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INTRODUCTION

Inequalities have played a dominant role in the development of all branches of mathematics, and they have a central place in the attention of many mathematicians. One reason for much of the successful mathematical development in the theory of ordinary and partial differential equations is the availability of some kinds of inequalities and variational principles involving functions and their derivatives. Differential and integral inequalities have become a major tool in the analysis of the differential and integral equations that occur in nature or are constructed by people. A good deal of information on this subject may be found in a number of monographs published during the last few years.

Integral inequalities play a fundamental role in the study of qualitative properties of differential and integral equations. They were introduced by Gronwall in 1919 [1], who gave their applications in the study of some problems concerning ordinary differential equation. The inequality of Gronwall was stated as follows.

Let $u : [\alpha, \alpha + h] \to \mathbb{R}$ be a continuous function satisfying the inequality

$$0 \leq u(t) \leq \int_{\alpha}^{t} [a + bu(s)]\, ds, \qquad \text{for } t \in [\alpha, \alpha + h]\,,$$

where a and b are nonnegative constants, then

$$0 \leq u(t) \leq ah\ e^{bh}, \ \text{for } t \in [\alpha, \alpha + h]\,.$$

This result is the prototype for the study of many integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. After the discovery of this integral inequality, a number of mathematicians have shown their considerable interest to generalize the original form of this inequality. Among the publications on this subject during the period 1919-1975, the papers of Bellman [2], Bihari [3] and Beesack [4] is well known and have found wide applications. Other names to be mentioned with the further development of the theory of integral inequalities are : R.P. Agarwal, Azbelev, Bainov, DEO, Dhongade, Lakshmikantham, Leela and Pachpatte in [8, 6, 33]. Gronwall-Bellman inequalities [1],[2] and their various generalizations can be used as tools in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations, integral equations, and integrodifferential equations.

The integral inequalities of various types have been widely studied in most subjects involving mathematical analysis. They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. In recent years, the application of integral inequalities has greatly expanded and they are now used not only in mathematics but also in the areas of physics, technology and biological sciences. The theory of differential and integral inequalities has gained increasing significance in the last century as is apparent from the large number of publications on the subject (see [22]-[30]).

Many nonlinear dynamical systems are too complicated to be effectively analized. In many situations, we are interested in knowing qualitative properties of solutions without explicit knowledge of the solution process. Having knowledge of the existence of solutions of the system, the integral inequalities with explicit estimates serve as an important tool in their analysis. In fact, the integral inequalities with explicit estimates and fixed point theorems are powerful tools in nonlinear analysis.

In the last few years, a number of nonlinear integral inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to literatures [4, 5, 8], and the references given therein.

The aim of the present work is to give an exposition of the classical results about integral inequalities with have appeared in the mathematical literature in recent years ; to establish new nonlinear integral inequalities and also many new nonlinear retarded integral inequalities. The results given here can be used in the qualitative theory of various classes of boundary value problems of partial differential equations, partial differential equations with a delay, differential equations, integral equations and integrodifferential equations.

The thesis consists of four chapters. Chapter 1 is devoted to presenting a number of classical facts in the domain of Gronwall inequalities and some nonlinear inequalities in the case of one variable, we collected a most of the these inequalities from [8] and [6].

The second chapter is devoted to establish some multidimensional Integral Inequalities Similar To Gronwall Inequalities. Some bidimensional inequalities obtained in [10, 11] and a new nonlinear integral inequalities for functions with n independent variables obtained by Denche and Khellaf [17]. These results extend the Gronwall type inequalities obtained by Pachpatte [12] and Oguntuase [16].

The third chapter is devoted to establish some integral inequalities in two independent variables with delay [20] and [21], our results generalize the integral inequalities obtained in [22, 23, 27, 29]. Three examples of applications are given to illustrate the usefulness of our results in the fourth chapter.

Chapitre 1

Some Classical Integral Inequalities

In this chapter we present a number of classical linear and nonlinear integral inequalities of Gornwall type for functions of one variable. we collected the most of these inequalities from [6] and [8].

1.1 Linear integral inequalities

1.1.1 Linear integral inequalities of Gronwall type

In a paper published in 1943, Bellman proved the following result [1].

Theorm 1.1 *Let $u(t)$ and $b(t)$ be nonegative continuous functions for $t \geq \alpha$, and let*

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)ds, \qquad t \geq \alpha, \tag{1.1}$$

where $a \geq 0$ is a constant. Then

$$u(t) \leq a \exp\left(\int_{\alpha}^{t} b(s)ds\right), \qquad t \geq \alpha. \tag{1.2}$$

Proof. Let $a > 0$. Then (1.1) implies the inequality

$$\frac{b(\tau)u(\tau)}{a + \int_\alpha^\tau b(s)u(s)ds} \leq b(\tau), \qquad \tau \geq \alpha.$$

Integrating this inequality from α to t yields

$$\ln\left[a + \int_\alpha^t b(s)u(s)ds\right] - \ln a \leq \int_\alpha^t b(s)ds.$$

Together with (1.1) this implies (1.2).

Let $a = 0$. Then $u(t) \leq \varepsilon + \int_\alpha^t b(s)u(s)ds$ for any $\varepsilon > 0$. Hence $u(t) \leq a \exp\left(\int_\alpha^t b(s)ds\right)$ and letting $\varepsilon \to 0$ we find $u(t) = 0$. ■

Lemma 1.2 *Let $b(t)$ and $f(t)$ be continuous functions for $t \geq \alpha$, let $v(t)$ be a differentiable function for $t \geq \alpha$, and suppose*

$$\begin{aligned} v'(t) &\leq b(t)v(t) + f(t), \qquad t \geq \alpha, \\ v(\alpha) &\leq v_{0.} \end{aligned} \tag{1.3}$$

Then, , for $t \geq \alpha$,

$$v(t) \leq v_{0.} \exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t f(s) \exp\left(\int_s^t b(\tau)d\tau\right) ds. \tag{1.4}$$

Proof. From (1.3) we have

$$[v'(s) - b(s)v(s)] \exp\left(\int_s^t b(\tau)d\tau\right) \leq f(s) \exp\left(\int_s^t b(\tau)d\tau\right), \quad s \geq \alpha,$$

or

$$\frac{d}{ds}\left[v(s) \exp\left(\int_s^t b(\tau)d\tau\right)\right] \leq f(s) \exp\left(\int_s^t b(\tau)d\tau\right).$$

Integrating over s from α to t gives

$$v(t) - v(\alpha)\exp\left(\int_s^t b(\tau)d\tau\right) \leq \int_\alpha^t f(s)\exp\left(\int_s^t b(\tau)d\tau\right) ds,$$

since $v(\alpha) \leq v_{0.}$ we obtain (1.4). ■

Remark 1.3 *Note that the right hand side of (1.4) coincides with the unique solution of the equation*

$$v'(t) = b(t)v(t) + f(t), \qquad t \geq \alpha, \tag{1.5}$$

for which

$$v(\alpha) = v_{0.} \tag{1.6}$$

Equation (1.5) is called the comparison differential equation of the inequality (1.3). The comparison initial value problem (1.5)-(1.6) is obtained by replacing in (1.3) $\leq$ by $=$.

Remark 1.4 *The result of lemma 1.2 remains valid if $\leq$ is replaced by $\geq$ in both (1.3) and (1.4).*

Remark 1.5 *If the functions $b(t)$ and $f(t)$ are continuous for $t \leq \alpha$ and*

$$v'(t) \leq b(t)v(t) + f(t), \qquad t \leq \alpha,$$

then

$$v(t) \geq v(\alpha)\exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t f(s)\exp\left(\int_s^t b(\tau)d\tau\right) ds. \qquad t \leq \alpha.$$

Moreover, this result remains valid if $\leq$ is replaced by $\geq$.

Theorm 1.6 *Let $a(t)$, $b(t)$ and $u(t)$ be continuous functions in $J = [\alpha, \beta]$ and let $b(t) \geq 0$, for $t \in J$. Suppose*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)ds, \qquad t \in J.$$

Then

$$u(t) \leq a(t) + \int_{\alpha}^{t} a(s)b(s) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds, \qquad t \in J. \tag{1.7}$$

Proof. Set $v(t) = \int_{\alpha}^{t} b(s)u(s)ds$, then

$$u(t) \leq a(t) + v(t), \tag{1.8}$$

and we have

$$\begin{aligned} v'(t) &= b(t)u(t) \leq b(t)v(t) + a(t)b(t), \qquad t \in J, \\ v(\alpha) &= 0, \end{aligned}$$

From lemma 1.2 we obtain

$$v(t) \leq \int_{\alpha}^{t} a(s)b(s)exp\left(\int_{s}^{t} b(\tau)d\tau\right).$$

From the last inequality and (1.8) yields (1.7) ■

Corollary 1.7 *Let, under the conditions of theorem 1.6, if $a(t)$ is nondecreasing in J. Then*

$$u(t) \leq a(t)exp\left(\int_{\alpha}^{t} b(\tau)d\tau\right), \quad t \in J.$$

Proof. (1.5) implies that

$$u(t) \leq a(t) + a(t)\int_{\alpha}^{t} b(s) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds = a(t)\left[1 - \int_{\alpha}^{t} \frac{d}{ds}\left(\exp\left(\int_{s}^{t} b(\tau)d\tau\right)\right)\right],$$

then

$$u(t) \leq a(t) \exp\left(\int_{\alpha}^{t} b(s)ds\right).$$

■

Corollary 1.8 *Let $b(t)$ and $u(t)$ be continuous functions in $J = [\alpha, \beta]$, let $b(t) \geq 0$, for $t \in J$, and suppose*

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)ds, \qquad t \in J,$$

where a is a constant. Then

$$u(t) \leq a \exp\left(\int_{\alpha}^{t} b(s)ds\right), \qquad t \in J.$$

Remark 1.9 *The conclusion of corollary 1.8 shows that in theorem 1.1 we may omit the requirement that $u(t)$ and a be nonegative.*

Corollary 1.10 *Let $u(t)$ be a continuous function in $J = [\alpha, \beta]$, and suppose*

$$u(t) \leq a + \int_{\alpha}^{t} bu(s)ds, \qquad t \in J,$$

where $b \geq 0$ and a are constants. Then

$$u(t) \leq ae^{b(t-\alpha)}, \qquad t \in J.$$

The following assertion is related to work of Giuliano, Kharlamov, Willet, and Beesack [4].

Theorm 1.11 *Let $u(t)$ and $k(t)$ be continuous functions in $J = [\alpha, \beta]$, and let $a(t)$ and $b(t)$ be Riemann integrable functions in J with $k(t)$ and $b(t)$ are nonnegative in J*

i) If

$$u(t) \leq a(t) + b(t)\int_{\alpha}^{t} k(s)u(s)ds, \qquad t \in J, \tag{1.9}$$

then

$$u(t) \leq a(t) + b(t)\int_{\alpha}^{t} a(s)k(s)\exp\left(\int_{s}^{t} k(\tau)b(\tau)d\tau\right)ds, \qquad t \in J. \tag{1.10}$$

Moreover, equality holds in (1.10) for a subinterval $J_1 = [\alpha_1, \beta_1]$ of J if equality holds in

(1.9) for $t \in J$.

ii) If " $\leq$ " is replaced by " $\geq$ " in both (1.9) and (1.10), the result remain valid.

iii) Both i) and ii) remain valid if $\int_\alpha^t$ is replaced by $\int_t^\beta$ and $\int_s^t$ by $\int_t^s$ throughout.

Corollary 1.12 *Let $u(t)$, $a(t)$, $b(t)$ and $k(t)$ be continuous functions in $J = [\alpha, \beta]$, let $c(t,s)$ be a continuous function for $\alpha \leq s \leq t \leq \beta$, let $b(t)$ and $k(t)$ be nonegative in J, and suppose*

$$u(t) \leq a(t) + \int_\alpha^t [k(t)b(s)u(s) + c(t,s)]\, ds, \qquad t \in J.$$

Then for $t \in J$,

$$\begin{aligned} u(t) \quad \leq & \quad a(t) + \int_\alpha^t c(t,s)ds \\ & + k(t) \int_\alpha^t b(s) \left[a(s) + \int_\alpha^s c(t,\tau)d\tau \right] \exp\left(\int_s^t b(\tau)k(\tau)d\tau \right) ds. \end{aligned}$$

Corollary 1.13 *Let $u(t)$, $a(t)$, $b_i(t)$ and $k_i(t)$ $(i = 1, ..., n)$ be continuous functions in $J = [\alpha, \beta]$, let $b_i(t)$ and $k_i(t)$ be nonegative in J, and suppose*

$$u(t) \leq a(t) + \sum_{i=1}^n k_i(t) \int_\alpha^t b_i(s)u(s)ds, \qquad t \in J.$$

Then for $t \in J$,

$$u(t) \leq a(t) + K(t) \int_\alpha^t a(s) \sum_{i=1}^n b_i(s) \exp\left(\int_s^t K(\tau) \sum_{i=1}^n b_i(\tau)d\tau \right) ds,$$

where, $K(t) = \sup_{i=1,\ldots,n} k_i(t)$.

1.1.2 GronwalI-Bellman linear inequalities

On the basis of various motivations, the Gronwall-Bellman inequality has been extended and used considerably in various contexts. This section gives some useful generalizations and variants of the Gronwall-Bellman inequality in one variable.

Gollwitzer [8] gave the following generalization of the Gronwall-Bellman inequality.

Theorm 1.14 *Let u, f, g and h be nonnegative continuous functions defined on $J = [\alpha, \beta]$, and*

$$u(t) \leq f(t) + g(t) \int_\alpha^t h(s)u(s)ds, \qquad t \in J.$$

Then

$$u(t) \leq f(t) + g(t) \int_\alpha^t h(s)f(s) \exp\left(\int_s^t h(\tau)g(\tau)d\tau\right) ds, \qquad t \in J.$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_\alpha^t h(s)u(s)ds,$$

then $z(\alpha) = 0$, $u(t) \leq f(t) + g(t)z(t)$ and

$$\begin{aligned} z'(t) &= h(t)u(t) \leq h(t)g(t)z(t) + h(t)f(t), \\ z(\alpha) &= 0 \end{aligned}$$

an application of lemma 1.2 to the last inequality we get

$$z(t) \leq \int_\alpha^t h(s)f(s) \exp\left(\int_s^t h(\tau)g(\tau)d\tau\right) ds,$$

substituting the last inequality in $u(t) \leq f(t) + g(t)z(t)$ we obtain the desired inequality.

■

Pachpatte in [9] employed the following variant of the inequality given in Theorem 14 in obtaining various generalizations of Bellman's inequality.

Theorm 1.15 *Let u, g and h be nonnegative continuous functions defined on $J = [\alpha, \beta]$, $n(t)$ be a continuous, positive and nondecreasing function defined on J and*

$$u(t) \leq n(t) + g(t) \int_\alpha^t h(s)u(s)ds, \qquad t \in J.$$

Then $u(t) \leq n(t) \left[1 + g(t) \int_\alpha^t h(s) \exp\left(\int_s^t h(\tau)g(\tau)d\tau\right) ds\right], \qquad t \in J.$

A fairly general version of theorem 1.14 is given in the following result.

Theorm 1.16 *Let u, p, q, f and g be nonnegative continuous functions defined on $J = [\alpha, \beta]$, and*

$$u(t) \leq p(t) + q(t) \int_\alpha^t [f(s)u(s) + g(s)]\, ds, \qquad t \in J.$$

Then

$$u(t) \leq p(t) + q(t) \int_\alpha^t [f(s)p(s) + g(s)] \exp\left(\int_s^t f(\tau)q(\tau)d\tau\right), \quad t \in J.$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_\alpha^t [f(s)u(s) + g(s)]\, ds,$$

Now by following the proof of theorem 1.14, we get the desired inequality ■

Remark 1.17 *By setting $q(t) = 1$ in theorem 1.16 we arrive at the inequality given by Chandirov in [6]. If we take $g(t) = 0$ in theorem 1.16 we get the inequality given in theorem 1.14.*

Gamidov proved the following inequalities and employed them to obtain bounds for the solutions of certain boundary value problems.

Theorm 1.18 *Let u, f, g_i, h_i $(i = 1, 2, ..., n)$ be continuous functions defined on $J = [\alpha, \beta]$, let g_i and h_i be nonnegative in J, and*

$$u(t) \leq f(t) + \sum_{i=1}^n g_i(t) \int_\alpha^t h_i(s)u(s)ds, \qquad t \in J.$$

Then for $t \in J$,

$$u(t) \leq f(t) + g(t) \int_\alpha^t f(s) \sum_{i=1}^n h_i(s) \exp\left(\int_s^t g(\sigma) \sum_{i=1}^n h_i(\sigma)d\sigma\right) ds,$$

where $g(t) = \sup_i \{g_i(t)\}$.

Proof. We observe that

$$u(t) \leq f(t) + g(t) \int_\alpha^t \left(\sum_{i=1}^n h_i(s) \right) u(s)ds, \qquad t \in J.$$

Now an application of theorem 1.14 gives the required inequality. ■

1.1.3 Volterra type integral inequalities

Integral inequalities which satisfies a Volterra integral inequality have wide applications in the theory of differential and integral equations. In this section we consider some Volterra type integral inequalities involving an unknown function of a single variable.

Chu and Metcalf [6] proved the following linear generalization of the Gronwall-Bellman inequality (with a kernel).

Theorm 1.19 *Let* $u(t)$ *and* $a(t)$ *be continuous functions in* $J = [\alpha, \beta]$, *let* $k(s,t)$ *be a nonnegative continuous function in the triangle* $\Delta = \{(t,s) \in \mathbb{R}^2 : \alpha \leq s \leq t \leq \beta\}$, *and suppose*

$$u(t) \leq a(t) + \int_\alpha^t k(t,s)u(s)ds, \qquad t \in J.$$

Then

i)

$$u(t) \leq v(t), \qquad t \in J,$$

where $v(t)$ *is solution of the equation*

$$v(t) = a(t) + \int_\alpha^t k(t,s)v(s)ds, \qquad t \in J.$$

ii) The solution $v(t)$ is unique and can be obtained as the sum of a Neumann series :

$$\begin{aligned} v(t) &= v_0(t) + ... + v_n(t) + ..., \\ \textit{where } v_0(t) &= a(t), v_n(t) = \int_\alpha^t k(t,s)v_{n-1}(s)ds, n = 1, 2, ... \end{aligned}$$

The following theorem presents slight variants of the inequality given by Norbury and Stuart [31] which are sometimes applicable more conveniently.

Theorm 1.20 *[8] Let $u(t)$ be a continuous function in $J = [\alpha, \beta]$ and $k(t,s)$ be nondecreasing function in t for each $s \in J$.*

(i) If

$$u(t) \leq c + \int_\alpha^t k(t,s)u(s)ds, \qquad t \in J, \tag{1.11}$$

where $c \geq 0$ is a constant. Then

$$u(t) \leq c \exp\left(\int_\alpha^t k(t,s)ds\right), \qquad t \in J, \tag{1.12}$$

(ii) Let $a(t)$ be a positive continuous and nondecreasing function for $t \in J$. If

$$u(t) \leq a(t) + \int_\alpha^t k(t,s)u(s)ds, \qquad t \in J, \tag{1.13}$$

then

$$u(t) \leq a(t) \exp\left(\int_\alpha^t k(t,s)ds\right), \qquad t \in J. \tag{1.14}$$

Proof. (i) Fix any T, $\alpha \leq T \leq \beta$. Then, for $\alpha \leq t \leq T \leq \beta$, we have

$$u(t) \leq c + \int_\alpha^t k(T,s)u(s)ds, \qquad t \in J. \tag{1.15}$$

Define a function $z(t)$ by the fight side of (1.15), then $z(\alpha) = c, u(t) \leq z(t)$ for $\alpha \leq t \leq T \leq \beta$ and

$$z'(t) = k(T,t)u(t) \leq k(T,t)z(t), \quad \alpha \leq t \leq T.$$

By setting $t = s$ in the last inequality and integrating it with respect to s from α to t we get

$$z(t) \leq c \exp\left(\int_\alpha^t k(T,s)ds\right).$$

Since T is arbitrary, with T replaced by t and $u(t) < z(t)$ we get the inequality (1.12).

(ii) Since $a(t)$ is a positive continuous and nondecreasing function for $t \in J$, from (1.13) we observe that

$$\frac{u(t)}{a(t)} \leq 1 + \int_\alpha^t k(t,s)\frac{u(s)}{a(s)}ds, \qquad t \in J.$$

Now an application of the inequality given in (i) yields the desired result (1.14). ■

Remark 1.21 *Note that the inequality given in part (i) was obtained by Norbury and Stuart [31] under the assumptions of the existence and nonnegativity of* $\frac{\partial}{\partial t}k(t,s)$.

Pachpatte in [8] proved the following inequality, which in turn is a further generalization of Norbury and Stuart's inequality [31].

Theorm 1.22 *Let* u, p, q, r *and* f *be nonnegative continuous functions defined on* $J = [\alpha, \beta]$. *Let* $k(t,s)$ *and its partial derivative* $\frac{\partial}{\partial t}k(t,s)$ *be nonnegative continuous functions for* $\alpha \leq s \leq t \leq \beta$, *and*

$$u(t) \leq p(t) + q(t)\int_\alpha^t k(t,s)\left[r(s)u(s) + f(s)\right]ds, \quad t \in J. \tag{1.16}$$

Then

$$u(t) \leq p(t) + q(t)\int_\alpha^t B(\sigma)\exp\left(\int_\sigma^t A(\tau)\,d\tau\right)d\sigma, \quad t \in J, \tag{1.17}$$

where

$$A(t) = k(t,t)r(t)q(t) + \int_\alpha^t \frac{\partial}{\partial t}k(t,s)r(s)q(s)ds, \quad t \in J, \tag{1.18}$$

$$B(t) = k(t,t)\left[r(t)p(t) + f(t)\right] + \int_\alpha^t \frac{\partial}{\partial t}k(t,s)\left[r(s)p(s) + f(s)\right]ds, \; t \in J. \tag{1.19}$$

Remark 1.23 *Note that the special version of the above inequality with* $r(t) = 1$ *and* $f(t) = 0$ *was obtained by Movlyankulov and Filatov [32].*

1.2 Nonlinear integral inequalities in one variable

One of the most useful methods available for studying a nonlinear system of ordinary differential equations, which is typical among investigations on this subject, is the use of nonlinear integral inequalities which provide explicit bounds on the unknown functions in the case of one or more than one variable. This section considers various nonlinear integral inequalities in the case of one variable discovered in the literature.

In the past few years many authors have obtained various generalizations and extensions of Gronwall-Bellman-Bihari inequalities [8]-[17]. In this section some generalizations of this inequalities containing combinations of the inequalities by taking a sum of two integrals will be given, one containing the unknown function of one variable in a linear form and the other in a non-linear form.

In what follow we need the following definition :

Definition 1.24 *A function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is said to be*

i) subadditive if $f(x+y) \leq f(x) + f(y), x, y \in \mathbb{R}_+$;

ii) submultiplicative if $f(xy) \leq f(x)f(y), x, y \in \mathbb{R}_+$.

Pachpatte [9] proved the following integral inequalities.

Theorm 1.25 *Let* u, f, g *and* h *be nonnegative continuous functions defined on* R^+. *Let* $w(u)$ *be a continuous nondecreasing and submultiplicative function defined on* R^+ *and* $w(u) > 0$ *on* $(0, \infty)$. *If*

$$u(t) \leq u_0 + g(t) \int_0^t f(s)u(s)ds + \int_0^t h(s)w(u(s))\,ds, \tag{1.20}$$

for all $t \in R^+$, *where* u_0 *is a positive constant, then for* $0 \leq t \leq t_1$,

$$u(t) \leq a(t)G^{-1}\left[G\left(u_0\right) + \int_0^t h(s)w\left(a(s)\right)ds\right], \tag{1.21}$$

where

$$a(t) = 1 + g(t)\int_0^t f(s)\exp\left(\int_0^t g(\sigma)f(\sigma)d\sigma\right)ds, \tag{1.22}$$

for $t \in R^+$ *and*

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)}, \qquad r > 0, r_0 > 0, \tag{1.23}$$

and G^{-1} *is the inverse function of* G, *and* $t_1 \in R^+$ *is chosen so that*

$$G\left(u_0\right) + \int_0^t h(s)w\left(a(s)\right)ds \in Dom\left(G^{-1}\right),$$

for all $t \in R^+$ *lying in the interval* $0 \leq t \leq t_1$.

Proof. Define a function $z(t)$ by

$$z(t) = u_0 + \int_0^t h(s)w\left(u(s)\right)ds, \tag{1.24}$$

then (1.20) can be restated as

$$u(t) \leq z(t) + g(t)\int_0^t f(s)u(s)ds.$$

Since $z(t)$ is positive monotonic nondecreasing on R^+, by applying theorem 3.14 we have

$$u(t) \leq a(t)z(t), \tag{1.25}$$

where $a(t)$ is defined by (1.22). From (1.24) and (1.25) we have

$$\begin{aligned} z'(t) &= h(t)w\left(u(t)\right) \\ &\leq h(t)w\left(a(t)z(t)\right) \\ &\leq h(t)w\left(a(t)\right)w\left(z(t)\right) \end{aligned} \tag{1.26}$$

From (1.23) and (1.26) we have

$$\frac{d}{dt}G\left(z(t)\right) = \frac{z'(t)}{w\left(z(t)\right)} \leq h(t)w\left(a(t)\right). \tag{1.27}$$

By taking $t = s$ in (1.27) and integrating it from 0 to t, we obtain

$$G\left(z(t)\right) \leq G\left(u_0\right) + \int_0^t h(s)w\left(a(s)\right)ds. \tag{1.28}$$

The desired bound in (1.21) follows from (1.25) and (1.28). The subinterval $0 \leq t \leq t_1$ is obvious ■

Pachpatte [9] also gave the following theorem, which can be used in more general situations.

Theorm 1.26 *Let u, f, g and h be nonnegative continuous functions defined on R^+. Let $w(u)$ be a continuous nondecreasing subadditive and submultiplicative function defined on R^+ and $w(u) > 0$ on $(0, \infty)$. Let $p(t) > 0, \phi(t) \geq 0$ be continuous and nondecreasing functions defined on R^+ and $\phi(0) = 0$. If*

$$u(t) \leq p(t) + g(t)\int_0^t f(s)u(s)ds + \phi\left(\int_0^t h(s)w\left(u(s)\right)ds\right), \tag{1.29}$$

for all $t \in R^+$*, then for* $0 \le t \le t_2$*,*

$$u(t) \le a(t)\left[p(t) + \phi\left(F^{-1}\left[F\left(A(t)\right) + \int\limits_0^t h(s)w\left(a(s)\right)ds\right]\right)\right], \tag{1.30}$$

where $a(t)$ *is defined by* (1.22) *and*

$$A(t) = \int\limits_0^t h(s)w\left(a(s)p(s)\right)ds,$$

$$F(r) = \int\limits_{r_0}^r \frac{ds}{w(\phi\left(s\right))}, \qquad r > 0, r_0 > 0, \tag{1.31}$$

where F^{-1} *is the inverse of* F*, and* $t_2 \in R^+$ *is chosen so that*

$$F\left(A(t)\right) + \int\limits_0^t h(s)w\left(a(s)\right)ds \in Dom\left(F^{-1}\right),$$

for all $t \in R^+$ *lying in the interval* $0 \le t \le t_2$*.*

Theorm 1.27 *[9] Let* u, f, g *and* h *be nonnegative continuous functions defined on* R^+*. Let* $w(t, u)$ *be a nonnegative continuous monotonic nondecreasing function in* $u \ge 0$*, for each fixed* $t \in R^+$*. Let* p *and* ϕ *be as defined in Theorem 25. If*

$$u(t) \le p(t) + g(t)\int\limits_0^t f(s)u(s)ds + \phi\left(\int\limits_0^t h(s)w\left(s, u(s)\right)ds\right),$$

for all $t \in R^+$*, then*

$$u(t) \le a(t)\left[p(t) + \phi\left(r(t)\right)\right], \quad t \in R^+,$$

where $a(t)$ *is defined by* (1.22) *and* $r(t)$ *is the maximal solution of*

$$r'(t) = h(t)w\left(t, a(t)\right)\left[p(t) + \phi\left(r(t)\right)\right], \ r(0) = 0,$$

existing on R^+.

Constantin [7] has given the following inequality, which can be used in certain applications.

Theorm 1.28 *Suppose*

(i) $u, k, g, h_1, h_2 : R^+ \to (0, \infty)$ *and continuous,*

(ii) $k, g \in C^1(R^+, R^+)$,

(iii) $H(u)$ *is a nonnegative monotonic nondecreasing and continuous function for* $u > 0$ *with* $H(0) = 0$.

If

$$u(t) \leq g(t) + k(t) \int_0^t h_1(s)u(s)ds + k(t) \int_0^t h_2(s)H(u(s))\,ds, \tag{1.32}$$

for $t \in R^+$, *then*

$$\begin{aligned} u(t) \quad \leq \quad & G^{-1}\left[G(g(0)) + \int_0^t [k(s)\max\{h_1(s), h_2(s)\}\right. \\ & \left. + \max\left\{0, \frac{k'(s)}{k(s)}\right\} + \max\left\{0, \frac{g'(s)}{g(s)}\right\}\right] ds \Bigg] \end{aligned} \tag{1.33}$$

for $t \in [0, t']$, *where*

$$G(r) = \int_{r_0}^r \frac{ds}{s + H(s)}, \qquad r > 0, r_0 > 0, \tag{1.34}$$

and t' *is defined so that the existence condition of the right-hand part of the inequality (1.33) should be assured.*

Chapitre 2

Multidimensional Integral Inequalities Similar To Gronwall Inequalities

This chapter gives some nonlinear multidimensional integral inequalities recently discovered in the literature. These inequalities can be used as ready and powerful tools in the study of various problems in the theory of certain partial differential, integral and integro-differential equations.

2.1 Bidimensional integral inequalities

2.1.1 Some linear inequalities in two independent variables

This subsection presents some linear inequalities given by Pachpatte which can be used in the study of qualitative properties of the solutions of certain integro-differential and integral equations.

Pachpatte [11] established the following inequality.

Theorm 2.1 *Let $u(x,y), f(x,y)$ and $g(x,y)$ be nonnegative continuous functions defined for $x,y \in R^+$. If*

$$u(x,y) \leq a(x)+b(y)+\int_0^x\int_0^y f(s,t)\left(u(s,t) + \int_0^s\int_0^t g(\sigma,\eta)u(\sigma,\eta)d\eta d\sigma\right)dtds, \tag{2.1}$$

for $x,y \in R^+$, where $a(x) > 0, b(y) > 0$ are continuous functions for $x,y \in R^+$, having derivatives such that $a'(x) \geq 0, b'(y) \geq 0$ for $x,y \in R^+$, then

$$u(x,y) \leq a(x)+b(y)+\int_0^x\int_0^y f(s,t)E(s,t) \times \exp\left(\int_0^s\int_0^t [f(\sigma,\eta)+g(\sigma,\eta)]\,d\eta d\sigma\right)dtds. \tag{2.2}$$

for $x,y \in R^+$, where

$$E(x,y) = \frac{[a(x)+b(0)]\,[a(0)+b(y)]}{a(0)+b(0)}. \tag{2.3}$$

Proof. Define a function $z(x,y)$ by the right-hand side of (2.1). Then $z(0,y) = a(0)+b(y), z(x,0) = a(x)+b(0), u(x,y) \leq z(x,y)$ and

$$\begin{aligned} z_{xy}(x,y) &= f(x,y)\left(u(x,y)+\int_0^x\int_0^y g(\sigma,\eta)u(\sigma,\eta)d\eta d\sigma\right) \\ &\leq f(x,y)\left(z(x,y)+\int_0^x\int_0^y g(\sigma,\eta)z(\sigma,\eta)d\eta d\sigma\right). \end{aligned} \tag{2.4}$$

Define a function $v(x,y)$ by

$$v(x,y) = z(x,y)+\int_0^x\int_0^y g(\sigma,\eta)z(\sigma,\eta)d\eta d\sigma. \tag{2.5}$$

Then $v(0,y) = a(0)+b(y), v(x,0) = a(x)+b(0), z_{xy}(x,y) \leq f(x,y)v(x,y), z(x,y) \leq$

$v(x,y)$ and

$$\begin{aligned} v_{xy}(x,y) &= z_{xy}(x,y) + g(x,y)z(x,y) \\ &\leq [f(x,y) + g(x,y)]\, z(x,y) \\ &\leq [f(x,y) + g(x,y)]\, v(x,y), \end{aligned}$$

then

$$\frac{v_{xy}(x,y)}{v(x,y)} \leq f(x,y) + g(x,y).$$

Since $v(x,y) > 0, v_x(x,y) > 0, v_y(x,y) > 0$, then the last inequality can be restated as

$$\frac{v_{xy}(x,y)}{v(x,y)} \leq [f(x,y) + g(x,y)] + \frac{v_x(x,y)v_y(x,y)}{v^2(x,y)},$$

or

$$\frac{\partial}{\partial y}\left(\frac{v_x(x,y)}{v(x,y)}\right) \leq f(x,y) + g(x,y).$$

Now keeping x fixed in the last inequality, set $y = t$ and integrate with respect to t from 0 to y to obtain the estimate

$$\frac{v_x(x,y)}{v(x,y)} - \frac{v_x(x,0)}{v(x,0)} \leq \int_0^y [f(x,t) + g(x,t)]\, dt.$$

Keeping y fixed in the last inequality, set $x = s$ and integrate with respect to s from 0 to x to obtain the estimate

$$\ln v(x,y) - \ln v(0,y) - \ln v(x,0) + \ln v(0,0) \leq \int_0^x \int_0^y [f(s,t) + g(s,t)]\, dtds,$$

then

$$v(x,y) \leq E(x,y) \exp\left(\int_0^x \int_0^y [f(s,t) + g(s,t)]\, dtds\right). \tag{2.6}$$

Using (2.6) in (2.4) we have

$$z_{xy}(x,y) \leq f(x,y)E(x,y)\exp\left(\int_0^x \int_0^y [f(s,t)+g(s,t)]\,dtds\right). \tag{2.7}$$

From (2.7) it follows that

$$\begin{aligned} z(x,y) \ \leq \ & a(x)+b(y)+\int_0^x \int_0^y f(s,t)E(s,t) \\ & \times \exp\left(\int_0^s \int_0^t [f(\sigma,\eta)+g(\sigma,\eta)]\,d\eta d\sigma\right) dtds. \end{aligned}$$

From $u(x,y) \leq z(x,y)$ we get the required inequality in (2.2) ■

Remark 2.2 *In the special case when $a(x)+b(y)=k$, for $x,y \in R^+$, where $k>0$ is a constant, then the bound obtained in the theorem 2.1 reduces to*

$$u(x,y) \leq k\left[1+\int_0^x \int_0^y f(s,t)\exp\left(\int_0^s \int_0^t [f(\sigma,\eta)+g(\sigma,\eta)]\,d\eta d\sigma\right) dtds\right].$$

Remark 2.3 *In the special case when $a(x)+b(y)=k, g=0$ for $x,y \in R^+$, where $k>0$ is a constant, then the bound obtained in the inequality (2.2) reduces to famous result of Gronwall-Bellman (1.1) in the case of two independent variables*

A useful generalization of theorem 2.1 is given in the following theorem.

Theorm 2.4 *[8] Let $u(x,y), f(x,y), g(x,y)$ and $c(x,y)$ be nonnegative continuous functions defined for $x,y \in R^+$, and let $c(x,y)$ be nondecreasing in each variable $x,y \in R^+$. If*

$$u(x,y) \leq c(x,y)+\int_0^x \int_0^y f(s,t)\left(u(s,t)+\int_0^s \int_0^t g(\sigma,\eta)u(\sigma,\eta)d\eta d\sigma\right) dtds,$$

for $x,y \in R^+$, then

$$u(x,y) \leq c(x,y)H(x,y),$$

for $x, y \in R^+$, where

$$H(x,y) = 1 + \int_0^x \int_0^y f(s,t) \exp\left(\int_0^s \int_0^t [f(\sigma,\eta) + g(\sigma,\eta)]\, d\eta d\sigma \right) dt ds,$$

for $x, y \in R^+$.

The proof of theorem 2.4 follows by the same argument as in the proof of theorem 2.1.

Theorm 2.5 *[8] Let $u(x,y), f(x,y), g(x,y), h(x,y)$ and $p(x,y)$ be nonnegative continuous functions defined for $x, y \in R^+$ and u_0 be nonnegative constant.*

(a_1) If

$$\begin{aligned} u(x,y) &\leq u_0 + \int_0^x \int_0^y [f(s,t)u(s,t) + p(s,t)]\, dt ds \\ &+ \int_0^x \int_0^y f(s,t) \left(\int_0^s \int_0^t g(\sigma,\eta) u(\sigma,\eta) d\eta d\sigma \right) dt ds, \end{aligned}$$

for $x, y \in R^+$, then

$$u(x,y) \leq \left(u_0 + \int_0^x \int_0^y p(s,t) dt ds \right) H(x,y),$$

for $x, y \in R^+$, where $H(x,y)$ is defined in theorem .

(a_2) If

$$\begin{aligned} u(x,y) &\leq u_0 + \int_0^x \int_0^y f(s,t)u(s,t) dt ds + \int_0^x \int_0^y f(s,t) \\ &\times \left(\int_0^s \int_0^t [g(\sigma,\eta)u(\sigma,\eta) + p(\sigma,\eta)]\, d\eta d\sigma \right) dt ds, \end{aligned}$$

for $x, y \in R^+$, then

$$u(x,y) \leq \left(u_0 + \int_0^x \int_0^y f(s,t) \left(\int_0^s \int_0^t p(\sigma,\eta) d\eta d\sigma \right) dt ds \right) H(x,y),$$

for $x, y \in R^+$, *where* $H(x, y)$ *is defined in theorem .*

(a_3) *If*

$$\begin{aligned} u(x,y) &\le u_0 + \int_0^x \int_0^y h(s,t)u(s,t)dtds + \int_0^x \int_0^y f(s,t) \\ &\quad \times \left(u(s,t) + \int_0^s \int_0^t [g(\sigma,\eta)u(\sigma,\eta) + p(\sigma,\eta)]\, d\eta d\sigma \right) dtds, \end{aligned}$$

for $x, y \in R^+$, *then*

$$u(x,y) \le u_0 \exp\left(\int_0^x \int_0^y h(s,t)dtds \right) H(x,y),$$

for $x, y \in R^+$, *where* $H(x, y)$ *is defined in theorem .*

(a_4) *If*

$$\begin{aligned} u(x,y) &\le h(x,y) + p(x,y) \int_0^x \int_0^y f(s,t)\left(u(s,t) + p(s,t)\right. \\ &\quad \left. \times \left(\int_0^s \int_0^t g(\sigma,\eta)u(\sigma,\eta)d\eta d\sigma \right) \right) dtds, \end{aligned}$$

for $x, y \in R^+$, *then*

$$\begin{aligned} u(x,y) &\le h(x,y) + p(x,y)M(x,y) \left[1 + \int_0^x \int_0^y f(s,t)p(s,t) \right. \\ &\quad \left. \times \exp\left(\int_0^s \int_0^t [f(\sigma,\eta) + g(\sigma,\eta)]\, p(\sigma,\eta)d\eta d\sigma \right) dtds, \right. \end{aligned}$$

for $x, y \in R^+$, *where*

$$M(x,y) = \int_0^x \int_0^y f(s,t) \left(h(s,t) + p(s,t) \int_0^s \int_0^t g(\sigma,\eta)h(\sigma,\eta)d\eta d\sigma \right) dtds$$

for $x, y \in R^+$.

Proof. (see [8]) ■

For other generalizations of linear inequalities in two independent variables of this form please consult the references [8, 6, 14].

2.1.2 Wendroff inequalities

The fundamental role played by Wendroff's inequality and its generalizations and variants in the development of the theory of partial differential and integral equations is well known. In this section we present some basic nonlinear generalizations of Wendroff's inequality established by Bondge and Pachpatte [10] and some new variants, which can be used as tools in the study of certain partial differential and integral equations.

First we present the basic inequality due to Wendroff given in Beckenbach and Bellman [5] and some of its variants which can be used in certain applications. The main result due to Wendroff [5] is embodied in the following theorem.

Theorm 2.6 *Let $u(x,y), c(x,y)$ be nonnegative continuous functions defined for $x, y \in R^+$. If*

$$u(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y c(s,t)u(s,t)dtds,$$

for $x, y \in R^+$, where $a(x), b(y)$ are positive continuous functions for $x, y \in R^+$, having derivatives such that $a'(x) \geq 0, b'(y) \geq 0$ for $x, y \in R^+$, then

$$u(x,y) \leq E(x,y) \exp\left(\int_0^x \int_0^y c(s,t)dtds \right),$$

for $x, y \in R^+$, where

$$E(x,y) = [a(x) + b(0)]\,[a(0) + b(y)]\,[a(0) + b(0)]\,,$$

for $x, y \in R^+$.

Proof. (See [8, Chapter 4]) ■

Bondge and Pachpatte [10] proved the following useful nonlinear generalization of Wendroff's inequality for functions in two independent variables.

Theorm 2.7 *Let $u(x, y)$ and $p(x, y)$ be nonnegative continuous functions defined for $x, y \in R^+$. Let $g(u)$ be a continuously differentiable function defined for $u \geq 0, g(u) > 0$ for $u > 0$ and $g'(u) \geq 0$ for $u \geq 0$. If*

$$u(x,y) \leq a(x) + b(y) + \int_0^x \int_0^y p(s,t) g\left(u(s,t)\right) dtds, \tag{2.8}$$

for $x, y \in R^+$, where $a(x) > 0, b(y) > 0, a'(x) \geq 0, b'(x) \geq 0$ are continuous functions defined for $x, y \in R^+$, then for $0 \leq x \leq x_1, 0 \leq y \leq y_1$,

$$\begin{aligned} u(x,y) \quad &\leq \quad \Omega^{-1} \left[\Omega\left(a(0) + b(y)\right) + \int_0^x \frac{a'(s)}{g\left(a(s) + b(0)\right)} ds \right. \\ &\qquad \left. + \int_0^x \int_0^y p(s,t) dtds \right], \end{aligned} \tag{2.9}$$

where

$$\Omega(r) = \int_{r_0}^r \frac{ds}{g(s)}, r > 0, r_0 > 0, \tag{2.10}$$

Ω^{-1} is the inverse function of Ω and x_1, y_1 are chosen so that

$$\Omega\left(a(0) + b(y)\right) + \int_0^x \frac{a'(s)}{g\left(a(s) + b(0)\right)} ds + \int_0^x \int_0^y p(s,t) dtds \in Dom\left(\Omega^{-1}\right),$$

for all x, y lying in the subintervals $0 \leq x \leq x_1, 0 \leq y \leq y_1$ of R^+.

Proof. We note that since $g\prime(u) \geq 0$ on R^+, the function $g(u)$ is monotonically increasing on $(0, \infty)$. Define a function $z(x, y)$ by the fight-hand side of (2.8), then

$z(x,0) = a(x) + b(0)$, $z(0,y) = a(0) + b(y)$, and

$$z_{xy}(x,y) = p(x,y)g(u(x,y)). \tag{2.11}$$

Using $u(x,y) \leq z(x, \leq y)$ in (2.11) and the fact that $z(x,y) > 0$, we observe that

$$\frac{z_{xy}(x,y)}{g(z(x,y))} \leq p(x,y). \tag{2.12}$$

From (2.12) and by using the facts that $z_x(x,y) \geq 0, z_y(x,y) \geq 0, z(x,y) > 0, g'(z(x,y)) \geq 0$, for $x, y \in R^+$, we observe that

$$\frac{z_{xy}(x,y)}{g(z(x,y))} \leq p(x,y) + \frac{z_x(x,y)g'(z(x,y))z_y(x,y)}{[g(z(x,y))]^2},$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{g(z(x,y))}\right) \leq p(x,y).$$

Keeping x fixed in the last inequality, we set $y = t$; then, integrating with respect to t from 0 to y and using the fact that $z(x,0) = a(x) + b(0)$, we have

$$\frac{z_x(x,y)}{g(z(x,y))} \leq \frac{a'(x)}{g\left(a(x) + b(0)\right)} + \int_0^y p(x,t)dt. \tag{2.13}$$

From (2.10) and (2.13) we observe that

$$\frac{\partial}{\partial x}\left(\Omega\left(z(x,y)\right)\right) = \frac{z_x(x,y)}{g(z(x,y))} \leq \frac{a'(x)}{g\left(a(x) + b(0)\right)} + \int_0^y p(x,t)dt. \tag{2.14}$$

Keeping y fixed in (2.14), set $x = s$; then, integrating with respect to s from 0 to x and using the fact that $z(0,y) = a(0) + b(y)$, we have

$$\Omega\left(z(x,y)\right) \leq \Omega\left(a(0) + b(y)\right) + \int_0^x \frac{a'(s)}{g\left(a(s) + b(0)\right)}ds + \int_0^x \int_0^y p(s,t)dtds. \tag{2.15}$$

Now substituting the bound on $z(x,y)$ from (2.15) in $u(x,y) \leq z(x,y)$, we obtain the desired bound in (2.9). The subintervals for x and y are obvious ■

Remark 2.8 *From the proof of theorem 2.7, it is easy to observe that, in addition to (2.9), we can conclude that*

$$u(x,y) \leq \Omega^{-1}\left[\Omega\left(a(x)+b(0)\right)+\int_0^y \frac{a'(s)}{g\left(a(s)+b(0)\right)}ds + \int_0^x\int_0^y p(s,t)dtds\right], \tag{2.16}$$

where the expression in the square bracket on the right-hand side of (2.16) belongs to the domain of Ω^{-1}.

Bondge and Pachpatte [11] gave the following generalization of Wendroff's inequality

Theorm 2.9 *Let* $u(x,y), a(x,y), b(x,y)$ *and* $c(x,y)$ *be nonnegative continuous functions defined for* $x, y \in R^+$. *Let* $g(u), h(u)$ *be continuously differentiable function defined for* $u \geq 0, g(u) > 0, h(u) > 0$ *for* $u > 0$ *and* $g'(u) \geq 0, h'(u) \geq 0$ *for* $u \geq 0$, *and let* $g(u)$ *be subadditive and submultiplicative for* $u \geq 0$. *If*

$$u(x,y) \leq a(x,y) + b(x,y)h\left(\int_0^x\int_0^y c(s,t)g\left(u(s,t)\right)dtds\right), \tag{2.17}$$

for $x, y \in R^+$, *then for* $0 \leq x \leq x_2, 0 \leq y \leq y_2$,

$$u(x,y) \leq a(x,y) + b(x,y)h\left(G^{-1}\left[G\left(A(x,y)\right)+\int_0^x\int_0^y c(s,t)g\left(b(s,t)\right)dtds\right]\right), \tag{2.18}$$

where

$$A(x,y) = \int_0^x\int_0^y c(s,t)g\left(a(s,t)\right)dtds, \tag{2.19}$$

$$G(r)=\int_{r_0}^{r}\frac{ds}{g\left(h\left(s\right)\right)},r>0,r_0>0, \tag{2.20}$$

G^{-1} is the inverse function of G and x_2, y_2 are chosen so that

$$G\left(A(x,y)\right)+\int_0^x\int_0^y c(s,t)g\left(b(s,t)\right)dtds\in Dom\left(G^{-1}\right),$$

for all x, y lying in the subinterval $0\le x\le x_2, 0\le y\le y_2$ of R^+.

Proof. From the hypotheses on g and h, we note that the functions g and h are monotonically increasing on $(0,\infty)$. Define a function $z(x,y)$ by

$$z(x,y)=\int_0^x\int_0^y c(s,t)g\left(u(s,t)\right)dtds, \tag{2.21}$$

From (2.21) and using the fact that $u(x,y)\le a(x,y)+b(x,y)h(z(x,y))$ from (2.17) and the hypotheses on g we have

$$z(x,y)\le A(x,y)+\int_0^x\int_0^y c(s,t)g\left(b(s,t)\right)g\left(h\left(z(s,t)\right)\right)dtds, \tag{2.22}$$

for $x,y\in R^+$, where $A(x,y)$ is defined by (2.21). Now fix $\alpha,\beta\in R^+$ such that $0\le x\le\alpha, 0\le y\le\beta$; then from (2.22) we observe that

$$z(x,y)\le A(\alpha,\beta)+\int_0^x\int_0^y c(s,t)g\left(b(s,t)\right)g\left(h\left(z(s,t)\right)\right)dtds, \tag{2.23}$$

for $0\le x\le\alpha, 0\le y\le\beta$. Define a function $v(x,y)$ by the right-hand side of (2.23); then $v(x,0)=v(0,y)=A(\alpha,\beta), z(x,y)\le v(x,y)$ and

$$\begin{aligned} v_{xy} &= c(x,y)g\left(b(x,y)\right)g\left(h\left(z(x,y)\right)\right) \\ &\le c(x,y)g\left(b(x,y)\right)g\left(h\left(v(x,y)\right)\right) \end{aligned} \tag{2.24}$$

Now first assume that $A(\alpha, \beta) > 0$; then from (2.24) we observe that

$$\frac{\partial}{\partial y}\left(\frac{v_x(x,y)}{g\left(h\left(v(x,y)\right)\right)}\right) \leq c(x,y)g\left(b(x,y)\right). \tag{2.25}$$

By keeping x fixed in (2.25), setting $y = t$, and then integrating with respect to t from 0 to β we have

$$\frac{v_x(x,\beta)}{g\left(h\left(v(x,\beta)\right)\right)} \leq \frac{v_x(x,0)}{g\left(h\left(v(x,0)\right)\right)} + \int_0^\beta c(x,t)g\left(b(x,t)\right)dt. \tag{2.26}$$

From (2.20) and (2.26) we observe that

$$\frac{\partial}{\partial x}G\left(v\left(x,\beta\right)\right) \leq \frac{\partial}{\partial x}G\left(v\left(x,0\right)\right) + \int_0^\beta c(x,t)g\left(b(x,t)\right)dt. \tag{2.27}$$

Now keeping y fixed in (2.27), setting $x = s$, and then integrating with resoect to s from 0 to α we get

$$G\left(v\left(x,\beta\right)\right) \leq G\left(v\left(x,0\right)\right) + \int_0^\alpha \int_0^\beta c(s,t)g\left(b(s,t)\right)dtds, \tag{2.28}$$

since $z(\alpha,\beta) \leq v(\alpha,\beta)$ and $\alpha, \beta \in R^+$ are arbitrary from (2.28) we have

$$z(x,y) \leq G^{-1}\left[G\left(A\left(x,y\right)\right) + \int_0^x \int_0^y c(s,t)g\left(b(s,t)\right)dtds,\right] \tag{2.29}$$

for $0 \leq x \leq x_2, 0 \leq y \leq y_2$. The desired bound in (2.18) follows by using (2.29) in $u(x,y) \leq a(x,y) + b(x,y)h(z(x,y))$.

If $A(\alpha,\beta)$ in (2.23) is nonnegative, we carry out the above procedure with $A(\alpha,\beta)+\epsilon$ instead of $A(\alpha,\beta)$, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.18) ■

The following theorem provides another useful generalization of Wendroff's inequality.

Theorm 2.10 *[8] Let $u(x,y), c(x,y)$ and $p(x,y)$ be nonnegative continuous functions defined for $x, y \in R^+$. Let $g(u), g'(u), a(x), a'(x), b(y)$ and $b'(y)$ be as in theorem and $g(u)$*

be submultiplicative on R^+*. If*

$$\begin{aligned} u(x,y) &\leq a(x)+b(y)+\int_0^x\int_0^y c(s,t)u(s,t)dtds \\ &\quad +\int_0^x\int_0^y p(s,t)g\left(u(s,t)\right)dtds, \end{aligned}$$

for $x,y\in R^+$*, then for* $0\leq x\leq x_3, 0\leq y\leq y_3$*,*

$$\begin{aligned} u(x,y) &\leq q(x,y)\left\{\Omega^{-1}\left[\Omega\left(a(0)+b(y)\right)+\int_0^x\frac{a'(s)}{g\left(a(s)+b(0)\right)}ds\right.\right. \\ &\quad \left.\left.+\int_0^x\int_0^y p(s,t)g\left(q(s,t)\right)dtds\right]\right\}, \end{aligned}$$

where

$$q(x,y)=\exp\left(\int_0^x\int_0^y c(s,t)dtds\right),$$

and Ω,Ω^{-1} *are defined in theorem 29 and* x_3,y_3 *are chosen so that*

$$\Omega\left(a(0)+b(y)\right)+\int_0^x\frac{a'(s)}{g\left(a(s)+b(0)\right)}ds+\int_0^x\int_0^y p(s,t)g\left(q(s,t)\right)dtds\in Dom\left(\Omega^{-1}\right),$$

for all x,y *lying in the subinterval* $0\leq x\leq x_3, 0\leq y\leq y_3$ *of* R^+*.*

2.2 Integral inequalities with several independents variables

Throughout this section, we assume that $I=[a,b]$ is any bounded open set in the dimensional euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n (n\geq 1)$, where $a=(a_1,a_2,...,a_n), b=(b_1,b_2,...,b_n)\in\mathbb{R}^n_+$. For $x=(x_1,x_2,...,x_n)=(x_1,x^1)$, where $x^1=(x_2,...,x_n)\in I$, we shall denote

$\int_a^x ...ds=\int_{a_1}^{x_1}...\int_{a_n}^{x_n}....ds_n...ds_1=\int_{a_1}^{x_1}\int_{a^1}^{x^1}....ds^1ds_1$.

Furthermore, for $x,t\in\mathbb{R}^n$, we shall write $t\leq x$ whenever $t_i\leq x_i, i=1,2,...,n$

and $0 \leq a \leq x \leq b$, for $x \in I$, and $D = D_1 D_2 ... D_n$, where $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, ..., n$. Let $C(I, \mathbb{R}_+)$ denote the class of continuous functions from I to $\mathbb{R}_+$. If $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}$, we say that $f(x)$ is a nondecreasing function in E if $x, y \in E$ and $x \leq y$ imply $f(x) \leq f(y)$.

2.2.1 linear inequalities in n independent variables

In this subsection we give some linear inequalities in n independent variables collected from [6].

Theorm 2.11 *Let $a, b \in \mathbb{R}^n, a < b$. Let $u(x), f(x)$ be nonnegative continuous functions for $x \in [a, b]$ satisfying the inequality*

$$u(x) \leq k + \int_a^x f(s)u(s)ds, \tag{2.30}$$

where $k \geq 0$ is a constant. Then

$$u(x) \leq k \exp\left(\int_a^x f(s)ds\right). \tag{2.31}$$

Proof. (2.30) implies

$$u(x) \leq k + \int_{a_1}^{x_1} \left(\int_{a^1}^{x^1} f(s_1, s^1)u(s_1, s^1)ds^1\right) ds_1 \equiv v(x_1, x^1). \tag{2.32}$$

For fixed $x^1 \in [a^1, b^1]$ the function $w(x_1) = v(x_1, x^1)$ satisfies the relations

$$\begin{aligned} w(a_1) &= k \\ w'(x_1) &= \int_{a^1}^{x^1} f(x_1, s^1) u(x_1, s^1) ds^1 \\ &\leq \int_{a^1}^{x^1} f(x_1, s^1) ds^1 . w(x_1), \end{aligned} \tag{2.33}$$

since $v(x_1, x^1)$ is nondecreasing in $[a, b]$ and $u(x_1, s^1) \leq v(x_1, s^1) \leq v(x_1, x^1) = w(x_1)$. lemma 1.2 and (2.33) imply

$$w(x_1) \leq k \exp\left(\int_{a_1}^{x_1} \left(\int_{a^1}^{x^1} f(s_1, s^1) ds^1 \right) ds_1 \right),$$

which together (2.32) implies (2.31). ■

Corollary 2.12 *If $k(x)$ is a nondecreasing function in $[a, b] \subset \mathbb{R}^n$ and*

$$u(x) \leq k(x) + \int_a^x f(s) u(s) ds,$$

then

$$u(x) \leq k(x) \exp\left(\int_a^x f(s) ds \right).$$

The following three theorems can be similarly proved.

Theorm 2.13 *Let $a, b \in \mathbb{R}^n, a < b$. Let $u(x), f(x)$ be nonnegative continuous functions for $x \in [a, b]$ satisfying the inequality*

$$u(x) \leq k + \int_x^b f(s) u(s) ds,$$

where $k \geq 0$ is a constant. Then

$$u(x) \leq k \exp\left(\int\limits_{x}^{b} f(s)ds\right).$$

Remark 2.14 *The result in theorem 2.13 is the generalization of Gronwall-Bellman theorem 1.1 in the case of n variables.*

Theorm 2.15 *Let $a, b \in \mathbb{R}^n, a < b$. Let $u(x), f(x)$ be nonnegative continuous functions for $x \in [a, b]$ satisfying the inequality*

$$u(x) \leq u(\tau) + \int\limits_{x}^{\tau} f(s)u(s)ds,$$

where $a \leq x \leq \tau \leq b$. Then

$$u(x) \geq u(a) \exp\left(-\int\limits_{a}^{x} f(s)ds\right).$$

Theorm 2.16 *Let $a, b \in \mathbb{R}^n, a < b$. Let $u(x), f(x)$ and $k(s, \tau)$ be nonnegative continuous functions for $a \leq \tau \leq s \leq b$ satisfying the inequality*

$$u(x) \leq k + \int\limits_{a}^{x} \left[f(s)u(s) + \int\limits_{a}^{s} k(s, \tau)u(\tau)d\tau\right] ds,$$

where $k \geq 0$ is a constant. Then

$$u(x) \leq k \exp\left(\int\limits_{a}^{x} \left[f(s) + \int\limits_{a}^{s} k(s, \tau)d\tau\right] ds\right).$$

Theorm 2.17 *Let $a, b \in \mathbb{R}^n, a < b$. Let $u(x), k(x), f(x)$ and $g(x)$ be nonnegative conti-*

nuous functions for $x \in [a,b]$. *Then the inequality*

$$u(x) \le k(x) + \int_a^x f(s) \left[u(s) + \int_a^s g(\tau)u(\tau)d\tau \right] ds, \tag{2.34}$$

implies

$$u(x) \le k(x) + \int_a^x k(s)f(s)\exp\left(\int_a^s \left[f(\tau)+g(\tau)\right] d\tau \right) ds. \tag{2.35}$$

Proof. We set $r(s) = u(s) + \int_a^s g(\tau)u(\tau)d\tau$. Then (2.34) takes the form

$$u(x) \le k(x) + \int_a^x f(s)r(s)ds. \tag{2.36}$$

Taking into account that $u(s) \le r(s)$ we obtain

$$r(x) = u(x) + \int_a^x g(s)u(s)ds \le k(x) + \int_a^x f(s)r(s)ds + \int_a^x g(s)r(s)ds.$$

Consequently, corollary 2.12 implies

$$r(x) \le k(x)\exp\left(\int_a^x \left[f(\tau)+g(\tau)\right] d\tau \right),$$

for $x \in [a,b]$, which together with (2.36) implies (2.35). ■

Corollary 2.18 *If* $k(x)$ *is nondecreasing in* $[a,b]$, *(2.35) implies*

$$u(x) \le k(x) \left[1 + \int_a^x f(s)\exp\left(\int_a^s \left[f(\tau)+g(\tau)\right] d\tau \right) ds \right],$$

for $x \in [a,b]$.

2.2.2 New non linear inequalities with n independents variables

Denche and Khellaf [17] established some nonlinear integral inequalities for functions with n independent variables. These results extend the Gronwall type inequalities obtained by Pachpatte [12] and Oguntuase [16]. This result can be applied to the nonlinear hyperbolic partial integrodifferential equation in n-independent variables.

The following theorem deals with n-independent variables versions of the inequalities established by Pachpatte [12, Theorem 2.3].

Theorm 2.19 *Let $u(x), f(x), a(x)$ be in $C(I, \mathbb{R}_+)$ and let $K(x,t), D_i k(x,t)$ be in $C(I \times I, \mathbb{R}_+)$ for all $i = 1, 2, ..., n,$ and let c be a nonnegative constant.*

(1) If

$$u(x) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(s,\tau) u(\tau) d\tau \right] ds, \tag{2.37}$$

for $x \in I$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq c \left[1 + \int_a^x f(t) \exp\left(\int_a^t \left[f(s) + k(b,s) \right] ds \right) dt \right]. \tag{2.38}$$

(2) If

$$u(x) \leq a(x) + \int_a^x f(s) \left[u(s) + \int_a^s k(s,\tau) u(\tau) d\tau \right] ds, \tag{2.39}$$

for $x \in I$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq a(x) + e(x) \left[1 + \int_a^x f(t) \exp\left(\int_a^t \left[f(s) + k(b,s) \right] ds \right) dt \right], \tag{2.40}$$

where

$$e(x) = \int_a^x f(s) \left[a(s) + \int_a^s k(s,\tau) a(\tau) d\tau \right] ds. \tag{2.41}$$

Proof. *(1) The inequality (2.37) implies the estimate*

$$u(x) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(b,\tau) u(\tau) d\tau \right] ds.$$

We define the function

$$z(x) = c + \int_a^x f(s)\left[u(s) + \int_a^s k(b,\tau)u(\tau)d\tau\right] ds.$$

Then $z(a_1, x_2, ...x_n) = c, u(x) \leq z(x)$ *and*

$$\begin{aligned} Dz(x) &= f(x)\left[u(x) + \int_a^x k(b,s)u(s)ds\right], \\ &\leq f(x)\left[z(x) + \int_a^x k(b,s)z(s)ds\right]. \end{aligned}$$

Define the function

$$v(x) = z(x) + \int_a^x k(b,s)z(s)ds,$$

then $v(a_1, x_2, ...x_n) = z(a_1, x_2, ...x_n) = c, Dz(x) \leq f(x)v(x)$ *and* $z(x) \leq v(x)$, *and we have*

$$Dv(x) = Dz(x) + k(b,x)z(x) \leq (f(x) + k(b,x))\, v(x). \tag{2.42}$$

Clearly $v(x)$ *is positive for all* $x \in I$, *hence the inequality (2.42) implies the estimate*

$$\frac{v(x)Dv(x)}{v^2(x)} \leq f(x) + k(b,x),$$

that is

$$\frac{v(x)Dv(x)}{v^2(x)} \leq f(x) + k(b,x) + \frac{D_n v(x) D_1 D_2 ... D_{n-1} v(x)}{v^2(x)},$$

hence

$$D_n\left(\frac{D_1 D_2 ... D_{n-1} v(x)}{v(x)}\right) \leq f(x) + k(b,x).$$

Integrating with respect to x_n *from* a_n *to* x_n, *we have*

$$\frac{D_1 D_2 ... D_{n-1} v(x)}{v(x)} \leq \int_{a_n}^{x_n} \left[f(x_1, x_2, ..., x_{n-1}, t_n) + k(b, x_1, x_2, ..., x_{n-1}, t_n)\right] dt_n,$$

thus

$$\frac{v(x)D_1D_2...D_{n-1}v(x)}{v^2(x)} \leq \int_{a_n}^{x_n} [f(x_1,x_2,...,x_{n-1},t_n)+k(b,x_1,x_2,...,x_{n-1},t_n)]\,dt_n + \frac{D_{n-1}v(x)D_1D_2...D_{n-2}v(x)}{v^2(x)}.$$

That is,

$$D_{n-1}\left(\frac{D_1D_2...D_{n-2}v(x)}{v(x)}\right) \leq \int_{a_n}^{x_n} [f(x_1,x_2,...,x_{n-1},t_n)+k(b,x_1,x_2,...,x_{n-1},t_n)]\,dt_n,$$

Integrating with respect to x_{n-1} from a_{n-1} to x_{n-1}, we have

$$\frac{D_1D_2...D_{n-2}v(x)}{v(x)} \leq \int_{a_{n-1}}^{x_{n-1}}\int_{a_n}^{x_n} [f(x_1,...,x_{n-2},t_{n-1},t_n)+k(b,x_1,...,x_{n-2},t_{n-1},t_n)]\,dt_n dt_{n-1}.$$

Continuing this process, we obtain

$$\frac{D_1v(x)}{v(x)} \leq \int_{a_2}^{x_2}...\int_{a_n}^{x_n} [f(x_1,t_2,...,t_n)+k(b,x_1,t_2,...,t_n)]\,dt_n...dt_2.$$

Integrating with respect to x_1 from a_1 to x_1, we have

$$\log\frac{v(x)}{v(a_1,x_2,...x_n)} \leq \int_a^x [f(t)+k(b,t)]\,dt,$$

that is

$$v(x) \leq c\exp\left(\int_a^x [f(t)+k(b,t)]\,dt\right). \tag{2.43}$$

Substituting (2.43) into $Dz(x) \leq f(x)v(x)$, we obtain

$$Dz(x) \leq cf(x)\exp\left(\int_a^x [f(t)+k(b,t)]\,dt\right), \tag{2.44}$$

integrating (2.44) with respect to x_n component from a_n to x_n, then with respect to x_{n-1} from a_{n-1} to x_{n-1}, and continuing until finally from a_1 to x_1, and noting that

$z(a_1, x_2, ...x_n) = c$, we have

$$z(x) \le c\left[1 + \int_a^x f(t)\exp\left(\int_a^t [f(s) + k(b,s)]\,ds\right)dt\right].$$

This completes the proof of the first part.

(2) We define a function $z(x)$ by

$$z(x) = \int_a^x f(s)\left[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\right]ds. \tag{2.45}$$

Then from (2.39), $u(x) \le a(x) + z(x)$ and using this in (2.45), we get

$$\begin{aligned} z(x) &\le \int_a^x f(s)\left[a(s) + z(s) + \int_a^s k(s,\tau)\,[a(\tau) + z(\tau)]\,d\tau\right]ds, \\ &\le e(x) + \int_a^x f(s)\left[z(s) + \int_a^s k(s,\tau)z(\tau)d\tau\right]ds, \end{aligned} \tag{2.46}$$

where $e(x)$ is defined by (2.41). Clearly $e(x)$ is positive, continuous and nondecreasing for all $x \in I$. From (2.46) it is easy to observe that

$$\frac{z(x)}{e(x)} \le 1 + \int_a^x f(s)\left[\frac{z(s)}{e(s)} + \int_a^s k(s,\tau)\frac{z(\tau)}{e(\tau)}d\tau\right]ds.$$

Now, by applying the inequality in part 1, we have

$$z(x) \le e(x)\left[1 + \int_a^x f(t)\exp\left(\int_a^t [f(s) + k(b,s)]\,ds\right)dt\right]. \tag{2.47}$$

The desired inequality in (2.40) follows from (2.47) and the fact that $u(x) \le a(x) + z(x)$.

■

In the following theorem we need the following lemma (see [18])

Lemma 2.20 *Let $u(x)$ and $b(x)$ be nonnegative continuous functions, defined for $x \in I$, and let $g \in S$. Assume that $a(x)$ is positive, continuous function, nondecreasing in each* *of the variables $x \in I$. Suppose that*

$$u(x) \leq c + \int_a^x b(t) g\left(u(t)\right) dt, \tag{2.48}$$

holds for all $x \in I$ with $x \geq a$, then

$$u(x) \leq G^{-1}\left[G(c) + \int_a^x b(t) dt\right], \tag{2.49}$$

for all $x \in I$ such that $G(c) + \int_a^x b(t)dt \in Dom\left(G^{-1}\right)$, where $G(u) = \int_{u_0}^u \frac{ds}{g(s)}, u > 0, u_0 > 0$.

Theorm 2.21 *Let $u(x), f(x), a(x)$ and $k(x,t)$ be as defined in theorem 2.19 and $g(u)$ be as in lemma 2.20. Let $\Phi(u(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive and submultiplicative function for $u(x) \geq 0$ and let $W(u(x))$ be real-valued, positive, continuous and non-decreasing function defined for $x \in I$. Assume that $a(x)$ is positive continuous function and nondecreasing for $x \in I$. If*

$$u(x) \leq a(x) + \int_a^x f(t) g\left(u(t)\right) dt + \int_a^x f(t) W\left(\int_a^t k(t,s)\Phi(u(s)) ds\right) dt, \tag{2.50}$$

for $a \leq s \leq t \leq x \leq b$, then for $a \leq x \leq x^*$,

$$\begin{aligned} u(x) \quad &\leq \quad \beta(x)\left\{a(x) + \int_a^x f(t) W\left[\Psi^{-1}\left(\Psi(\eta)\right.\right.\right. \\ &\qquad \left.\left.\left. + \int_a^t k(b,s)\Phi\left[\beta(s)\int_a^s f(\tau)d\tau\right] ds\right)\right]\right\}, \end{aligned} \tag{2.51}$$

where

$$\beta(x) = G^{-1}\left(G(1) + \int_a^x f(s) ds\right), \tag{2.52}$$

$$\eta = \int_a^b k(b,s)\Phi\left(\beta(s) a(s)\right) ds, \tag{2.53}$$

$$G(u) = \int_{u_0}^u \frac{ds}{g(s)}, u > 0, u_0 > 0, \tag{2.54}$$

$$\Psi(x) = \int_{x_0}^{x} \frac{ds}{\Phi\left(W(s)\right)}, x \geq x_0 > 0. \tag{2.55}$$

Here G^{-1} is the inverse function of G, and Ψ^{-1} is the inverse function of Ψ, x^* is chosen so that $G(1) + \int_a^x f(s)ds \in Dom\left(G^{-1}\right),$ and $\Psi(\eta) + \int_a^t k(b,s)\Phi\left[\beta(s)\int_a^s f(\tau)d\tau\right] ds \in Dom\left(\Psi^{-1}\right)$.

Chapitre 3

Generalization Of Some New Retarded Integral Inequalities

In this chapter, we give some linear integral inequalities from the article of Pachpatte [6] and we establish some new delay non-linear integral inequalities in two independent variables [20, 21], which generalize some integral inequalities with delay obtained by Ma-Picaric [22] and Ferreira-Torres [23], which can be used as handy tools in the study of certain partial differential equations and integral equations with delay. An application is given to illustrate the usefulness of our results in the last chapter.

Throughout this chapter, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$. $C^i(A, B)$ denotes the class of all i times continuously differentiable functions defined on a set A with range in the set B $(i = 1, 2, ...)$ and $C^0(A, B) = C(A, B)$.The partial derivative of a function $z(x, y), x, y \in \mathbb{R}$ with respect to x and y are denoted by $D_1z(x, y)$ and $D_2z(x, y)$ respectively.

3.1 Linear retarded integral inequalities

In this section we present some explicit bounds for linear retarded integral inequalities, established by Pachpatte in [15].

Let $I = [t_0, T), J_1 = [x_0, X), J_2 = [y_0, Y)$ are the given subsets of $\mathbb{R}, \triangle = J_1 \times J_2$.

Theorm 3.1 *Let $u(t), a(t) \in C(I, \mathbb{R}_+), b(t,s) \in C(I^2, \mathbb{R}_+)$ for $t_0 \leq s \leq t \leq T$ and $\alpha(t) \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I and $k \geq 0$ be a constant. If*

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)u(\sigma)d\sigma \right] ds, \tag{3.1}$$

for $t \in I$, then

$$u(t) \leq k \exp\left(A(t)\right), \tag{3.2}$$

for $t \in I$, where

$$A(t) = \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)d\sigma \right]. \tag{3.3}$$

Proof. From the hypotheses, we observe that $\alpha'(t) \geq 0$ for $t \in I$.let $k > 0$ and define a function $z(t)$ by the right hand side of (3.1). Then $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= \left[a(\alpha(t))u(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),\sigma)u(\sigma)d\sigma \right] \alpha'(t) \\ &\leq \left[a(\alpha(t))z(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),\sigma)z(\sigma)d\sigma \right] \alpha'(t). \end{aligned} \tag{3.4}$$

From (3.4) it is easy to observe that

$$\frac{z'(t)}{z(t)} \leq \left[a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),\sigma)d\sigma \right] \alpha'(t). \tag{3.5}$$

Integrating (3.5) from t_0 to $t, t \in I$ and by making the change of variables yields

$$z(t) \leq k \exp\left(A(t)\right), \tag{3.6}$$

for $t \in I$. Using (3.6) in $u(t) \leq z(t)$ we get the inequality in (3.2). If $k \geq 0$, we carry out the above procedure with $k + \epsilon$ instead of k, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to limit as $\epsilon \to 0$ to obtain (3.2). ■

Remark 3.2 *For $b = 0$, (3.1) reduce to Gronwall-Bellman inequality (1.1) with delay.*

Theorm 3.3 *Let $u(x,y), a(x,y) \in C(\triangle, \mathbb{R}_+), b(x,y,s,t) \in C(\triangle^2, \mathbb{R}_+)$, for $x_0 \leq s \leq x \leq X, y_0 \leq t \leq y \leq Y, \alpha(x) \in C^1(J_1, J_1), \beta(y) \in C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \leq x$ on $J_1, \beta(y) \leq y$ on J_2 and $k \geq 0$ be a constant. If*

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t)u(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta) u(\sigma,\eta)\, d\eta d\sigma \right] dtds,$$

for $(x,y) \in \triangle$, then

$$u(x,y) \leq k \exp(A(x,y)),$$

for $(x,y) \in \triangle$, where

$$A(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta) d\eta d\sigma \right] dtds,$$

for $(x,y) \in \triangle$.

3.2 New non linear retarded integral inequalities in two independent variables

In [22], Ma and Picaric (2008) have established the following useful nonlinear retarded Volterra-Fredholm integral inequalities under suitable conditions.

If $u(x,y)$ satisfies

$$\begin{aligned} u(x,y) \leq\ & k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \sigma_1(s,t) \left[f(s,t)\omega(u(s,t)) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} \sigma_2(\tau,\xi)\omega(u(\tau,\xi))\, d\xi d\tau \right] dtds \\ & + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} \sigma_1(s,t) \left[f(s,t)\omega(u(s,t)) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} \sigma_2(\tau,\xi)\omega(u(\tau,\xi))\, d\xi d\tau \right] dtds, \end{aligned}$$

then

$$u(x,y) \le G^{-1}\left\{ G\left[H^{-1}\left(\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)} \sigma_1(s,t)\left[f(s,t) + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} \sigma_2(\tau,\xi)d\xi d\tau \right] dtds \right)\right] \right.$$
$$\left. + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} \sigma_1(s,t)\left[f(s,t) + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} \sigma_2(\tau,\xi)d\xi d\tau \right] dtds \right\}.$$

In [23], Ferreira and Torres (2009) have discussed the following useful nonlinear retarded integral inequality.

If $u(t)$ satisfies

$$\phi(u(t)) \le c(t) + \int_0^{\alpha(t)} \left[f(t,s)\eta\left(u(s)\right)\omega(u(s)) + g(t,s)\eta\left(u(s)\right)\right] ds,$$

then, we have

$$u(t) \le \phi^{-1}\left\{ G^{-1}\left(\Psi^{-1}\left[\Psi\left(p(t)\right) + \int_0^{\alpha(t)} f(s,t)ds \right]\right)\right\}.$$

Motivated by the results mentioned above we establish a general two independent variables retarded version which can be used as a tool to study the boundedness of solutions of differential and integral equations.

Let $I_1 = [0,M]\,, I_2 = [0,N]$ are the given subsets of $\mathbb{R}$, and $\Delta = I_1 \times I_2$.

Lemma 3.4 *Let $u(x,y), f(x,y), \sigma(x,y) \in C(\Delta, \mathbb{R}_+)$ and $a(x,y) \in C(\Delta, \mathbb{R}_+)$ be nondecreasing with respect to $(x,y) \in \Delta$, let $\alpha \in C^1(I_1, I_1), \beta \in C^1(I_2, I_2)$ be nondecreasing with $\alpha(x) \le x$ on I_1, $\beta(y) \le y$ on I_2. Further let $\psi, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\psi,\omega\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \psi(u) = +\infty$. If $u(x,y)$ satisfies*

$$\psi\left(u(x,y)\right) \le a(x,y) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma(s,t)f(s,t)\omega\left(u(s,t)\right)dtds \qquad (3.7)$$

for $(x, y) \in \Delta$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}G\left(a\left(x,y\right)\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma(s,t) f(s,t) dtds\right] \tag{3.8}$$

for $0 \le x \le x_1, 0 \le y \le y_1$, *where*

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega\left(\psi^{-1}(s)\right)}, v \ge v_0 > 0,\ G(+\infty) = \int_{v_0}^{+\infty} \frac{ds}{\omega\left(\psi^{-1}(s)\right)} = +\infty \tag{3.9}$$

and $(x_1, y_1) \in \Delta$ *is chosen so that* $\left(G\left(a\left(x,y\right)\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dtds\right) \in Dom\left(G^{-1}\right)$.

Proof. First we assume that $a\left(x,y\right) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x, y)$ by

$$z(x,y) = a(x_0, y_0) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma(s,t) f(s,t) \omega\left(u(s,t)\right) dtds$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(0, y) = z(x, 0) = a(x_0, y_0)$ and

$$u(x,y) \le \psi^{-1}\left(z(x,y)\right) \tag{3.10}$$

and then we have

$$\begin{aligned}
\frac{\partial z(x,y)}{\partial x} &= \alpha'(x)\int_0^{\beta(y)} \sigma(\alpha(x),t) f(\alpha(x),t)\omega\left(u(\alpha(x),t)\right) dt \\
&\le \alpha'(x)\int_0^{\beta(y)} \sigma(\alpha(x),t) f(\alpha(x),t)\omega\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right) dt \\
&\le \omega\left(\psi^{-1}\left(z(\alpha(x),\beta(y))\right)\right)\alpha'(x)\int_0^{\beta(y)} \sigma(\alpha(x),t) f(\alpha(x),t) dt
\end{aligned}$$

or

$$\frac{\frac{\partial z(x,y)}{\partial x}}{\omega\left(\psi^{-1}\left(z(x,y)\right)\right)} \le \alpha'(x)\int_0^{\beta(y)} \sigma(\alpha(x),t) f(\alpha(x),t) dt.$$

Keeping y fixed, setting $x = s$, integrating the last inequality with respect to s from 0 to

x, and making the change of variable $s = \alpha(x)$ we get

$$\begin{aligned} G\left(z(x,y)\right) &\leq G\left(z(0,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t) f(s,t) dt ds \\ &\leq G\left(a(x_0,y_0)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t) f(s,t) dt ds. \end{aligned}$$

Since $(x_0, y_0) \in \Delta$ is chosen arbitrary,

$$z(x,y) \leq G^{-1}\left[G\left(a(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t) f(s,t) dt ds\right].$$

So from the last inequality and (3.10) we obtain (3.8). If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. ■

Theorm 3.5 *Let u, a, f, α and β be as in lemma 3.4. Let $\sigma_1(x,y), \sigma_2(x,y) \in C(\Delta, \mathbb{R}_+)$. Further $\psi, \omega, \eta \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\psi, \omega, \eta\}(u) > 0$ for $u > 0$, and $\lim\limits_{u \to +\infty} \psi(u) = +\infty$.*

(A_1) If $u(x,y)$ satisfies

$$\begin{aligned} \psi\left(u(x,y)\right) &\leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t)\omega\left(u(s,t)\right)\right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t)\omega\left(u(\tau,t)\right) d\tau\right] dt ds \end{aligned} \tag{3.11}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \psi^{-1}\left\{G^{-1}\left(p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dt ds\right)\right\} \tag{3.12}$$

for $0 \leq x \leq x_1, 0 \leq y \leq y_1$, where G is defined by (3.9) and

$$p(x,y) = G\left(a(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau\right) dt ds \tag{3.13}$$

and $(x_1, y_1) \in \Delta$ is chosen so that $\left(p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dt ds\right) \in Dom\left(G^{-1}\right)$.

(A_2) *If* $u(x,y)$ *satisfies*

$$\begin{aligned}\psi\left(u(x,y)\right) \;\; &\leq \;\; a(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\left[f(s,t)\omega\left(u(s,t)\right)\eta\left(u(s,t)\right)\right. \\ &\left. + \int_0^{s}\sigma_2(\tau,t)\omega\left(u(\tau,t)\right)d\tau\right]dtds \qquad (3.14)\end{aligned}$$

for $(x,y)\in\Delta$, *then*

$$u(x,y)\leq\psi^{-1}\left\{G^{-1}\left(F^{-1}\left[F\left(p(x,y)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\right)\right\} \qquad (3.15)$$

for $0\leq x\leq x_1, 0\leq y\leq y_1$, *where* G *and* p *are as in* (A_1), *and*

$$F(v)=\int_{v_0}^{v}\frac{ds}{\eta\left(\psi^{-1}\left(G^{-1}(s)\right)\right)}, v\geq v_0>0, \qquad F(+\infty)=+\infty \qquad (3.16)$$

and $(x_1,y_1)\in\Delta$ *is chosen so that* $\left[F\left(p(x,y)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\in Dom\left(F^{-1}\right)$.

(A_3) *If* $u(x,y)$*satisfies*

$$\begin{aligned}\psi\left(u(x,y)\right) \;\; &\leq \;\; a(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\left[f(s,t)\omega\left(u(s,t)\right)\eta\left(u(s,t)\right)\right. \\ &\left. + \int_0^{s}\sigma_2(\tau,t)\omega\left(u(\tau,t)\right)\eta\left(u(\tau,t)\right)d\tau\right]dtds \qquad (3.17)\end{aligned}$$

for $(x,y)\in\Delta$, *then*

$$u(x,y)\leq\psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\right)\right\} \qquad (3.18)$$

for $0\leq x\leq x_1$, $0\leq y\leq y_1$ *where*

$$p_0(x,y)=F(G\left(a(x,y)\right))+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\left(\int_0^{s}\sigma_2(\tau,t)d\tau\right)dtds$$

and $(x_1,y_1)\in\Delta$ *is chosen so that* $\left[p_0(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\in Dom\left(F^{-1}\right)$.

Proof. (A_1) By the same steps of the proof of lemma 3.4 we can obtain (3.12), with suitable changes.

(A_2) Assume that $a(x,y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= a(x_0,y_0) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t)\omega\left(u(s,t)\right)\eta\left(u(s,t)\right)\right. \\ &\quad \left. + \int_0^{s} \sigma_2(\tau,t)\omega\left(u(\tau,t)\right)d\tau\right] dtds \end{aligned}$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(0,y) = z(x,0) = a(x_0,y_0)$ and

$$u(x,y) \le \psi^{-1}\left(z(x,y)\right) \tag{3.19}$$

$$\begin{aligned} \frac{\partial z(x,y)}{\partial x} &= \alpha'(x) \int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\omega\left(u(\alpha(x),t)\right)\eta\left(u(\alpha(x),t)\right)\right. \\ &\qquad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)\omega\left(u(\tau,t)\right)d\tau\right] dt \\ &\le \alpha'(x) \int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\omega\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\eta\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\ &\quad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)\omega\left(\psi^{-1}\left(z(\tau,t)\right)\right)d\tau\right] dt \\ &\le \alpha'(x).\omega\left(\psi^{-1}\left(z(\alpha(x),\beta(y))\right)\right) \times \\ &\quad \int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\eta\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right) + \int_0^{\alpha(x)} \sigma_2(\tau,t)d\tau\right] dt \end{aligned}$$

then

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\omega\left(\psi^{-1}\left(z(x,y)\right)\right)} &\le \alpha'(x) \int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\eta\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\ &\qquad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)d\tau\right] dt. \end{aligned}$$

Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from 0 to x, and making the change of variable $s = \alpha(x)$ we get

$$\begin{aligned} G\left(z(x,y)\right) &\leq G\left(z(0,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) \eta\left(\psi^{-1}\left(z(s,t)\right)\right) \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t) d\tau \right] dtds \\ &\leq G\left(a(x_0,y_0)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) \eta\left(\psi^{-1}\left(z(s,t)\right)\right) \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t) d\tau \right] dtds. \end{aligned}$$

Since $(x_0, y_0) \in \Delta$ is chosen arbitrarily, the last inequality can be rewritten as

$$G\left(z(x,y)\right) \leq p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) \eta\left(\psi^{-1}\left(z(s,t)\right)\right) dtds. \tag{3.20}$$

Since $p(x,y)$ is a nondecreasing function, an application of lemma 3.4 to (3.20) gives us

$$z(x,y) \leq G^{-1}\left(F^{-1}\left[F\left(p(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dtds \right] \right). \tag{3.21}$$

From (3.19) and (3.21) we obtain the desired inequality (3.15).

Now we take the case $a(x,y) = 0$ for some $(x,y) \in \Delta$. Let $a_\epsilon(x,y) = a(x,y) + \epsilon$, for all $(x,y) \in \Delta$, where $\epsilon > 0$ is arbitrary, then $a_\epsilon(x,y) > 0$ and $a_\epsilon(x,y) \in C(\Delta, R_+)$ be nondecreasing with respect to $(x,y) \in \Delta$. We carry out the above procedure with $a_\epsilon(x,y) > 0$ instead of $a(x,y)$, and we get

$$u(x,y) \leq \psi^{-1}\left\{ G^{-1}\left(F^{-1}\left[F\left(p_\epsilon(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dtds \right] \right) \right\}$$

where

$$p_\epsilon(x,y) = G\left(a_\epsilon(x,y)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau \right) dtds.$$

Letting $\epsilon \to 0^+$, we obtain (3.15).

(A_3) Assume that $a(x,y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= a(x_0,y_0) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t)\omega\left(u(s,t)\right)\eta\left(u(s,t)\right) \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t)\omega\left(u(\tau,t)\right)\eta\left(u(\tau,t)\right) d\tau \right] dtds \end{aligned}$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(0,y) = z(x,0) = a(x_0,y_0)$, and

$$u(x,y) \le \psi^{-1}\left(z(x,y)\right). \tag{3.22}$$

By the same steps as the proof of (A_2), we obtain

$$\begin{aligned} z(x,y) &\le G^{-1} \left\{ G\left(a(x_0,y_0)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t)\eta\left(\psi^{-1}\left(z(s,t)\right)\right) \right. \right. \\ &\quad \left. \left. + \int_0^s \sigma_2(\tau,t)\eta\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau \right] dtds \right\}. \end{aligned}$$

We define a nonnegative and nondecreasing function $v(x,y)$ by

$$\begin{aligned} v(x,y) &= G\left(a(x_0,y_0)\right) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[\left[f(s,t)\eta\left(\psi^{-1}\left(z(s,t)\right)\right) \right] \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t)\eta\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau \right] dtds \end{aligned}$$

then $v(0,y) = v(x,0) = G\left(a(x_0,y_0)\right)$,

$$z(x,y) \le G^{-1}\left[v(x,y)\right] \tag{3.23}$$

and then

$$\begin{aligned}
\frac{\partial v(x,y)}{\partial x} &\leq \alpha'(x)\int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\eta\left(\psi^{-1}\left(G^{-1}\left(v(\alpha(x),y)\right)\right)\right)\right. \\
&\qquad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)\eta\left(\psi^{-1}\left(G^{-1}\left(v(\tau,y)\right)\right)\right) d\tau\right] dt \\
&\leq \alpha'(x).\eta\left(\psi^{-1}\left(G^{-1}\left(v(\alpha(x),\beta(y))\right)\right)\right)\int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\right. \\
&\quad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)d\tau\right] dt
\end{aligned}$$

or

$$\begin{aligned}
\frac{\frac{\partial v(x,y)}{\partial x}}{\eta\left(\psi^{-1}\left(G^{-1}\left(v(x,y)\right)\right)\right)} &\leq \alpha'(x)\int_0^{\beta(y)} \sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\right. \\
&\qquad \left. + \int_0^{\alpha(x)} \sigma_2(\tau,t)d\tau\right] dt.
\end{aligned}$$

Fixing y and integrating the last inequality with respect to $s = x$ from 0 to x and using a change of variables yield the inequality

$$F(v(x,y)) \leq F(v(0,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t) + \int_0^{s} \sigma_2(\tau,t)d\tau\right] dtds$$

or

$$\begin{aligned}
v(x,y) &\leq F^{-1}\left\{F(G\left(a(x_0,y_0)\right)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t)\right.\right. \\
&\qquad \left.\left. + \int_0^{s} \sigma_2(\tau,t)d\tau\right] dtds\right\}. \qquad (3.24)
\end{aligned}$$

From (3.22) - (3.24), and since $(x_0, y_0) \in \Delta$ is chosen arbitrarily, we obtain the desired inequality (3.18). If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. ∎

Remark 3.6 *If we take $\sigma_2(x,y) = 0$, then theorem 3.5 (A_1) reduces to lemma 3.4.*

Corollary 3.7 *Let the functions $u, f, \sigma_1, \sigma_2, a, \alpha$ and β be as in theorem 3.5. Further $q > p > 0$ are constants.*

(B_1) If $u(x,y)$ satisfies

$$\begin{aligned} u^p(x,y) \;&\leq\; a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) u^p(s,t) \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t) u^p(\tau,t) d\tau \right] dtds \end{aligned} \tag{3.25}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq (a(x,y))^{\frac{1}{p}} \exp\left(\frac{1}{p} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) + \int_0^s \sigma_2(\tau,t) d\tau \right] dtds \right). \tag{3.26}$$

(B_2) If $u(x,y)$ satisfies

$$\begin{aligned} u^q(x,y) \;&\leq\; a(x,y) + \frac{q}{q-p} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) u^p(s,t) \right. \\ &\quad \left. + \int_0^s \sigma_2(\tau,t) u^p(\tau,t) d\tau \right] dtds \end{aligned} \tag{3.27}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \left\{ p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t) dtds \right\}^{\frac{1}{q-p}} \tag{3.28}$$

where

$$p(x,y) = (a(x,y))^{\frac{q-p}{q}} + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau \right) dtds.$$

Proof. (B_1) In theorem 3.5 (A_1), by letting $\psi(u) = \omega(u) = u^p$, we obtain

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega\left(\psi^{-1}(s)\right)} = \int_{v_0}^{v} \frac{ds}{s} = \ln \frac{v}{v_0},$$

and hence

$$G^{-1}(v) = v_0 \exp(v), \qquad v \geq v_0 > 0.$$

From equation (3.13), we obtain the inequality (3.26).

(B_2) In theorem 3.5 (A_1), by letting $\psi(u) = u^q, \omega(u) = u^p$ we have

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega\left(\psi^{-1}(s)\right)} = \int_{v_0}^{v} \frac{ds}{s^{\frac{p}{q}}} = \frac{q}{q-p}\left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}}\right), v \geq v_0 > 0$$

and

$$G^{-1}(v) = \left\{v_0^{\frac{q-p}{q}} + \frac{q-p}{q}v\right\}^{\frac{1}{q-p}}$$

we obtain the inequality (3.28). ■

Theorm 3.8 *Let $u, f, \sigma_1, \sigma_2, a, \alpha, \beta, \psi, \omega$ and η be as in theorem 3.5. If $u(x, y)$ satisfies*

$$\begin{aligned} \psi\left(u(x,y)\right) \quad \leq \quad & a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\eta\left(u\left(s,t\right)\right) \times \\ & \left[f(s,t)\omega\left(u(s,t)\right) + \int_0^s \sigma_2(\tau,t)d\tau\right] dtds \end{aligned} \tag{3.29}$$

for $(x, y) \in \Delta$, then

$$u\left(x,y\right) \leq \psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\left(p_1\left(x,y\right)\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)dtds\right]\right)\right\} \tag{3.30}$$

for $0 \leq x \leq x_2, 0 \leq y \leq y_2$, where

$$G_1(v) = \int_{v_0}^{v} \frac{ds}{\eta\left(\psi^{-1}(s)\right)}, v \geq v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{ds}{\eta\left(\psi^{-1}(s)\right)} = +\infty \tag{3.31}$$

$$F_1(v) = \int_{v_0}^{v} \frac{ds}{\omega\left[\psi^{-1}\left(G_1^{-1}\left(s\right)\right)\right]}, v \geq v_0 > 0, F_1(+\infty) = +\infty \tag{3.32}$$

$$p_1\left(x,y\right) = G_1(a(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\left(\int_0^s \sigma_2(\tau,t)d\tau\right) dtds \tag{3.33}$$

and $(x_2, y_2) \in \Delta$ is chosen so that $\left[F_1\left(p_1\left(x,y\right)\right) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)dtds\right] \in Dom\left(F_1^{-1}\right)$.

Proof. Suppose that $a(x, y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z\left(x,y\right) &= a(x_0,y_0)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\eta\left(u\left(s,t\right)\right)\left[f(s,t)\omega\left(u(s,t)\right)\right. \\ &\quad \left. +\int_0^s\sigma_2(\tau,t)d\tau\right]dtds \end{aligned}$$

for $0\le x\le x_0\le x_2, 0\le y\le y_0\le y_2$, then $z\left(0,y\right)=z(x,0)=a(x_0,y_0)$,

$$u(x,y)\le\psi^{-1}\left(z(x,y)\right) \tag{3.34}$$

and

$$\begin{aligned} \frac{\partial z(x,y)}{\partial x} &\le \alpha'(x)\int_0^{\beta(y)}\sigma_1(\alpha(x),t)\eta\left[\psi^{-1}\left(z(\alpha(x),t)\right)\right]\left[f(\alpha(x),t)\omega\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\ &\quad \left.+\int_0^{\alpha(x)}\sigma_2(\tau,t)d\tau\right]dt \\ &\le \alpha'(x)\eta\left[\psi^{-1}\left(z\left(\alpha(x),\beta(y)\right)\right)\right]\int_0^{\beta(y)}\sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\omega\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\ &\quad \left.+\int_0^{\alpha(x)}\sigma_2(\tau,t)d\tau\right]dt \end{aligned}$$

then

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\eta\left[\psi^{-1}\left(z\left(x,y\right)\right)\right]} &\le \alpha'(x)\int_0^{\beta(y)}\sigma_1(\alpha(x),t)\left[f(\alpha(x),t)\omega\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\ &\quad \left.+\int_0^{\alpha(x)}\sigma_2(\tau,t)d\tau\right]dt. \end{aligned}$$

Keeping y fixed, setting $x=s$ and integrating the last inequality with respect to s from

0 to x and making the change of variable, we obtain

$$\begin{aligned} G_1\left(z\left(x,y\right)\right) &\leq G_1\left(z\left(0,y\right)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\left[f(s,t)\omega\left(\psi^{-1}\left(z(s,t)\right)\right)\right.\\ &\quad\left.+\int_0^s\sigma_2(\tau,t)d\tau\right]dtds \end{aligned}$$

then

$$\begin{aligned} G_1\left(z\left(x,y\right)\right) &\leq G_1\left(a(x_0,y_0)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\left[f(s,t)\omega\left(\psi^{-1}\left(z(s,t)\right)\right)\right.\\ &\quad\left.+\int_0^s\sigma_2(\tau,t)d\tau\right]dtds. \end{aligned}$$

Since $(x_0,y_0)\in\Delta$ is chosen arbitrary, the last inequality can be restated as

$$G_1\left(z\left(x,y\right)\right)\leq p_1\left(x,y\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)\omega\left(\psi^{-1}\left(z(s,t)\right)\right)dtds \tag{3.35}$$

It is easy to observe that $p_1\left(x,y\right)$ is positive and nondecreasing function for all $(x,y)\in\Delta$, then an application of lemma 3.4 to (3.35) yields the inequality

$$z\left(x,y\right)\leq G_1^{-1}\left(F_1^{-1}\left[F_1\left(p_1\left(x,y\right)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\right). \tag{3.36}$$

From (3.36) and (3.34) we get the desired inequality (3.30).

If $a(x,y)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(x,y)$ and subsequently let $\epsilon\to 0$. ■

Theorm 3.9 *Let u,f,σ_1,$\sigma_2,a,\alpha,\beta,\psi$ and ω be as in theorem 3.5, and $p>0$ a constant. If $u(x,y)$ satisfies*

$$\begin{aligned} \psi\left(u(x,y)\right) &\leq a(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)u^p\left(s,t\right)\times\\ &\quad\left[f(s,t)\omega\left(u(s,t)\right)+\int_0^s\sigma_2(\tau,t)d\tau\right]dtds \end{aligned} \tag{3.37}$$

for $(x,y)\in\Delta$*, then*

$$u(x,y)\leq\psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\left(p_1\left(x,y\right)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\right)\right\} \quad (3.38)$$

for $0\leq x\leq x_2, 0\leq y\leq y_2$*, where*

$$G_1(v)=\int_{v_0}^{v}\frac{ds}{\left[\psi^{-1}(s)\right]^p}, v\geq v_0>0, G_1(+\infty)=\int_{v_0}^{+\infty}\frac{ds}{\left[\psi^{-1}(s)\right]^p}=+\infty \quad (3.39)$$

and F_1, p_1 *are as in theorem 3.8 and* $(x_2,y_2)\in\Delta$ *is chosen so that*

$$\left[F_1\left(p_1\left(x,y\right)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\in Dom\left(F_1^{-1}\right).$$

Proof. An application of theorem 3.8, with $\eta(u)=u^p$ yields the desired inequality (3.38). ■

Remark 3.10 *When* $p=1$, $a(x,y)=b(x)+c(y), \sigma_1(s,t)f(s,t)=h(s,t)$*, and* $\sigma_1(s,t)\left(\int_0^s\sigma_2(\tau,t)d\tau\right)=g(s,t)$*, then inequality established in theorem 3.8 generalizes [24, Theorem 1].*

Corollary 3.11 *Let* $u, f, \sigma_1, \sigma_2, a, \alpha, \beta$ *and* ω *be as in theorem 3.5 and* $q>p>0$ *be constants. If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^q(x,y) \quad &\leq \quad a(x,y)+\frac{p}{p-q}\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)u^p(s,t)\times \\ &\left[f(s,t)\omega\left(u(s,t)\right)+\int_0^s\sigma_2(\tau,t)d\tau\right]dtds \end{aligned} \quad (3.40)$$

for $(x,y)\in\Delta$*, then*

$$u(x,y)\leq\left\{F_1^{-1}\left[F_1\left(p_1\left(x,y\right)\right)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)dtds\right]\right\}^{\frac{1}{q-p}} \quad (3.41)$$

for $0 \leq x \leq x_2$, $0 \leq y \leq y_2$, *where*

$$p_1(x,y) = [a(x,y)]^{\frac{q-p}{q}} + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau \right) dt ds$$

and F_1 *is defined in theorem 3.8.*

Remark 3.12 *Setting* $a(x,y) = b(x) + c(y), \sigma_1(s,t) f(s,t) = h(s,t)$, *and* $\sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau \right) = g(s,t)$ *in corollary 3.11 we obtain [25, Theorem 1].*

Remark 3.13 *Setting* $a(x,y) = c^{\frac{p}{p-q}}, \sigma_1(s,t) f(s,t) = h(t)$, *and* $\sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t) d\tau \right) = g(t)$ *and keeping* y *fixed in corollary 3.11, we obtain [26, Theorem 2.1].*

3.3 Further generalizations

In this section, we establish new nonlinear retarded integral inequalities of Gronwall-Bellman type [21]. These inequalities generalize some famous inequalities and can be used as handy tools to study the qualitative properties of solutions of some nonlinear partial differential and integral equations. The purpose of this section is to extend certain results which proved by Wang [30] and Abdeldaim [29]. Some applications are also given to illustrate the usefulness of our results in the last chapter.

Let $I_1 = [x_0, M], I_2 = [y_0, N]$ are the given subsets of $\mathbb{R}$, and $\Delta = I_1 \times I_2$.

Theorm 3.14 *Let* $u(x,y), f(x,y) \in C(\Delta, \mathbb{R}_+)$, *and* $c(x,y) \in C(\Delta, \mathbb{R}_+^*)$ *be nondecreasing with respect to* $(x,y) \in \Delta$, *let* $\alpha \in C^1(I_1, I_1), \beta \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha(x) \leq x$ *on* I_1, $\beta(y) \leq y$ *on* I_2. *Further* $\psi, \varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions with*

$\{\psi, \varphi\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \psi(u) = +\infty$. If $u(x,y)$ satisfies

$$\begin{aligned}
\psi(u(x,y)) \leq \; & c(x,y) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\, dtds \right)^2 \\
& + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t)) \left[\varphi(u(s,t)) \right. \\
& \left. + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\varphi(u(\tau,\sigma))\, d\sigma d\tau \right] dtds, \qquad (3.42)
\end{aligned}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \psi^{-1}\left\{ F^{-1}\left[F(c(x,y)) + A(x,y) + B(x,y) \right] \right\}, \qquad (3.43)$$

for $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1$, where

$$A(x,y) = \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) dtds \right)^2,$$

$$B(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, d\sigma d\tau \right] dtds,$$

$$F(r) = \int_{r_0}^{r} \frac{ds}{\varphi^2(\psi^{-1}(s))}, r \geq r_0 > 0, F(+\infty) = \int_{r_0}^{+\infty} \frac{ds}{\varphi^2(\psi^{-1}(s))} = +\infty, \qquad (3.44)$$

and $(x_1, y_1) \in \Delta$ is chosen so that

$$F(c(x,y)) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) dtds \right)^2$$

$$+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, d\sigma d\tau \right] dtds \in Dom\left(F^{-1}\right)$$

Proof. Fixing an arbitrary $(X,Y) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= c(X,Y) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\,dtds\right)^2 \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\,[\varphi(u(s,t)) \\ &+ \left.\int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\varphi(u(\tau,\sigma))\,d\sigma d\tau\right] dtds, \end{aligned}$$

for $x_0 \le x \le X \le x_1, y_0 \le y \le Y \le y_1$, then $z(x_0,y) = z(x,y_0) = c(X,Y)$ and

$$u(x,y) \le \psi^{-1}(z(x,y)), \tag{3.45}$$

then we have

$$\begin{aligned} \frac{\partial z}{\partial x} &\le 2\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}(z(s,t))\right) dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}(z(\alpha(x),t))\right) dt \\ &+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}(z(\alpha(x),t))\right)\left[\varphi\left(\psi^{-1}(z(\alpha(x),t))\right)\right. \\ &+ \left.\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\varphi\left(\psi^{-1}(z(\tau,\sigma))\right) d\sigma d\tau\right] dt \\ &\le \varphi^2\left(\psi^{-1}(z(\alpha(x),\beta(y)))\right)\left\{2\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt\right. \\ &\left.+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\left[1+\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\,d\sigma d\tau\right] dt\right\}, \end{aligned}$$

or

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^2\left(\psi^{-1}(z(x,y))\right)} &\le \frac{\partial}{\partial x}\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \\ &+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\left[1+\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\,d\sigma d\tau\right] dt. \end{aligned}$$

Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from x_0

to x, making the change of variable $s=\alpha(x)$ we get

$$\begin{aligned} F\left(z(x,y)\right) &\leq F\left(c\left(X,Y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \\ &\quad +\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f\left(s,t\right)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds, \end{aligned}$$

Since $(X,Y)\in\Delta$ is chosen arbitrary, then

$$\begin{aligned} z(x,y) &\leq F^{-1}\left[F\left(c\left(x,y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2\right. \\ &\quad \left.+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f\left(s,t\right)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\right]. \end{aligned} \tag{3.46}$$

From (3.45) and (3.46) we obtain the desired inequality (3.43). ■

Remark 3.15 *If $\psi\left(u\right)=u$ and $c\left(x,y\right)=u_0>0$ is a constant, $x_0=0$ and for y fixed, then theorem 3.14 reduces Theorem 1 in [27].*

In the case $\psi\left(u\right)=u$, we obtain the following corollary.

Corollary 3.16 *Let $u(x,y), f(x,y), c(x,y), \alpha, \beta$ and φ be as in theorem 3.14. If $u(x,y)$ satisfies*

$$\begin{aligned} u(x,y) &\leq c\left(x,y\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)dtds\right)^2 \\ &\quad +\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)\left[\varphi\left(u(s,t)\right)\right. \\ &\quad \left.+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\varphi\left(u(\tau,\sigma)\right)d\sigma d\tau\right]dtds, \end{aligned} \tag{3.47}$$

for $(x,y)\in\Delta$, *then*

$$\begin{aligned} u(x,y) &\leq G^{-1}\Bigg\{G\left(c\left(x,y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \\ &\qquad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\Bigg\}, \end{aligned} \tag{3.48}$$

for $x_0\leq x\leq x_1, y_0\leq y\leq y_1$, *where*

$$G(r)=\int_{r_0}^{r}\frac{ds}{\varphi^2(s)}, r\geq r_0>0,\ G(+\infty)=\int_{r_0}^{+\infty}\frac{ds}{\varphi^2(s)}=+\infty, \tag{3.49}$$

and $(x_1,y_1)\in\Delta$ *is chosen so that*

$$G\left(c\left(x,y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\times$$

$$\times\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\in Dom\left(G^{-1}\right).$$

Theorm 3.17 *Let* $u(x,y), f(x,y), c(x,y), \alpha, \beta, \varphi$ *and* ψ *be as in theorem 3.14. Further* $\varphi_1\in C(R_+,R_+)$*be nondecreasing function with* $\varphi_1(u)>0$ *for* $u>0$*. If* $u(x,y)$ *satisfies*

$$\begin{aligned} \psi\left(u(x,y)\right) &\leq c\left(x,y\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)dtds\right)^2 \\ &\qquad +\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi_1\left(u(s,t)\right)\left[\varphi_1\left(u(s,t)\right)\right. \\ &\qquad \left.+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\varphi_1\left(u(\tau,\sigma)\right)d\sigma d\tau\right]dtds, \end{aligned} \tag{3.50}$$

for $(x,y)\in\Delta$, *then*

(i) *In case* $\varphi(u) \leq \varphi_1(u)$,

$$u(x,y) \leq \psi^{-1}\left\{F_1^{-1}\left[F_1(c(x,y)) + \left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f(\tau,\sigma)\,d\sigma d\tau r f b\right]dtds\right]\right\}, \quad (3.51)$$

for $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1$, *where*

$$F_1(r) = \int_{r_0}^{r}\frac{ds}{\varphi_1^2(\psi^{-1}(s))}, r \geq r_0 > 0, F_1(+\infty) = \int_{r_0}^{+\infty}\frac{ds}{\varphi_1^2(\psi^{-1}(s))} = +\infty, \quad (3.52)$$

(ii) *In case* $\varphi_1(u) \leq \varphi(u)$, *we obtain (3.43).*

Theorm 3.18 *Let* $u(x,y), f(x,y), c(x,y), \alpha, \beta, \varphi$ *and* ψ *be as in theorem 3.14. Let* $g(x,y), h(x,y) \in C(\Delta, R_+)$. *If* $u(x,y)$ *satisfies*

$$\begin{aligned}\psi(u(x,y)) \leq\ & c(x,y) + \left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\,dtds\right)^2 \\ & + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)\varphi(u(s,t))\left[\varphi(u(s,t))\right. \\ & \left. + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} h(\tau,\sigma)\,\varphi(u(\tau,\sigma))\,d\sigma d\tau\right] dtds, \end{aligned} \quad (3.53)$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \psi^{-1}\left\{F^{-1}\left[F(c(x,y)) + \left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} h(\tau,\sigma)\,d\sigma d\tau\right]dtds\right]\right\}, \quad (3.54)$$

for $x_0 \le x \le x_1, y_0 \le y \le y_1,$ *and* $(x_1, y_1) \in \Delta$ *is chosen so that*

$$F\left(c\left(x,y\right)\right) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t)\times$$

$$\times\left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} h\left(\tau,\sigma\right) d\sigma d\tau\right] dtds \in Dom\left(F^{-1}\right).$$

Corollary 3.19 *Let* $u(x,y), f(x,y), c(x,y), \alpha$ *and* β *be as in theorem 3.14,* $p > 0$ *is a constant. If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^p(x,y) \;\; &\le \;\; c\left(x,y\right) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)u^p(s,t)dtds\right)^2 \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)u^p(s,t)\left[u^p(s,t)\right. \\ &\left.+ \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) u^p(\tau,\sigma)d\sigma d\tau\right] dtds, \end{aligned}$$

for $(x,y) \in \Delta,$ *then*

$$\begin{aligned} u(x,y) \;\; &\le \;\; \left\{\frac{1}{c\left(x,y\right)} - \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2\right. \\ &\left.- \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) d\sigma d\tau\right] dtds\right\}^{-\frac{1}{p}}, \end{aligned}$$

for $x_0 \le x \le x_1, y_0 \le y \le y_1,$ *where* $(x_1, y_1) \in \Delta$ *is chosen so that*

$$\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) d\sigma d\tau\right] dtds < \frac{1}{c\left(x,y\right)}.$$

Corollary 3.20 *Let* $u(x,y), f(x,y), c(x,y), \alpha,$ *and* β *be as in theorem 3.14,* $p > 2q > 0$

are constants. If $u(x,y)$ *satisfies*

$$\begin{aligned} u^p(x,y) &\leq c(x,y) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)u^q(s,t)dtds\right)^2 \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)u^q(s,t)\left[u^q(s,t)\right. \\ &\quad \left. + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, u^q(\tau,\sigma)d\sigma d\tau\right] dtds, \end{aligned}$$

for $(x,y) \in \Delta$, *then*

$$\begin{aligned} u(x,y) &\leq \left\{ (c(x,y))^{\frac{p-2q}{p}} + \frac{p-2q}{p}\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \right. \\ &\quad \left. + \frac{p-2q}{p}\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, d\sigma d\tau\right] dtds \right\}^{\frac{1}{p-2q}}. \end{aligned}$$

Remark 3.21 *If* $p = q + 1$, $c(x,y) = u_0 > 0$ *is a constant,* $x_0 = 0, \alpha(x) = x$ *and for* y *fixed, then corollary 3.20 reduces Theorem 3.4 in [29].*

Theorm 3.22 *Let* $u(x,y), f(x,y), c(x,y), g(x,y), h(x,y), \alpha, \beta, \varphi$ *and* ψ *be as in theorem 3.18,* $d(x,y) \in C(\Delta, R_+)$. *If* $u(x,y)$ *satisfies*

$$\begin{aligned} \psi(u(x,y)) &\leq c(x,y) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\, dtds\right)^2 \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} g(s,t)\left[d(s,t)\,\varphi(u(s,t))\right. \\ &\quad \left. + \int_{\alpha(x_0)}^{s} h(\tau,t)\,\varphi(u(\tau,t))\, d\tau\right] dtds, \end{aligned} \tag{3.55}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \psi^{-1}\left\{\Phi^{-1}\left[\Omega^{-1}\left(\Omega(A(x,y)) + 2\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2\right)\right]\right\}, \tag{3.56}$$

where

$$A\left(x,y\right)=\Phi\left(c\left(x,y\right)\right)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)\left[d\left(s,t\right)+\int_{\alpha(x_0)}^{s}h\left(\tau,t\right)d\tau\right]dtds, \qquad (3.57)$$

$$\Phi(r)=\int_{r_0}^{r}\frac{ds}{\varphi(\psi^{-1}\left(s\right))}, r\geq r_0>0, \Phi(+\infty)=\int_{r_0}^{+\infty}\frac{ds}{\varphi(\psi^{-1}\left(s\right))}=+\infty, \qquad (3.58)$$

$$\Omega\left(r\right)=\int_{r_0}^{r}\frac{ds}{\varphi(\psi^{-1}\left(\Phi^{-1}\left(s\right)\right))}, r\geq r_0>0, \Omega(+\infty)=\int_{r_0}^{+\infty}\frac{ds}{\varphi(\psi^{-1}\left(\Phi^{-1}\left(s\right)\right))}=+\infty, \qquad (3.59)$$

for $x_0\leq x\leq x_1, y_0\leq y\leq y_1,$ *and* $(x_1,y_1)\in\Delta$ *is chosen so that*

$$\left(\Omega\left(A\left(x,y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\right)^2\right)\in Dom\left(\Omega^{-1}\right).$$

Proof. Fixing an arbitrary $(X,Y)\in\Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z\left(x,y\right) &= c\left(X,Y\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi\left(u(s,t)\right)dtds\right)^2 \\ &\quad +\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)\left[d\left(s,t\right)\varphi\left(u(s,t)\right)\right. \\ &\quad \left.+\int_{\alpha(x_0)}^{s}h\left(\tau,t\right)\varphi\left(u(\tau,t)\right)d\tau\right]dtds, \end{aligned}$$

for $x_0\leq x\leq X\leq x_1, y_0\leq y\leq Y\leq y_1$, then $z\left(x_0,y\right)=z\left(x,y_0\right)=c\left(X,Y\right)$ and

$$u\left(x,y\right)\leq\psi^{-1}\left(z\left(x,y\right)\right), \qquad (3.60)$$

then we have

$$\begin{aligned}
\frac{\partial z}{\partial x} &\leq 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)dt + \\
&\quad \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} g(\alpha(x),t)\left[d\left(\alpha(x),t\right)\varphi\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right) + \int_{\alpha(x_0)}^{\alpha(x)} h\left(\tau,t\right)\varphi\left(\psi^{-1}\left(z(\tau,t)\right)\right)d\tau\right]dt \\
&\leq \varphi\left(\psi^{-1}\left(z(\alpha(x),\beta(y))\right)\right)\left\{2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right)\times\right. \\
&\quad \left.\times\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt + \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} g(\alpha(x),t)\left[d\left(\alpha(x),t\right) + \int_{\alpha(x_0)}^{\alpha(x)} h\left(\tau,\sigma\right)d\tau\right]dt\right\},
\end{aligned}$$

or

$$\begin{aligned}
\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi\left(\psi^{-1}\left(z(x,y)\right)\right)} &\leq 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt \\
&\quad +\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} g(\alpha(x),t)\left[d\left(\alpha(x),t\right) + \int_{\alpha(x_0)}^{\alpha(x)} h\left(\tau,t\right)d\tau\right]dt.
\end{aligned}$$

Since $\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds$ is a nonnegative and nondecreasing function with respect to $(x,y)\in\Delta$, then we get

$$\begin{aligned}
\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi\left(\psi^{-1}\left(z(x,y)\right)\right)} &\leq 2\left(\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt \\
&\quad +\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} g(\alpha(x),t)\left[d\left(\alpha(x),t\right) + \int_{\alpha(x_0)}^{\alpha(x)} h\left(\tau,t\right)d\tau\right]dt,
\end{aligned}$$

for $x_0 \leq x \leq X \leq x_1, y_0 \leq y \leq Y \leq y_1$. Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from x_0 to x, making the change of variable $s = \alpha(x)$

we get

$$\begin{aligned}\Phi\left(z(x,y)\right) &\leq \Phi\left(c\left(X,Y\right)\right)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)\left[d\left(s,t\right)+\int_{\alpha(x_0)}^{s} h\left(\tau,t\right)d\tau\right]dtds \\ &\quad +2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\left(\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right).\end{aligned}$$

Since $(X,Y)\in\Delta$ is chosen arbitrary, then

$$\Phi\left(z(x,y)\right)\leq A\left(x,y\right)+k\left(x,y\right)\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right),$$

where $k\left(x,y\right)=2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds$ is a nonegative and nondecreasing function with respect to $(x,y)\in\Delta$, then we get

$$\Phi\left(z(x,y)\right)\leq A\left(X,Y\right)+k\left(X,Y\right)\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right).$$

Define a positive and nondecreasing function $v\left(x,y\right)$ by

$$v\left(x,y\right)=A\left(X,Y\right)+k\left(X,Y\right)\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right),$$

for $x_0\leq x\leq X\leq x_1, y_0\leq y\leq Y\leq y_1$, then $v\left(x_0,y\right)=A\left(X,Y\right)$ and

$$z(x,y)\leq\Phi^{-1}\left(v\left(x,y\right)\right), \tag{3.61}$$

and

$$\begin{aligned}\frac{\partial v}{\partial x} &\leq k\left(X,Y\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}\left(\Phi^{-1}\left(v\left(\alpha(x),t\right)\right)\right)\right)dt \\ &\leq \varphi\left(\psi^{-1}\left(\Phi^{-1}\left(v\left(\alpha(x),\beta(y)\right)\right)\right)\right)k\left(X,Y\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt,\end{aligned}$$

or

$$\frac{\frac{\partial v}{\partial x}}{\varphi\left(\psi^{-1}\left(\Phi^{-1}\left(v\left(x,y\right)\right)\right)\right)} \leq k\left(X,Y\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt.$$

Keeping y fixed, setting $x=s$ integrating the last inequality with respect to s from x_0 to x, making the change of variable $s=\alpha(x)$ we get

$$\Omega\left(v\left(x,y\right)\right) \leq \Omega\left(A\left(X,Y\right)\right) + k\left(X,Y\right)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds.$$

Since $(X,Y)\in\Delta$ is chosen arbitrary, then we get

$$v\left(x,y\right) \leq \Omega^{-1}\left(\Omega\left(A\left(x,y\right)\right) + 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2\right). \tag{3.62}$$

From (3.60) - (3.62) we obtain (3.56). The proof is complete. ■

Remark 3.23 *If $f(x,y)=0$ for all $(x,y)\in\Delta$ and $x_0=0$, then theorem 3.22 reduces Theorem 2.2 in [20].*

Theorm 3.24 *Let $u(x,y), f(x,y), c(x,y), \alpha, \beta, \varphi$ and ψ be as in theorem 3.14. If $u(x,y)$ satisfies*

$$\begin{aligned}\psi\left(u(x,y)\right) \;\leq\; & c\left(x,y\right) + \left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)dtds\right)^2 \\ & + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)\left[u\left(s,t\right)\varphi\left(u(s,t)\right)\right. \\ & \left. + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)u\left(\tau,\sigma\right)\varphi\left(u(\tau,\sigma)\right)d\sigma d\tau\right] dtds, \end{aligned} \tag{3.63}$$

for $(x,y)\in\Delta$, then

$$\begin{aligned} u(x,y) \;\leq\; & \psi^{-1}\left\{F^{-1}\left[H^{-1}\left(H\left(B\left(x,y\right)\right)\right.\right.\right. \\ & \left.\left.\left. + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\right)\right]\right\}, \end{aligned} \tag{3.64}$$

for $x_0 \le x \le x_1, y_0 \le y \le y_1$, *where* F *is defined by (3.44) and*

$$B(x,y) = F(c(x,y)) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds \right)^2, \tag{3.65}$$

$$H(r) = \int_{r_0}^{r} \frac{ds}{\psi^{-1}(F^{-1}(s))}, r \ge r_0 > 0,\ H(+\infty) = \int_{r_0}^{+\infty} \frac{ds}{\psi^{-1}(F^{-1}(s))} = +\infty, \tag{3.66}$$

and $(x_1, y_1) \in \Delta$ *is chosen so that*

$$H(B(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, d\sigma d\tau \right] dtds \in Dom\left(H^{-1}\right).$$

Proof. Fixing an arbitrary $(X,Y) \in \Delta$, and define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) &= c(X,Y) + \left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t))\, dtds \right)^2 \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi(u(s,t)) \left[u(s,t)\,\varphi(u(s,t)) \right. \\ &\quad \left. + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f(\tau,\sigma)\, u(\tau,\sigma)\, \varphi(u(\tau,\sigma))\, d\sigma d\tau \right] dtds, \end{aligned}$$

for $x_0 \le x \le X \le x_1, y_0 \le y \le Y \le y_1$, then $z(x_0,y) = z(x,y_0) = c(X,Y)$ and

$$u(x,y) \le \psi^{-1}(z(x,y)), \tag{3.67}$$

then we have

$$\begin{aligned}
\frac{\partial z}{\partial x} &\leq 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(z(s,t)\right)\right)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)dt \\
&+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\left[\psi^{-1}\left(z(\alpha(x),t)\right)\varphi\left(\psi^{-1}\left(z(\alpha(x),t)\right)\right)\right. \\
&\left.+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\psi^{-1}\left(z(\tau,\sigma)\right)\varphi\left(\psi^{-1}\left(z(\tau,\sigma)\right)\right)d\sigma d\tau\right]dt \\
&\leq \varphi^2\left(\psi^{-1}\left(z(\alpha(x),\beta(y))\right)\right)\left\{2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)dt\right. \\
&\left.+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\left[\psi^{-1}\left(z(\alpha(x),t)\right)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\psi^{-1}\left(z(\tau,\sigma)\right)d\sigma d\tau\right]dt\right\},
\end{aligned}$$

or

$$\begin{aligned}
\frac{\frac{\partial z}{\partial x}}{\varphi^2\left(\psi^{-1}\left(z(x,y)\right)\right)} &\leq \frac{\partial}{\partial x}\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \\
&+\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t)\left[\psi^{-1}\left(z(\alpha(x),t)\right)\right. \\
&\left.+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\psi^{-1}\left(z(\tau,\sigma)\right)d\sigma d\tau\right]dt.
\end{aligned}$$

Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from x_0 to x, making the change of variable $s = \alpha(x)$ we get

$$\begin{aligned}
F\left(z(x,y)\right) &\leq F\left(c\left(X,Y\right)\right)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2 \\
&+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[\psi^{-1}\left(z(s,t)\right)+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right)\psi^{-1}\left(z(\tau,\sigma)\right)d\sigma d\tau\right]dtds.
\end{aligned}$$

Since $(X, Y) \in \Delta$ is chosen arbitrary, then

$$\begin{aligned} F\left(z(x,y)\right) &\leq B\left(x,y\right) \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[\psi^{-1}\left(z(s,t)\right) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) \psi^{-1}\left(z(\tau,\sigma)\right) d\sigma d\tau \right] dtds. \end{aligned}$$

Since $B\left(x,y\right)$ is nondecreasing, we define a positive and nondecreasing function $v(x,y)$

$$v(x,y) = B\left(X,Y\right) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[\psi^{-1}\left(z(s,t)\right) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) \psi^{-1}\left(z(\tau,\sigma)\right) d\sigma d\tau \right] dtds,$$

for $x_0 \leq x \leq X \leq x_1, y_0 \leq y \leq Y \leq y_1$, then $v\left(x_0,y\right) = v\left(x,y_0\right) = B\left(X,Y\right)$ and

$$z\left(x,y\right) \leq F^{-1}\left(v\left(x,y\right)\right), \tag{3.68}$$

then we have

$$\begin{aligned} \frac{\partial v}{\partial x} &\leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t) \left[\psi^{-1}\left(F^{-1}\left(v\left(\alpha(x),t\right)\right)\right) \right. \\ &\quad \left. + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) \psi^{-1}\left(F^{-1}\left(v\left(\tau,\sigma\right)\right)\right) d\sigma d\tau \right] dt \\ &\leq \psi^{-1}\left(F^{-1}\left(v\left(\alpha(x),\beta(y)\right)\right)\right) \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t) \left[1 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) d\sigma d\tau \right] dt, \end{aligned}$$

or

$$\frac{\frac{\partial v}{\partial x}}{\psi^{-1}\left(F^{-1}\left(v\left(x,y\right)\right)\right)} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(\alpha(x),t) \left[1 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) d\sigma d\tau \right] dt.$$

Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from x_0 to x, making the change of variable $s = \alpha(x)$ we get

$$H\left(v\left(x,y\right)\right) \leq H\left(B\left(X,Y\right)\right) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(s,t) \left[1 + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} f\left(\tau,\sigma\right) d\sigma d\tau \right] dtds.$$

Since $(X,Y)\in\Delta$ is chosen arbitrary, then

$$v(x,y)\leq H^{-1}\left(H(B(x,y))+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t}f(\tau,\sigma)\,d\sigma d\tau\right]dtds\right). \tag{3.69}$$

From (3.67) - (3.69) we obtain the desired inequality (3.64). ■

Sometimes we need to study these inequalities with a continuous function $p(x,y)\in C(\Delta,R_+)$ instead of nondecreasing function $c(x,y)$, so we get the followig theorem.

Theorm 3.25 *Let $u(x,y),f(x,y),\alpha,\beta,\varphi$ and ψ be as in theorem 3.14, and suppose that $\varphi\circ\psi^{-1}$ is a subadditive function. Let $p(x,y)\in C(\Delta,R_+^*)$. If $u(x,y)$ satisfies*

$$\begin{aligned}\psi(u(x,y)) \;\leq\;& p(x,y)+\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi(u(s,t))\,dtds\right)^2+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\times\\ &\left[\varphi(u(s,t))+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t}f(\tau,\sigma)\varphi(u(\tau,\sigma))\,d\sigma d\tau\right]dtds,\end{aligned} \tag{3.70}$$

for $(x,y)\in\Delta$, then

$$u(x,y)\leq\psi^{-1}\left\{p(x,y)+\Phi^{-1}\left[\Omega^{-1}\left(\Omega(E(x,y))+2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\right)^2\right)\right]\right\}, \tag{3.71}$$

for $x_0\leq x\leq x_1, y_0\leq y\leq y_1$, where Φ and Ω are defined by (3.58) and (3.59) resp, and

$$\begin{aligned}E(x,y) \;=\;& \Phi(D(x,y))+2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds.\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi\left(\psi^{-1}(p(s,t))\right)dtds\\ &+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\left[1+\int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t}f(\tau,\sigma)\,d\sigma d\tau\right]dtds,\end{aligned} \tag{3.72}$$

$$
\begin{aligned}
D(x,y) &= \left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\left(p(s,t)\right)\right)dtds\right)^2 \\
&\quad + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\left[\varphi\left(\psi^{-1}\left(p(s,t)\right)\right)\right. \\
&\quad \left. + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} f(\tau,\sigma)\varphi\left(\psi^{-1}\left(p(\tau,\sigma)\right)\right)d\sigma d\tau\right]dtds, \qquad (3.73)
\end{aligned}
$$

and $(x_1, y_1) \in \Delta$ *is chosen so that*

$$
\left(\Omega\left(E(x,y)\right) + 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\right)^2\right) \in Dom\left(\Omega^{-1}\right).
$$

Chapitre 4

Applications

The applications of integral inequalities are crucial in the discussion of the existence, uniqueness, continuation, boundedness, oscillation, stability and other qualitative properties of solutions of differential equations, integral equations and integro differential equations. In this chapter we present some applications on the results obtained in chapters 1, 2 and 3 to illustrate the usefulness of our results.

4.1 Application 1

In this section, we present some immediate applications of the inequality (3.7) in theorem 3.1 to study the uniqueness of solutions of the integrodifferential equation

$$x'(t) = F\left(t, x\left(t-h(t)\right), \int_{t_0}^{t} f\left(t, \sigma, x\left(\sigma - h(\sigma)\right)\right) d\sigma\right), \tag{4.1}$$

with the given initial condition

$$x(t_0) = x_0, \tag{4.2}$$

where $f \in C\left(I^2 \times \mathbb{R}, \mathbb{R}\right), F \in C\left(I \times \mathbb{R}^2, \mathbb{R}\right)$, x_0 is a real constant and $h \in C\left(I, I\right)$ be nondecreasing with $t - h(t) \geq 0, h'(t) < 1, h(t_0) = 0$.

Theorm 4.1 *Suppose that the functions f, F in (4.1) satisfy the conditions*

$$|f(t,s,x) - f(t,s,y)| \leq b(t,s)\,|x-y|\,, \tag{4.3}$$

$$|F(t,x,\overline{x}) - F(t,y,\overline{y})| \leq a(t)\,|x-y| + |\overline{x}-\overline{y}|\,, \tag{4.4}$$

where $a(t), b(t,s)$ are defined as in theorem 3.1 and let $M = \max_{t\in I} \frac{1}{1-h'(t)}$. Then the problem (P)-(Po) has at most one solution on I.

Proof. let $x(t)$ and $\overline{x}(t)$ be two solutions of (4.1)-(4.2) on I, then we have

$$\begin{aligned} x(t) - \overline{x}(t) &= \int_{t_0}^{t} \left\{ F\left(s, x\left(s-h(s)\right), \int_{t_0}^{s} f\left(s,\sigma,x\left(\sigma-h(\sigma)\right)\right) d\sigma\right) \right. \\ &\quad \left. - F\left(s, \overline{x}\left(s-h(s)\right), \int_{t_0}^{s} f\left(s,\sigma,\overline{x}\left(\sigma-h(\sigma)\right)\right) d\sigma\right) \right\} ds. \end{aligned} \tag{4.5}$$

Using (4.3), (4.4) in (4.5) and making the change of variables we have

$$\begin{aligned} |x(t) - \overline{x}(t)| &\leq \int_{t_0}^{t-h(t)} \left[Ma\left(s+h(\eta)\right)|x(s)-\overline{x}(s)| \right. \\ &\quad \left. + \int_{t_0}^{s} M^2 b\left(s+h(\eta), \sigma+h(\tau)\right)|x(\sigma)-\overline{x}(\sigma)|\,d\sigma \right] ds, \end{aligned}$$

for $t, \eta, \tau \in I$. A suitable application of the inequality (3.7) given in theorem 3.1 yields $|x(t) - \overline{x}(t)| \leq 0$. Therefore $x(t) = \overline{x}(t), i.e.$, there is at most one solution of (4.1)-(4.2). ■

4.2 Application 2

In this section, we present an application of our results obtained in chapter 3 to the qualitative analysis of solutions to the retarded integro differential equations. We study the boundedness of the solutions of the initial boundary value problem for partial delay integro differential equations of the form

$$D_1D_2z^q(x,y) = A\left(x, y, z\left(x-h_1(x), y-h_2(y)\right), \int_0^x B\left(s, y, z(s-h_1(s), y)\right) ds\right) \quad (4.6)$$

$$z(x,0) = a_1(x), z(0,y) = a_2(y), a_1(0) = a_2(0) = 0$$

for $(x,y) \in \Delta$, where $z, b \in C\left(\Delta, R_+\right), A \in C(\Delta \times R^2, R), B \in C\left(\Delta \times R, R\right)$ and $h_1 \in C^1\left(I_1, R_+\right), h_2 \in C^1\left(I_2, R_+\right)$ are nondecreasing functions such that $h_1(x) \leq x$ on I_1, $h_2(y) \leq y$ on I_2, and $h_1'(x) < 1$, $h_2'(y) < 1$.

Theorm 4.2 *Assume that the functions b, A, B in (4.6) satisfy the conditions*

$$|a_1(x) + a_2(y)| \leq a(x,y) \quad (4.7)$$

$$|A\left(s,t,z,u\right)| \leq \frac{q}{q-p}\sigma_1(s,t)\left[f(s,t)\left|z\right|^p + |u|\right] \quad (4.8)$$

$$|B\left(\tau,t,z\right)| \leq \sigma_2(\tau,t)\left|z\right|^p \quad (4.9)$$

where $a(x,y), \sigma_1(s,t), f(s,t)$, and $\sigma_2(\tau,t)$ are as in theorem 3.5, $q > p > 0$ are constants. If $z(x,y)$ satisfies (4.6), then

$$|z(x,y)| \leq \left\{p(x,y) + M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)} \bar{\sigma_1}(s,t)\bar{f}(s,t)dtds\right\}^{\frac{1}{q-p}} \quad (4.10)$$

where

$$\begin{aligned} p(x,y) &= \left(a(x,y)\right)^{\frac{q-p}{q}} \\ &\quad + M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)} \bar{\sigma_1}(s,t)\left(M_1\int_0^s \bar{\sigma_2}(\tau,t)d\tau\right) dtds \end{aligned}$$

and

$$M_1 = \underset{x\in I_1}{Max}\frac{1}{1-h_1'(x)}, \qquad M_2 = \underset{y\in I_2}{Max}\frac{1}{1-h_2'(y)}$$

and $\bar{\sigma_1}(\gamma,\xi) = \sigma_1\left(\gamma + h_1(s), \xi + h_2(t)\right), \bar{\sigma_2}\left(\mu,\xi\right) = \sigma_2\left(\mu, \xi + h_2(t)\right),$

$\bar{f}(\gamma,\xi) = f\left(\gamma + h_1(s), \xi + h_2(t)\right)$.

Proof. If $z(x,y)$ is any solution of (4.6), then

$$z^q(x,y) = a_1(x) + a_2(y)$$
$$+ \int_0^x \int_0^y A\left(s,t,z\left(s-h_1(s), t-h_2(t)\right), \int_0^s B\left(\tau,t,z(\tau - h_1(\tau),t)\right) d\tau\right) dtds. \quad (4.11)$$

Using the conditions (4.7)-(4.9) in (4.11) we obtain

$$\begin{aligned} |z(x,y)|^q &\leq a(x,y) + \frac{q-p}{q}\int_0^x \int_0^y \sigma_1(s,t)\left[f(s,t)\left|z\left(s-h_1(s), t-h_2(t)\right)\right|^p \right. \\ &\left. + \int_0^s \sigma_2(\tau,t)\left|z(\tau,t)\right|^p d\tau\right] dtds. \end{aligned} \quad (4.12)$$

Now making a change of variables on the right side of (4.12), $s - h_1(s) = \gamma, t - h_2(t) = \xi, x - h_1(x) = \alpha(x)$ for $x \in I_1, y - h_2(y) = \beta(y)$ for $y \in I_2$ we obtain the inequality

$$\begin{aligned} |z(x,y)|^q &\leq a(x,y) + \frac{q-p}{q}M_1M_2\int_0^{\alpha(x)} \int_0^{\beta(y)} \bar{\sigma_1}(\gamma,\xi)\left[\bar{f}(\gamma,\xi)\left|z\left(\gamma,\xi\right)\right|^p \right. \\ &\left. + M_1\int_0^{\gamma} \bar{\sigma_2}\left(\mu,\xi\right)\left|z(\mu,t)\right|^p d\mu\right] d\xi d\gamma. \end{aligned} \quad (4.13)$$

We can rewrite the inequality (4.13) as follows

$$\begin{aligned} |z(x,y)|^q &\leq a(x,y) + \frac{q-p}{q}M_1M_2\int_0^{\alpha(x)} \int_0^{\beta(y)} \bar{\sigma_1}(s,t)\left[\bar{f}(s,t)\left|z\left(s,t\right)\right|^p \right. \\ &\left. + M_1\int_0^{s} \bar{\sigma_2}\left(\tau,t\right)\left|z(\tau,t)\right|^p d\tau\right] dtds. \end{aligned} \quad (4.14)$$

As an application of corollary 3.7 (B_2) to (4.14) with $u(x,y) = |z(x,y)|$ we obtain the desired inequality (4.10). ■

Corollary 4.3 *If $z(x,y)$ satisfies the equation*

$$D_1D_2z^p(x,y) = A\left(x,y,z\left(x-h_1(x), y-h_2(y)\right), \int_0^x B\left(s,y,z(s-h_1(s),y)\right) ds\right) \quad (4.15)$$

$$z(x,0)=a_1(x), z(0,y)=a_2(y), a_1(0)=a_2(0)=0$$

and we suppose that the conditions (4.7)-(4.9) are satisfied, then we have the inequality

$$\begin{aligned} |z(x,y)|^p \leq\; & a(x,y)+M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)}\bar{\sigma_1}(s,t)\left[\bar{f}(s,t)\left|z\left(s,t\right)\right|^p\right. \\ & \left.+M_1\int_0^s\bar{\sigma_2}\left(\tau,t\right)\left|z(\tau,t)\right|^p d\tau\right]dtds \end{aligned} \tag{4.16}$$

then we obtain

$$\begin{aligned} |z(x,y)| \leq\; & (a(x,y))^{\frac{1}{p}}\exp\left(\frac{1}{p}M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)}\bar{\sigma_1}(s,t)\left[\bar{f}(s,t)\right.\right. \\ & \left.\left.+M_1\int_0^s\bar{\sigma_2}(\tau,t)d\tau\right]dtds\right) \end{aligned} \tag{4.17}$$

where $\bar{\sigma_1}, \bar{f}, \bar{\sigma_2}, M_1$ *and* M_2 *are as in theorem 4.2.*

Proof. By an application of corollary 3.7 (B_1) to (4.16) we obtain the desired inequality (4.17) ■

4.3 Application 3

We shall in this section illustrate how the results in Section 3 of chapter 3 can be applied to study the boundedness of the solutions of certain Volterra equation. Consider the following integral equation in two independent variables.

$$z^{\frac{p}{2}}(x,y)=h^p(x)+g^q(y)+p\int_{x_0}^{x}\int_{y_0}^{y}F\left(s,t,z(\alpha(s),\beta(t))\right)dtds, \tag{4.18}$$

Where $p\geq 2,\ F\in C(\Delta\times IR,IR),\ \ h\in C(I_1,IR),\ g\in C(I_2,IR)$ and $\alpha\in C^1\left(I_1,IR_+\right),\beta\in C^1\left(I_2,IR_+\right)$ are nondecreasing functions such that $\alpha(x)\leq x$ on I_1, $\beta(y)\leq y$ on I_2, with $\alpha\prime<1$ and $\beta\prime<1$.

Our first result deals with the boundedness of solutions.

Theorm 4.4 *Consider the problem (4.18). If*

$$|F(x,y,v)| \leq b(x,y)\,|v|^p\,, \tag{4.19}$$

and

$$|h^p(x)+g^q(y)| \leq \frac{1}{2}, \tag{4.20}$$

where $b \in C(\Delta, IR_+)$, *then all solutions* $z(x,y)$ *of (4.18) satisfy*

$$|z(x,y)| \leq \{\exp(C(x,y)) + A(x,y)\}^{-\frac{1}{p}}\,, \tag{4.21}$$

where

$$A(x,y) = 2\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} B(s,t)dtds\right)^2,$$

$$C(x,y) = \log\frac{1}{4} + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} B(s,t)dtds,$$

and

$$B(x,y) = b\left(\alpha^{-1}(x), \beta^{-1}(y)\right) \tag{4.22}$$

In particular, if B *is bounded on* Δ *, then every solution* $z(x,y)$ *of (4.18) is bounded on* Δ*.*

Proof. By (4.20) and (4.18), we obtain

$$|z(x,y)|^p \leq \frac{1}{4} + \left|\int_{x_0}^{x}\int_{y_0}^{y} F\left(s,t,z(\alpha(s),\beta(t)\right)dtds\right|^2 + \left|\int_{x_0}^{x}\int_{y_0}^{y} F\left(s,t,z(\alpha(s),\beta(t)\right)dtds\right|. \tag{4.23}$$

Hence by (4.19) we have

$$|z(x,y)|^p \leq \frac{1}{4} + \left[\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\left|z\left(\alpha(s),\beta(t)\right)\right|^p dtds\right]^2 + \left|\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\left|z\left(\alpha(s),\beta(t)\right)\right|^p dtds\right|, \tag{4.24}$$

by a change of variables $\sigma = \alpha(s), \quad \tau = \beta(t)$, in (4.24) we have

$$\begin{aligned}
|z(x,y)|^p &\leq \frac{1}{4} + \left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(\alpha^{-1}(\sigma), \beta^{-1}(\tau)) \left|z\left(\sigma,\tau\right)\right|^p d\sigma d\tau\right]^2 \\
&\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} b(\alpha^{-1}(\sigma), \beta^{-1}(\tau)) \left|z\left(\sigma,\tau\right)\right|^p d\sigma d\tau, \\
&\leq \frac{1}{4} + \left[\int_{\alpha(x_0)}^{\alpha(x)} B(\sigma,\tau) \left|z\left(\sigma,\tau\right)\right|^p d\sigma d\tau\right]^2 + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} B(\sigma,\tau) \left|z\left(\sigma,\tau\right)\right|^p d\sigma d\tau,
\end{aligned}$$

where B define in (4.22). An application of theorem 3.22 ($\varphi(u) = \psi(u) = u^p,\ d(s,t) = f(s,t), g(s,t) = 1,\ h(s,t) = 0$ and $c(x,y) = \frac{1}{4}$) to the function $|z(x,y)|$ now gives the assertion immediately. ■

Remark 4.5 *Our results in this work (Chapter 3) can be also applied to study the uniqueness, and continuous dependence of the solutions of certain initial boundary value problems for hyperbolic partial differential equations given in 4.18.*

Remark 4.6 *We can replace the condition (4.20) by*

$$|h^p(x) + g^q(y)| \leq \frac{1}{p},$$

we obtain a similar estimation above.

Finally, we give an open problem here : how to get the estimates of the solutions of (4.18) when $\alpha\prime < 1$ and $\beta\prime < 1$ are replaced by unknown functions. and when we replace the equation (4.18) by

$$z^{\frac{p}{q}}(x,y) = h^p(x) + g^q(y) + p \int_{x_0}^{x} \int_{x_0}^{y} F\left(s,t,z(\alpha(s),\beta(t))\right) dtds$$

where $q > 0$ under some suitable conditions on $F\left(s,t,z(\alpha(s),\beta(t))\right)$.

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