REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEUR
ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITE MENTOURI CONSTANTINE

FACULTE DES SCIENCES
DEPARTEMENT DE MATHEMATIQUES

N° d'ordre : 12/T.E/2006
N° série : 03/MATH/2006

THESE

Présentée pour l'obtention du diplôme de

DOCTORAT D'ETAT EN MATHEMATIQUES

Thème

PROBLEMES AUX LIMITES AVEC CONDITIONS AUX BORD NON LOCALES

Option

MATHEMATIQUES APPLIQUEES

Par

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Soutenue le 22-02-06

ACKNOWLEDGEMENTS

I would like to express my special thanks to my supervisor Mr. **Marhoune A. L.**, Lecturer at Mentouri University Constantine, who have direct this work, for his valuable advice, his encouragements and his big availability and open-mindedness that helped me a lot, to achieve this work.

I do thank strongly Mr. **Merouani B.**, Professor at Farhat Abbes University Sétif, that honored me by accepting to lead the examination jury.

I address my most cordial acknowledgements to Mr. **Zerzaïhi T.**, Lecturer at M. S. Benyahia University Jijel; Mr. **Messaoudi K.**, Lecturer at Hadj Lakhdar University Batna; Mr. **Benkafadar M.N.**, Professor at Mentouri University Constantine and Mr. **Djezar S.**, Lecturer at Mentouri University Constantine; who accepted to be part of the jury and to accept judging this work.

I would not forget all those who provided me with a continuous moral support, all along this work, in particular Mr. **Denche M**. Professor at Mentouri University Constantine and Mr. **Zarza T**., Lecturer at Mentouri University Constantine, once more I thank them for their incessant encouragements.

Contents

INTRODUCTION

The method of energy inequalities, known also as the method of functional analysis, has its origin in the works of A. A. Dezin [15]. It was then applied and developed in many works such as : I. G. Petrovsky [25] for the resolution of the Cauchy problem related to equations of hyperbolic type, O. A. Ladysenskaja [23],], K. Fridricks [16] and N. I. Yurchuk [38, 39, 40, 41, 42, 43, 44, 45]. Afterwards the method has known important developments due to J. Leray [24] and L. Garding [17]. It was also used for the resolution of different problems in the domain of the theory of thermo conduction [7, 9, 20] and the physic of plasmas [32].

The present work is the object of an extension of the method of energy inequalities to a new class of problems with non local boundary conditions and integral conditions, it is also considered as an extension of the results obtained in [39].

The mixed problem with integral condition takes more and more interest as a result of the fundamental reason which is the basis of the physical significance of the integral condition as an average, a flux, a total energy, a moment, etc… These are mathematical models encountered in the theory of thermo conduction [7, 8, 19], in the memory materials [31] and in the semiconductors [1]. Such problems were studied in [2, 3, 4, 5, 7, 8, 9, 10, 12, 20, 21, 22, 35, 42] for parabolic equations, in [26, 27, 28, 29, 37] for the hyperbolic equations and in [12, 13, 14] for mixed type equations.

Description of the method

The method of energy inequalities is based on the research of an operator *Mu* known as multiplier. This last one depends on the function u, its derivatives and some weight function. We are then conducted to take integrations over the considered domain with a view to equipping E and F with appropriate norms in order to show the existence and uniqueness of the solution, said strong, of the considered problem once it has been made into the form

$$Lu=\mathscr{F}, \tag{1}$$

where $L:E\rightarrow F$ is the operator generated by the considered problem, E is an Banach space, F a Hilbert space, $u\in E$ and $\mathscr{F}\in F$.

The method is presented into two aspects :

1st aspect

We demonstrate two sided a priori inequalities

$$\|Lu\|_F \le C\|u\|_E \quad \forall u \in D(L) \tag{2}$$

$$\|u\|_E \le c\|Lu\|_F \quad \forall u \in D(L) \tag{3}$$

Where C and c are constants.

The uniqueness of the solution, said strong, of the considered problem results from these two inequalities.

From the inequality (2) results that the operator L is continuous and from the inequality (3) results that it has a continuous inverse and that the image $R(L)$ of L is closed. The operator L is then a linear homeomorphism from E in the closed $R(L)$, which proves the uniqueness of the solution. Its existence is ensured by the fact that $R(L)$ is dense in F.

2nd aspect

We demonstrate the energy inequality of the type

$$\|u\|_E \le C\|Lu\|_F \quad \forall u \in D(L), \tag{4}$$

where C is constant.

By passing to the limit, we extend the inequality (4) to $D(\overline{L})$. Since the image $R(\overline{L})$ of the operator $\overline{L}$, which plays an important role in this research, is closed in F and such that $R(\overline{L}) = \overline{R(L)}$, it is sufficient to show that $R(L)$ is dense in *F*, this can be done using the regularly operators which we choose according to the nature of the considered problem.

In this work, we used the first aspect, and the regularly operators were chosen with respect to the variable *t* introduce in [42].

The method of energy inequalities shows the advantages and disadvantages.

Advantages

- It is efficient for many problems where certain number of then is cited above.
- Its theoretical aspect is strong and its development is done in an abstract and elegant frame.
- The actuality of the problems treated by this method.

Disadvantages

A lot of difficulties are encountered during the search for

- The solution space.
- The multiplier.
- The regularly operator.

The elaboration of a technique which eliminates these difficulties is not yet available; this is due to the variety and actuality of the treated problems by this method.

In fact the application of the method requires a special study for each considered problem.

CHAPTER 1

DISSIPATIVE OPERATORS AND REGULARLY DISSIPATIVE OPERATORS

1. 1. DISSIPATIVE OPERATORS

Dissipative operators in Hilbert space

Let *H* be a Hilbert space, its inner product and norm will be denoted by (.,.) and ||.||, respectively and *A* an operator its domain *D*(*A*) is assumed to be dense in *H*.

Definition 1 The *operator A is called a dissipative operator if*

$$\mathrm{Re}(Au,u) \leq 0 \qquad \forall u \in D(A). \tag{1.1}$$

Definition 2 *The operator A is called an accretive operator if* (-*A*) *is a dissipative operator, i.e.*

$$\mathrm{Re}(Au,u) \geq 0 \qquad \forall u \in D(A). \tag{1.2}$$

Definition 3 *A dissipative operator which extends a dissipative operator A is called a dissipative extension of A.*

Definition 4 *An operator A is said to be maximal dissipative if its only dissipative extension is A itself.*

Accretive extensions and maximal accretive operators are defined similarly.

Proposition 1 *An operator A with its domain dense is dissipative if and only if*

$$\|(A+1)u\| \leq \|(A-1)u\| \qquad \forall u \in D(A). \tag{1.3}$$

Proposition 2 *Let A: D(A)⊂H → H be a linear operator with its domain dense. Then the following three statements are equivalent.*

(a) A is dissipative operator.

(b) $\|(A-\lambda)u\| \geq \mathrm{Re}\lambda\|u\|$ *for all u*∈*D*(*A*) *and all* λ *satisfying* Reλ*>0.*

(c) $\|(A-\lambda)u\| \geq \lambda\|u\|$ *for all u*∈*D*(*A*) *and all* λ*>0.*

Remark 1 *If a dissipative operator A is closed, it follows from (b) that R(A-*λ*) is closed subspace for all* λ *satisfying Re*λ*>0.*

Theorem 1 [36] *Any dissipative operator has a closed extension. The minimum closed extension of a dissipative operator is again a dissipative operator. Hence, a maximal dissipative operator is closed.*

Proposition 3 *If A is a dissipative operator and R(A-λ)=H for some* λ *satisfying* Reλ>0, *then A is maximal dissipative.*

Theorem 2 [36] *Every dissipative operator has a maximal dissipative extension.*

Proposition 4 *When A is a dissipative operator, then the following three conditions are equivalent*

(a) *A is a maximal dissipative operator.*

(b) *R(A-λ)=H for all $\lambda \in C$ satisfying* Reλ>0.

(c) *R(A-λ)=H for same $\lambda \in C$ satisfying* Reλ>0.

Theorem 3 [36] *Let A be a densely-defined linear operator. A is maximal dissipative if and only if it is closed, its resolvent set $\rho(A)$ contains the half plane* $\{\lambda : \mathrm{Re}\,\lambda > 0\}$ *and* $\|(A-\lambda)\| \le (\mathrm{Re}\,\lambda)^{-1}$ *holds there.*

Theorem 4 [36] *Let $A : D(A) \subset H \to H$ be a maximal dissipative operator with its domain dense and let* A_ε^{-1} *denote the operator* $(I-\varepsilon A)^{-1}$, *then the following three conditions are equivalent*

1- $A_\varepsilon^{-1} \in L(H)$.

2- $\|A_\varepsilon^{-1}\| \le 1$.

3- $\lim_{\varepsilon \to 0} A_\varepsilon^{-1} u = u \quad \forall u \in H$.

Dissipative operators in Banach space

The collection of all continuous linear functional defined on the whole of *X* (*X* is a real or a complex linear space) constitutes a space conjugate to *X* which is denoted *X**.

The space *X** is a Banach space with the norm

$$\|f\| = \sup_{\|u\| \le 1} |f(x)| \qquad f \in X^*, u \in X . \tag{1.4}$$

Definition 5 *Let X be a normed space and X* its conjugate. The set of all f*$\in X$ *which satisfy*

$$(u, f) = \|u\|^2 = \|f\|^2, \qquad (1.5)$$

for every u$\in X$ *is denoted by Fu. The F is called a duality mapping from X into X*.*

A generalization of dissipative operators in Hilbert space to those in Banach space is explained.

Let *X* be a complex Banach space and let *F* denote the duality mapping in *X*. We have

$$\|u\|^2 = \|f\|^2 = (u, f) \quad \forall u \in X, f \in Fu \subset X^*. \qquad (1.6)$$

Definition 6 *Let A be a linear operator in X. If for any u*$\in$ *D(A) there exists an f*$\in$ *Fu satisfying* $\mathrm{Re}(Au, f) \le 0$, *A is called a dissipative operator. If –A is dissipative, A is called an accretive operator.*

Proposition 5 *For any linear operator A, the following three conditions are equivalent*

(a) A is dissipative operator.

(b) $\|(A-\lambda)u\| \ge \mathrm{Re}\,\lambda\|u\| \quad \forall u \in D(A)$ *and all* λ *satisfying* $\mathrm{Re}\lambda>0$.

(c) $\|(A-\lambda)u\| \ge \lambda\|u\| \quad \forall u \in D(A)$ *and all* $\lambda>0$.

Theorem 5 [36] *Let A be a closed dissipative operator. If* $R(A-\lambda)=X$ *for some* λ *satisfying* $\mathrm{Re}\lambda>0$, *then the same is true for all* λ *satisfying* $\mathrm{Re}\lambda>0$.
If, in addition, D(A) is dense, then

$$\mathrm{Re}(Au,f) \le 0 \quad \forall u \in D(A), \forall f \in Fu. \qquad (1.7)$$

Theorem 6 [36] *Let A be a closed operator with its domain D(A) dense. Both A and A* are dissipative if and only if the half-plane* (λ / $\mathrm{Re}\lambda>0$) *is contained in* $\rho(A)$ *and* $\|(A-\lambda)^{-1}\| \le \dfrac{1}{\mathrm{Re}\,\lambda}$ *holds in the half-space.*

1. 2. REGULARLY DISSIPATIVE OPERATORS

Let X be a complex Hilbert space, its inner product and norm will be denoted by (.,.) and |.|, respectively.

Let V be another Hilbert space with inner product and norm denoted by ((.,.)) and ||.||, respectively.

We assume that V is embedded in X as a dense subspace and that V has a stronger topology than X. Therefore, there exists an M_0 such that

$$|u|<M_0\|u\| \qquad \forall u\in V.$$

Let $a(u,v)$ be a quadratic form defined on $V\times V$. That is, to each $u,v\in V$ there corresponds a complex number $a(u,v)$ which is linear in u an anti linear in v :

$$a(u_1+u_2, v)= a(u_1,v)+a(u_2, v) ,$$
$$a(u, v_1+ v_2)= a(u,v_1)+a(u,v_2),$$
$$a(\lambda u, v) = \lambda a(u, v),$$
$$a(u, \lambda v) = \overline{\lambda} a(u, v).$$

We assume that $a(u,v)$ is bounded, i.e., there exists a certain number M such that

$$|a(u,v)| < M\|u\|\cdot\|v\| \quad \forall u,v\in V\ . \tag{1.8}$$

We further assume that there exists a positive number δ and a real number k such that

$$\operatorname{Re} a(u,v) > \delta\|u\|^2 - k\|u\|^2 \quad \forall u\in V\ . \tag{1.9}$$

This inequality is called Garding's inequality.

In the particular case k=0, we obtain

$$\operatorname{Re} a(u,u) > \delta\|u\|^2 \quad \forall u\in V\ . \tag{1.10}$$

Using $a(u,v)$, an operator A is defined as follows

Given $u\in V$. If there exists an element f of X so that $a(u,v)=(f,v)$ for all $v\in V$, then $u\in D(A)$ and $Au=f$. (1.11)

The quadratic form $a(u,v)$, considered as a functional of v, is continuous in V-topology. If, in particular, it is also continuous in topology of V induced by X, $a(u,v)$ can be extended to X as a continuous functional. Hence, by the Riesz theorem, there exists an element f of X so that

$$a(u,v)=(f,v) \qquad \forall v\in V$$

In this case, we interpret $u\in D(A)$ and $Au=f$.

In studying such operator A, it is often convenient to extend it in the following way.

The space of all continuous anti linear functional defined on V and X are denoted by V^* and X^*, respectively.

That is, V^* and X^* are the spaces of all continuous functional l on V and X, which satisfy

$$l(u+v)=l(u)+l(v),$$
$$l(\lambda u)=\overline{\lambda}l'u).$$

for all $u,v\in V$ and X, and for all complex λ respectively.

For any element of V^* or X^*, its norm is defined similarly to be a continuous linear functional.

That is, the norm of l as element of V^* and X^* are given by

$$\|l\|_* = \sup_{\|v\|<1}|l(v)| \quad \text{and} \quad |l|_* = \sup_{|f|<1}|l(f)|,$$

respectively.

Let $l|_V$ denote the restriction of $l\in X^*$ to V, then

$$\left|l|_V(v)\right| = |l(v)| \le |l|\cdot|v| \le |l|\cdot M_0\|v\|, \tag{1.12}$$

hence, $l|_V \in V^*$.

Since V is dense in X, the correspondence $l \to l|_V$ is one-to-one, so that by identifying l with $l|_V$ we may consider $X^*\subset V^*$. Since $\|l_V\|_* \le M_0|l|_*$ by (1.12), X^* has a stronger topology than V^*.

Furthermore, the embedding $V \to X$ and $X \to V^*$ are both continuous. We can show that V is dense in V^* as follows.

If $v \in V$ satisfies $(u,v)=0$ for all u, it follows by taking $u=v$ that $v=0$. Accordingly, by the reflexivity of V and a result of functional analysis, V is dense in V^*, and, hence, X is also dense in V^*.

For $l \in V^*$ the value $l(v)$ of l at v is also denoted by (l,v). The use of this notation is convenient, because if, in particular, $l=f \in X$, it is seen from the meaning of $X \subset V^*$ that the notation represents just the inner product of f and v in X. We denote elements of V^* by f, g and so on, and sometimes $\overline{(f,v)}$ by (v,f). When $a(u,v)$ with $u \in V$ fixed is considered as a functional of v, it is an element of V^* by (1.8).

Therefore, using an element $f \in V^*$, we can express $a(u,v)=(f,v)$. Since f so obtained is determined by u, we write

$$\tilde{A}u = f \;.$$

That is, $\tilde{A}$ is an operator defined by

$$a(u,v) = (\tilde{A}u, v) \quad \forall u,v \in V \;. \tag{1.13}$$

It is obvious that $\tilde{A}$ is an extension of the operator A defined by (1.11). More precisely

$$D(A) = \left\{ u \in V : \tilde{A}u \in X \right\}. \tag{1.14}$$

Lemma 1 *Let be H a Hilbert space, whose inner product and norm will be denoted by (.,.) and ||.||, respectively. Assume that B[u,v] is a quadratic form defined on $H \times H$ and that there exist positive constants C and c such that*

$$|B[u,v]| \le C\|u\| \cdot \|v\|, \tag{1.15}$$

$$|B[u,v]| \ge c\|u\|^2 \qquad \forall u,v \in H \;. \tag{1.16}$$

Under these conditions, if $F \in H^*$, i.e. if F is a continuous anti linear functional on H, there exists an element u such that

$$F(v)=B[u,v] \qquad \forall v \in H.$$

Furthermore, u is uniquely determined by F.

Lemma 2 *$D(A)$ is dense in V. Therefore, it is also dense in X.*

Definition 7 *An operator A defined by (1.11), using a quadratic form satisfying* (1.8) *and* (1.10) *is called a regularly accretive operator. If –A is regularly accretive, A is called a regularly dissipative operator.*

Definition 8 *The quadratic form a*(u,v) defined by* $a^*(u,v) = \overline{a(v,u)}$ *is called an adjoint quadratic form.*

If $a(u,v)$ satisfies (1.8), (1.9) or (1.10), so, correspondingly, does $a^*(u,v)$.

Let $\tilde{A}$ be operator defined by

$$a(u,v) = (\tilde{A}u, v) \quad \forall u, v \in V, \tag{1.17}$$

and let A' and $\tilde{A}'$ be operators defined by $a^*(u,v)$ in ways similar to (1.11) and (1.17), respectively.

Let $u \in H$. If there exists an $f \in X$ such that $a^*(u,v)=(f,v)$ for all $v \in V$, then

$$u \in D(A') \text{ and } A'u=f. \tag{1.18}$$

$$a^*(u,v)=(\tilde{A}' u,v) \quad \forall u,v \in V. \tag{1.19}$$

Lemma 3 *Let A* be an adjoint of A when the latter is viewed as an operator in X. Then A'=A*.*

Theorem 7 [36] *A regularly accretive operator is maximal accretive.*

Theorem 8 [36] *Let a(u,v) be a quadratic form on* $V \times V$ *satisfying* (1.8) *and* (1.9), *and let A be the operator defined by* (1.11). *The domain D(A) is dense in V and also in X, and* $0 \in \rho(A+k)$. *Also let* $\tilde{A}$ *be the operator defined by* (1.17). *Then* $\tilde{A} + k$ *is an isomorphism from V onto V*.*

Let a(u,v) be the adjoint of a(u,v) and A' the operator defined by* (1.18). *Then the adjoint operator A* of A in X coincides with A'. A+k is a regularly accretive operator.*

Both A and $\tilde{A}$ are denoted simply by A. We also denote $\tilde{A}'$ by A^*. Therefore, we have

$$a(u,v)= (Au,v), \quad a^*(u,v)= (A^*u,v) \qquad \forall u,v \in V. \tag{1.20}$$

This notation will not cause any confusion.

When a*(u,v)= $a(u,v)$ holds for all $u,v\in V$, the quadratic form $a(u,v)$ is said to by symmetric. In this case, by theorem 8, an operator A in X is self-adjoint. It is evident that $a(u,v)$ is a real number for each $u\in V$, Since, by (1.9), we have

$$(Au,u) = a(u,u) \geq -k|u|^2 \quad \forall u\in D(A).$$

Theorem 9 [36] *If $a(u,v)$ is a symmetric quadratic form satisfying* (1.8) *and* (1.10), *then A is positive definite and self-adjoint, $D(A^{1/2})=V$, and*

$$a(u,v)= (A^{1/2}u, A^{1/2}v), \qquad \forall u,v\in V . \qquad (1.21)$$

Example Let be $A = \dfrac{\partial}{\partial t}$ where

$$D(A) = \{u \in L_2(\Omega) / u(0,t) = 0\} \text{ and } \Omega = (0,1)\times(0,T),$$

then A is an accretive operator.

Proof. $(Au,u) = \int_\Omega \frac{\partial u}{\partial t}\bar{u}\,dxdt = \int_0^1 u\bar{u}\Big|_0^T dx - \int_\Omega u\frac{\partial \bar{u}}{\partial t}dxdt$

Thus $(Au,u) + \overline{(Au,u)} = \int_0^1 |u|^2\Big|_T dx - \int_0^1 |u|^2\Big|_0 dx$

$2\,\mathrm{Re}(Au,u) = \int_0^1 |u(x,T)|^2 dx$ car $u(0,x)$=0, From where $\mathrm{Re}(Au,u) \geq 0$.

CHAPTER 2

MIXED PROBLEM WITH NON LOCAL BOUNDARY CONDITION FOR A HIGH ORDER PARTIAL DIFFERENTIAL EQUATION

2.1 Introduction

In this chapter we study a problem for a high-order differential equation with no classical boundary condition. The existence and uniqueness of the strong solution in functional weighted Sobolev space are proved. The proof is based in two sided a priori estimates and the fact that the range of operator generalized by the considered problem is dense.

2.2 Position of the problem

Let α be a positive integer and Ω be the set $(0,T)\times(0,1)$, we consider the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} + (-1)^{\alpha}\left(\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} + \frac{\alpha}{x}\frac{\partial^{2\alpha-1}u}{\partial x^{2\alpha-1}}\right) = f(t,x). \tag{2.1}$$

To equation (2.1) we attach the initial condition

$$lu = u(0,x) = \varphi(x) \qquad x \in (0,1), \tag{2.2}$$

the boundary conditions

$$\frac{\partial^i u(t,1)}{\partial x^i} = 0 \quad \text{for} \quad 0 \le i \le \alpha - 1 \qquad t \in (0,T), \tag{2.3}$$

$$\frac{\partial^i u(t,0)}{\partial x^i} = 0 \quad \text{for} \quad 0 \le i \le \alpha - 2 \qquad t \in (0,T), \tag{2.4}$$

and integral condition

$$\int_0^1 u(t,\xi)d\xi = 0 \qquad \text{for} \quad t \in (0,T), \tag{2.5}$$

Were φ is a known function which satisfy the compatibility conditions given in (2.3), (2.4) and (2.5).

Remark that the boundary value problems with integrals conditions are mainly motivated by the work of Samarskii [32]. Regular case of this problem for second order equations is studied in [19]. The problem where the equation of mixed type contains an operator of the form $a(t)\frac{\partial^{2\alpha+1}u}{\partial x^{2\alpha}\partial t}$ is treated in [12], the operator of the form $\frac{\partial}{\partial x}\left(a(t,x)\frac{\partial u}{\partial x}\right)$ and $\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(a(t,x)\frac{\partial^{\alpha}u}{\partial^{\alpha}x}\right)$ is treated in [22] and [13]. Similar problems for second order parabolic equations are investigated by the potential method in [21]. Two-point boundary value problems for parabolic equations, with an integral condition, are investigated using the energy inequalities method in [4, 5, 6, 8, 14, 42] and the Fourier method [18]. Three-point boundary value problem with an integral condition for parabolic equations with the Bessel operator is studied in [11].

2.3 Preliminaries

In this work, we prove the existence and the uniqueness of a solution of problem (2.1)-(2.5). For this, we consider the problem (2.1)-(2.5) as a solution of the operator equation

$$Lu=\mathscr{F},$$

where $L=(\mathcal{L},l)$. The operator L is acting from E to F, where E is the Banach space consisting of functions $u\in L_2(\Omega)$, satisfying (2.3), (2.4) and (2.5), with finite norm

$$\|u\|_E^2=\int_\Omega x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\int_\Omega\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 dxdt+\sup_{0\le t\le T}\int_0^1 x^2\left\{\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2+|u|^2\right\}dx \qquad (2.6)$$

and F is the Hilbert space of vector-valued functions $\mathscr{F}=(f,\varphi)$ obtained by completing the space $L_2(\Omega)\times W_2^{2\alpha}(0,1)$ with the following norm

$$\|\mathscr{F}\|_F^2=\|(f,\varphi)\|_F^2=\int_\Omega x^2|f(t,x)|^2 dxdt+\int_0^1 x^2\left\{\left|\frac{\partial^\alpha\varphi}{\partial x^\alpha}\right|^2+|\varphi|^2\right\}dx \qquad (2.7)$$

Using the energy inequalities method proposed in [22], we establish two-sided a priori estimates. Then we prove that the operator L is a linear homeomorphism between the space E and F.

Lemma 4 *For any function $u\in E$, we have*

$$\exp(-cT)\int_0^1 x^2|u(\tau,x)|^2 dx\le\int_0^1 x^2|\varphi|^2 dx+\int_0^1\int_0^\tau x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt \qquad (2.8)$$

with the constant c satisfying $c\ge 1$.

Proof. Integrating by part $\int_0^1\int_0^\tau \exp(-ct)x^2 u\frac{\partial \overline{u}}{\partial t}dtdx$ and using elementary inequalities yields (2.8). ■

Theorem 10 *The following a priori estimate*

$$\|Lu\|_F \le c\|u\|_E, \tag{2.9}$$

holds for any function $u\in E$*, where c is constant.*

Proof. Using equation (2.1) and initial condition (2.2) we obtain

$$\int_\Omega x^2 |\mathcal{L}|^2 dxdt \le 2\int_\Omega \left[x^2\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2\right]dxdt \tag{2.10}$$

$$\int_0^1 x^2\left|\frac{\partial^\alpha lu}{\partial x^\alpha}\right|^2 dx \le \sup_{0\le t\le T}\int_0^1 x^2\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2 dx \tag{2.11}$$

and

$$\int_0^1 x^2|lu|^2 dx \le \sup_{0\le t\le T}\int_0^1 x^2|u|^2 dx \tag{2.12}$$

Combining the inequalities (2.10), (2.11) and (2.12) we obtain (2.9) for $u\in E$. ■

Theorem 11 *For any function* $u\in E$*, we have the inequality*

$$\|u\|_E \le C\|Lu\|_F, \tag{2.13}$$

where the constant $C = \dfrac{3+2\alpha}{\inf(3, e^{-cT})}$.

Proof. Let

$$Jg = \int_x^1 g(t,\xi)d\xi$$

and

$$Mu = x^2\frac{\partial u}{\partial t} + \alpha x J\frac{\partial u}{\partial t}.$$

We consider the quadratic form

$$\operatorname{Re}\int_0^\tau\int_0^1 \mathcal{L}uM\bar{u}dxdt, \tag{2.14}$$

which is obtained by multiplying (2.1) by $\overline{Mu}$.

We have

$$\mathcal{L}uM\bar{u} = \left[\frac{\partial u}{\partial t} + (-1)^\alpha\left(\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} + \frac{\alpha}{x}\frac{\partial^{2\alpha-1}u}{\partial x^{2\alpha-1}}\right)\right]\left(x^2\frac{\partial\bar{u}}{\partial t} + \alpha xJ\frac{\partial\bar{u}}{\partial t}\right).$$

Integrating by report to x, we obtain

$$\int_0^1 \mathcal{L}uM\bar{u}dx = \int_0^1 x^2\frac{\partial u}{\partial t}\frac{\partial\bar{u}}{\partial t}dx + \int_0^1\frac{\partial u}{\partial t}\alpha xJ\frac{\partial\bar{u}}{\partial t}dx + (-1)^\alpha\int_0^1 x\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\frac{\partial\bar{u}}{\partial t}dx$$

$$+(-1)^\alpha\int_0^1\alpha\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)J\frac{\partial\bar{u}}{\partial t}dx \tag{2.15}$$

We have

$$\int_0^1 x^2\frac{\partial u}{\partial t}\frac{\partial\bar{u}}{\partial t}dx = \int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx$$

Integrating by parts the terms of the second member of (2.15) and by taking into account of the boundary conditions, we obtain

$$\int_0^1\frac{\partial u}{\partial t}\alpha xJ\frac{\partial\bar{u}}{\partial t}dx = -\alpha\int_0^1 x\frac{\partial}{\partial x}\left(J\frac{\partial u}{\partial t}\right)J\frac{\partial\bar{u}}{\partial t}dx$$

$$= -\left[J\frac{\partial u}{\partial t}xJ\frac{\partial\bar{u}}{\partial t}\right]_0^1 + \alpha\int_0^1 J\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(xJ\frac{\partial\bar{u}}{\partial t}\right)dx$$

$$\int_0^1\frac{\partial u}{\partial t}\alpha xJ\frac{\partial\bar{u}}{\partial t}dx = \alpha\int_0^1 J\frac{\partial u}{\partial t}J\frac{\partial\bar{u}}{\partial t}dx + \alpha\int_0^1 J\frac{\partial u}{\partial t}x\frac{\partial}{\partial x}J\frac{\partial\bar{u}}{\partial t}dx$$

$$= \alpha\int_0^1\left|J\frac{\partial u}{\partial t}\right|^2 dx - \alpha\int_0^1 J\frac{\partial u}{\partial t}x\frac{\partial\bar{u}}{\partial t}dx$$

From where

$$\operatorname{Re}\int_0^1\alpha x\frac{\partial u}{\partial t}J\frac{\partial\bar{u}}{\partial t}dx = \frac{\alpha}{2}\int_0^1\left|J\frac{\partial u}{\partial t}\right|^2 dx \tag{2.16}$$

$$(-1)^{\alpha}\int_0^1 x\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\bar{u}}{\partial t}dx=(-1)^{\alpha}\left[\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial\bar{u}}{\partial t}\right]_0^1$$

$$+(-1)^{\alpha+1}\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial}{\partial x}\left(x\frac{\partial\bar{u}}{\partial t}\right)dx$$

$$=(-1)^{\alpha+1}\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\bar{u}}{\partial t}dx+(-1)^{\alpha+1}\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^2\bar{u}}{\partial x\partial t}dx$$

$$=(-1)^{\alpha+1}\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\bar{u}}{\partial t}\right]_0^1+(-1)^{\alpha+2}\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}dx$$

$$+(-1)^{\alpha+1}\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^2\bar{u}}{\partial x\partial t}\right]_0^1+(-1)^{\alpha+2}\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial}{\partial x}\left(x\frac{\partial^2\bar{u}}{\partial x\partial t}\right)dx$$

$$=(-1)^{\alpha+2}\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}dx+(-1)^{\alpha+2}\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}dx$$

$$+(-1)^{\alpha+2}\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx$$

$$=(-1)^{\alpha+2}2\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}dx+(-1)^{\alpha+2}\int_0^1 x\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx$$

$$=(-1)^{\alpha+2}2\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}\right]_0^1+(-1)^{\alpha+3}2\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx$$

$$+(-1)^{\alpha+2}\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^3\bar{u}}{\partial x^2\partial t}\right]_0^1+(-1)^{\alpha+3}\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial}{\partial x}\left(x\frac{\partial^3\bar{u}}{\partial x^2\partial t}\right)dx$$

$$=(-1)^{\alpha+3}2\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx+(-1)^{\alpha+3}\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx$$

$$+(-1)^{\alpha+3}\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^4\bar{u}}{\partial x^3\partial t}dx$$

$$=(-1)^{\alpha+3}3\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\bar{u}}{\partial x^2\partial t}dx+(-1)^{\alpha+3}\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)x\frac{\partial^4\bar{u}}{\partial x^3\partial t}dx$$

Reasoning by recurrence, we obtain

$$(-1)^{\alpha}\int_0^1 x\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\overline{u}}{\partial t}dx=(-1)^{2\alpha-1}(\alpha-1)\left[\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^{\alpha-1}\overline{u}}{\partial x^{\alpha-2}\partial t}\right]_0^1$$

$$+(-1)^{2\alpha}(\alpha-1)\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}dx+(-1)^{2\alpha-1}\left[x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}x\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}\right]_0^1$$

$$+(-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial}{\partial x}\left(x\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha}}\right)dx$$

$$=(-1)^{2\alpha}(\alpha-1)\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}dx+(-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}dx$$

$$+(-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}x\frac{\partial^{\alpha+1}\overline{u}}{\partial x^{\alpha}\partial t}dx$$

$$=(-1)^{2\alpha}\alpha\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}dx+(-1)^{2\alpha}\int_0^1 x^2\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)dx$$

From where

$$(-1)^{\alpha}\int_0^1 x\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\overline{u}}{\partial t}dx=\alpha\int_0^1 x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{u}}{\partial x^{\alpha-1}\partial t}dx+\int_0^1 x^2\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)dx \qquad (2.17)$$

$$(-1)^{\alpha}\int_0^1\alpha\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)J\frac{\partial\overline{u}}{\partial t}dx=(-1)^{\alpha}\alpha\left[\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)J\frac{\partial\overline{u}}{\partial t}\right]_0^1$$

$$+(-1)^{\alpha+1}\alpha\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial}{\partial x}\left(J\frac{\partial\overline{u}}{\partial t}\right)dx$$

$$=(-1)^{\alpha+2}\alpha\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\overline{u}}{\partial t}dx$$

$$=(-1)^{\alpha+2}\alpha\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial\overline{u}}{\partial t}\right]_0^1+(-1)^{\alpha+3}\alpha\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\overline{u}}{\partial x\partial t}dx$$

$$=(-1)^{\alpha+3}\alpha\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^2\overline{u}}{\partial x\partial t}\right]_0^1+(-1)^{\alpha+4}\alpha\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\overline{u}}{\partial x^2\partial t}dx$$

$$=(-1)^{\alpha+4}\alpha\left[\frac{\partial^{\alpha-4}}{\partial x^{\alpha-4}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^3\overline{u}}{\partial x^2\partial t}\right]_0^1+(-1)^{\alpha+5}\alpha\int_0^1\frac{\partial^{\alpha-4}}{\partial x^{\alpha-4}}\left(x\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)\frac{\partial^4\overline{u}}{\partial x^3\partial t}dx$$

Reasoning by recurrence, we obtain

$$(-1)^{\alpha}\int_0^1 \alpha \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right) J\,\frac{\partial \bar{u}}{\partial t}\,dx = (-1)^{\alpha}\alpha\left[x\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial^{\alpha-1}\bar{u}}{\partial x^{\alpha-1}\partial t}\right]_0^1 + (-1)^{2\alpha+1}\alpha\int_0^1 x\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha-1}\partial t}dx$$

$$= -\alpha\int_0^1 x\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha-1}\partial t}dx\,.$$

From where

$$\mathrm{Re}\int_0^1 \mathcal{L}uM\bar{u}dx = \int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{\alpha}{2}\int_0^1\left|J\,\frac{\partial u}{\partial t}\right|^2 dx + \mathrm{Re}\int_0^1 x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\right)dx$$

$$+\,\mathrm{Re}\int_0^1 x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\right)dx\,. \qquad (2.18)$$

Integrating by parts the third term of (2.18)

$$\int_0^{\tau}\int_0^1 x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\right)dxdt = \left[\int_0^1\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right]_0^{\tau} - \int_0^{\tau}\int_0^1\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right)dxdt$$

$$= \int_0^1\frac{\partial^{\alpha}\bar{u}(\tau,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(\tau,x)}{\partial x^{\alpha}}dx - \int_0^1\frac{\partial^{\alpha}\bar{u}(0,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(0,x)}{\partial x^{\alpha}}dx$$

$$-\int_0^{\tau}\int_0^1\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\,x^2\,\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\right)dxdt,$$

from where

$$\mathrm{Re}\int_0^{\tau}\int_0^1 x^2\,\frac{\partial^{\alpha} u}{\partial x^{\alpha}}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha}\bar{u}}{\partial x^{\alpha}}\right)dxdt = \frac{1}{2}\int_0^1\frac{\partial^{\alpha}\bar{u}(\tau,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(\tau,x)}{\partial x^{\alpha}}dx$$

$$-\frac{1}{2}\int_0^1\frac{\partial^{\alpha}\bar{u}(0,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(0,x)}{\partial x^{\alpha}}dx\,.$$

Replacing in (2.18), we obtain

$$\mathrm{Re}\int_0^{\tau}\int_0^1 \mathcal{L}uM\bar{u}dx = \int_0^{\tau}\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{\alpha}{2}\int_0^{\tau}\int_0^1\left|J\,\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{1}{2}\int_0^1\frac{\partial^{\alpha}\bar{u}(\tau,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(\tau,x)}{\partial x^{\alpha}}dx$$

$$-\frac{1}{2}\int_0^1\frac{\partial^{\alpha}\bar{u}(0,x)}{\partial x^{\alpha}}\,x^2\,\frac{\partial^{\alpha} u(0,x)}{\partial x^{\alpha}}dx - \frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha} u(0,x)}{\partial x^{\alpha}}\right|^2 dx$$

$$= \int_0^\tau \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{\alpha}{2}\int_0^\tau \int_0^1 \left|J\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$+\frac{1}{2}\int_0^1 \frac{\partial^\alpha \overline{u}(\tau,x)}{\partial x^\alpha} x^2 \frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha} dx - \frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx$$

$$= \int_0^\tau \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{\alpha}{2}\int_0^\tau \int_0^1 \left|J\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$+\frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx - \frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx$$

By using the proprieties of the modules and of the ε-inequality, we obtain

$$\operatorname{Re}\int_0^\tau \int_0^1 \mathcal{L}uM\overline{u}dx \le \left|\int_0^\tau \int_0^1 \mathcal{L}uM\overline{u}dx\right|$$

$$\le \int_0^\tau \int_0^1 |\mathcal{L}u|\,|M\overline{u}|dxdt$$

$$\le \int_0^\tau \int_0^1 |\mathcal{L}u|\left|x^2\frac{\partial \overline{u}}{\partial t} + \alpha x J\frac{\partial \overline{u}}{\partial t}\right| dxdt$$

$$\le \int_0^\tau \int_0^1 |\mathcal{L}u|\, x^2 \left|\frac{\partial \overline{u}}{\partial t}\right| dxdt + \alpha\int_0^\tau \int_0^1 |\mathcal{L}u|\,|x|\left|J\frac{\partial \overline{u}}{\partial t}\right| dxdt$$

$$\le \frac{1}{2}\int_0^\tau \int_0^1 x^2 \left(|\mathcal{L}u|^2 + \left|\frac{\partial \overline{u}}{\partial t}\right|^2\right) dxdt$$

$$+\frac{\alpha}{2}\int_0^\tau \int_0^1 \left(x^2|\mathcal{L}u|^2 + \left|J\frac{\partial \overline{u}}{\partial t}\right|^2\right) dxdt$$

$$\le \left(\frac{1+\alpha}{2}\right)\int_0^\tau \int_0^1 x^2\,|\mathcal{L}u|^2 dxdt + \frac{1}{2}\int_0^\tau \int_0^1 x^2 \left|\frac{\partial \overline{u}}{\partial t}\right|^2 dxdt$$

$$+\frac{\alpha}{2}\int_0^\tau \int_0^1 \left|J\frac{\partial \overline{u}}{\partial t}\right|^2 dxdt$$

From where

$$\left(\frac{1+\alpha}{2}\right)\int_0^\tau \int_0^1 x^2\,|\mathcal{L}u|^2 dxdt + \frac{1}{2}\int_0^\tau \int_0^1 x^2 \left|\frac{\partial \overline{u}}{\partial t}\right|^2 dxdt$$

$$+\frac{\alpha}{2\varepsilon_2}\int_0^\tau\int_0^1\left|J\frac{\partial\bar{u}}{\partial t}\right|^2 dxdt \geq \int_0^\tau\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\frac{\alpha}{2}\int_0^\tau\int_0^1\left|J\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx-\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx\,.$$

We obtain

$$(\frac{1+\alpha}{2})\int_0^\tau\int_0^1 x^2\,|\mathcal{L}u|^2 dxdt+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx$$

$$\geq \frac{1}{2}\int_0^\tau\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx\,. \qquad (2.19)$$

We have

$$\int_0^1 x^2 e^{-ct}\left|u(\tau,x)\right|^2 dx-\int_0^1 x^2\left|\varphi\right|^2 dx+c\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\right|^2 dxdt \leq 2\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\frac{\partial\bar{u}}{\partial t}\right|dxdt\,.$$

From the relation

$$\left(\left|u\right|-\left|\frac{\partial\bar{u}}{\partial t}\right|\right)^2 \geq 0$$

we obtain

$$\left|u\right|^2-2\left|u\frac{\partial\bar{u}}{\partial t}\right|+\left|\frac{\partial\bar{u}}{\partial t}\right|^2 \geq 0\,.$$

From where

$$-\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\right|^2 dxdt-\int_0^\tau\int_0^1 x^2 e^{-ct}\left|\frac{\partial\bar{u}}{\partial t}\right|dxdt \leq -2\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\frac{\partial\bar{u}}{\partial t}\right|dxdt$$, c is constant

$$\int_0^1 x^2 e^{-ct}\left|u(\tau,x)\right|^2 dx-\int_0^1 x^2\left|\varphi\right|^2 dx+(c-1)\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\right|^2 dxdt$$

$$-\int_0^\tau\int_0^1 x^2 e^{-ct}\left|\frac{\partial\bar{u}}{\partial t}\right|^2 dxdt \leq 0$$

$$\int_0^1 x^2 e^{-ct}\left|u(\tau,x)\right|^2 dx-\int_0^1 x^2\left|\varphi\right|^2 dx+(c-1)\int_0^\tau\int_0^1 x^2 e^{-ct}\left|u\right|^2 dxdt$$

$$-\int_0^\tau\int_0^1 x^2 e^{-ct}\left|\frac{\partial\bar{u}}{\partial t}\right|^2 dxdt \leq 0$$

For $c \geq 1$, using lemma 4 we obtain

$$\int_0^1 x^2 e^{-ct} |u(\tau,x)|^2 dx \leq \int_0^1 x^2 |\varphi|^2 dx + \int_0^\tau \int_0^1 x^2 e^{-ct} \left|\frac{\partial \bar{u}}{\partial t}\right|^2 dxdt\,.$$

From where

$$\frac{e^{-cT}}{8} \int_0^1 x^2 |u(\tau,x)|^2 dx \leq \frac{1}{8}\int_0^1 x^2 |\varphi|^2 dx + \frac{1}{8}\int_0^\tau \int_0^1 x^2 e^{-ct} \left|\frac{\partial \bar{u}}{\partial t}\right|^2 dxdt\,. \tag{2.20}$$

From the equation (2.1) we have

$$\mathcal{L}u - \frac{\partial u}{\partial t} = (-1)^\alpha \frac{1}{x}\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)$$

$$(\mathcal{L}u - \frac{\partial u}{\partial t})^2 = \frac{1}{x^2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2$$

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 \leq x^2 \,|\, \mathcal{L}u + \frac{\partial u}{\partial t}\,|^2 \leq 2x^2 |\mathcal{L}u|^2 + 2x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt\,.$$

From where

$$\frac{1}{8}\int_0^\tau \int_0^1 \left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right|^2 dxdt \leq \frac{1}{4}\int_0^\tau \int_0^1 x^2\, |\mathcal{L}u|^2 dxdt + \frac{1}{4}\int_0^\tau \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dxdt\,. \tag{2.21}$$

Adding inequalities (2.19), (2.20) and (2.21) member with member we obtain

$$\frac{3+2\alpha}{4}\int_0^\tau \int_0^1 x^2\, |\mathcal{L}u|^2 dxdt + \frac{1}{8}\int_0^1 x^2 |\varphi|^2 dx + \frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx \geq \frac{3}{8}\int_0^\tau \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dx$$

$$+\frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx + \frac{e^{-cT}}{8}\int_0^1 x^2 |u(\tau,x)|^2 dx\,.$$

Raising the left-hand side, we obtain

$$\frac{3+2\alpha}{4}\int_0^T \int_0^1 x^2\, |\mathcal{L}u|^2 dxdt + \frac{1}{8}\int_0^1 x^2 |\varphi|^2 dx + \frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha \varphi}{\partial x^\alpha}\right|^2 dx \geq \frac{3}{8}\int_0^\tau \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dx$$

$$+\frac{1}{2}\int_0^1 x^2 \left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx + \frac{e^{-cT}}{8}\int_0^1 x^2 |u(\tau,x)|^2 dx$$

$$\frac{3+2\alpha}{4}\int_0^T\int_0^1 x^2\ |\mathcal{L}u|^2 dxdt+\frac{1}{8}\int_0^1 x^2|\varphi|^2 dx+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha\varphi}{\partial x^\alpha}\right|^2 dx\ge\frac{3}{8}\int_0^\tau\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx$$

$$+\frac{1}{2}\sup_{0\le\tau\le T}\int_0^1 x^2\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2 dx+\frac{e^{-cT}}{8}\sup_{0\le\tau\le T}\int_0^1 x^2|u|^2 dx\,.$$

From where

$$2(3+2\alpha)\int_\Omega x^2\ |\mathcal{L}u|^2 dxdt+\int_0^1 x^2|\varphi|^2 dx+4\int_0^1 x^2\left|\frac{\partial^\alpha\varphi}{\partial x^\alpha}\right|^2 dx\ge 3\int_0^\tau\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx$$

$$+4\sup_{0\le\tau\le T}\int_0^1 x^2\left|\frac{\partial^\alpha u(\tau,x)}{\partial x^\alpha}\right|^2 dx+e^{-cT}\sup_{0\le\tau\le T}\int_0^1 x^2|u(\tau,x)|^2 dx\,.$$

While posing

$$||(f,\varphi)||_F^2=||\mathscr{F}||_F^2=\int_\Omega x^2|f|^2 dxdt+\int_0^1 x^2|\varphi|^2 dx+\int_0^1 x^2\left|\frac{\partial^\alpha\varphi}{\partial x^\alpha}\right|^2 dx$$

$$\|u\|_E^2=\int_\Omega x^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\sup_{0\le\tau\le T}\int_0^1 x^2\left|\frac{\partial^\alpha u}{\partial x^\alpha}\right|^2 dx+\frac{e^{-cT}}{8}\sup_{0\le\tau\le T}\int_0^1 x^2|u|^2 dx\,.$$

We obtain

$$||u||_E\le C\,||Lu||_F\quad,\quad C=\frac{3+2\alpha}{\inf(3,e^{-cT})}\quad.$$

This ends the proof of the theorem. ■

2.4 Solvability of the problem

From estimates (2.9) and (2.13), it follows that the operator $L:E\to F$ is continuous and its range is closed in F. To prove the solvability of (2.1)-(2.5), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 5 *Let* $D_0(L)=\{u\in D(L)\ /\ lu=0\}$. *If for* $u\in D_0(L)$ and *some* ω *such that* $\omega\in L_2(\Omega)$, *we have*

$$\int_\Omega x^2 \mathcal{L}u\overline{\omega}dxdt = 0\,, \tag{2.22}$$

then $\omega=0$.

Proof. The equality (2.22) can be written as follows

$$-\int_\Omega x^2 \frac{\partial u}{\partial t}\overline{\omega}dxdt = (-1)^\alpha \int_\Omega x \frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)\overline{\omega}dxdt\,. \tag{2.23}$$

For $\omega(x,t)$ given, we introduce the function

$$v(x,t) = x^{\alpha-1}\int_x^1 \frac{\frac{\partial\omega(\xi,t)}{\partial x}}{\xi^{\alpha-1}}d\xi + x^{\alpha-1}\int_x^1 \frac{\omega(\xi,t)}{\xi^\alpha}d\xi\,,$$

then we have $\int_x^1 v(x,t)dx = 0$ and $x^2\omega= x^2v+\alpha xJv=Nv$. Then from equality (2.23) we have

$$-\int_\Omega \frac{\partial u}{\partial t}N\overline{v}dxdt = (-1)^\alpha\int_\Omega \frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)x\overline{v}dxdt$$

$$+\alpha(-1)^\alpha\int_\Omega \frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^\alpha u}{\partial x^\alpha}\right)J\overline{v}dxdt\,. \tag{2.24}$$

Integrating by parts the second member of the right hand side of (2.24), we get

$$-\int_\Omega \frac{\partial u}{\partial t}N\overline{v}dxdt = \int_\Omega Au\overline{v}dxdt\,. \tag{2.25}$$

Where

$$Au = (-1)^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\left(x^2\frac{\partial^\alpha u}{\partial x^\alpha}\right).$$

When we introduce the smoothing operators J_ε^{-1} and $\left(J_\varepsilon^{-1}\right)^*$, with respect to [42] then these operators provide the solution of the problems

$$\varepsilon\frac{dg_\varepsilon(t)}{dt}+g_\varepsilon(t)=g(t), \qquad (2.26)$$

$$g_\varepsilon(t)\big|_{t=0}=0,$$

And

$$-\varepsilon\frac{dg_\varepsilon^*(t)}{dt}+g_\varepsilon^*(t)=g(t), \qquad (2.27)$$

$$g_\varepsilon^*(t)\big|_{t=T}=0,$$

The solution have the following properties: for $g\in L_2(0,T)$, the functions $g_\varepsilon=\left(J_\varepsilon^{-1}\right)g$ and $g_\varepsilon^*=\left(J_\varepsilon^{-1}\right)^*g$ are in $W_2^1(0,T)$ such that $g_\varepsilon(t)\big|_{t=0}=0$ and $g_\varepsilon^*(t)\big|_{t=T}=0$. Moreover, $\left(J_\varepsilon^{-1}\right)$ commutes with $\frac{\partial}{\partial t}$, so $\int_0^T\left|g_\varepsilon-g\right|^2dt\to 0$ and $\int_0^T\left|g_\varepsilon^*-g\right|^2dt\to 0$, for $\varepsilon\to 0$.

Replacing in (2.25) u by the smoothed function $\left(J_\varepsilon^{-1}\right)u$, using the relation $AJ_\varepsilon^{-1}=J_\varepsilon^{-1}A$, and using properties of the smoothing operators we get

$$\int_\Omega u\overline{N\left(\frac{\partial v_\varepsilon^*}{\partial t}\right)}dxdt=\int_\Omega Au\overline{v_\varepsilon^*}dxdt. \qquad (2.28)$$

Passing to the limit, (2.28) is satisfied for all functions satisfying the conditions (2.2)-(2.5) such that

$$\frac{\partial^i}{\partial x^i}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)\in L_2(\Omega),\ \frac{\partial^i u}{\partial x^i}\in L_2(\Omega)\ \text{for}\ 0\le i\le\alpha.$$

The left-hand side of (2.28) is a continuous linear functional of u. Hence the function v_ε^* has the derivatives

$$\frac{\partial^i v_\varepsilon^*}{\partial x^i}\in L_2(\Omega),\ \frac{\partial^i}{\partial x^i}\left(\frac{\partial^\alpha v_\varepsilon^*}{\partial x^\alpha}\right)\in L_2(\Omega),\ i=\overline{0,\alpha},$$

and the following conditions are satisfied

$$\left.\frac{\partial^i v_\varepsilon^*}{\partial x^i}\right|_{x=0} = \left.\frac{\partial^i v_\varepsilon^*}{\partial x^i}\right|_{x=1} = 0, \quad i = \overline{0,\alpha-1}. \tag{2.29}$$

In addition v_ε^* satisfies the integral condition (2.5).

Putting $u = \int_0^t v_\varepsilon^*(x,\tau)d\tau$ in (2.25), and using (2.27), we obtain

$$-\int_\Omega v_\varepsilon^* \overline{Nu} dxdt = \int_\Omega Au \frac{\partial \overline{u}}{\partial t} dxdt - \varepsilon \int_\Omega Au \frac{\partial \overline{v_\varepsilon^*}}{\partial t} dxdt. \tag{2.30}$$

Integrating by parts each term in the right-hand side of (2,30), we have

$$\operatorname{Re}\int_\Omega Au \frac{\partial \overline{u}}{\partial t} dxdt \geq 0, \tag{2,31}$$

$$\operatorname{Re}\left(-\varepsilon \int_\Omega Au \frac{\partial \overline{v_\varepsilon^*}}{\partial t} dxdt\right) = \varepsilon \int_\Omega x^2 \left|\frac{\partial^\alpha v_\varepsilon^*}{\partial x^\alpha}\right|^2 dxdt. \tag{2.32}$$

Now, using (2.31) and (2.32) in (2.30) we have

$$\operatorname{Re}\int_\Omega v_\varepsilon^* \overline{Nv} dxdt \leq 0,$$

then $\operatorname{Re}\int_\Omega v\overline{Nv} dxdt \leq 0$ as ε approaches zero.

Since $\int_\Omega x^2 |v|^2 dxdt = 0$, we conclude that v=0, hence $\omega=0$, what finishes the proof of the lemma. ■

Theorem 12 *The range R(L) of the operator L coincides with F.*

Proof. Since F is a Hilbert space, we have $R(L)=F$ if and only if the following implication is satisfied:

$$\int_\Omega x^2 \mathcal{L}u \overline{f} dxdt + \int_0^1 x^2 \left(\frac{\partial^\alpha lu}{\partial x^\alpha} \frac{\partial^\alpha \overline{\varphi}}{\partial x^\alpha} + lu\overline{\varphi}\right) dx = 0, \tag{2.33}$$

for arbitrary $u \in E$ and $\mathcal{F}=(f,\varphi) \in F$, implies that f and φ are zero.

Putting $u \in D(L_0)$ in (2.33), we obtain

$$\int_\Omega x^2 \mathcal{L} u \overline{f} dxdt = 0.$$

Using lemma 5 we obtain that *f=0*.

Consequently, we have

$$\int_0^1 x^2 \left(\frac{\partial^\alpha lu}{\partial x^\alpha} \frac{\partial^\alpha \overline{\varphi}}{\partial x^\alpha} + lu\overline{\varphi} \right) dx = 0 \ . \tag{2.34}$$

The range of the trace operator *l* is everywhere dense in a Hilbert space with norm

$$\left[\int_0^1 x^2 \left(\left| \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right|^2 + |\varphi|^2 \right) dx \right]^{\frac{1}{2}},$$

therefore $\varphi=0$, and the present proof is completed. ■

CHAPTER 3

MIXED PROBLEM WITH NON LOCAL BOUNDARY CONDITION FOR A HIGH ORDER PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE

3.1 Introduction

In this chapter we study a mixed problem for a high-order differential equation of mixed type with no classical boundary condition. The existence and uniqueness of the strong solution in functional weighted Sobolev space are proved. The proof is based in two sided a priori estimates and the fact that the range of operator generalized by the considered problem is dense.

3.2 Position of the problem

Let α be a positive integer and Ω be the set $(0,T)\times(0,1)$ we consider the equation

$$\mathcal{L}u=\frac{\partial^2 u}{\partial t^2}+(-1)^{\alpha}\frac{1}{x}\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)=f(t,x)\,. \tag{3.1}$$

To equation (3.1) we attach the initial condition

$$lu=u(0,x)=\varphi(x)\qquad x\in(0,1)\,, \tag{3.2}$$

$$qu=\frac{\partial u(0,x)}{\partial t}=\psi(x)\qquad x\in(0,1), \tag{3.3}$$

the boundary conditions

$$\frac{\partial^i u(t,1)}{\partial x^i}=0\quad \text{for}\quad 0\le i\le \alpha-1\qquad t\in(0,T)\,, \tag{3.4}$$

$$\frac{\partial^i u(t,0)}{\partial x^i}=0\quad \text{for}\quad 0\le i\le \alpha-2\qquad t\in(0,T)\,, \tag{3.5}$$

and integral condition

$$\int_0^1 u(t,\xi)d\xi=0\qquad \text{for}\quad t\in(0,T)\,, \tag{3.6}$$

were φ and ψ are two known functions which satisfy the compatibility conditions given in (3.4), (3.5) and (3.6).

3.3 Preliminaries

In this work, we prove the existence and the uniqueness of a solution of problem (3.1)-(3.5). For this, we consider the problem (3.1)-(3.5) as a solution of the operator equation

$$Lu=\mathcal{F},$$

where $L=(\mathcal{L},l,q)$, the operator L is acting from E to F, where E is the Banach space consisting of functions $u\in L_2(\Omega)$, satisfying (3.3), (3.4) and (3.5), with finite norm

$$\|u\|_E^2=\int_\Omega x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt+\int_\Omega\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2 dxdt+\sup_{0\le t\le T}\int_0^1 x^2\left\{\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right|^2+\left|\frac{\partial u}{\partial t}\right|^2+|u|^2\right\}dx \quad (3.7)$$

and F is the Hilbert space of vector-valued functions $\mathcal{F}=(f,\varphi,\psi)$ obtained by completing the space $L_2(\Omega)\times W_2^{2\alpha+1}(0,1)$ with the following norm

$$\|\mathcal{F}\|_F^2=\|(f,\varphi,\psi)\|_F^2=\int_\Omega x^2|f(t,x)|^2 dxdt+\int_0^1 x^2\left\{\left|\frac{\partial^\alpha\psi}{\partial x^\alpha}\right|^2+|\psi|^2++|\psi|^2\right\}dx. \quad (3.8)$$

Using the energy inequalities method proposed in [22], we establish two-sided a priori estimates. Then we prove that the operator L is a linear homeomorphism between the space E and F.

Theorem 13 *The following a priori estimate*

$$\|Lu\|_F\le c\|u\|_E, \quad (3.9)$$

holds for any function $u\in E$, where c is constant.

Proof. Using equation (3.1) and initial condition (3.2) we obtain

$$\int_\Omega x^2|\mathcal{L}u|^2\,dxdt\le 2\int_\Omega\left[x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2+\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2\right]dxdt \quad (3.10)$$

$$\int_0^1 x^2\left|\frac{\partial^\alpha qu}{\partial x^\alpha}\right|^2 dx\le\sup_{0\le t\le T}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right|^2 dx, \quad (3.11)$$

and $$\int_0^1 x^2 |qu|^2 dx \le \sup_{0\le t\le T} \int_0^1 x^2 \left|\frac{\partial u}{\partial t}\right|^2 dx. \tag{3.12}$$

We have $$\int_0^1\int_0^\tau e^{-ct} u \frac{\partial \bar{u}}{\partial t} dxdt = \int_0^1 \left[|u|^2 e^{-ct}\right]_0^\tau dx + c\int_0^1\int_0^\tau |u|^2 e^{-ct} dxdt - \int_0^1\int_0^\tau e^{-ct} \bar{u} \frac{\partial u}{\partial t} dxdt$$

$$2\,\mathrm{Re}\left(\int_0^1\int_0^\tau e^{-ct} u \frac{\partial \bar{u}}{\partial t} dxdt\right) = \int_0^1 |u(x,\tau)|^2 e^{-c\tau} dx - \int_0^1 |\varphi|^2 dx + c\int_0^\tau\int_0^1 e^{-ct} |u|^2 dxdt$$

By using the ε-inequality for ε=1 at the first member, we obtain

$$2\,\mathrm{Re}\left(\int_0^1\int_0^\tau e^{-ct} u \frac{\partial \bar{u}}{\partial t} dxdt\right) \le \int_0^\tau\int_0^1 e^{-ct} |u|^2 dxdt + \int_0^\tau\int_0^1 e^{-ct} \left|\frac{\partial u}{\partial t}\right|^2 dxdt\,.$$

By substitution, we obtain

$$\int_0^1 |u(x,\tau)|^2 e^{-ct} dx + c\int_0^\tau\int_0^1 e^{-ct}|u|^2 dxdt + \le \int_0^\tau\int_0^1 e^{-ct}|u|^2 dxdt + \int_0^\tau\int_0^1 e^{-ct}\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_0^1 |\varphi|^2 dx$$

For c≥1, we have $$\int_0^1 e^{-ct}|u(x,\tau)|^2 dx \le \int_0^\tau\int_0^1 e^{-ct}\left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_0^1 |\varphi|^2 dx \tag{3.13}$$

While multiplying by x^2 and integrating and combining the inequalities (3.10), (3.11), (3.12) and (3.13), we obtain (3.9) for $u\in E$. ■

Theorem 14 *For any function u∈E, we have the inequality*

$$\|u\|_E \le C\|Lu\|_F\,, \tag{3.14}$$

where the constant $C = \dfrac{\sup(3+2\alpha,4)}{\inf(3, e^{-cT})}$.

Proof. Let $Jg = \int_x^1 g(t,\xi)d\xi$ and $Mu = x^2 \dfrac{\partial^2 u}{\partial t^2} + \alpha x J\dfrac{\partial^2 u}{\partial t^2}$.

We consider the quadratic form $\mathrm{Re}\int_0^\tau\int_0^1 \mathcal{L}uM\bar{u}dxdt$,

which is obtained by multiplying (3.1) by $\overline{Mu}$. We have

$$\mathcal{L}uM\bar{u} = \left[\frac{\partial^2 u}{\partial t^2} + (-1)^\alpha \frac{1}{x}\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\right]\left(x^2\frac{\partial^2 \bar{u}}{\partial t^2} + \alpha x J\frac{\partial^2 \bar{u}}{\partial t^2}\right).$$

Integrating by report to x, we obtain

$$\int_0^1 \mathcal{L}uM\bar{u}dx = \int_0^1 x^2 \frac{\partial^2 u}{\partial t^2}\frac{\partial^2 \bar{u}}{\partial t^2}dx + \int_0^1 \frac{\partial^2 u}{\partial t^2}\alpha xJ\frac{\partial^2 \bar{u}}{\partial t^2}dx + (-1)^\alpha \int_0^1 x\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^2 \bar{u}}{\partial t^2}dx$$

$$+(-1)^\alpha \int_0^1 \alpha \frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)J\frac{\partial^2 \bar{u}}{\partial t^2}dx. \tag{3.15}$$

We have

$$\int_0^1 x^2 \frac{\partial^2 u}{\partial t^2}\frac{\partial^2 \bar{u}}{\partial t^2}dx = \int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx\ .$$

Integrating by parts the terms of the second member of (3.15) and by taking into account of the boundary conditions, we obtain

$$\int_0^1 \frac{\partial^2 u}{\partial t^2}\alpha xJ\frac{\partial^2 \bar{u}}{\partial t^2}dx = -\alpha\int_0^1 x\frac{\partial}{\partial x}\left(J\frac{\partial^2 u}{\partial t^2}\right)J\frac{\partial^2 \bar{u}}{\partial t^2}dx$$

$$= -\left[J\frac{\partial^2 u}{\partial t^2}xJ\frac{\partial^2 \bar{u}}{\partial t^2}\right]_0^1 + \alpha\int_0^1 J\frac{\partial^2 u}{\partial t^2}\frac{\partial}{\partial x}\left(xJ\frac{\partial^2 \bar{u}}{\partial t^2}\right)dx$$

$$= \alpha\int_0^1 J\frac{\partial^2 u}{\partial t^2}J\frac{\partial^2 \bar{u}}{\partial t^2}dx + \alpha\int_0^1 J\frac{\partial^2 u}{\partial t^2}x\frac{\partial}{\partial x}J\frac{\partial^2 \bar{u}}{\partial t^2}dx$$

$$= \alpha\int_0^1 \left|J\frac{\partial^2 u}{\partial t^2}\right|^2 dx - \alpha\int_0^1 J\frac{\partial^2 u}{\partial t^2}x\frac{\partial^2 \bar{u}}{\partial t^2}dx.$$

From where

$$\operatorname{Re}\int_0^1 \alpha x\frac{\partial^2 u}{\partial t^2}J\frac{\partial^2 \bar{u}}{\partial t^2}dx = \frac{\alpha}{2}\int_0^1\left|J\frac{\partial^2 u}{\partial t^2}\right|^2 dx \tag{3.16}$$

$$(-1)^\alpha\int_0^1 x\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial^2 \bar{u}}{\partial t^2}dx = (-1)^\alpha\left[\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)x\frac{\partial^2 \bar{u}}{\partial t^2}\right]_0^1$$

$$+(-1)^{\alpha+1}\int_0^1 \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial}{\partial x}\left(x\frac{\partial^2 \bar{u}}{\partial t^2}\right)dx$$

$$=(-1)^{\alpha+1}\int_0^1 \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial^2 \bar{u}}{\partial t^2}dx + (-1)^{\alpha+1}\int_0^1 \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)x\frac{\partial^2 \bar{u}}{\partial x\partial t}dx$$

$$=(-1)^{\alpha+1}\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial^2 \bar{u}}{\partial t^2}\right]_0^1 + (-1)^{\alpha+2}\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial^3 \bar{u}}{\partial x\partial t^2}dx$$

$$+(-1)^{\alpha+1}\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)x\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\right]_0^1 + (-1)^{\alpha+2}\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\frac{\partial}{\partial x}\left(x\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\right)dx$$

$$=(-1)^{\alpha+2}\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\,dx+(-1)^{\alpha+2}\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\,dx$$

$$+(-1)^{\alpha+2}\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)x\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx$$

$$=(-1)^{\alpha+2}2\int_0^1 \frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\,dx+(-1)^{\alpha+2}\int_0^1 x\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx$$

$$=(-1)^{\alpha+2}2\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^3 \bar{u}}{\partial x\partial t^2}\right]_0^1+(-1)^{\alpha+3}2\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx$$

$$+(-1)^{\alpha+2}\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)x\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\right]_0^1+(-1)^{\alpha+3}\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial}{\partial x}\left(x\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\right)dx$$

$$=(-1)^{\alpha+3}2\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx+(-1)^{\alpha+3}\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx$$

$$+(-1)^{\alpha+3}\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)x\frac{\partial^5 \bar{u}}{\partial x^3\partial t^2}\,dx$$

$$=(-1)^{\alpha+3}3\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^4 \bar{u}}{\partial x^2\partial t^2}\,dx+(-1)^{\alpha+3}\int_0^1 \frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)x\frac{\partial^5 \bar{u}}{\partial x^3\partial t^2}\,dx\,.$$

Reasoning by recurrence, we obtain

$$(-1)^{\alpha}\int_0^1 x\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^2 \bar{u}}{\partial t^2}\,dx=(-1)^{2\alpha-1}(\alpha-1)\left[\left(x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\right)\frac{\partial^{\alpha} \bar{u}}{\partial x^{\alpha-2}\partial t^2}\right]_0^1$$

$$+(-1)^{2\alpha}(\alpha-1)\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\frac{\partial^{\alpha+1} \bar{u}}{\partial x^{\alpha-1}\partial t^2}\,dx+(-1)^{2\alpha-1}\left[x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}x\frac{\partial^{\alpha+1} \bar{u}}{\partial x^{\alpha-1}\partial t^2}\right]_0^1$$

$$+(-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^{\alpha}\partial t}\frac{\partial}{\partial x}\left(x\frac{\partial^{\alpha+1} \bar{u}}{\partial x^{\alpha}\partial t}\right)dx.$$

$$= (-1)^{2\alpha}(\alpha-1)\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}\,dx + (-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}dx$$

$$+ (-1)^{2\alpha}\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\, x\frac{\partial^{\alpha+2}\bar{u}}{\partial x^\alpha \partial t^2}dx$$

$$= (-1)^{2\alpha}\alpha\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}\,dx + (-1)^{2\alpha}\int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)dx\,.$$

From where

$$(-1)^\alpha\int_0^1 x\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+2}u}{\partial x^\alpha \partial t^2}\right)\frac{\partial^2\bar{u}}{\partial t^2}dx = \alpha\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}\,dx + \int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+2}u}{\partial x^\alpha \partial t^2}\right)dx \qquad (3.17)$$

$$(-1)^\alpha\int_0^1 \alpha\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)J\frac{\partial^2\bar{u}}{\partial t^2}\,dx = (-1)^\alpha\alpha\left[\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)J\frac{\partial^2\bar{u}}{\partial t^2}\right]_0^1$$

$$+ (-1)^{\alpha+1}\alpha\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial}{\partial x}\left(J\frac{\partial^2\bar{u}}{\partial t^2}\right)dx$$

$$= (-1)^{\alpha+2}\alpha\int_0^1\frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^2\bar{u}}{\partial t^2}\,dx$$

$$= (-1)^{\alpha+2}\alpha\left[\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^2\bar{u}}{\partial t^2}\right]_0^1 + (-1)^{\alpha+3}\alpha\int_0^1\frac{\partial^{\alpha-2}}{\partial x^{\alpha-2}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^2\bar{u}}{\partial x\partial t}\,dx$$

$$= (-1)^{\alpha+3}\alpha\left[\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^3\bar{u}}{\partial x\partial t^2}\right]_0^1 + (-1)^{\alpha+4}\alpha\int_0^1\frac{\partial^{\alpha-3}}{\partial x^{\alpha-3}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^4\bar{u}}{\partial x^2\partial t^2}\,dx$$

$$= (-1)^{\alpha+4}\alpha\left[\frac{\partial^{\alpha-4}}{\partial x^{\alpha-4}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^4\bar{u}}{\partial x^2\partial t^2}\right]_0^1 + (-1)^{\alpha+5}\alpha\int_0^1\frac{\partial^{\alpha-4}}{\partial x^{\alpha-4}}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)\frac{\partial^5\bar{u}}{\partial x^3\partial t^2}\,dx\,\cdot$$

Reasoning by recurrence, we obtain

$$(-1)^\alpha\int_0^1 \alpha\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\right)J\frac{\partial^2\bar{u}}{\partial t^2}\,dx = (-1)^\alpha\alpha\left[x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^\alpha\bar{u}}{\partial x^{\alpha-1}\partial t^2}\right]_0^1 + (-1)^{2\alpha+1}\alpha\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}dx$$

$$= -\alpha\int_0^1 x\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial^{\alpha+1}\bar{u}}{\partial x^{\alpha-1}\partial t^2}dx\,.$$

From where

$$\mathrm{Re}\int_0^1 \mathcal{L}uM\bar{u}dx = \int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx + \frac{\alpha}{2}\int_0^1\left|J\frac{\partial^2 u}{\partial t^2}\right|^2 dx + \mathrm{Re}\int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}\bar{u}}{\partial x^\alpha \partial t}\right)dx$$

$$+ \mathrm{Re}\int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}\bar{u}}{\partial x^\alpha \partial t}\right)dx\,. \qquad (3.18)$$

Integrating by parts the third term of (3.18)

$$\int_0^\tau\int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}\overline{u}}{\partial x^\alpha\partial t}\right)dxdt=\left[\int_0^1\frac{\partial^{\alpha+1}\overline{u}}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right]_0^\tau-\int_0^\tau\int_0^1\frac{\partial^{\alpha+1}\overline{u}}{\partial x^\alpha\partial t}\frac{\partial}{\partial t}\left(x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)dxdt$$

$$=\int_0^1\frac{\partial^{\alpha+1}\overline{u}(\tau,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}dx-\int_0^1\frac{\partial^{\alpha+1}\overline{u}(0,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(0,x)}{\partial x^\alpha\partial t}dx$$

$$-\int_0^\tau\int_0^1\frac{\partial^{\alpha+1}\overline{u}}{\partial x^\alpha\partial t}x^2\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)dxdt.$$

From where

$$\mathrm{Re}\int_0^\tau\int_0^1 x^2\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\frac{\partial}{\partial t}\left(\frac{\partial^{\alpha+1}\overline{u}}{\partial x^\alpha\partial t}\right)dxdt=\frac{1}{2}\int_0^1\frac{\partial^{\alpha+1}\overline{u}(\tau,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}dx$$

$$-\frac{1}{2}\int_0^1\frac{\partial^{\alpha+1}\overline{u}(0,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(0,x)}{\partial x^\alpha\partial t}dx.$$

Replacing in (3.18), we obtain

$$\mathrm{Re}\int_0^\tau\int_0^1 \mathcal{L}uM\overline{u}dx=\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2dxdt+\frac{\alpha}{2}\int_0^\tau\int_0^1\left|J\frac{\partial^2 u}{\partial t^2}\right|^2dxdt+\frac{1}{2}\int_0^1\frac{\partial^{\alpha+1}\overline{u}(\tau,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}dx$$

$$-\frac{1}{2}\int_0^1\frac{\partial^{\alpha+1}\overline{u}(0,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(0,x)}{\partial x^\alpha\partial t}dx-\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u(0,x)}{\partial x^\alpha\partial t}\right|^2dx$$

$$=\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2dxdt+\frac{\alpha}{2}\int_0^\tau\int_0^1\left|J\frac{\partial^2 u}{\partial t^2}\right|^2dxdt$$

$$+\frac{1}{2}\int_0^1\frac{\partial^{\alpha+1}\overline{u}(\tau,x)}{\partial x^\alpha\partial t}x^2\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}dx-\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha\psi}{\partial x^\alpha}\right|^2dx$$

$$=\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2dxdt+\frac{\alpha}{2}\int_0^\tau\int_0^1\left|J\frac{\partial^2 u}{\partial t^2}\right|^2dxdt$$

$$+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}\right|^2dx-\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^\alpha\psi}{\partial x^\alpha}\right|^2dx.$$

By using the proprieties of the modules and of the ε-inequality, we obtain

$$\mathrm{Re}\int_0^\tau\int_0^1 \mathcal{L}uM\overline{u}dx\le\int_0^\tau\int_0^1 |\mathcal{L}u|\,|M\overline{u}|dxdt$$

$$\leq \int_0^\tau \int_0^1 |\mathcal{L}u| \left| x^2 \frac{\partial^2 \overline{u}}{\partial t^2} + \alpha x J \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt$$

$$\leq \int_0^\tau \int_0^1 |\mathcal{L}u| \, x^2 \left| \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt + \alpha \int_0^\tau \int_0^1 |\mathcal{L}u| \, |x| \left| J \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt$$

$$\leq \frac{1}{2} \int_0^\tau \int_0^1 x^2 (\varepsilon_1 |\mathcal{L}u|^2 + \frac{1}{\varepsilon_1} \left| \frac{\partial^2 \overline{u}}{\partial t^2} \right|^2) dxdt$$

$$+ \frac{\alpha}{2} \int_0^\tau \int_0^1 (\varepsilon_2 x^2 |\mathcal{L}u|^2 + \frac{1}{\varepsilon_2} \left| J \frac{\partial^2 \overline{u}}{\partial t^2} \right|^2) dxdt$$

$$\leq \left(\frac{\varepsilon_1 + \alpha \varepsilon_2}{2} \right) \int_0^\tau \int_0^1 x^2 \, |\mathcal{L}u|^2 \, dxdt + \frac{1}{2\varepsilon_1} \int_0^x \int_0^1 x^2 \left| \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt$$

$$+ \frac{\alpha}{2\varepsilon_2} \int_0^\tau \int_0^1 \left| J \frac{\partial^2 \overline{u}}{\partial t^2} \right|^2 dxdt \, .$$

From where

$$\left(\frac{\varepsilon_1 + \alpha \varepsilon_2}{2} \right) \int_0^\tau \int_0^1 x^2 \, |\mathcal{L}u|^2 dxdt + \frac{1}{2\varepsilon_1} \int_0^x \int_0^1 x^2 \left| \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt$$

$$+ \frac{\alpha}{2\varepsilon_2} \int_0^\tau \int_0^1 \left| J \frac{\partial^2 \overline{u}}{\partial t^2} \right|^2 dxdt \geq \int_0^\tau \int_0^1 x^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dxdt + \frac{\alpha}{2} \int_0^\tau \int_0^1 \left| J \frac{\partial^2 u}{\partial t^2} \right|^2 dxdt$$

$$+ \frac{1}{2} \int_0^1 x^2 \left| \frac{\partial^{\alpha+1} u(\tau, x)}{\partial x^\alpha \partial t} \right|^2 dx - \frac{1}{2} \int_0^1 x^2 \left| \frac{\partial^\alpha \psi}{\partial x^\alpha} \right|^2 dx \, .$$

While taking $\varepsilon_1 = \varepsilon_2 = 1$, we obtain

$$\left(\frac{1+\alpha}{2} \right) \int_0^\tau \int_0^1 x^2 \, |\mathcal{L}u|^2 dxdt + \frac{1}{2} \int_0^1 x^2 \left| \frac{\partial^\alpha \psi}{\partial x^\alpha} \right|^2 dx$$

$$\geq \frac{1}{2} \int_0^\tau \int_0^1 x^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dxdt + \frac{1}{2} \int_0^1 x^2 \left| \frac{\partial^{\alpha+1} u(\tau, x)}{\partial x^\alpha \partial t} \right|^2 dx \, . \qquad (3.19)$$

We have

$$\int_0^1 x^2 e^{-ct} \left| \frac{\partial u(\tau, x)}{\partial t} \right|^2 dx - \int_0^1 x^2 |\psi|^2 dx + c \int_0^\tau \int_0^1 x^2 e^{-ct} \left| \frac{\partial u}{\partial t} \right|^2 dxdt \leq 2 \int_0^\tau \int_0^1 x^2 e^{-ct} \left| \frac{\partial u}{\partial t} \frac{\partial^2 \overline{u}}{\partial t^2} \right| dxdt \, .$$

From where

$$-\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}\right|^2 dxdt-\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial^2 \overline{u}}{\partial t^2}\right|dxdt\le -2\int_0^\tau\int_0^1 x^2e^{-ct}\left|u\frac{\partial^2 \overline{u}}{\partial t^2}\right|dxdt\,,$$

c is constant.

$$\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx-\int_0^1 x^2|\psi|^2 dx+(c-1)\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$-\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial^2 \overline{u}}{\partial t^2}\right|^2 dxdt\le 0$$

$$\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx-\int_0^1 x^2|\psi|^2 dx+(c-1)\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$-\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial^2 \overline{u}}{\partial t^2}\right|^2 dxdt\le 0.$$

For $c\ge 1$, using lemma 4 we obtain

$$\int_0^1 x^2e^{-ct}\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx\le\int_0^1 x^2|\psi|^2 dx+\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial^2 \overline{u}}{\partial t^2}\right|^2 dxdt\,,$$

from where

$$\frac{e^{-cT}}{8}\int_0^1 x^2\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx\le\frac{1}{8}\int_0^1 x^2|\psi|^2 dx+\frac{1}{8}\int_0^\tau\int_0^1 x^2e^{-ct}\left|\frac{\partial^2 \overline{u}}{\partial t^2}\right|^2 dxdt\,. \qquad (3.20)$$

From the equation (3.1) we have

$$\mathcal{L}u-\frac{\partial^2 u}{\partial t^2}=(-1)^\alpha\frac{1}{x}\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)$$

$$(\mathcal{L}u-\frac{\partial^2 u}{\partial t^2})^2=\frac{1}{x^2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2$$

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2\le x^2|\,\mathcal{L}u+\frac{\partial^2 u}{\partial t^2}\,|^2\le 2x^2|\,\mathcal{L}u|^2+2x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt\,,$$

from where

$$\frac{1}{8}\int_0^\tau\int_0^1\left|\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\right|^2 dxdt\le\frac{1}{4}\int_0^\tau\int_0^1 x^2\,|\,\mathcal{L}u|^2dxdt+\frac{1}{4}\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt\,. \qquad (3.21)$$

Adding inequalities (3.19), (3.20) and (3.21) member with member we obtain

$$\left(\frac{3+2\alpha}{4}\right)\int_0^\tau\int_0^1 x^2\ |\mathcal{L}u|^2 dxdt+\frac{1}{8}\int_0^1 x^2|\psi|^2 dx+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}\varphi}{\partial x^\alpha dt}\right|dx\geq\frac{3}{8}\int_0^x\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx$$

$$+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}\right|^2 dx+\frac{e^{-cT}}{8}\int_0^1 x^2\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx\,.$$

Raising the left-hand side, we obtain

$$\left(\frac{3+2\alpha}{4}\right)\int_0^T\int_0^1 x^2\ |\mathcal{L}u|^2 dxdt+\frac{1}{8}\int_0^1 x^2|\psi|^2 dx+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}\varphi}{\partial x^\alpha\partial t}\right|^2 dx\geq\frac{3}{8}\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx$$

$$+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}\right|^2 dx+\frac{e^{-cT}}{8}\int_0^1 x^2\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx$$

$$\left(\frac{3+2\alpha}{4}\right)\int_0^T\int_0^1 x^2\ |\mathcal{L}u|^2 dxdt+\frac{1}{8}\int_0^1 x^2|\psi|^2 dx+\frac{1}{2}\int_0^1 x^2\left|\frac{\partial^{\alpha}\psi}{\partial x^\alpha}\right|^2 dx\geq\frac{3}{8}\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx$$

$$+\frac{1}{2}\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right|^2 dx+\frac{e^{-cT}}{8}\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx\,.$$

From where

$$2(3+2\alpha)\int_\Omega x^2\ |\mathcal{L}u|^2 dxdt+\int_0^1 x^2|\psi|^2 dx+4\int_0^1 x^2\left|\frac{\partial^{\alpha}\psi}{\partial x^\alpha}\right|^2 dx\geq 3\int_0^\tau\int_0^1 x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dx$$

$$+4\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u(\tau,x)}{\partial x^\alpha\partial t}\right|^2 dx+e^{-cT}\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx\,.$$

While posing

$$\|(f,\varphi,\psi)\|_F^2=\|\mathscr{F}\|_F^2=\int_\Omega x^2|f|^2 dxdt+\int_0^1 x^2|\psi|^2 dx+\int_0^1 x^2\left|\frac{\partial^{\alpha}\psi}{\partial x^\alpha}\right|^2 dx$$

$$\|u\|_E^2=\int_\Omega x^2\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt+\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right|^2 dx+\frac{e^{-cT}}{8}\sup_{0\leq\tau\leq T}\int_0^1 x^2\left|\frac{\partial u}{\partial t}\right|^2 dx\,,$$

we obtain

$$\|u\|_E\leq C\|L\ u\|_F\quad,\quad C=\frac{\sup(3+2\alpha,4)}{\inf(3,e^{-cT})}\,.$$

This ends the proof of the theorem. ■

3.4 Solvability of the problem

From estimates (3.9) and (3.13), it follows that the operator $L:E\to F$ is continuous and its range is closed in F. To prove the solvability of (3.1)-(3.5), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 6 *Let* $D_0(L)=\{u\in D(L)\ /\ lu=0\ \Lambda\ qu=0\}$. *If for* $u\in D_0(L)$ *and some* ω *such that* $\omega\in L_2(\Omega)$, *we have*

$$\int_\Omega x^2 \mathcal{L}u\overline{\omega}dxdt=0\,, \tag{3.22}$$

then $\omega=0$.

Proof. The equality (3.22) is can be written as follows

$$-\int_\Omega x^2\frac{\partial^2 u}{\partial t^2}\overline{\omega}dxdt=(-1)^\alpha\int_\Omega x\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)\overline{\omega}dxdt\,, \tag{3.23}$$

for $\omega(x,t)$ given, we introduce the function

$$v(x,t)=x^{\alpha-1}\int_x^1\frac{\frac{\partial\omega(\xi,t)}{\partial x}}{\xi^{\alpha-1}}d\xi+x^{\alpha-1}\int_x^1\frac{\omega(\xi,t)}{\xi^\alpha}d\xi\,.$$

Then we have $\int_x^1 v(x,t)dx=0$ and $x^2\omega= x^2v+\alpha xJv=Nv$. Then from equality (3.23) we have

$$-\int_\Omega\frac{\partial^2 u}{\partial t^2}N\overline{v}dxdt=(-1)^\alpha\int_\Omega\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)x\overline{v}dxdt$$

$$+\alpha(-1)^\alpha\int_\Omega\frac{\partial^\alpha}{\partial x^\alpha}\left(x\frac{\partial^{\alpha+1}u}{\partial x^\alpha\partial t}\right)J\overline{v}dxdt\,. \tag{3.24}$$

Integrating by parts the second member of the right-hand side of (3.24), we get

$$-\int_\Omega\frac{\partial^2 u}{\partial t^2}N\overline{v}dxdt=\int_\Omega A\frac{\partial\overline{u}}{\partial t}\overline{v}dxdt\,, \tag{3.25}$$

where

$$Au = (-1)^{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(x^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} \right),$$

Then

$$A \frac{\partial u}{\partial t} = (-1)^{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(x^2 \frac{\partial^{\alpha+1} u}{\partial x^{\alpha} \partial t} \right).$$

Using properties of the smoothing operators J_{ε}^{-1} and $\left(J_{\varepsilon}^{-1}\right)^*$ that we introduced into chapter 2, replacing in (3.25) u by the smoothed function $\left(J_{\varepsilon}^{-1}\right)u$ and using the relation $AJ_{\varepsilon}^{-1} = J_{\varepsilon}^{-1}A$, we get

$$\int_{\Omega} \frac{\partial u}{\partial t} \overline{N\left(\frac{\partial v_{\varepsilon}^*}{\partial t} \right)} dxdt = \int_{\Omega} A \frac{\partial u}{\partial t} \overline{v_{\varepsilon}^*} dxdt . \qquad (3.26)$$

Passing to the limit, (3.26) is satisfied for all functions satisfying the conditions (3.2)- (3.5) such that

$$\frac{\partial^i}{\partial x^i} \left(\frac{\partial^{\alpha+1} u}{\partial x^{\alpha} \partial t} \right) \in L_2(\Omega), \ \frac{\partial^{i+1} u}{\partial x^i \partial t} \in L_2(\Omega) \text{ for } 0 \le i \le \alpha .$$

The left hand of (3.26) is a continuous linear functional of $\frac{\partial u}{\partial t}$. Hence the function v_{ε}^* has the derivatives

$$\frac{\partial^i v_{\varepsilon}^*}{\partial x^i} \in L_2(\Omega), \ \frac{\partial^i}{\partial x^i} \left(\frac{\partial^{\alpha} v_{\varepsilon}^*}{\partial x^{\alpha}} \right) \in L_2(\Omega), \ i = \overline{0, \alpha},$$

and the following conditions are satisfied

$$\left. \frac{\partial^i v_{\varepsilon}^*}{\partial x^i} \right|_{x=0} = \left. \frac{\partial^i v_{\varepsilon}^*}{\partial x^i} \right|_{x=1} = 0, \quad i = \overline{0, \alpha - 1}, \qquad (3.27)$$

in addition v_{ε}^* satisfies the integral condition (3.5).

Putting $u = \int_0^t \int_0^{\tau} v_{\varepsilon}^*(x, \eta) d\eta d\tau$ in (3.25), and using (2.27), we obtain

$$- \int_{\Omega} v_{\varepsilon}^* \overline{Nu} dxdt = \int_{\Omega} A \frac{\partial u}{\partial t} \frac{\partial^2 \overline{u}}{\partial t^2} dxdt - \varepsilon \int_{\Omega} A \frac{\partial u}{\partial t} \frac{\partial \overline{v_{\varepsilon}^*}}{\partial t} dxdt . \qquad (3.28)$$

Integrating by parts each term in the right-hand side of (3.30), we have

$$\operatorname{Re}\int_{\Omega} A\frac{\partial u}{\partial t}\frac{\partial^2 \overline{u}}{\partial t^2}dxdt \geq 0, \tag{3.29}$$

$$\operatorname{Re}\left(-\varepsilon\int_{\Omega} A\frac{\partial u}{\partial t}\frac{\partial \overline{v_{\varepsilon}^{*}}}{\partial t}dxdt\right) = \varepsilon\int_{\Omega} x^2\left|\frac{\partial^{\alpha} v_{\varepsilon}^{*}}{\partial x^{\alpha}}\right|^2 dxdt\,. \tag{3.30}$$

Now, using (3.29) and (3.30) in (3.28) we have

$$\operatorname{Re}\int_{\Omega} v_{\varepsilon}^{*}\overline{Nv}dxdt \leq 0,$$

then $\operatorname{Re}\int_{\Omega} v\overline{Nv}dxdt \leq 0$ as ε approaches zero.

Since $\int_{\Omega} x^2|v|^2 dxdt = 0$, we conclude that v=0, hence $\omega =0$, what finishes the proof of the lemma. ■

Theorem 15 *The range R(L) of the operator L coincides with F.*

Proof. Since F is a Hilbert space, we have $R(L)=F$ if and only if the following implication is satisfied:

$$\int_{\Omega} x^2 \mathcal{L} u\overline{f}dxdt + \int_0^1 x^2\left(\frac{\partial^{\alpha} qu}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{\psi}}{\partial x^{\alpha}} + qu\overline{\psi} + lu\overline{\varphi}\right)dx = 0$$

$$\int_0^1 x^2\left(\frac{\partial^{\alpha} lu}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{\psi}}{\partial x^{\alpha}} + lu\overline{\psi} + lu\overline{\varphi}\right)dx = 0 \tag{3.31}$$

for arbitrary $u\in E$ and $\mathcal{F}=(f,\varphi,\psi)\in F$, implies that f and φ are zero.

Putting $u\in D(L_0)$ in (3.31), we obtain

$$\int_{\Omega} x^2 \mathcal{L}\, u\overline{f}dxdt = 0\,.$$

Using lemma 6 we obtain that $f=0$. Consequently, we have

$$\int_0^1 x^2\left(\frac{\partial^{\alpha} lu}{\partial x^{\alpha}}\frac{\partial^{\alpha}\overline{\psi}}{\partial x^{\alpha}} + lu\overline{\psi} + lu\overline{\varphi}\right)dx = 0\cdot \tag{3.32}$$

The range of the trace operator l is everywhere dense in a Hilbert space with norm

$$\left[\int_0^1 x^2\left(\left|\frac{\partial^{\alpha}\psi}{\partial x^{\alpha}}\right|^2 + |\psi|^2 + |\varphi|^2\right)dx\right]^{\frac{1}{2}}$$

therefore φ=0, ψ=0, and the present proof is completed. ■

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[illegible]

[illegible] ...

[illegible] u [illegible] Mu [illegible] F [illegible] E [illegible]

$$Lu = \mathscr{F},$$

[illegible] $u \in E$ [illegible] F [illegible] E [illegible] $L: E \to F$ [illegible] $\mathscr{F} \in F$.

[illegible]:

$$\|Lu\|_F \leq C\|u\|_E \quad \forall u \in D(L),$$
$$\|u\|_E \leq c\|Lu\|_F \quad \forall u \in D(L),$$

[illegible] C [illegible] c [illegible].

[illegible] F [illegible] $R(L)$ [illegible].

[illegible].

Résumé

Le présent travail est l'objet d'une extension de la méthode des inégalités énergétiques à de nouveaux problèmes mixtes pour équations aux dérivées partielles et équations aux dérivées partielles de type mixte avec conditions aux bords non classiques de type intégral. Ces problèmes sont les modèles mathématiques rencontrés en théorie de la conduction thermique, mémoire des matériaux, semi-conducteurs et en électrochimie etc…

La méthode utilisée est la méthode des inégalités énergétiques qui est basée sur la recherche d'un opérateur Mu, dit multiplicateur, qui dépend de la fonction u, ses dérivées et d'une certaine fonction poids. On est ramené par la suite à effectuer des intégrations sur le domaine considéré en vu de doter E et F de normes adéquates afin de pouvoir montrer l'existence et l'unicité de la solution, dite forte, du problème considéré après l'avoir mis sous la forme

$$Lu=\mathscr{F},$$

où $L:E\rightarrow F$ est l'opérateur engendré par le problème considéré, E est un espace de Banach, F est un espace de Hilbert, $u\in E$ et $\mathscr{F}\in F$.

On démontre deux inégalités à priori:

$$\|Lu\|_F \leq C\|u\|_E \quad \forall u \in D(L),$$

$$\|u\|_E \leq c\|Lu\|_F \quad \forall u \in D(L),$$

où C et c sont des constantes.

L'unicité de la solution du problème considéré résulte de ces deux inégalités. Son existence est assurée par le fait que $R(L)$ est dense dans F, chose faisable moyennant des opérateurs de régularisation que l'on choisira suivant la nature de problème.

Il convient de noter que l'absence d'une théorie générale a nécessité une étude spéciale pour chaque problème considéré.

Abstract

The present work is the object of an extension of the method of energy inequalities to new mixed problems for high-order differential equations and high-order differential of mixed type with non classical boundary conditions of integral type. These problems are mathematical models encountered in the theory of thermo conduction, memory materials, semiconductors and the electrochemistry ect…

The mixed problems with integral conditions takes more and more interest as a result of the fundamental reason which is the basis of the physical significance of the integral condition as an average, a flux, a total energy, a moment, etc…

The existence and uniqueness of the strong solutions in functional weighted Sobolev space are proved. The used method is the energy equalities method which is based on the research of an operator *Mu* known as multiplier. This last one depends on the function *u*, its derivatives and some weight function. We are then conducted to take integrations over the considered domain with a view to equipping *E* and *F* with appropriate norms in order to show the existence and uniqueness of the solution of the considered problem once it has been made into the form

$$Lu=\mathscr{F},$$

where $L:E\to F$ is the operator generated by the considered problem, *E* is an Banach space, *F* a Hilbert space, $u\in E$ and $\mathscr{F}\in F$.

We demonstrate two sided a priori inequalities

$$\|Lu\|_F \le C\|u\|_E \quad \forall u\in D(L),$$

$$\|u\|_E \le c\|Lu\|_F \quad \forall u\in D(L),$$

where *C* and *c* are constants.

The uniqueness of the solution, said strong, of the considered problems results from these two inequalities. Its existence is ensured by the fact that *R*(*L*) is dense in *F*, which can be proved by the regularly operators, according to the nature of the considered problem.

It is convenient to note that the absence of a general theory made it necessarily to investigate each problem separately.