4\sum_{i=1}^{n}\mathbb{E}\sup_{t\geq t_0}\left[\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|\sigma_{ij}(x_j(s))|\,dw_j\left(s\right)\right]^2.$$

Therefore,

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t\geq m(t_0)}|\left(\mathcal{P}_i x_i\right)(t)|^2\right]$$

$$\leq 2\sum_{i=1}^{n}|\varphi_i(t_0)|^2\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 e^{-2\int_{t_0}^t a_i(\xi)d\xi}$$

$$+4\left[\sum_{i=1}^{n}\left(\mathbb{E}\sup_{s\geq m(t_0)}|x_j(s))|^2\right)\right]\Bigg\{\sum_{i=1}^{n}\sup_{t\geq t_0}\Bigg[\left(\sum_{j=1}^{n}|q_{ij}(t)|\right)$$

$$+\int_{t_0}^{t}|a_i\left(s\right)|\,e^{-\int_s^t a_i(\xi)d\xi}\left(\sum_{j=1}^{n}|q_{ij}(s)|\right)ds+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|\overline{a_{ij}}(s)|\,ds$$

$$+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|b_{ij}(s)|\,\alpha_j ds+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}|c_{ij}(s)|\,\beta_j ds\Bigg]^2$$

$$+4\sum_{j=1}^{n}\int_{t_0}^{t}L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi}\Bigg\}$$

$$\leq 2\sum_{i=1}^{n}|\varphi_i(t_0)|^2\left[1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right]^2 e^{-2\int_{t_0}^t a_i(\xi)d\xi}+4\gamma\sum_{i=1}^{n}\left(\mathbb{E}\sup_{s\geq m(t_0)}|x_j(s))|^2\right)$$

$$\leq 2\delta\sum_{i=1}^{n}\left(1+\sum_{j=1}^{n}|q_{ij}(t_0)|\right)^2 e^{-2\int_{t_0}^t a_i(\xi)d\xi}+4\gamma l^2.$$

By applying (4.18), we see that $\sum_{i=1}^{n}\left(\mathbb{E}\sup_{t\geq m(t_0)}|\left(\mathcal{P}_i x_i\right)(t)|^2\right)\leq l^2\left(1-4\gamma\right)+4\gamma l^2=l^2$. Hence, $\|\mathcal{P}x\|_X\leq l$ for $t\in\left[m\left(t_0\right),\infty\right)$ because $\|\mathcal{P}x\|_X=\|\varphi\|\leq l$ for $t\in\left[m\left(t_0\right),t_0\right]$.

We will prove that $\mathbb{E}\sum_{i=1}^{n}|\left(\mathcal{P}_i\left(x_i\right)\right)(t)|^2\to 0$ as $t\to\infty$. Indeed, $\mathbb{E}\left|x_i\left(t\right)\right|^2\to 0$ as $t\to\infty$. Then, for any $\varepsilon>0$, there exists $T_1>0$, such that $t\geq T_1$ we have $\mathbb{E}\left|x_i\left(t\right)\right|^2<\varepsilon$, for $i=1,2,...,n$. Hence

$$\mathbb{E}\left|Q_{i7}\left(t\right)\right|^2\leq\mathbb{E}\int_{t_0}^{T_1}e^{-2\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left|\sigma_{ij}^2(x_j(s))\right|ds$$

$$+\mathbb{E}\int_{T_1}^{t} e^{-2\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left|\sigma_{ij}^2(x_j(s))\right| ds$$
$$\leq \sum_{j=1}^{n} L_{ij}^2 \mathbb{E}\left(\sup_{s>m(t_0)} |x_j(s)|\right)^2 e^{-2\int_{T_1}^t a_i(\xi)d\xi}\left(\int_{t_0}^{T_1} e^{-2\int_s^{T_1} a_i(\xi)d\xi} ds\right)$$
$$+\sum_{j=1}^{n} L_{ij}^2 \varepsilon \left(\int_{T_1}^{t} e^{-2\int_s^t a_i(\xi)d\xi} ds\right).$$

By using condition (4.12), there is $T_2 \geq T_1$ such that when $t \geq T_2$ we have
$$\sum_{j=1}^{n} L_{ij}^2 \mathbb{E}\left(\sup_{s>m(t_0)} |x_j(s)|\right)^2 e^{-2\int_{T_1}^t a_i(\xi)d\xi}\left(\int_{t_0}^{T_1} e^{-2\int_s^{T_1} a_i(\xi)d\xi} ds\right) \leq (1-\gamma)\,\varepsilon.$$

By condition (4.11) we have $\mathbb{E}\left|Q_{i7}(t)\right|^2 \leq \gamma\varepsilon + (1-\gamma)\varepsilon = \varepsilon$. Thus $\mathbb{E}\left(\left|Q_{i7}(t)\right|^2\right) \to 0$ as $t \to \infty$. Similarly, we can show that $\mathbb{E}\left(\left|Q_{im}(t)\right|^2\right) \to 0$ $(m = 1, 2, ..., 7)$ as $t \to \infty$. This implies $\mathbb{E}\left|(\mathcal{P}_i(x_i))(t)\right|^2 \to 0$ as $t \to \infty$, and hence, $\mathcal{P}_i\left(X_{\varphi_i}^{l_i}\right) \subset X_{\varphi_i}^{l_i}$, for $i = 1, 2, ..., n$. Then $\mathcal{P}\left(X_{\varphi}^{l}\right) \subset X_{\varphi}^{l}$.

Second step: Now we will show that $\mathcal{P}$ has a unique fixed point x in X_{φ}^{l}. For any $x = (x_1, x_2, ..., x_n)^T \in X_{\varphi}^{l}$, $y = (y_1, y_2, ..., y_n)^T \in X_{\varphi}^{l}$, we have
$$\mathbb{E}\left(\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|(\mathcal{P}_i x_i)(t) - (\mathcal{P}_i y_i)(t)\right|^2\right)$$
$$\leq \mathbb{E}\left(\sum_{i=1}^{n}\sup_{t\geq t_0}\left|\sum_{j=1}^{n} q_{ij}(t)\left[x_j(t-\tau_j(t)) - y_j(t-\tau_j(t))\right]\right.\right.$$
$$-\int_{t_0}^{t} a_i(s)\, e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n} q_{ij}(s)\left[x_j(s-\tau_j(s)) - y_j(s-\tau_j(s))\right] ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\overline{a_{ij}}(s)\left[x_j(s) - y_j(s)\right] ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n} b_{ij}(s)\left[f_j(x_j(s)) - f_j(y_j(s))\right] ds$$
$$+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n} c_{ij}(s)\left[g_j(x_j(s-\delta_j(s))) - g_j(y_j(s-\delta_j(s)))\right] ds$$
$$\left.\left.+\int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left[\sigma_{ij}(x_j(s)) - \sigma_{ij}(y_j(s))\right] dw_j(s)\right|\right)^2.$$

By using the Doob L^p-inequality (see Karatzas & Shreve, 1991),

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t\geq t_0}\left|\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\sum_{j=1}^{n}\left[\sigma_{ij}(x_j(s))-\sigma_{ij}(y_j(s))\right]dw_j\left(s\right)\right|\right]^2\\
\leq\;& 4\mathbb{E}\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t a_i(\xi)d\xi}\left|\sigma_{ij}(x_j(s))-\sigma_{ij}(y_j(s))\right|^2 ds\right)\\
\leq\;& 4\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}^2\sup_{t\geq t_0}\left(\int_{t_0}^{t}e^{-2\int_s^t a_i(\xi)d\xi}\mathbb{E}\sum_{j=1}^{n}\left(\sup_{s\geq m(t_0)}\left|x_j(s))-y_j(s))\right|^2\right)ds\right).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\left\{\mathbb{E}\sum_{i=1}^{n}\sup_{t\geq m(t_0)}\left|\left(\mathcal{P}_i x_i\right)(t)-\left(\mathcal{P}_i y_i\right)(t)\right|^2\right\}^{\frac{1}{2}}\\
&\leq\sqrt{2}\left\{\left[\mathbb{E}\sum_{i=1}^{n}\left(\sup_{t\geq m(t_0)}\left|x_i\left(t\right)-y_i\left(t\right)\right|^2\right)\right]\right\}^{\frac{1}{2}}\\
&\times\left\{\sum_{i=1}^{n}\sup_{t\geq t_0}\left[\sum_{j=1}^{n}\left(\left|q_{ij}(t)\right|+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\left|\overline{a_{ij}}(s)\right|ds\right.\right.\right.\\
&\left.\left.+\int_{t_0}^{t}e^{-\int_s^t a_i(\xi)d\xi}\left|q_{ij}(s)\right|\left|a_i\left(s\right)\right|ds+\left|b_{ij}(s)\right|\alpha_j ds+\left|c_{ij}(s)\right|\beta_j\right)ds\right)\right]^2\\
&\left.+4\sum_{j=1}^{n}\int_{t_0}^{t}L_{ij}^2 e^{-2\int_s^t a_i(\xi)d\xi}\right\}^{\frac{1}{2}}.
\end{aligned}
$$

By condition (4.11), $\mathcal{P}$ is a contraction mapping with constant $\sqrt{2\gamma}$. Thanks to the contraction mapping principle (Smart, 1974, p. 2), we deduce that $\mathcal{P}: X_{\varphi}^{l}\rightarrow X_{\varphi}^{l}$ possesses a unique fixed point $x(t)=\left(x_1\left(t\right),x_2\left(t\right),...,x_n\left(t\right)\right)$ in X_{φ}^{l}, which is the unique solution of (4.1) with $x\left(s\right)=\varphi\left(s\right)$ on $s\in\left[m\left(t_0\right),t_0\right]$ and $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t,t_0,\varphi_i\right)\right|^2\rightarrow 0$ as $t\rightarrow\infty$.

Referring to (Burton, 2006; Dib, Maroun & Raffoul, 2005; Raffoul, 2004), except for the fixed point method, we know of another way to prove that solutions of (4.1) are stable. Let $\varepsilon>0$ be given such that $0<\varepsilon<l$. Replacing l by ε in X_{φ}^{l}, we obtain that there is $\delta>0$ such that $\|\varphi\|<\delta$ implies that the unique solution u of (4.1) with $x=\varphi$ on $\left[m\left(t_0\right),t_0\right]$ satisfies

$\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t,t_0,\varphi_i\right)\right|^2<\varepsilon$ for all $t\geq m\left(t_0\right)$. Moreover $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t,t_0,\varphi_i\right)\right|^2\to 0$ as $t\to\infty$. This also shows that the zero solution of (4.1) is asymptotically stable if (4.12) holds.

Third step: We will prove that the zero solution of (4.1) is mean-square asymptotically stable. Let $\varepsilon>0$ be given and choose $\delta>0$ ($\delta<\varepsilon$) satisfying

$$4\delta\sum_{i=1}^{n}\left[1+\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\right]^2 M_i^2<(1-2\gamma)\,\varepsilon, \tag{4.19}$$

where γ is the left hand side of (4.11). If $x\left(t\right)=x\left(t,t_0,\varphi\right)$ is a solution of (4.1) with the initial condition (4.3) satisfying $\left\|\varphi\right\|^2<\delta$, then $x\left(t\right)=\left(\mathcal{P}x\right)\left(t\right)$ as defined in (4.16). We claim that $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t\right)\right|^2<\varepsilon$ for all $t\geq t_0$. Notice that $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t\right)\right|^2<\varepsilon$ on $t\in\left[m\left(t_0\right),t_0\right]$, we suppose that there exists $t^*>t_0$ such that $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t^*\right)\right|^2=\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t\right)\right|^2<\varepsilon$ for $m\left(t_0\right)\leq t\leq t^*$. Then, it follows from (4.19) and (4.16) that

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t^*\right)\right|^2\leq 4\mathbb{E}\sum_{i=1}^{n}\left|\varphi_i\left(t_0\right)\right|^2\left[1+\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\right]^2 e^{-2\int_{t_0}^{t^*}a_i(\xi)d\xi}\\
&+2\varepsilon\sum_{i=1}^{n}\left\{\left[\sum_{j=1}^{n}\left(\left|q_{ij}(t^*)\right|+\int_{t_0}^{t^*}e^{-\int_s^{t^*}a_i(\xi)d\xi}\left|\overline{a_{ij}}(s)\right|ds\right.\right.\right.\\
&\left.\left.\left.+\int_{t_0}^{t^*}e^{-\int_s^{t^*}a_i(\xi)d\xi}\left|q_{ij}(s)\right|\left|a_i\left(s\right)\right|ds\right)\right]^2+L_{ij}^2\int_{t_0}^{t^*}e^{-2\int_s^{t^*}a_i(\xi)d\xi}ds\right\}\\
&\leq 4\delta\sum_{i=1}^{n}\left[1+\sum_{j=1}^{n}\left|q_{ij}(t_0)\right|\right]^2 M_i+2\gamma\varepsilon<(1-2\gamma)\,\varepsilon+2\gamma\varepsilon=\varepsilon,
\end{aligned}$$

which contradicts that $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t^*\right)\right|^2=\varepsilon$. Thus $\mathbb{E}\sum_{i=1}^{n}\left|x_i\left(t\right)\right|^2<\varepsilon$ for all $t\geq t_0$, and the zero solution of (4.1) is stable. This shows that the zero solution of (4.1) is asymptotically stable if (4.12) holds.

Conversely, we suppose that (4.12) fails. For each i fixed, $i \in \{1, 2, ..., n\}$. From (4.10), there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} a_i(s)ds = \xi_i$ for some $\xi_i \in \mathbb{R}$. We may also choose a positive constant J_i satisfying

$$-J_i \leq \int_0^{t_n} a_i(s)ds \leq J_i, \tag{4.20}$$

for all $n \geq 1$. To simplify the expression, we define

$$F_i(s) := \sum_{j=1}^{n} \left[|\overline{a_{ij}}(s)| + |q_{ij}(s)a_i(s)| + |b_{ij}(s)|\,\alpha_j + |c_{ij}(s)|\,\beta_j \right],$$

for all $s \geq 0$. From (4.11), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} a_i(\xi)d\xi} F_i(s)\,ds \leq \sqrt{\gamma}, \tag{4.21}$$

wich implies that

$$\int_0^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\,ds \leq \sqrt{\gamma} e^{\int_0^{t_n} a_i(\xi)d\xi} \leq \sqrt{\gamma} e^{M_i}. \tag{4.22}$$

The sequence $\left\{ \int_0^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\,ds \right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty} \int_0^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\,ds = \theta_i, \tag{4.23}$$

for some $\theta_i \in \mathbb{R}^+$ and choose a positive integer m large enough that

$$\int_{t_m}^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\,ds \leq \frac{\delta_0}{8M_i}, \tag{4.24}$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies

$$2\delta_0^2 M_i^2 e^{2J_i} \left(1 + \sum_{j=1}^{n} |q_{ij}(t_m)| \right)^2 \leq (1 - 4\gamma).$$

Now we consider the solution $x_i(t) = x_i(t, t_m, \varphi_i)$ of (4.1) with $\|\varphi_i(t_m)\| = \delta_0$ and $\|\varphi_i(s)\| \leq \delta_0$ for $s < t_m$. If we replace l_i by 1 in the proof of

$\|\mathcal{P}_i(x_i)\|_X \le l_i$, we have $\mathbb{E}|x_i(t)|^2 < 1$ for $t \ge t_m$. We may choose φ_i so that

$$G_i(t_m) := \varphi_i(t_m) - \sum_{j=1}^{n} q_{ij}(t_m)\varphi_j(t_m - \tau_j(t_m)) \ge \frac{\delta_0}{2}. \tag{4.25}$$

It follows from (4.16), (4.24) and (4.25) with $x_i(t) = (\mathcal{P}x_i)(t)$ that for $n \ge m$,

$$\begin{aligned}
&\mathbb{E}\left|x_i(t_n) - \sum_{j=1}^{n} q_{ij}(t_n)x_i(t_n - \tau_j(t_n))\right|^2 \\
\ge\;& G_i^2(t_m)\, e^{-2\int_{t_m}^{t_n} a_i(\xi)d\xi} - 2G_i(t_m)\, e^{-\int_{t_m}^{t_n} a_i(\xi)d\xi} \int_{t_m}^{t_n} e^{-\int_s^{t_n} a_i(\xi)d\xi} F_i(s)\, ds \\
\ge\;& \frac{\delta_0}{2} e^{-2\int_{t_m}^{t_n} a_i(\xi)d\xi} \left(\frac{\delta_0}{2} - 2M_i \int_{t_m}^{t_n} e^{\int_0^s a_i(\xi)d\xi} F_i(s)\, ds\right) \\
\ge\;& \frac{\delta_0^2}{8} e^{-2M_i} > 0.
\end{aligned} \tag{4.26}$$

If the zero solution of (4.1) is mean square asymptotically stable, then $\mathbb{E}|x_i(t)|^2 = \mathbb{E}|x_i(t, t_m, \varphi_i)|^2 \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$, for $j = 1, 2, ..., n$ and condition (4.11) holds, we have

$$\mathbb{E}\left|x_i(t_n) - \sum_{j=1}^{n} q_{ij}(t_n)x_i(t_n - \tau_j(t_n))\right|^2 \to 0,$$

as $n \to \infty$, which contradicts (4.26). Hence condition (4.12) is necessary in order that (4.1) has a solution $\mathbb{E}|x_i(t, t_0, \varphi_i)|^2 \to 0$ as $t \to \infty$. The proof is complete.

Remark 4.1 *When $a_i(t) = a_i$ and a_i are positive scalars, then Theorem 4.1 becomes Theorem A, which was recently stated in (Guo et al., 2017). Therefore, paper (Guo et al., 2017) is a particular case of ours. But we would like to emphasize that the proof in (Guo et al., 2017) is not completely correct since they claim that the spaces denoted by X or X^n with the norm*

$\|x_i\|_{[0,t]} = \left\{ \mathbb{E}(\sup_{s\in[0,t]} |x_i(s,\omega)|^2 \right\}^{\frac{1}{2}}$ *are Banach spaces and they use this fact in the proof (Guo et al., 2017, p. 1557), but this statement is not correct. However, in our investigation we use a different space which is indeed a complete metric space.*

Remark 4.2 *It follows from the first part of the proof of Theorem 4.1 that the zero solution of (4.1) is stable under (4.11). Moreover, Theorem 4.1 still holds if (4.11) is satisfied for $t \geq t_\rho$ for some $t_\rho \in \mathbb{R}^+$.*

Application Example for a system of SDDEs

In this section, we analyze an example to illustrate two facts. On the one hand, we will show how to apply our main result in this paper, Theorem 4.1. On the other hand and most importantly, we will highlight the real interest and importance of our result because the previous theory developed by Guo et al. (2017) cannot be applied to this example.

Example 4.1 [26] *Consider the following two-dimensional stochastic delay differential equation*

$$\begin{aligned} d\left[x(t) - Q(t)\,x(t-\tau(t))\right] &= \left[A(t)\,x(t) + B(t)\,x(t-\tau(t))\right]dt \\ +G(t)\,x(t-\tau(t))dw(t)\,,t &\geq 0, \end{aligned} \tag{4.27}$$

where

$$Q(t) = \begin{pmatrix} -\dfrac{\sin t}{8} & 0 \\ 0.025 & \dfrac{3\sin t}{200} \end{pmatrix}, A(t) = \begin{pmatrix} -\dfrac{0.112}{t+1} & 0 \\ 0 & -\dfrac{0.125}{t+1} \end{pmatrix},$$

$$B(t) = \begin{pmatrix} -\dfrac{0.112}{20(t+1)^2} & 0 \\ -\dfrac{5\times10^{-3}}{(t+1)^2} & -\dfrac{3\times10^{-2}}{4(t+1)^2} \end{pmatrix}, G(t) = \sqrt{\dfrac{0.001}{2(t+1)}}I.$$

By straightforward computations, we can check that condition (4.11) in Theorem 4.1 holds true, where $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous functions which satisfies $t - \tau(t) \to \infty$ and $t \to \infty$ and choosing $a_1(t) = \dfrac{0.112}{t+1}, a_2(t) = \dfrac{0.125}{t+1}$, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg\{\Bigg[\sum_{j=1}^{2}\Bigg(|q_{ij}(t)| + \int_0^t e^{-\int_t^s a_i(\xi)d\xi}\,|\overline{a_{ij}}(s)|\,ds \\
&+ \int_0^t e^{-\int_t^s a_i(u)du}\,|a_i(s)|\,|q_{ij}(s)|\,ds + \int_0^t e^{-\int_t^s a_i(u)du}\,|b_{ij}(s)|\,ds\Bigg)\Bigg]^2 \\
&+ 4\sum_{j=1}^{2}\int_0^t \left|g_{ij}^2(s)\right| e^{-2\int_t^s a_i(\xi)d\xi}ds\Bigg\} < 0.1224 + 0.017 < \frac{1}{4},
\end{aligned} \tag{4.28}$$

and since $\int_0^t a_1(s)\,ds = \int_0^t \frac{0.112}{s+1}ds = 0.112\ln(t+1) \to \infty$ and $\int_0^t a_2(s)\,ds$ $= \int_0^t \frac{0.125}{s+1}ds = 0.125\ln(t+1) \to \infty$ as $t \to \infty$. It is easy to see that all the conditions of Theorem 4.1 hold for $\gamma \simeq 0.1394 < 0.25$. Thus, Theorem 4.1 implies that the zero solution of (4.27) is asymptotically stable. ■

Remark 4.3 *Observe that Example 4.1 cannot be analyzed by applying Theorem A (see also Theorem 4.1 in Guo et al.,* 2017*). Indeed, in order to apply Theorem A, we need to check that there exist positive constants* a_1, a_2 *such that (4.9) holds. However, notice that, for any (fixed)* $a_1 > 0$*, if we set* $\overline{a_{11}}(t) = a_1 - \dfrac{0.112}{t+1}$*, we have that there exists* $T_0 > 0$ *such that* $\overline{a_{11}}(t) > \dfrac{3a_1}{4}$ *for all* $t \geq T_0$*. Consequently for one of the integrals appearing in (4.9) we deduce, for* $t > T_0$*,*

$$\begin{aligned}
\int_0^t e^{-a_1(t-s)}\left|a_1 - \frac{0.112}{s+1}\right| ds &\geq \int_{T_0}^t e^{-a_1(t-s)}\left|a_1 - \frac{0.112}{s+1}\right| ds \\
&> \frac{3}{4}a_1\int_{T_0}^t e^{-a_1(t-s)}ds \\
&= \frac{3}{4}\left[1 - e^{-a_1(t-T_0)}\right].
\end{aligned}$$

Then, it is clear that there exists $T_1 \geq T_0$ such that for $t \geq T_1$,

$$\frac{3}{4}\left[1 - e^{-a_1(t-T_0)}\right] > \frac{1}{4},$$

which implies that (4.9) cannot hold true.

Remark 4.4 *Theorem 4.1 contains all the stability results for* (4.1) *discussed in* [22, 54, 82]*. Note that, in addition, in our results the delays can be unbounded and that the coefficients can change sign. See Examples* $2.4 - 2.8$ *and Example* 3.1*. Our results are news attempt of applying the fixed point theory to the stability analysis of SDDEs with variable delays, which is different from the existing relevant publications where Lyapunov theory is the main technique. From what have been discussed above, we see that the contraction mapping principle is effective for not only the investigation of the existence and uniqueness of solution but also for the stability analysis of trivial equilibrium.In the future, we will continue to explore the application of other kinds of fixed point theorems to the stability research of complex stochastic delay differential equations.*

CHAPTER 5

Others Works

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5.4 Stability results for neutral differential equations by Krasnoselskii fixed point theorem 103

This chapter collects the others works have been published in [23], [24] , [25] , [27] . The approach we used in these papers is based on a fixed point method to show the stability and asymptotic stability of the zero solution.

5.1 Mean square asymptotic stability in nonlinear stochastic neutral Volterra-Levin equations with Poisson jumps and variable delays

Keywords: Fixed points theory, Poisson jumps, asymptotically stable in mean square, neutral stochastic differential equations, variable delays.

The goal of this paper is to present a very recent work published in [24], namely, M. Benhadri, H. Zeghdoudi, *Mean square asymptotic stability in nonlinear stochastic neutral Volterra-Levin equations with Poisson jumps and variable delays, Functiones et Approximatio Commentarii Mathematici Volume* 58,*Number* $2, 157-176, (2018)$.

In this section, we use the contraction mapping principle to obtain mean square asymptotic stability results of a nonlinear stochastic neutral Volterra-Levin equation with Poisson jumps and variable delays. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Burton (2004), Becker and Burton (2006) and Jin and Luo (2008), Ardjouni and Djoudi (2015).

Model description

Now, we consider the following stochastic neutral Volterra-Levin equation with variable delays and Poisson jumps

$$
\begin{aligned}
&d\left[x(t)-Q\left(t,x\left(t-\tau_{1}\left(t\right)\right)\right)\right]=-\left(\int_{t-\tau_{1}(t)}^{t}a(t,s)x(s)ds\right)dt\\
&+G\left(t,x\left(t\right),x(t-\tau_{2}\left(t\right))\right)dw\left(t\right)\\
&+\int_{-\infty}^{+\infty}h\left(t,x\left(t\right),x(t-\tau_{3}\left(t\right)),u\right)\widetilde{N}\left(dt,du\right),\quad t\geq0,
\end{aligned}
\tag{5.1}
$$

with the intial condition

$$
x(t)=\psi\left(t\right)\ \text{ for } t\in\left[m\left(0\right),0\right],
$$

where $\psi\in C\left(\left[m\left(0\right),0\right],\mathbb{R}\right)$,

$$
m\left(0\right)=\min\left\{\inf\left(s-\tau_{j}\left(s\right),s\geq0\right),j=1,2,3\right\},
$$

and $\left\{N\left(dt,du\right),t\in\mathbb{R}^{+},u\in\mathbb{R}\right\}$ is a centred Poisson randon measure with parameter $\pi\left(du\right)dt$.

Where $Q:\mathbb{R}^{+}\times\mathbb{R}\rightarrow\mathbb{R}, G:\mathbb{R}^{+}\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}, h:\mathbb{R}^{+}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{+}\rightarrow\mathbb{R}$ are continuous, $x:\left[m\left(0\right),\infty\right[\times\Omega\rightarrow\mathbb{R}$, and $a\in C\left(\mathbb{R}^{+}\times\left[m\left(0\right),\infty\right[,\mathbb{R}\right)$, and $\tau_{j}\in C\left(\mathbb{R}^{+},\mathbb{R}^{+}\right)$ satisfy $t-\tau_{j}\left(t\right)\rightarrow\infty$ as $t\rightarrow\infty$ for $j=1,2,3$. $\widetilde{N}\left(dt,du\right)=N\left(dt,du\right)-\pi\left(du\right)dt$ is a compensated Poisson random measure which is independent of $\left\{W\left(t\right)\right\}$. Suppose $\int_{-\infty}^{+\infty}\pi\left(du\right)<\infty$ and the follwing conditions are satisfied:

(i) There exists a positive constant $K_{1}>0$ such that for all $x,y\in\mathbb{R}$,

$$
\left|Q\left(t,x\right)-Q\left(t,y\right)\right|\leq K_{1}\left|x-y\right|.
$$

We also assume that

$$
Q\left(t,0\right)=0.
$$

(ii) The global Lipschitz condition: there exists a positive constant $K_{2}>0$ such that

$$
\begin{aligned}
|G(t,x_1,y_1)-G(t,x_2,y_2)|^2 \vee \int_{-\infty}^{+\infty} |h(t,x_1,y_1,u)-h(t,x_2,y_2,u)|^2 \pi(du) \\
\leq K_2\left(|x_1-x_2|^2+|y_1-y_2|^2\right),
\end{aligned}
$$

where $x_1, y_1, x_2, y_2 \in \mathbb{R}$. We also assume that

$$
G(t,0,0) = h(t,0,0,u) = 0.
$$

Lemma 5.1. [24] *Equation* (5.1) *is equivalent to*

$$
\begin{aligned}
d\left[x(t)-Q(t,x(t-\tau_1(t)))\right] &= B(t,t-\tau_1(t))(1-\tau_1'(t))x(t-\tau_1(t)) \\
&+\frac{d}{dt}\int_{t-\tau_1(t)}^{t} B(t,s)x(s)ds \\
&+G(t,x(t),x(t-\tau_2(t)))\,dw(t) \\
&+\int_{-\infty}^{+\infty} h(t,x(t),x(t-\tau_3(t)),u)\,\widetilde{N}(dt,du),\, t\geq 0,
\end{aligned}
$$

where

$$
B(t,s)=\int_t^s a(u,s)du \quad \text{and} \quad B(t,t-\tau_1(t))=\int_t^{t-\tau_1(t)} a(u,t-\tau_1(t))du.
$$

Theorem 5.1. [24] *Suppose that τ_1 is differentiable, and there exist continuous functions $H : [m(0),\infty[\to \mathbb{R}$ and a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$
\liminf_{t\to\infty} \int_0^t H(s)ds > -\infty,
$$

$$\begin{aligned}
&K_1 + \int_{s-\tau_1(s)}^{s} |H(z) + B(s,z)|\, dz \\
&+ \int_0^s e^{-\int_z^s H(u)du} \left[|(H(z-\tau_1(z)) + B(z, z-\tau_1(z))\,(1-\tau_1'(z)))| + K_1 \,|H\,(z)| \right] dz \\
&+ \int_0^s e^{-\int_z^s H(u)du} \,|H\,(z)| \left(\int_{z-\tau_1(z)}^{z} |H(u) + B(z,u)|\, du \right) dz \\
&+2 \left(2K_2 \int_0^s e^{-2\int_z^s H(u)du} dz \right)^{\frac{1}{2}} \leq \alpha,
\end{aligned}$$

and for $s \in [0,t]$ and a positive constant $\alpha < 1$, the following inequality holds, where

$$B(t,s) = \int_t^s a(u,s)du \quad with \quad B(t, t-\tau_1\,(t)) = \int_t^{t-\tau_1(t)} a(u, t-\tau_1\,(t))du.$$

Then the zero solution of (5.1) *is mean square asymtotically stable if and only if*

$$\int_0^t H\,(s)\, ds \to \infty \;\; as \;\; t \to \infty.$$

For more details see Benhadri and Zeghdoudi (2018).

5.2 Stability analysis of neutral stochastic differential equations with Poisson jumps and variable delays

Keywords: Fixed points theory, asymptotic stability, neutral differential equations, variable delays.

In this work, we expose the work cited in [25] as follow:

M. Benhadri, H. Zeghdoudi,Stability analysis of neutral stochastic differential equations with Poisson jumps and variable delays, Applied Mathemat-

ics E-Notes, $476-496, 19(2019)$.

This section proofs some results on the mean square asymptotic stability of the zero solution for a class of neutral stochastic differential with Poisson jumps and variable delays by using a contraction mapping principle. A mean square asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Zhao (2011).

Model description

We consider the linear neutral stochastic differential equation with variable delays and Poisson jumps:

$$\begin{aligned} & d\left(x(t)-\frac{c\left(t\right)}{1-\tau_1'\left(t\right)}x\left(t-\tau_1\left(t\right)\right)\right) \\ = & \left(-a(t)x(t-\tau_1\left(t\right))-\frac{d}{dt}\left(\frac{c\left(t\right)}{1-\tau_1'\left(t\right)}\right)x\left(t-\tau_1\left(t\right)\right)\right)dt \\ & +\Sigma\left(t\right)x(t-\tau_2\left(t\right))dw\left(t\right)+\Gamma\left(t\right)x(t-\tau_3\left(t\right))d\widetilde{N}\left(t\right),\ t\geq t_0, \quad (5.2) \end{aligned}$$

denote $x\left(t\right)\in\mathbb{R}$ the solution to (5.2) with the intial condition

$$x(t)=\psi\left(t\right) \text{ for } t\in\left[m\left(t_0\right),t_0\right],$$

and $\psi\in C\left(\left[m\left(t_0\right),t_0\right],\mathbb{R}\right)$, where $a,b,\Sigma,\Gamma\in C\left(\mathbb{R}^+,\mathbb{R}\right)$, $c\in C^1\left(\mathbb{R}^+,\mathbb{R}\right)$ and $\tau_i\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ satisfy $t-\tau_i\left(t\right)\rightarrow\infty$ as $t\rightarrow\infty, i=1,2,3$ and for each $t_0\geq 0$,

$$m_i\left(t_0\right)=\inf\left\{t-\tau_i\left(t\right),t\geq t_0\right\},m\left(t_0\right)=\min\left\{m_i\left(t_0\right),i=1,2,3\right\}.$$

Theorem 5.2. [25] *Let τ_1 be twice differentiable and suppose that $\tau_1'\left(t\right)\neq 1$ for all $t\in\left[m\left(t_0\right),\infty\right[$. Suppose that*

(i) there exists a bounded function $p:\left[m\left(t_0\right),\infty\right[\rightarrow(0,\infty)$ with $p(t)=1$ for $t\in\left[m\left(t_0\right),t_0\right]$ and $p'\left(t\right)$ exists for all $t\in\left[m\left(t_0\right),\infty\right[$, and there exists an

arbitrary continuous function $h : [m(t_0), \infty[\to \mathbb{R}$ *and a constant* $\gamma \in \left(0, \frac{1}{4}\right)$ *such that for any* $t \geq t_0$,

$$\Bigg[\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{(1-\tau_1'(t))}\right| + \int_{t-\tau_1(t)}^{t}\left|h(s) - \frac{p'(s)}{p(s)}\right| ds$$

$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du}|h(s)|\left(\int_{s-\tau_1(s)}^{s}\left|h(u) - \frac{p'(u)}{p(u)}\right| du\right) ds$$

$$+\int_{t_0}^{t} e^{-\int_s^t h(u)du}\left|-\bar{b}(s) + \left(h(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \bar{k}(s)\right| ds\Bigg]^2$$

$$+4\int_{t_0}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Sigma(s)\, p(s-\tau_2(s))}{p(s)}\right|^2 ds$$

$$+4\beta\int_{t_0}^{t} e^{-2\int_s^t h(u)du}\left|\frac{\Gamma(s)\, p(s-\tau_3(s))}{p(s)}\right|^2 ds \leq \gamma,$$

where

$$\bar{k}(t) = \frac{[C(t)\, h(t) + C'(t)]\,(1-\tau_1'(t)) + C(t)\,\tau_1''(t)}{(1-\tau_1'(t))^2},$$

and

$$\bar{b}(t) = \frac{a(t)\, p(t-\tau_1(t)) - c(t)p'(t-\tau_1(t))}{p(t)}, C(t) = \frac{c(t)\, p(t-\tau_1(t))}{p(t)}.$$

ii) and such that

$$\liminf_{t\to\infty}\int_{t_0}^{t} h(s)ds > -\infty.$$

Then the zero solution of (5.2) *is mean-square asymptotic stable if and only if*

$$\int_{t_0}^{t} h(s)ds \to \infty \text{ as } t \to \infty.$$

The technique for constructing a contraction mapping comes from an idea in Zhao (2011). For an extensive presentation, the reader is referred to Benhadri and Zeghdoudi (2019) of which study the stability analysis of neutral stochastic differential equations with Poisson jumps and variable delays.

5.3 Existence of solutions and stability for impulsive neutral stochastic functional differential equations

Keywords: Fixed points theory, Asymptotic stability in mean square, Neutral stochastic differential equations, Variable delays, Impulses.

The goal of this paper is to present a very recent work published in [27], namely, M. Benhadri, T. Caraballo, H. Zeghdoudi, *Existence of solutions and stability for impulsive neutral stochastic functional differential equations, Stochastic analysis and applications,. no.*$5, 777-798, 37(2019)$.

This section proofs some results on the existence of solutions and the mean square asymptotic stability for a class of impulsive neutral stochastic differential systems with variable delays by using a contraction mapping principle. Namely, a sufficient condition ensuring the asymptotic stability is proved. The assumptions do not impose any restrictions neither on boundedness nor on the differentiability of the delay functions. In particular, the results improve some previous ones in the literature.

Model description

We consider the following class of impulsive neutral stochastic differential systems with variable delays:

$$d\left[u_i(t)-\sum_{j=1}^{n} q_{ij}(t)u_j(t-\tau_j(t))\right]=\left[\sum_{j=1}^{n} a_{ij}(t)u_j(t)+\sum_{j=1}^{n} b_{ij}(t)f_j(u_j(t))\right.$$
$$\left.+\sum_{j=1}^{n} c_{ij}(t)g_j(u_j(t-\delta_j(t)))\right]dt+\sum_{j=1}^{n}\sigma_{ij}(u_j(t))dw_j(t),\, t\geq t_0, t\neq t_k,$$
$$\Delta u_i(t_k)=u_i(t_k+0)-u_i(t_k)=I_{ik}(u_i(t_k)),\, k=1,2,..., \tag{5.3}$$

This can be written in a vector–matrix form as follows:

$$d\left[u(t)-Q\left(t\right)u(t-\tau\left(t\right))\right]=\left[A\left(t\right)u(t)+B(t)f\left(u(t)\right)+C(t)g\left(u(t-\delta\left(t\right))\right]dt$$
$$+\sigma\left(u(t)\right)dw\left(t\right),t\geq t_0,t\neq t_k,$$
$$\Delta u_i\left(t_k\right)=u_i\left(t_k+0\right)-u_i\left(t_k\right)=I_{ik}\left(u_i\left(t_k\right)\right),k=1,2,...,$$

for $i=1,2,3,...,n$, where $u(t)=\left[u_1(t),u_2(t),..,u_n(t)\right]^T\in\mathbb{R}^n$, and a_{ij},b_{ij}, c_{ij}, $q_{ij}\in C\left(\mathbb{R}^+,\mathbb{R}\right)$, are continuous functions, $A\left(t\right)=\left(a_{ij}(t)\right)_{n\times n}$, $B\left(t\right)=\left(b_{ij}(t)\right)_{n\times n}$, $Q\left(t\right)=\left(q_{ij}(t)\right)_{n\times n}$, are real matrices and $\sigma(\cdot)=\left(\sigma_{ij}(\cdot)\right)_{n\times n}$ is the diffusion coefficient matrix, $f(u\left(t\right))=\left[f_1\left(u_1(t)\right),f_2\left(u_2(t)\right),...,f_n\left(u_n(t)\right)\right]^T\in\mathbb{R}^n$, $g(u\left(t\right))=\left[g_1\left(u_1(t)\right),g_2\left(u_2(t)\right),...,g_n\left(u_n(t)\right)\right]^T\in\mathbb{R}^n$, and $\tau_j,\delta_j,j=1,...,n$, which are the variable delays, are continuous functions satisfying appropriate conditions described below.

Before proceeding, we firstly introduce some assumptions:

(A1) The delay functions $\tau_j,\delta_j\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ satisfy

$$t-\delta_j\left(t\right)\to\infty \text{ and } t-\tau_j\left(t\right)\to\infty \text{ as } t\to\infty \text{ for } j=1,2,...,n.$$

(A2) there exist nonnegative constants α_j such that for all $x,y\in\mathbb{R}$,

$$\left|f_j\left(x\right)-f_j\left(y\right)\right|\leq\alpha_j\left|x-y\right|,\ j=1,2,...,n.$$

(A3) there exist nonnegative constants β_j such that for all $x,y\in\mathbb{R}$,

$$\left|g_j\left(x\right)-g_j\left(y\right)\right|\leq\beta_j\left|x-y\right|,\ j=1,2,...,n.$$

(A4) there exist nonnegative constants L_{ij} such that for all $x,y\in\mathbb{R}$,

$$\left|\sigma_{ij}\left(x\right)-\sigma_{ij}\left(y\right)\right|\leq L_{ij}\left|x-y\right|,\ i,j=1,2,...,n.$$

(A5) there exist nonnegative constants p_{ik} such that for all $x,y\in\mathbb{R}$,

$$\left|I_{ik}\left(x\right)-I_{ik}\left(y\right)\right|\leq p_{ik}\left|x-y\right|,\ i=1,2,...,n,k=1,2,...$$

Throughout this paper, we always assume that

$$f_j(0) = g_j(0) = I_{ij}\,(0) = \sigma_{ik}(0) = 0,\ \text{for } i, j = 1, 2, \ldots, n, k = 1, 2...,$$

which imply that problem (5.3) admits a trivial equilibrium $u = 0$.

Now, we can state our main result.

Theorem 5.3. [27] *Suppose that assumptions (A1)–(A5) hold and there exist continuous functions $a_i : [t_0, \infty) \to \mathbb{R}^+$ such that:*

1) there exists a constant μ satisfying $\inf\{t_k - t_{k-1}\} \geq \mu$, for $k = 1, 2, ...$;

2) there exist constants p_i such that $p_{ik} \leq p_i\mu$ for $i = 1, 2, 3, ..., n$ and $k = 1, 2, ...$;

3) there exist constants $n_i > 0$ such that $a_i\,(t) \geq n_i, t \in \mathbb{R}^+$ for $i = 1, 2, 3, ..., n$;

4) for $t \in \mathbb{R}^+$, $t \geq t_0$ and a positive constant $\gamma < \dfrac{1}{4}$, the following inequality holds:

$$\begin{aligned}
&\sum_{i=1}^{n}\Bigg\{\Bigg[\sum_{j=1}^{n}\Bigg(|q_{ij}(t)| + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\,|\overline{a_{ij}}(s)|\,ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\,|q_{ij}(s)|\,|a_i\,(s)|\,ds + \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\,|b_{ij}(s)|\,\alpha_j ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t a_i(\xi)d\xi}\,|c_{ij}(s)|\,\beta_j ds\Bigg)\Bigg]^2 \\
&+ 4\sum_{j=1}^{n}\int_{t_0}^{t} L_{ij}^2 e^{-2\int_t^s a_i(\xi)d\xi} ds + p_i^2\left(\frac{1}{n_i} + \mu\right)^2\Bigg\} \leq \gamma < \frac{1}{4},
\end{aligned}$$

where $\overline{a_{ij}}(t) = a_{ij}(t)(i \neq j)$, $\overline{a_{ii}}(t) = a_{ii}\,(t) + a_i\,(t)$. Then for any $\varphi \in C\left([m\,(t_0)\,, t_0]\,, \mathbb{R}^n\right)$ there exists a unique global solution $u\,(t, t_0, \varphi)$. Moreover, the zero solution of (5.3) is mean-square asymptotically stable. For more details see Benhadri et al.(2019).

5.4 Stability results for neutral differential equations by Krasnoselskii fixed point theorem

Keywords: Fixed points theory, Stability, Neutral differential equations, Integral equation,Variable delays.

The goal of this paper is to present a very recent work published in [23], namely, M. Benhadri, *Stability results for neutral differential equations by Krasnoselskii fixed point theorem, Differential equations and dynamical systems, (2019)*.

This section considers a neutral differential equation with variable delays and give some new conditions for the boundedness and stability results by using of Krasnoselskii's fixed point theorem. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved, which improves and generalizes some results due to Burton and Furumochi (2002), and Jin and Luo (2008).

Model description

In this section, we consider the following class of neutral differential equations with variable delays,

$$x'(t) = -a(t)x(t-\tau_1(t)) + c(t)\,x'(t-\tau_1(t)) + b(t)\,x^{\sigma}(t-\tau_2(t)),\ t \geq t_0, \tag{5.4}$$

denote $x(t) \in \mathbb{R}$ the solution to (5.4) with the intial condition

$$x(t) = \psi(t) \ \text{ for } t \in [m(t_0), t_0],$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$, $\sigma \in (0,1)$ is a quotient with odd positive integer denominator. We assume that $a, b \in C(\mathbb{R}^+, \mathbb{R})$, $c \in C^1(\mathbb{R}^+, \mathbb{R})$ and

$\tau_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfy $t - \tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2$ and for each $t_0 \geq 0$,

$$m_i(t_0) = \inf\{t - \tau_i(t), t \geq t_0\}, m(t_0) = \min\{m_i(t_0), i = 1, 2\}.$$

For each $(t_0, \psi) \in \mathbb{R}^+ \times C([m(t_0), t_0], \mathbb{R})$, a solution of (5.4) through (t_0, ψ) is a continuous function $x : [m(t_0), t_0 + \rho) \to \mathbb{R}$ for some positive constant $\rho > 0$ such that x satisfies (5.4) on $[t_0, t_0 + \rho)$ and $x = \psi$ on $[m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \psi)$. We define $\|\psi\| = \max\{|\psi(t)| : m(t_0) \leq t \leq t_0\}$.

As we mentioned previously, one of our objectives in this section is to generalize the work carried out in Jin and Luo (2008) to the case in which the neutral term $c(t)x'(t - \tau_1(t))$ is taken into account in the problem, and allowing the coefficients to be more general. In other words, we will establish and prove a necessary and sufficient condition ensuring the boundedness of solutions and the asymptotic stability of the zero solution to Equation (5.4). However, the mathematical analysis used in this research to construct the mapping to employ Krasnoselskii's fixed point theorem is different than that of Jin and Luo (2008).

Now, we can state our main result.

Theorem 5.4. [23] *Let τ_1 be twice differentiable and suppose that $\tau_1'(t) \neq 1$ for all $t \in [m(t_0), \infty[$.* Suppose that

(i) there exists a bounded function $p : [m(t_0), \infty[\to (0, \infty)$ with $p(t) = 1$ for $t \in [m(t_0), t_0]$ such that $p'(t)$ exists for all $t \in [m(t_0), \infty[$, and that there are constants $\alpha \in (0, 1)$, $k_1, k_2 > 0$, and an arbitrary continuous function $g \in C([m(t_0), \infty), \mathbb{R}^+)$ such that for $|t_1 - t_2| \leq 1$,

$$\left|\int_{t_1}^{t_2} \left|b(u)\frac{p^\sigma(u - \tau_2(u))}{p(u)}\right| du\right| \leq k_1 |t_1 - t_2|,$$

and

$$\left|\int_{t_1}^{t_2} g(u)\, du\right| \leq k_2 |t_1 - t_2|,$$

while for $t \geq t_0$

$$\begin{aligned}
&\left|\frac{p(t-\tau_1(t))}{p(t)}\frac{c(t)}{1-\tau_1'(t)}\right| + \int_{t-\tau_1(t)}^{t}\left|g(u) - \frac{p'(u)}{p(u)}\right| du \\
&+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}\left\{\left|-\overline{\mu}(s) + \left(g(s-\tau_1(s)) - \frac{p'(s-\tau_1(s))}{p(s-\tau_1(s))}\right)(1-\tau_1'(s)) - \overline{\beta}(s)\right|\right\} ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}|g(s)|\left(\int_{s-\tau_1(s)}^{s}\left|g(u) - \frac{p'(u)}{p(u)}\right| du\right) ds \\
&+ \int_{t_0}^{t} e^{-\int_s^t g(u)du}|b(s)|\left|\frac{p^{\sigma}(s-\tau_2(s))}{p(s)}\right| ds < \alpha,
\end{aligned}$$

where

$$\overline{\mu}(t) = \frac{a(t)\,p(t-\tau_1(t)) - c(t)p'(t-\tau_1(t))}{p(t)}, C(t) = \frac{c(t)\,p(t-\tau_1(t))}{p(t)}.$$

and

$$\overline{\beta}(t) = \frac{[C(t)\,h(t) + C'(t)]\,(1-\tau_1'(t)) + C(t)\,\tau_1''(t)}{(1-\tau_1'(t))^2}.$$

If ψ is a given continuous initial function which is sufficiently small, then there is a solution $x(t, t_0, \psi)$ of (5.4) *on $\mathbb{R}^+$ with* $|x(t, t_0, \psi)| \leq 1$. For more details see Benhadri (2019).

Conclusion and perspective

One of the most important qualitative aspects of differential equations is determining the stability of a given model. Stability of solution to neutral differential equations has been studied using several methods; among them we have Lyapunov functionals [28, 29], fixed points [20] and characteristic equations [33, 81]. Each of these methods has its own advantages and disadvantages. This thesis studies a stability of determinsitic and stochastic delay differential equations. The approach used in our project is based on fixed point technique, particularly, using the Banach contraction principle. This approach relies mainly on three principles: an elementary variation of parameters formula, a complete metric space and a contraction mapping principle. The benefit of this approach is that the fixed point arguments can yield existence, uniqueness and stability of a system in one step. The main difficulty of this approach is to define a suitable complete metric space and a suitable mapping. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction with respect to the norm. What are the limitations of this technique?. We will point out the important limitation that the Banach fixed point theorem gives uniqueness of solutions only within the complete metric space where it is defined.

If the metric space onto which we apply the contraction mapping principle is too small, then we are not obtaining a satisfactory uniqueness result. The results in this thesis extend and improve some exist results in the literature in some ways. Recently, Benhadri et al in [24, 25, 27] have addressed this technique to investigate a wider class of stochastic neutral differential equations with impulsive effects, poisson jumps. However, there are many problems to be solved for the stochastic differential equations, such as stability of these equations driven by factional Brownian motion, persistence, and so on. We leave these for our future work.

Annexe

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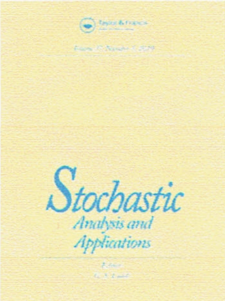
Annex



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Existence of solutions and stability for impulsive neutral stochastic functional differential equations

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MEAN SQUARE ASYMPTOTIC STABILITY IN NONLINEAR STOCHASTIC NEUTRAL VOLTERRA-LEVIN EQUATIONS WITH POISSON JUMPS AND VARIABLE DELAYS

Mimia Benhadri, Halim Zeghdoudi

Abstract: In this paper, we use the contraction mapping principle to obtain mean square asymptotic stability results of a nonlinear stochastic neutral Volterra-Levin equation with Poisson jumps and variable delays. An asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Burton [5], Becker and Burton [4] and Jin and Luo [10], Ardjouni and Djoudi [1]. Finally, an illustrative example is given.

Keywords: fixed points theory, Poisson jumps, asymptotically stable in mean square, neutral stochastic differential equations, variable delays.

1. Introduction

In recent years, the stability of stochastic differential equations has been studied by using Lyapunov functions, which has led to a lot of better results, see for example, [18–20] and so on. Unfortunately, a number of difficulties have been encountered in the study of stability by means of Lyapunov's direct method. Luckily, Luo [14] and Burton et al. [2,3, 5–7] have successfully solved these problems by the application of fixed point theory. Since the method is in its initial stages, we are sure that the investigators will obtain much better results than by using the method of Lyapunov functions which is old and has been previously made in the literature.

Very recently, many scholars have begun to deal with the stability of stochastic delay differential equations by using fixed point theory (see, for example, [1, 6, 14–26]). More precisely, Appleby [3] and Burton [6] (see pp. 315–328) considered the almost sure stability for some classical equations by splitting the stochastic

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ORIGINAL RESEARCH

Stability Results for Neutral Differential Equations by Krasnoselskii Fixed Point Theorem

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Abstract
In this paper we consider a neutral differential equation with variable delays and give some new conditions for the boundedness and stability results by using of Krasnoselskii's fixed point theorem. Namely, a necessary and sufficient condition ensuring the asymptotic stability is proved, which improves and generalizes some results due to Burton and Furumochi (Dyn Syst Appl 11:499–519, 2002), and Jin and Luo (Nonlinear Anal 68:3307–3315, 2008). Finally, an example is exhibited to illustrate the effectiveness of the proposed results.

Keywords Fixed points theory · Stability · Neutral differential equations · Integral equation · Variable delays

Mathematics Subject Classification 34K20 · 34K30 · 34B40

Introduction

Liapunov's direct method has long been viewed the main classical method to study stability problems in several areas of differential equations. The success of Liapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. However, it may be difficult to look for such a good Liapunov functional for some classes of delay differential equations. Therefore, an alternative approach may be explored to overcome such difficulties. To this end, Burton and other authors have applied the fixed point theory to investigate the stability of deterministic systems during the last years, and have obtained some more applicable conclusions which can be found, for example, in the monograph [8] and the works [1–3,5–7,10–12,15,16,30]. In addition, there are some papers where the fixed point theory is used to investigate the stability of stochastic (delayed) differential equations (see for instance [20,21,29]). For example, Luo [20] studied the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of Banach's fixed point theory. It turns out that the fixed point method is becoming a powerful technique in dealing with stability problems for deterministic and stochastic differential equations with delays.

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Stability Analysis Of Neutral Stochastic Differential Equations With Poisson Jumps And Variable Delays*

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Abstract

In this paper, we prove some results on the mean square asymptotic stability of the zero solution for a class of neutral stochastic differential with Poisson jumps and variable delays by using a contraction mapping principle. A mean square asymptotic stability theorem with a necessary and sufficient condition is proved, which improves and generalizes some previous results due to Dianli Zhao [35]. Finally, an example is exhibited to illustrate the effectiveness of the proposed results.

1 Introduction

In nature, physics, society, engineering, and so on we always meet two kinds of functions with respect to time: one is deterministic and another is random. Stochastic differential equations were first initiated and developed by K. Itô [9]. Today they have become a very powerful tool applied to mathematics, physics, biology, finance, and so forth. Real systems depend on not only present and past states but also involve derivatives with delays. As a result, these systems are often built in the form of neutral differential equations. Practical examples of neutral delay differential systems include the distributed networks containing lossless transmission lines [4], population ecology [15], and other engineering systems [13]. For neutral stochastic delay differential equations, we refer to [14, 22, 27].

It is well known that Lyapunov's method has been the classical technique to study stability of deterministic and stochastic differential equations and functional differential equations for more than 100 years, for example [26, 27]. However, there are a lot of difficulties to construct Lyapunov functions for examining stability. Burton in the monograph [3] and the works [1, 2, 5, 10, 11, 29, 35, 37, 38] have successfully applied fixed point theory to overcome these problems. In addition, there are some papers where the fixed point theory is used to investigate the stability of stochastic (delayed) differential equations (see for instance [17, 18, 20, 21, 33]). More precisely, Luo [17] studied the mean square asymptotic stability for a class of linear scalar neutral stochastic differential equations by means of fixed point theory. Furthermore, Luo [18–19]

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