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Présentée par

Mme. **BOUHALI Keltoum**

Membres du jury:

Président	Mr Berkane Abdelhak	MCA	Université Constantine 1
Rapporteur	Mr Ellaggoune Fateh	Prof	Université Guelma
Examinateur	Mr Dalah Mohamed	Prof	Université Constantine 1
Examinateur	Mr Mennouni Abdelaziz	Prof	Université Batna 2
Examinateur	Mr Debbar Rabah	MCA	Université de Guelma

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Qualitative study of some dissipative systems for partial differential equations of hyperbolic-type

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By: Keltoum Bouhali

Advisor: Pr. Fateh Ellaggoune

Dedication

To the memory of my father - To My family

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Abstract

In this thesis, we study the existence/nonexistence in time as well as the asymptotic behavior of some viscoelastic problems in the presence of different damping and different nonlinearities in the sources. In this regard, we prove several decay results under appropriate assumptions on the kernels and the structural parameters of the equations. We use the multiplier method, the well depth method to establish the desired stability results of the problems. Our results generalize many results existing in the literature.

Keywords: Wave equation, Viscoelastic, Damping, Coupled system, Existence, Decay rate, Thermoelasticity, Transmission system, Infinite memory, Blow up.

Mathematics Subject Classification: 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20

Résumé

Dans cette thèse, nous étudions l'existence / non-existence dans le temps ainsi que le comportement asymptotique de certains problèmes viscoélastiques en présence d'amortissement différent et de non-linéarités différentes dans les sources. À cet regard, nous prouvons plusieurs résultats sous des hypothèses appropriées sur les noyaux et les paramètres structurels des équations. Nous utilisons la méthode du multiplicateur, la méthode de la profondeur du puits, pour établir les résultats souhaités en termes de stabilité des problèmes. Nos résultats généralisent de nombreux résultats existants dans la littérature.

Mots-clés: Equation des ondes, Viscoélastique, Amortissement, Système couplé, Existence, Taux de la décroissance, Thermoélasticité, Système de transmission, Mémoire infinie, Explosion en temps fini.

Mathematics Subject Classification: 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20

Contents

Introduction

Motivation

The problem of stabilization and control of PDEs play a pivotal role in the current paradigm of fundamental sciences. Evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only in many fields of mathematics, but also in other branches of science such as physics, mechanics and material science. For example, Navier-Stokes and Euler equations of fluid mechanics, nonlinear reaction-diffusion equations of heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrodinger equations of quantum mechanics and Cahn-Hilliard equations of material science, to name just a few, are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. see [8], [30], [48], [49], [55], [57].

The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.

The problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization:

1. Strong stabilization: $E(t) \to 0, \text{as } \ t \to \infty.$
2. Uniform stabilization: $E(t) \leq Cexp(-\delta t), \forall t > 0, \ \ (C, \delta > 0).$
3. Polynomial stabilization: $E(t) \leq Ct^{-\delta}, \forall t > 0, \ \ (C, \delta > 0).$
4. Logarithmic stabilization: $E(t) \leq C(ln(t))^{-\delta}, \forall t > 0, \ \ (C, \delta > 0).$

In recent years, an increasing interest has been developed to study the dynamical behavior of several thermoelastic problems so as to describe the thermo-mechanical interactions in elastic materials. In the begining, people mainly considered the dynamical problems of classical thermoelastic systems, the $1-D$ linear model of wich is given as follows:

$$\begin{cases} u_{tt} - u_{xx} - b\theta_x = 0, & x \in (0, L), t > 0 \\ \theta_t + \theta_{xx} + bu_{xt} = 0, & x \in (0, L), t > 0 \end{cases} \tag{1}$$

Where $u(x, t)$ denotes the deplacement of the rod at time t, and $\theta(x, t)$ is the temperature difference with respect to a fixed reference temperature. In 1960s,Dafermos in [16] discussed the existence of solution of the classical thermoelastic system and showed the asymptotic stability of the system under certain condition.Rivera further proved that the solution of this kind of thermoelastic system decays exponentially.
The classical thermoelasticity is mainly modeled based on the Fourier's law, in wich the speed of thermal propagation is infinite. This violates practical conditions, since the whole materials will not fell instantly at a sudden didturbance in some point(see[23]). In order to eliminate this paradox, Lord and Shulman in [34] employed the modified Fourier's law, proposed by Cattaneo (named Cattaneo's law), and developed what now is known as extended thermoelasticity. Based on this nonclassical thermoelastic theory, many nice results on large time behavior of the thermoelastic systems.
In 1990s, three thermoelastic theories, known as typeI, typeII and type III, respectively, were proposed by Green and Naghdi[26]. They developed their theories by introducing the thermal displacement τ satisfying the following equation.

$$\tau(., t) = \int_0^t \theta(., s)ds + \tau(., 0) \tag{2}$$

The typeI theory is consistent with the classical thermoelasticity. the type II is also named thermoelasticity without dissipation, that is, the energy is conservative. these two theories, type I and type II, are restricted cases of the type III given as follows.

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ c\tau'' + lu'_x - (\beta\theta_x + k\tau_x)_x = 0. \end{cases} \tag{3}$$

When $k = 0$, the above system becomes(1), the so-called type I thermoelasticity (classical one), and when $b = 0$, the following thermoelastic system is obtained, named type II

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ c\tau'' + lu'_x - k\tau_{xx} = 0. \end{cases} \tag{4}$$

Based on these three types of thermoelasticity, there has been an extensive literature on the decay rate for thermoelastic systems in recent years. We refer for instance, ([60]) for the exponential decay and polynomial decay of multi-dimensional thermoelasticity of type III by observability estimates; [32] for the exponential decay for thermoelasticity of type II with porous damping based on frequency domain analysis; ([42]-[41]) for the stability analysis of thermoelastic Timoshenko-type systems of type III by energy multiplier method; [42] for the spectral properties of thermoelasticity of type II and for the stability analysis of transmission problem between thermoelasticity and pure elasticity at the interfaces; and [39]for analyticity of solution of thermoelasticities.
From the above results on asymptotic behavior of the systems, we find that for the linear $1-D$ thermoelastic models of type I and type III, the thermal effects are all always strong enough to stabilize the system exponentially, while the one of type II is a conservative system in which there is no dissipation. Thus, an interesting issue is roused that whether or not the system can achieve exponential decay rate when mixing two of them (type I, type II, type III) together, that is, in one part of the domain we have a type of thermoelasticity, but in the other part of the domain, we have another type of thermoelasticity coupling with certain transmission condition at the interface. The dynamical behavior of this kind of transmission problem is difficult to analyze, since coupling exist not only between the therm and elasticity but also at the interface. Liu and Quintanilla in [37] considered the asymptotic behavior of the mixed type II and type III thermoelastic system. They proved that the system is lack of exponential decay rate bat achieves polynomial decay under certain condition. However, the sharpness of the polynomial decay rate for this kind of system is still unknown, wich is very tough issue due to the complex couplings.

Conserved and dissipated quantities

The notion of dissipative - of number, energy, mass, momentum - is a fundamental principle that can be used to derive many partial differential equations.
Any function, especially one with several independent variables, carries a huge amount of information. The questions we want to answer about PDEs are often simple, however. Complete knowledge of the details of an equation's solution are frequently unavailable, and would be overkill in any event. It is therefore useful to study coarse grained quantities that arise in PDEs in order to circumvent a complete analysis of these problems. Notice this philosophy has a long history in science: physicists and chemists like to talk about a system's energy or entropy, which can be understood without any intimate knowledge of the microscopic details.
For some solution of a PDE $u(x,t)$, we can define a coarse-grained quantity as a functional,

which is a mapping from u to the real numbers. For example,

$$\int_\Omega u dx, \qquad \int_\Omega u_x^2 dx, \qquad \int_\Omega u_{xx}^4 dx,$$

are all examples of functionals. It often happens that functionals represent quantities of physical interest-mass, energy, momentum, etc. But such an interpretation is not essential for these objects to be useful.
Suppose E is some functional of $u(x,t)$ of the form

$$E[u] = \int_\Omega f(u, u_x, \dots) dx.$$

so that E depends on t, but not on the variable x which has been integrated out. There are two common properties which depend on the time evolution of E. If $E' = 0$, then E is called conserved. If $E' \leq 0$, then E is called dissipated.
Suppose u solves the wave equation and boundary conditions

$$u'' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional (essentially the sum of kinetic and potential energy)

$$E(t) = \frac{1}{2}\int_0^L u'^2 + u_x^2 dx,$$

is conserved. Indeed,

$$E'(t) = \int_0^L u'u'' + u_x u'_x dx = [u_x u']_0^L + \int_0^L u'u'' + u'u_{xx} dx = 0$$

where integration by parts and the boundary condition was used for the second equality. The fact that E remains the same for all t has profound qualitative implications. Any solution which has wave oscillations initially (so that the energy is positive) must continue to have oscillations for all time - they never die out, for example. Conversely, if the initial conditions are quiescent, so that $E = 0$, then this must happen forever. Notice we learn these things without ever finding a solution of the equation.
As another example, suppose u solves the diffusion equation

$$u' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional

$$E(t) = \frac{1}{2}\int_0^L u_x^2 dx,$$

is dissipated, since

$$E'(t) = \int_0^L u_x u'_x dx = -\int_0^L u' u_{xx} dx = -\int_0^L u_{xx}^2 dx < 0$$

where again integration by parts and the boundary condition was used.
We can interpret E as follows. The arclength of $x-$cross sections of u can be approximated for small u_x as

$$\int_0^L \sqrt{1+u_x^2}dx \equiv \int_0^L 1+\frac{1}{2}u_x^2 dx.$$

Since $E' \leq 0$, the approximate arclength must also diminish over time. This means the graph of $u(x,.)$ gradually becomes smoother, and oscillations die away. This statement will be made perfectly quantitative by solving the equation outright using separation of variables.

Overview of the dissertation and target problems

The thesis divided in to four chapters beginning by a general introduction.

The first chapter

This chapter summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this study. In the next four chapters, we develop our main results for nonlinear evolution problems of hyperbolic type

The second chapter

In this chapter, in any spaces dimension, we use weighted spaces to establish a general decay rate of solution of viscoelastic wave equations with logarithmic nonlinearities. Furthermore, we establish, under convenient hypotheses on g and the initial data, the existence of weak solution associated to the equations.This is subject of publictain in J. Part. Diff. Eq., Vol. 30, No. 1, pp. 47-63, doi: 10.4208/jpde.v30.n1.4

The third chapter

This chapter, we establish a general decay rate properties of solutions for a coupled system of viscoelastic wave equations in $\mathbb{R}^n$ under some assumptions on g_1, g_2 and linear forcing terms. We exploit a density function to introduce weighted spaces for solutions and using an appropriate perturbed energy method. The questions of global existence in the nonlinear cases is also proved in Sobolev spaces using the well known Galerkin method.This is subject of publication in Bol. Soc. Paran. Mat. (2018) doi:10.5269/bspm.41175

The fourth chapter

This paper describes a polynomial decay rate of solution for a transmission problem with $1-d$ mixed type I and type II thermoelastic system with infinite memories acting in the first and second parts. The main contribution here is to show that the t^{-1} is the sharp decay rate of our problem (4.1). That is to show that for this types of materials the dissipation produced by the infinite memories are not strong enough to produce an exponential decay of the solution. This is subject of submitted paper.

The fifth chapter

In this last chapter, a coupled system of nonlinear Love equations with infinite memories is considered. The nonexistence of weak solution is proved. This is subject of submitted paper.

Chapter 1

Preliminaries- Technical tools

The aim of this chapter is to recall the essential notions and results used throughout this work. First, we recall some definitions and results on Sobolev spaces and the spaces $L^p(0,T,X)$ and give the statement of some important theorems in the analysis of problems to be studied and eventually some notations used throughout this study.

1.1 Function Analysis

Normed spaces, Banach spaces and their properties

Let V be linear space.

Definition 1.1 *A* non-negative, degree-1 homogeneous, subadditive functional $\|.\|_V$: $V \longrightarrow \mathbb{R}$ is called a norm if it vanishes only at 0, often, we will write briefly $\|.\|$ instead of $\|.\|_V$ if the following properties are satisfying respectively

$$\begin{cases} \|v\| \geq 0 \\ \|av\| = |a|\|v\| \\ \|u+v\| \leq \|u\| + \|v\| \\ \|v\| = 0 \longrightarrow v = 0. \end{cases}$$

for any $v \in V$ and $a \in \mathbb{R}$.

A linear space equipped with a norm is called a normed linear space. If the last $(i.e.\|v\|_v = 0 \longrightarrow v = 0)$is missing, we call such a functional a semi-norm.

Definition 1.2 A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.

Example of Banach spaces

1. $C[\alpha, \beta]$
   Let $[\alpha, \beta]$ be closed interval $-\infty \leq \alpha < \beta \leq \infty$. Let $C[\alpha, \beta]$ denote the set of all bounded continuous complex-valued functions $x(t)$ on $[\alpha, \beta]$ (If the interval is not bounded, we assume further that $x(t)$ is uniformly continuous). Define $x + y$ and αx by
   $$(x + y)(t) = x(x) + y(t)$$
   $$(\alpha x)(t) = \alpha.x(x)$$
   $C[\alpha, \beta]$ is a Banach space with the norm given by
   $$\|x\| = \sup_{t\in[\alpha,\beta]} |x(t)|.$$
   Converges in this metric is nothing but uniform convergence on the whole space.

2. $L^p(\alpha, \beta), (1 \leq p \leq \infty)$.
   This is the space of all real or complex valued Lebesgue functions f on the open interval (α, β) for which $|f(t)|^p$ is Lebesgue summable over (α, β); two functions f and g which are equal almost everywhere are considered to define the same vector of $L^p(\alpha, \beta)$. $L^p(\alpha, \beta)$ is a Banach space with the norm:
   $$\|f\|\Big(\int_\alpha^\beta |f(t)|^p dt\Big)^{1/p}.$$
   The fact that $\|Vert$ thus defined is a norm follows from Minkowski's inequality; the Riesz-Fischer theorem asserts the completeness of L^p.

3. $L^\infty(\alpha, \beta)$.
   This is the space of all measurable (complex valued) functions f on (α, β) which are essentially bounded, i.e., for every $f \in L^\infty(\alpha, \beta)$ there exists $a > 0$ such that $|f(t)| \leq a$ almost everywhere. Define $\|f\|$ to be the infinitum of such a. (Here also we identify two functions which are equal almost everywhere).

4. V' equipped with the norm $\|.\|_{V'}$ defined by
   $$\|u\|_{V'} = sup|u(x)| : \|x\| \leq 1,$$
   is also a Banach space.
   If V is a Banach space such that, for any
   $$v \in V, V \longrightarrow \mathbb{R} : u \longrightarrow \|u + v\|^2 - \|u - v\|^2,$$

is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by

$$(u, v) = \frac{1}{4}\|u + v\|^2 - \frac{1}{4}\|u - v\|^2.$$

Definition 1.3 Since u is linear we see that

$$u : V \longrightarrow V'',$$

is a linear isometry of V onto a closed subspace of V'' , we denote this by

$$V \longrightarrow V''.$$

Let V be a Banach space and $u \in V'$. Denote by

$$\phi_u : V \longrightarrow \mathbb{R}$$

$$x \longmapsto \phi_u(V),$$

when u covers V', we obtain a family of applications to $V \in \mathbb{R}$.

Definition 1.4 The weak topology on V, denoted by $\sigma(V, V')$, is the weakest topology on V for which every $(\phi_u)_{u \in V'}$ is continuous. We will define the third topology on V', the weak star topology, denoted by $\sigma(V', V)$. For all $x \in V$, denote by

$$\begin{array}{rcl} \phi_x : V' & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \phi_x(u) =< u, x >_{V', V} \end{array}$$

when x cover V, we obtain a family $(\phi_x)_{x \in V}$, of applications to V' in $\mathbb{R}$.

Theorem 1.1 *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 1.1 *Every weakly y^* convergent sequence in V' must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Definition 1.5 Let V be a Banach space and let $(u_n)_{n\in N}$ be a sequence in V. Then u_n converges strongly to u in V if and only if

$$\lim_{t\longrightarrow\infty} \|u_n - u\|_V = 0$$

and this is denoted by $u_n \longrightarrow u$, or

$$\lim_{t\longrightarrow\infty} u_n = u$$

Remark 1.1 The weak convergence does not imply strong convergence in general

Example 1.1 We shall now show by an example that weak convergence does not imply strong convergence in general. Consider the sequence $\sin n\pi t$ in $L^2(0,1)$ (real). This sequence converges weakly to zero. Since, by the Riesz theorem, any linear functional is given by the scalar product with a function we have to show that

$$\int_0^t f(t)\sin n\pi t dt \to 0, \quad foreach \quad f \in L^2(0,1).$$

But By Bessel's inequality

$$\sum_{n=1}^{\infty} |\int_0^1 f(t)\sin n\pi t dt|^2 \leq \int_0^1 |f(t)|^2 dt,$$

so $\int_0^t f(t)\sin n\pi t dt \to 0$ as $n \to \infty$. But $\sin n\pi t$ is not strongly convergent, since

$$\begin{aligned} \|\sin n\pi t - \sin m\pi t\|^2 &= \int_0^1 |\sin n\pi t - \sin m\pi t|^2 dt \\ &= 2 \quad for \quad n \neq m. \end{aligned}$$

Functional spaces

The $L^p(\Omega)$ spaces

Definition 1.6 Let $1 \leq p \leq \infty$; and let Ω be an open domain in $\mathbb{R}^n$; $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$; by:

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \quad is \quad measurable \quad and \int_\Omega |f|^p dx < \infty\}.$$

Notation 1.1 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$, denote by:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}},$$

if $p = \infty$, we have

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f\, measurable \quad and\, \exists C \in \mathbb{R}_+, |f(x)| \leq C \quad {}_{a.e}\}.$$

Theorem 1.2 *It is well known that $L^p(\Omega)$ equipped with the norm $\|.\|_p$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2 In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$< f, g >_{L^2(\Omega)} = \int_\Omega f(x).g(x)dx,$$

is a Hilbert space.

Theorem 1.3 *For $1 \leq p \leq \infty$, $L^p(\Omega)$ is a reflexive space.*

Definition 1.7 ([27], [49]) We define the function spaces of our problem and its norm as follows.

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n . \right\} \tag{1.1}$$

Note that $\mathcal{H}(\mathbb{R}^n)$ can be embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. The space $L^2_\rho(\mathbb{R}^n)$ we define to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{1.2}$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{1.3}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.1 *Let Ω be an open domain in $\mathbb{R}^n$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$if there exists a function $u \in L^p(\Omega)$ such that*

$$< T, \phi > = \int_\Omega u(x)\phi(x)dx, \forall \phi \in D(\Omega),$$

where $1 \leq p \leq \infty$, and it's well-known that u is unique.

Definition 1.8 Let $m \geq 2$ be an integer and let p be a real number with $1 \leq p < \infty$. we define by induction $W^{m,p}(\Omega)$ is the space of all $u \in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega), \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \forall i = 1, 2...N \right\}$$

Alternatively, these sets could also be introduced as

$$W^{m,p}(\Omega) = \Big\{ u \in L^p(\Omega),\ \forall \alpha \leq m, \exists v_\alpha \in L^p(\Omega) \quad such \quad that \quad \textstyle\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi, \forall \varphi \in C^\alpha(\Omega) \Big\}$$

Theorem 1.4 *$W^{m,p}(\Omega)$ is a Banach space with its usual norm*

$$\|u\|_{W_{m,p}(\Omega)} = \sum_{\alpha < m} \|\partial^\alpha u\|_{L^p(\Omega)}, 1 \leq p < \infty \quad \forall u \in W^{m,p}(\Omega).$$

Notation 1.2 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Space $H^m(\Omega)$

Definition 1.9 When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$
and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ endowed with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{\alpha < m} (\|\partial^\alpha f\|)^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

which renders $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$< u, v >_{H^m(\Omega)} = \sum_{\alpha < m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Theorem 1.5 *1)$H^m(\Omega)$ endowed with inner product $< .,. >_{H^m(\Omega)}$ is a Hilbert space.*
2) If $m < m'$, $H^m(\Omega) \longrightarrow H^{m'}(\Omega)$, with continuous embedding.

Lemma 1.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω and we have*

$$D(\Omega) \longrightarrow H_0^m(\Omega) \longrightarrow L^2(\Omega) \longrightarrow H_0^{-m}(\Omega) \longrightarrow D'(\Omega)$$

1.2 Useful technical lemmas

Lemma 1.2 *For any $v \in C^1(0,T,H^1(\mathbb{R}^n))$ we have*

$$\begin{aligned}
& -\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)Av(s)v'(t)dsdx \\
= & \frac{1}{2}\frac{d}{dt}\alpha(t)\left(g \circ A^{1/2}v\right)(t) \\
- & \frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds\right] \\
& -\frac{1}{2}\alpha(t)\left(g'^{1/2}v\right)(t) + \frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds \\
& -\frac{1}{2}\alpha'(t)\left(g \circ A^{1/2}v\right)(t) + \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds.
\end{aligned}$$

Proof.

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)Av(s)v'(t)dsdx \\
= & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'^{1/2}v(s)dxds \\
= & \alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'(t)\left[A^{1/2}v(s) - A^{1/2}v(t)\right]dxds \\
& +\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} A^{1/2}v'^{1/2}v(t)dxds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)Av(s)v'(t)dsdx \\
= & -\frac{1}{2}\alpha(t)\int_0^t g(t-s)\frac{d}{dt}\int_{\mathbb{R}^n}\left|A^{1/2}v(s) - A^{1/2}v(t)\right|^2 dxds \\
& +\alpha(t)\int_0^t g(s)\left(\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dx\right)ds
\end{aligned}$$

which implies,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)Av(s)v'(t)dsdx \\
= \ & -\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} \left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds\right] \\
&+\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dxds\right] \\
&+\frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_{\mathbb{R}^n} \left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\
&-\frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dxds. \\
&+\frac{1}{2}\alpha'(t)\int_0^t g(t-s)\int_{\mathbb{R}^n} \left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\
&-\frac{1}{2}\alpha'(t)\int_0^s g(s)ds\int_{\mathbb{R}^n} \left|A^{1/2}v(t)\right|^2 dxds.
\end{aligned}$$

■

The next Lemma can be easily shown (see [30], [31]).

Lemma 1.3 *[8] Let ρ satisfy (2.3), then for any $u \in D(A^{1/2})$, we have*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|A^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

with

$$s = \frac{2n}{2n-qn+2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

Corollary 1.2 *If $q = 2$, the Lemma 1.3, yields*

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^{n/2}(\mathbb{R}^n)}\|\nabla u\|_{L^2(\mathbb{R}^n)},$$

where we can assume $\|\rho\|_{L^{n/2}(\mathbb{R}^n)} = c > 0$ to get

$$\|u\|_{L^2_\rho(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^2(\mathbb{R}^n)}. \tag{1.5}$$

Lemma 1.4 *Let ρ satisfy (2.3), then for any $u \in \mathcal{H}(\mathbb{R}^n)$, for $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$ we have*

$$\|u\|_{L^p_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{1.6}$$

with $s = \frac{2n}{2n-pn+2p}, 2 \leq p \leq \frac{2n}{n-2}$.

the following Lemma concerning Logarithmic Sobolev inequality.

Lemma 1.5 *(see [11], [25]) Let* $u \in \mathcal{H}(\mathbb{R}^n)$ *be any function and* $c_1, c_2 > 0$ *be any numbers. Then*

$$2\int_{\mathbb{R}^n} \rho(x)|u|^2 \ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)dx + n(1+c_1)\|u\|^2_{L^2_\rho}$$
$$\leq c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla_x u\|^2_2$$

Some algebraic and integral inequalities

We give here some important integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Theorem 1.6 *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \leq p < \infty$,*then* $fg \in L^1(\Omega)$ *and*

$$\|fg\|_{L^1(\Omega)} \quad \leq \|f\|_{L^p(\Omega)}.\|g\|_{L^{p'}(\Omega)}$$

when $p = p' = 2$ *one finds the Cauchy-Schwarz inequality.*

Assume $f \in L^p(\Omega) \cap L^q(\Omega)$ *then* $f \in L^r(\Omega)$ *for* $r \in [p, q]$ *and*

$$\|f\|_{L^r(\Omega)} \leq \|f\|^\alpha_{L^p(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)},$$

with

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad for\, some \quad 0 \leq \alpha \leq 1.$$

Theorem 1.7 *Let* a *and* b *be strictly positive realities* p *and* q *such as,* $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $1 < p < \infty$, *we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The function f defined by:

$$f(x) = \frac{x^p}{p} - x$$

reached its minimum point $x = 1$ indeed :

$$y' = x^{p-1} \quad et \quad y'' = (p-1)x^{p-2} > 0$$

from where

$$f(ab^{1-q}) \geq f(1)$$

which gives

$$\frac{(ab^{1-q})^p}{p} - ab^{1-q} \geq \frac{1}{p} - 1 = -\frac{1}{q}$$

so that

$$\frac{a^p}{p} b^{(1-q)p} - ab^{1-q} + \frac{1}{q} \quad \geq 0$$

By dividing the two members by $b^{(1-q)p}$ we obtain :

$$\frac{a^p}{p} - ab^{(1-q)-p+pq} + \frac{b^q}{q} \geq 0$$

which yields

$$\frac{a^p}{p} - ab + \frac{b^q}{q} \geq 0$$

so that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

■

Remark 1.3 A simple case of Young's inequality is the inequality for $p = q = 2$:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

which also gives Young's inequality for all $\delta > 0$:

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2$$

Theorem 1.8 *(Young) Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.*
Then for a.e.$x \in \mathbb{R}^N$ the function is integrable on $\mathbb{R}^n$ and we define:

$$(f * g) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

In addition

$$(f * g) \in L^p(\mathbb{R}^n)$$

and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

The following is an extension of Theorem 1.8.

Theorem 1.9 *(Young) Assume* $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n$ *with* $1 \leq p \leq \infty$, $1 \leq p \leq \infty$ *and* $x \in \mathbb{R}^N$ *the function is integrable on* $\mathbb{R}^n$ *and* $\frac{1}{p}+\frac{1}{q}-1 = \frac{1}{r}$. *Therefore*

$$(f * g) \in L^r(\mathbb{R}^n)$$

and

$$\|f * g\|_r \leq \|f\|_p \|g\|_p.$$

Remark 1.4 Young's inequality can sometimes be written in the form :

$$ab \leq \delta a^p + C(\delta) b^q, \qquad C(\delta) = \delta^{-\frac{1}{p-1}}$$

Hölder's inequalities

Theorem 1.10 *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \leq p < \infty$,*Then* $fg \in L^1(\Omega)$ *and:*

$$\|fg\|_{L^1(\Omega)} \quad \leq \|f\|_{L^p(\Omega)}.\|g\|_{L^{p'}(\Omega)},$$

when $p = p' = 2$, *we get the inequality of Cauchy-Schwartz inequality*

Corollary 1.3 *(Hölder's inequality general form) Let* $f_1, f_2, ...f_k$ *be* k *functions such that,* $f_i \in L^{p_i}(\Omega)$, $1 \leq i \leq k$, *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then, the product $f_1, f_2, ...f_k \in L^p(\Omega)$ *and* $\|f_1 f_2 ... f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_k\|_{p_k}$.

Lemma 1.6 *(Minkowski inequality) For* $1 \leq p \leq \infty$, *we have*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

Lemma 1.7 *(Cauchy-Schwarz inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$< x_1, x_2 > \leq \|x_1\| \|x_2\|.$$

The equality sign holds if and only if x_1 *and* x_2 *are dependent.*

Will give here some integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Lemma 1.8 *let* $1 \leq p \leq r \leq q, \frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}$ *and* $1 \leq \alpha \leq 1$. *Then*

$$\|u\|_r \leq \|u\|_p^{\alpha} \|u\|_q^{1-\alpha}$$

Chapter 2

Viscoelastic wave equation with logarithmic nonlinearities in $\mathbb{R}^n$

In this chapter, we use weighted spaces to establish a general decay rate of solution of viscoelastic wave equations with logarithmic nonlinearities. Furthermore, we establish, under convenient hypotheses on g and the initial data, the existence of weak solution associated to the equations. (see [5]).

2.1 Introduction

It is well known that from a class of nonlinearities, the logarithmic nonlinearity is distinguished by several interesting physical properties (nuclear physics, optics, and geophysics...). We consider the following semilinear equation with logarithmic nonlinearitiy

$$u'' - \phi(x)\left(\Delta_x u - \int_0^t g(t-s)\Delta_x u(s)ds\right) = u \ln |u|^k \tag{2.1}$$

where $x \in \mathbb{R}^n, t > 0, n \geq 2, p > 1$ and the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (A1). The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.
Eq. (2.1) is equipped by the following initial data.

$$u\,(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n), \quad u'\,(0,x) = u_1(x) \in L^2_\rho(\mathbb{R}^n), \tag{2.2}$$

where the weighted spaces $\mathcal{H}$ is given in Definition 1.7 and the density function $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \tag{2.3}$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.
First, the following initial boundary value problem

$$u'' - \Delta_x u + \int_0^t g(t-s)\Delta_x u(s)ds + h(u') = f(u), \quad x \in \Omega, t > 0 \tag{2.4}$$

has been studied widely. For example [8], [30], [44], [49], [55], [57], the authors investigated global existence, decay rate and blow-up of the solutions.

Studies in $\mathbb{R}^n$, we quote essentially the results of [1], [27], [28], [29], [47]. In [28], authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (2.1), (2.2) with $\rho(x) = 1$ is polynomial. The finite-speed propagation is used to compensate for the lack of Poincare's inequality. In [27], author looked into a linear Cauchy viscoelastic problem with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincare's inequality. The same problem treated in [27], was considered in [29], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.
The problem (2.1),(2.2) without term source, for the case $\rho(x) = 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered in [47], where they established an explicit and general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, \quad H(0) = 0 \tag{2.5}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. This improves the conditions considered in [1] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \quad \chi(0) = \chi'(0) = 0 \tag{2.6}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$.
The goal of the present paper is to establish the existence of a weak solution to the problem (2.1)-(2.2). We obtain also, a fast decay results.

2.2 Material, Assumptions and technical lemmas

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. We denote by $|\nabla_x u|^2 = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2, \Delta_x u = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.
We recall and make use the following hypothesis on the function g as:
(A1) We assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$1 - \overline{g} = l > 0, \quad g(0) = g_0 > 0 \tag{2.7}$$

where $\overline{g} = \int_0^\infty g(t)dt$.
(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g'(t) + H(g(t)) \leq 0, t \geq 0, \quad H(0) = 0 \tag{2.8}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.
(A3) According to results in [47], we have
1- We can deduce that there exists $t_1 > 0$ large enough such that:
<u>1) $\forall t \geq t_1$</u>: We have $\lim_{s\to+\infty} g(s) = 0$, which implies that $\lim_{s\to+\infty} -g'(s)$ cannot be positive, so $\lim_{s\to+\infty} -g'(s) = 0$. Then $g(t_1) > 0$ and

$$\max\{g(s), -g'(s)\} < \min\{r, H(r), H_0(r)\}, \tag{2.9}$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$\int_0^{+\infty} g(s)H_0(-g'(s))ds < +\infty.$$

<u>2) $\forall t \in [0, t_1]$</u>: As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$ then $g(t) > 0$ and

$$g(0) \geq g(t) \geq g(t_1) > 0.$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b$$

for some positive constants a and b. Consequently,

$$g'(t) \leq -H(g(t)) \leq -kg(t), \quad k > 0$$

which gives

$$g'(t) \leq -kg(t), k > 0 \tag{2.10}$$

2- Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [3], pages 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r]. \tag{2.11}$$

Definition 2.1 By the weak solution of (2.1) over $[0,T]$ we mean a function

$$u \in C([0,T], \mathcal{H}(\mathbb{R}^n)) \cap C^1([0,T], L^2_\rho(\mathbb{R}^n)) \cap C^2([0,T], \mathcal{H}^{-1}(\mathbb{R}^n))$$

with $u' \in L^2([0,T], \mathcal{H}(\mathbb{R}^n))$, such that $u(0) = u_0, u'(0) = u_1$ and for all $v \in \mathcal{H}, t \in [0,T]$,

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(x) u \ln |u|^k v dx &= \int_{\mathbb{R}^n} \rho(x) u'' v dx + \int_{\mathbb{R}^n} \nabla_x u \nabla_x v dx \\ &- \int_{\mathbb{R}^n} \int_0^t g(t-s) \nabla_x u(s) ds \nabla_x v dx \end{aligned} \tag{2.12}$$

Multiplying the equation (2.1) by $\rho(x)u'$, and integrating by parts over $\mathbb{R}^n$, we have the energy of u at time t is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\Big(\|u'\|^2_{L^2_\rho} + \Big(1 - \int_0^t g(s) ds\Big)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u) - \int_{\mathbb{R}^n} \rho(x) u^2 \ln |u|^k dx\Big) \\ &+ \frac{k}{4}\|u\|^2_{L^2_\rho} \end{aligned} \tag{2.13}$$

and the following energy functional law holds:

$$E'(t) = \frac{1}{2}(g' \circ \nabla_x u)(t) - \frac{1}{2} g(t) \|\nabla_x u(t)\|_2^2, \quad \text{for all} \quad t \geq 0. \tag{2.14}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g \circ \nabla_x u)(t) = \int_0^t g(t-\tau) \|\nabla_x u(t) - \nabla_x u(\tau)\|_2^2 \, d\tau, \tag{2.15}$$

for $u(t) \in \mathcal{H}(\mathbb{R}^n), t \geq 0$.

2.2. Material, Assumptions and technical lemmas

2.3 Global existence in time

According to logarithmic Sobolev inequality and by using Galerkin method combined with compact theorem, similar to the proof in ([10], [20], [11], [25]), we have the following result.

Theorem 2.1 *(Local existence) Let $u_0(x) \in \mathcal{H}(\mathbb{R}^n), u_1(x) = L^2_\rho(\mathbb{R}^n)$ be given. Then, under hypothesis $(A1), (A2)$ and (2.3), the problem (2.1) has a unique local solution*

$$u \in C([0,T], \mathcal{H}(\mathbb{R}^n)) \cap C^1([0,T], L^2_\rho(\mathbb{R}^n))$$

Now, we introduce two functionals

$$\begin{aligned} J(t) &= \frac{1}{2}\Big(\Big(1-\int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u) - \int_{\mathbb{R}^n} \rho(x) u^2 \ln|u|^k dx\Big) \\ &+ \frac{k}{4}\|u\|^2_{L^2_\rho} \end{aligned} \tag{2.16}$$

and

$$I(t) = \Big(1-\int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u) - \int_{\mathbb{R}^n} \rho(x) u^2 \ln|u|^k dx \tag{2.17}$$

Then,

$$J(t) = \frac{1}{2}I(t) + \frac{k}{4}\|u\|^2_{L^2_\rho} \tag{2.18}$$

As in ([36]) to establish the corresponding method of potential wells which is related to the logarithmic nonlinear term, we introduce the stable set as follows:

$$W = \{u \in \mathcal{H}(\mathbb{R}^n) : I(t) > 0, J(t) < d\} \cup \{0\} \tag{2.19}$$

Remark 2.1 We notice that the mountain pass level d given in (2.19) defined by

$$d = \inf\{\sup_{u \in \mathcal{H}(\mathbb{R}^n)\setminus\{0\} \mu \geq 0} J(\mu u)\}, \tag{2.20}$$

Also, by introducing the so called "Nehari manifold"

$$\mathcal{N} = \{u \in \mathcal{H}(\mathbb{R}^n)\setminus\{0\} : I(t) = 0\}$$

Similar to results in [21], it is readily seen that the potential depth d is also characterized by

$$d = \inf_{u \in \mathcal{N}} J(t). \tag{2.21}$$

This characterization of d shows that

$$dist\,(0,\mathcal{N}) = \min_{u\in\mathcal{N}} \|u\|_{\mathcal{H}(\mathbb{R}^n)} \tag{2.22}$$

By the fact that (2.14), we will prove the invariance of the set W. That is if for some $t_0 > 0$ if $u(t_0) \in W$, then $u(t) \in W$, $\forall t \geq t_0$, let us beginning by giving the existence Lemma of the potential depth. (See [11] Lemma 2.4)

Lemma 2.1 *d is positive constant.*

Lemma 2.2 *Let $u \in \mathcal{H}(\mathbb{R}^n)$ and $\beta = e^{\frac{1}{2}n(1+c_1)}$. if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|_2^2 \neq 0$, then $\|u\|^2_{L^2_\rho} > \beta$.*

Proof. By (A1), (2.17) and Lemma1.5, we have

$$\begin{aligned}
I(t) &= \Big(1 - \int_0^t g(s)ds\Big)\|\nabla_x u\|_2^2 + (g\circ\nabla_x u) - \int_{\mathbb{R}^n} \rho(x)u^2 \ln|u|^k dx \\
&\geq l\|\nabla_x u\|_2^2 - k\int_{\mathbb{R}^n} \rho(x)u^2\Big(\ln\frac{|u|}{\|u\|^2_{L^2_\rho}} + \ln\|u\|^2_{L^2_\rho}\Big)dx \\
&\geq \left(l - \frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}\right)\|\nabla_x u\|_2^2 + \frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho} - k\|u\|^2_{L^2_\rho}\ln\|u\|^2_{L^2_\rho}
\end{aligned}$$

Choosing c_2 such that $l > \frac{kc_2}{2\pi}\|\rho\|^2_{L^2_\rho}$, then

$$I(t) \geq k\Big(\frac{1}{2}n(1+c_1) - \ln\|u\|^2_{L^2_\rho}\Big)\|u\|^2_{L^2_\rho}$$

Therefore, if $0 < \|u\|^2_{L^2_\rho} < \beta$, then $I(t) > 0$; if $I(t) = 0, \|u\|_2^2 \neq 0$, we have $\beta < \|u\|^2_{L^2_\rho}$ then, $\|u\|^2_{L^2_\rho} > \beta$. ■

Theorem 2.2 *(Global Existence) Let $u_0(x) \in \mathcal{H}(\mathbb{R}^n), u_1(x) \in L^2_\rho(\mathbb{R}^n)$ and $0 < E(0) < d, I(0) > 0$. Then, under hypothesis $(A1),(A2)$ and conditions (2.3), the problem (2.1) has a global solution in time.*

Proof. From the definition of energy for the weak solution and by (2.14), we have

$$\frac{1}{2}\|u'\|^2_{L^2_\rho} + J(t) \leq \frac{1}{2}\|u_1\|^2_{L^2_\rho} + J(0), \quad \forall t \in [0, T_{max}) \tag{2.23}$$

where T_{max} is the maximal existence time of weak solution of u. Then, by the definition of the stable set and using Lemma 2.2, we have $u \in W, \quad \forall t \in [0, T_{max})$

■

2.4 Decay estimates

We apply the multiplier techniques to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions. For this purpose, we introduce the functionals

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) u u' dx, \tag{2.24}$$

Lemma 2.3 *Under the hypothesis (A1) and (A2), the functional ψ_1 satisfies, along the solution of (2.1),(2.2)*

$$\begin{aligned}
\psi_1'(t) &\leq \|u'\|^2_{L^2_\rho} + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) \\
&+ \left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right) + k\|\rho\|^2_{L^2}\left(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\right]\|\nabla u\|^2_2.
\end{aligned}$$

Proof. From (2.24), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x)|u'|^2 dx + \int_{\mathbb{R}^n} \rho(x) u u'' dx \\
&= \int_{\mathbb{R}^n} \left(\rho(x)|u'|^2 + u\Delta_x u - u\int_0^t g(t-s)\Delta_x u(s,x) ds\right) dx \\
&+ \int_{\mathbb{R}^n} \rho(x) u^2 \ln |u|^k dx \\
&\leq \|u'\|^2_{L^2_\rho(\mathbb{R}^n)} - l\|\nabla_x u\|^2_2 + k\int_{\mathbb{R}^n} \rho(x) u^2 \left(\ln\left(\frac{|u|}{\|u\|^2_{L^2_\rho}}\right) + \ln \|u\|^2_{L^2_\rho}\right) dx \\
&+ \int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(s) - \nabla_x u(t)) ds dx.
\end{aligned}$$

We have by using the Logarithmic Sobolev inequality in Lemma 1.5 and generalized version of Poincare's inequality in Lemma1.3 Using Young's inequality and Lemma 3.1 for $\theta = 1/2$, we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \|u'\|^2_{L^2_\rho} + \left(\frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right)\|\nabla_x u\|^2_2 + k\|u\|^2_{L^2_\rho} \ln \|u\|^2_{L^2_\rho} \\
&+ \sigma\|\nabla_x u\|^2_2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)|\nabla_x u(s) - \nabla_x u(t)| ds\right)^2 dx - \frac{1}{2}kn(1+c_1)\|u\|^2_{L^2_\rho} \\
&\leq \|u'\|^2_{L^2_\rho} + \left(\sigma + \frac{kc_2}{2\pi}\|\rho\|^2_{L^2} - l\right)\|\nabla_x u\|^2_2 \\
&+ \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) + k\left(\ln \|u\|^2_{L^2_\rho} - \frac{1}{2}n(1+c_1)\right)\|u\|^2_{L^2_\rho}.
\end{aligned}$$

Then

$$\begin{aligned}\psi_1'(t) &\leq \|u'\|_{L^2_\rho}^2 + \frac{(1-l)}{4\sigma}(g\circ\nabla_x u)\\ &+ \left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|_{L^2}^2 - l\right) + k\|\rho\|_{L^2}^2\left(\ln\|u\|_{L^2_\rho}^2 - \frac{1}{2}n(1+c_1)\right)\right]\|\nabla u\|_2^2.\end{aligned}$$

■ The existence of the memory term forces us to make second modification of the associate energy functional. Set

$$\psi_2(t) = -\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))dsdx. \tag{2.25}$$

Lemma 2.4 *Under the hypothesis (A1) and (A2), the functional ψ_2 satisfies, along the solution of (2.1),(2.2), for any $\sigma\in(0,1)$*

$$\begin{aligned}\psi_2'(t) &\leq \left[\sigma + k\left(\sigma\frac{c_2}{2\pi} + \ln\|u\|_{L^2_\rho}^2 - \frac{n(1+c_1)}{2}\right)\right]\|\nabla_x u\|_2^2\\ &+ c_\sigma(1+(k\frac{c_2}{2\pi}+1)\|\rho\|_{L^2}^2)(g\circ\nabla_x u) - c_\sigma\|\rho\|_{L^2}^2(g'\circ\nabla_x u)\\ &+ \left(\sigma - \int_0^t g(s)ds\right)\|u'\|_{L^2_\rho}^2.\end{aligned}$$

Proof. Exploiting Eq. (2.1), (2.25) to get

$$\begin{aligned}\psi_2'(t) &= -\int_{\mathbb{R}^n}\rho(x)u''\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &- \int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx\\ &- \int_0^t g(s)ds\|u'\|_{L^2_\rho}^2\\ &= \int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx\\ &- \int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &- \int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla_x u(s,x)ds\right)\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)dx\\ &- \int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx\\ &- \int_0^t g(s)ds\|u'\|_{L^2_\rho}^2\end{aligned}$$

By (A1), we have

$$\begin{aligned}\psi_2'(t) &= \left(1-\int_0^t g(s)ds\right)\int_{\mathbb{R}^n}\nabla_x u\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx \\ &+ \int_{\mathbb{R}^n}\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)^2 dx \\ &- \int_{\mathbb{R}^n}\rho(x)u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &- \int_0^t g(s)ds\|u'\|^2_{L^2_\rho}+c(g\circ\nabla_x u)(t).\end{aligned}$$

By Hölder's and Young's inequalities and Lemma 1.3, we estimate

$$\begin{aligned}&-\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &\le \left(\int_{\mathbb{R}^n}\rho(x)|u'|^2dx\right)^{1/2}\times \\ &\left(\int_{\mathbb{R}^n}\rho(x)\left|\int_0^t g'(t-s)(u(t)-u(s))ds\right|^2\right)^{1/2} \\ &\le \sigma\|u'\|^2_{L^2_\rho}+c_\sigma\left\|\int_0^t -g'(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho} \\ &\le \sigma\|u'\|^2_{L^2_\rho}-c_\sigma\|\rho\|^2_{L^2}(g'\circ\nabla_x u)(t).\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(x)u'\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &\le \sigma\|u'\|^2_{L^2_\rho}+c_\sigma\|\rho\|^2_{L^2}(g\circ\nabla_x u)(t).\end{aligned}$$

and by Lemma 1.3 and Lemma 1.5 and conditions in Lemma 2.2, we have

$$\begin{aligned}&-\int_{\mathbb{R}^n}\rho(x)\ln|u|^k u\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &\le k\int_{\mathbb{R}^n}\rho(x)\Big(\ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)+\ln\|u\|^2_{L^2_\rho}\Big)u\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &\le k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|u\|^2_{L^2_\rho}+k\frac{c_2}{2\pi}\Big\|u\int_0^t g(t-s)(u(t)-u(s))ds\Big\|^2_{L^2_\rho} \\ &\le k\Big(\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2}\|\nabla_x u\|^2_2 \\ &+k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}\Big\|\nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big\|^2_{L^2_\rho} \\ &\le k\Big(\sigma\frac{c_2}{2\pi}+\ln\|u\|^2_{L^2_\rho}-\frac{n(1+c_1)}{2}\Big)\|\rho\|^2_{L^2}\|\nabla_x u\|^2_2+c_\sigma k\frac{c_2}{2\pi}\|\rho\|^2_{L^2}(g\circ\nabla_x u).\end{aligned}$$

Using Young's and Poincare's inequalities and Lemma 3.1 for $\theta = 1/2$, we obtain

$$\begin{aligned}\psi_2'(t) &\leq \left[\sigma + k\Big(\sigma\frac{c_2}{2\pi} + \ln \|u\|^2_{L^2_\rho} - \frac{n(1+c_1)}{2}\Big)\right] \|\nabla_x u\|_2^2 \\ &+ c_\sigma(1 + (k\frac{c_2}{2\pi} + 1)\|\rho\|^2_{L^2})(g \circ \nabla_x u) - c_\sigma \|\rho\|^2_{L^2}(g' \circ \nabla_x u) \\ &+ \left(\sigma - \int_0^t g(s)ds\right) \|u'\|^2_{L^2_\rho}.\end{aligned}$$

■ Now, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2\psi_2(t) \tag{2.26}$$

for $\xi_1, \xi_2 > 1$. We need the next Lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{2.27}$$

holds for two positive constants β_1 and β_2.

Lemma 2.5 *For $\xi_1, \xi_2 > 1$, we have*

$$L(t) \sim E(t). \tag{2.28}$$

Proof. By (2.26) we have

$$\begin{aligned}|L(t) - \xi_1 E(t)| &\leq |\psi_1(t)| + \xi_2|\psi_2(t)| \\ &\leq \int_{\mathbb{R}^n} |\rho(x)uu'|\, dx \\ &+ \xi_2 \int_{\mathbb{R}^n} \left|\rho(x)u' \int_0^t g(t-s)(u(t)-u(s))ds\right| dx.\end{aligned}$$

Thanks to Hölder and Young's inequalities, we have by using Lemma 1.3

$$\begin{aligned}\int_{\mathbb{R}^n} |\rho(x)uu'|\, dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}^n} \rho(x)|u'|^2 dx\right)^{1/2} \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}^n} \rho(x)|u|^2 dx\right) + \frac{1}{2}\left(\int_{\mathbb{R}^n} \rho(x)|u'|^2 dx\right) \\ &\leq c\|u'\|^2_{L^2_\rho} + c\|\rho\|^2_{L^2}\|\nabla_x u\|_2^2 \end{aligned}\tag{2.29}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n} \left|\left(\rho(x)^{\frac{1}{2}}u'\right)\left(\rho(x)^{\frac{1}{2}} \int_0^t g(t-s)(u(t)-u(s))ds\right)\right| dx \\ &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^2 dx\right)^{1/2} \times \left(\int_{\mathbb{R}^n} \rho(x)\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2 dx\right)^{1/2} \\ &\leq \frac{1}{2}\|u'\|^2_{L^2_\rho} + \frac{1}{2}\left\|\int_0^t g(t-s)(u(t)-u(s))ds\right\|^2_{L^2_\rho} \\ &\leq \frac{1}{2}\|u'\|^2_{L^2_\rho} + \frac{1}{2}\|\rho\|^2_{L^2}(g \circ \nabla_x u).\end{aligned}$$

Then,

$$|L(t) - \xi_1 E(t)| \leq cE(t).$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{2.30}$$

■

Lemma 2.6 *For all $t \geq t_1 > 0$, we have*

$$\int_{t_1}^{t} (g \circ \nabla_x u)(s) ds \leq H_0^{-1}\Big(- \int_{t_1}^{t} H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \Big).$$

where H_0 introduced in (2.9).

Proof. By (2.14) and (A3), we have for all $t \geq t_1$

$$\begin{aligned} \int_{\mathbb{R}^n} \int_0^{t_1} g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 ds dx &\leq -\frac{1}{k} \int_{\mathbb{R}^n} \int_0^{t_1} g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 ds dx \\ &\leq -cE'(t). \end{aligned}$$

Now, we define

$$I(t) = \int_{t_1}^{t} H_0(-g'(s))(g \circ \nabla_x u)(t) ds. \tag{2.31}$$

Since $\int_0^{+\infty} H_0(-g'(s))g(s)ds < +\infty$, from (2.14) we have

$$\begin{aligned} I(t) &= \int_{t_1}^{t} H_0(-g'(s)) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \\ &\leq 2 \int_{t_1}^{t} H_0(-g'(s))g(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dx ds \\ &\leq cE(0) \int_{t_1}^{t} H_0(-g'(s))g(s) ds < 1. \end{aligned} \tag{2.32}$$

We define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t) = - \int_{t_1}^{t} H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds. \tag{2.33}$$

From (A1)-(A3) and, we get

$$H_0(-g'(s))g(s) \leq H_0(H(g(s)))g(s) = D(g(s))g(s) \leq k_0.$$

for some positive constant k_0. Then, for all $t \geq t_1$

$$
\begin{aligned}
\lambda(t) &\leq -k_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \\
&\leq -k_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dx ds \\
&\leq -cE(0) \int_{t_1}^{t} g'(s) ds \\
&\leq cE(0) g(t_1) \\
&< \min\{r, H(r), H_0(r)\}.
\end{aligned} \tag{2.34}
$$

Using the properties of H_0 (strictly convex in $(0,r], H_0(0) = 0$), then for $x \in (0,r], \theta \in [0,1]$

$$H_0(\theta x) \leq \theta H_0(x).$$

Using hypothesis in (A3), (2.32), (2.34) and Jensen's inequality leads to

$$
\begin{aligned}
\lambda(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g'(s))] H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \\
&\geq \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g'(s))] H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \\
&\geq H_0 \left(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g'(s)) H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \right) \\
&\geq H_0 \left(\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \right)
\end{aligned}
$$

which implies

$$\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dx ds \leq H_0^{-1}(\lambda(t)).$$

■ Our next main result reads as follows.

Theorem 2.3 *Let $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^n) \times L^2_\rho(\mathbb{R}^n)$ and suppose that (A1)- (A2) hold. Then there exist positive constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that the energy of solution given by (2.1),(2.2) satisfies,*

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \textit{for all} \quad t \geq 0,$$

where

$$H_1(t) = \int_t^1 (s H_0'(\alpha_0 s))^{-1} ds$$

Proof. From (2.14), results of Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2\psi_2'(t) \\ &\leq (\frac{1}{2}\xi_1 - c_\sigma\|\rho\|_{L^2}^2\xi_2)(g'\circ\nabla_x u) + M_0(g\circ\nabla_x u) \\ &- M_1\|u'\|_{L^2_\rho}^2 - M_2\|\nabla_x u\|_2^2 \end{aligned}$$

where

$$M_0 = \left(\xi_2 c_\sigma(1 + (k\frac{c_2}{2\pi} + 1)\|\rho\|_{L^2}^2) + \frac{(1-l)}{4\sigma}\right) > 0,$$

$$M_1 = \left(\xi_2\left(\int_0^{t_1} g(s)ds - \sigma\right) - 1\right),$$

$$\begin{aligned} M_2 &= \frac{1}{2}\xi_1 g(t_1) \\ &- \left[\left(\sigma + \frac{kc_2}{2\pi}\|\rho\|_{L^2}^2 - l\right) + k\|\rho\|_{L^2}^2\left(\ln\|u\|_{L^2_\rho}^2 - \frac{1}{2}n(1+c_1)\right)\right] \\ &- \xi_2\left[\sigma + k\left(\sigma\frac{c_2}{2\pi} + \ln\|u\|_{L^2_\rho}^2 - \frac{n(1+c_1)}{2}\right)\right] \end{aligned}$$

and t_1 was introduced in (A3).
We choose σ so small that

$$\xi_1 > 2c_\sigma\|\rho\|_{L^2}^2\xi_2.$$

Whence σ is fixed, we can choose

$$\xi_2 > \left(\int_0^{t_1} g(s)ds - \sigma\right)^{-1}$$

and ξ_1 large enough so that $M_2 > 0$, which yields

$$L'(t) \leq M_0(g\circ\nabla_x u) - cE'(t), \forall t \geq t_1.$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then,

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^t g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 ds dx, \quad \text{for all} \quad t \geq t_1. \end{aligned} \tag{2.35}$$

Using Lemma(2.6), we obtain

$$F'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)), \quad \text{for all} \quad t \geq t_1.$$

Now, we will following the steps in ([47]) and using the fact that $E' \leq 0, 0 < H_0', 0 < H_0''$ on $(0, r]$ to define the functional

$$F_1(t) = H_0'\left(\alpha_0\frac{E(t)}{E(0)}\right)F(t) + cE(t), \quad \alpha_0 < r, 0 < c,$$

where $F_1(t) \sim E(t)$ and

$$\begin{aligned} F_1'(t) &= \alpha_0 \frac{E'(t)}{E(0)} H_0'' \left(\alpha_0 \frac{E(t)}{E(0)} \right) F(t) + H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) F'(t) + cE'(t) \\ &\leq -cE(t) H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) + cH_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) H_0^{-1}(\lambda(t)) + cE'(t). \end{aligned}$$

Let H_0^* given in (A3) and using Young's inequality (2.11) with $A = H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right), B = H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned} F_1'(t) &\leq -cE(t) H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) + cH_0^* \left(H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) \right) + c\lambda(t) + cE'(t) \\ &\leq -cE(t) H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) + c\alpha_0 \frac{E(t)}{E(0)} H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) - c'E'(t) + cE'(t). \end{aligned}$$

Choosing α_0, c, c', such that for all $t \geq t_1$ we have

$$\begin{aligned} F_1'(t) &\leq -k \frac{E(t)}{E(0)} H_0' \left(\alpha_0 \frac{E(t)}{E(0)} \right) \\ &= -kH_2 \left(\frac{E(t)}{E(0)} \right), \end{aligned}$$

where $H_2(t) = tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives on $(0, 1]$, then

$$R(t) = \tau \frac{k_1 F_1(t)}{E(0)} \sim E(t), \quad \tau \in (0, 1) \tag{2.36}$$

and

$$R'(t) \leq -\tau k_0 H_2(R(t)), \quad k_0 \in (0, +\infty), t \geq t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t) \leq H_1^{-1}(\alpha_1 t + \alpha_2), \quad \alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

here $H_1(t) = \int_t^1 H_2^{-1}(s) ds$. From (2.36), for a positive constant α_3, we have

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0, 1]$ and due to properties of H_2, we have

$$\lim_{t \to 0} H_1(t) = +\infty.$$

Then

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \quad \text{for all} \quad t \geq 0.$$

This completes the proof of Theorem 2.3. ■

Remark 2.2 Noting that, we have obtained all results without any conditions on the exponent k in the logarithmic nonlinearities.

Chapter 3

Existence and decay of solution to coupled system of viscoelastic wave equations with strong damping in $\mathbb{R}^n$

In this chapter, we establish a general decay rate properties of solutions for a coupled system of viscoelastic wave equations in $\mathbb{R}^n$ under some assumptions on g_1, g_2 and linear forcing terms. We exploit a density function to introduce weighted spaces for solutions and using an appropriate perturbed energy method. The questions of global existence in the nonlinear cases is also proved in Sobolev spaces using the well known Galerkin method (see [6]).

3.1 Introduction and previous results

In this paper, we consider the following problem:

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + \alpha u_2 - \phi(x)\Delta\left(u_1 - \int_0^t g_1(t-s)u_1(s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + \alpha u_1 - \phi(x)\Delta\left(u_2 - \int_0^t g_2(t-s)u_2(s,x)ds + u_2'\right) = 0, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{3.1}$$

where $\alpha \neq 0, x \in \mathbb{R}^n, t \in \mathbb{R}^+_*$ where the space $D(\mathbb{R}^n)$ defined in (1.1) and $l, n \geq 2$, $\phi(x) > 0, \forall x \in \mathbb{R}^n, (\phi(x))^{-1} = \rho(x)$ defined in $(A2)$.
This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics, it was first considered by Dafermos in [16], where the general decay was discussed. The problems related to (3.1) attract a great deal of attention

in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results are by now rather developed, especially in any space dimension when it comes to nonlinear problems. The term $\int_0^t g_i(t-s)\Delta u_i(s)ds$ corresponds to the memories terms and the scalar functions $g_i(t)$ (so-called relaxation kernel) is assumed to satisfy (3.9)-(3.11). The energy of (u_1, u_2) at time t is defined by

$$\begin{aligned} E(t) &= \frac{(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \frac{1}{2}\sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|\nabla u_i\|_2^2 \\ &+ \frac{1}{2}\sum_{i=1}^{2}(g_i\circ\nabla u_i) + \alpha\int_{\mathbb{R}^n}\rho u_1u_2dx. \end{aligned} \tag{3.2}$$

For α small enough we use Lemma 1.3, we deduce that:

$$\begin{aligned} E(t) \geq \frac{1}{2}(1-c|\alpha|\|\rho\|^{-1}_{L^{n/2}})\Big[\frac{2(l-1)}{l}\sum_{i=1}^{2}\|u_i'\|^l_{L^l_\rho} + \sum_{i=1}^{2}\left(1-\int_0^t g_i(s)ds\right)\|\nabla u_i\|_2^2 \\ + \sum_{i=1}^{2}(g_i\circ\nabla u_i)\Big], \end{aligned} \tag{3.3}$$

and the following energy functional law holds

$$E'(t) \leq \frac{1}{2}\sum_{i=1}^{2}(g_i'\circ\nabla u_i)(t) - \sum_{i=1}^{2}\|\nabla u_i'\|_2^2, \forall t\geq 0. \tag{3.4}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories.

In the present paper we consider the solutions in an appropriate spaces weighted by the density function $\rho(x)$ in order to compensate the lack of Poincare's inequality which play a decisive role in the proof. To motivate our work, we start with some results related to viscoelastic plate equations with strong damping in [35]:

$$u_{tt} + \Delta^2 u - \Delta_p u - \int_0^t g(t-s)\Delta u(s,x)ds - \Delta u_t + f(u) = 0, \quad x\in\Omega\times\mathbb{R}^+,$$

supplemented with the following conditions:

$$u(t,x) = \Delta u = 0, \text{ on } \partial\Omega\times\mathbb{R}^+, \quad u(0,x) = u_0, u_t(0,t) = u_1, \text{ on } \Omega. \tag{3.5}$$

In this paper, Liu and *all* extend the exponential rate result obtained in [2] to the general case and show that the rate of decay for the solution is similar to that of the memory term under the following assumption for the function g is

$$g'(t) \leq -\xi(t)g(t), \quad \text{where } \xi(t) \text{ satisfies } \xi'(t)\leq 0, \quad \int_0^t \xi(t)dt = \infty.$$

Paper [18] is concerned with a class of plate equations with memory in a history space setting and perturbations of $p-$Laplacian type

$$u_{tt} + \alpha\Delta^2 u - \Delta_p u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s,x)ds - \Delta u_t + f(u) = h, \quad x \in \Omega \times \mathbb{R}^+ \tag{3.6}$$

and results on the well-posedness and asymptotic stability of the problem were proved.

In many existing works on this field, the following conditions on the kernel

$$g'(t) \geq -\lambda g^p(t), \quad t \geq 0, p \geq 0, \tag{3.7}$$

is crucial in the proof of the stability. For a viscoelastic systems with oscillating kernels, we mention the work by Rivera and *all*[44], the authors proved that if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, that is for $p = 1$ in (3.7), then the solution also decays exponentially to zero. On the other hand, if the kernel decays polynomially, i.e. $(p > 1)$ in the inequality (3.7), then the solution also decays polynomially with the same rate of decay. Recently the problem related to (3.1) in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered as equation in [47], where they established an explicit and very general decay rate result for relaxation functions satisfying:

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0,$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.

For the literature, In $\mathbb{R}^n$, we quote essentially the results of [4], [1], [8], [27]-[31], [47]-[51] and the references therein. In [28], authors showed for one equation that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem (3.1) without strong damping in the case $l = 2, \rho(x) = 1$, is polynomial. The finite-speed propagation is used to compensate the lack of Poincare's inequality. In the case $l = 2$, in [27], author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate the lack of Poincare's inequality in the absence of strong damping. The same problem treated in [27], was considered in [29], where under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. The conditions which used, on the relaxation function g and its derivative g' are different from the usual ones. Coupled systems in $\mathbb{R}^n$, we mention, for instance, the work of *[Takashi Narazaki, 2009. Global solutions* *to the Cauchy problem for the weakly coupled system of damped wave equations. Discrete And Continuous Dynamical Systems, 592-601]*, where the following weakly coupled system of a damped wave equations has considered:

$$\begin{cases} u'' - \Delta u + u' = f(v), \quad t > 0, x \in \mathbb{R}^n, \\ v'' - \Delta v + v' = f(u), \quad t > 0, x \in \mathbb{R}^n, \\ (u(0,x), v(0,x)) = (\phi_0(x), \psi_0(x)), \quad x \in \mathbb{R}^n, \\ (u'(0,x), v'(0,x)) = (\phi_1(x), \psi_1(x)), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.8}$$

Authors have shown the sufficient condition under which the Cauchy problem (3.8) admits global solutions when $n = 1, 2, 3$ provided that the initial data are sufficiently small in an associate space. Moreover, they have also shown the asymptotic behavior of the solutions and to generalize the existence result in [54] to the case $n = 1, 2, 3$ and improve time decay estimates when $n = 3$.

3.2 Function spaces and statements

In this section we introduce some notation and results needed for our work. We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denotes $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real-valued. Here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. We denote by B_R the open ball of $\mathbb{R}^n$ with center 0 and radius R.
First we recall and make use the following assumptions on the functions ρ and g for $i = 1, 2$ as:
(A1) We assume that the function $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$(for $i = 1, 2$) is of class C^1 satisfying:

$$1 - \int_0^\infty g_i(t)dt \geq k_i > 0, g_i(0) = g_{i0} > 0, \tag{3.9}$$

and there exist nonincreasing continuous functions ξ_1,ξ_2: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\xi'(t) \leq 0, \quad \forall t > 0, \quad \int_0^\infty \xi(t) = \infty, \quad \xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \tag{3.10}$$

where

$$g_i'(t) + \xi(t)g_i(t) \leq 0. \tag{3.11}$$

(A2) The function $\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.
The following technical Lemma will play an important role in the sequel.

Lemma 3.1 *[9] (Lemma1.1) For any two functions g, $v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have*

$$\left|\int_0^t g(t-s)(v(t)-v(s))ds\right|^2 \leq \left(\int_0^t |g(s)|^{2(1-\theta)}ds\right)\int_0^t |g(t-s)|^{2\theta}|v(t)-v(s)|^2 ds.$$

To study the properties of the operator $\phi\Delta$, we consider as in [31], the equation

$$\phi(x)\Delta u(x) = \eta(x), \quad x \in \mathbb{R}^n, \tag{3.12}$$

without boundary conditions. Since for every u, v in $C_0^\infty(\mathbb{R}^n)$

$$(\phi\Delta u, v)_{L^2_\rho} = \int_{\mathbb{R}^n} \nabla u \nabla v dx, \tag{3.13}$$

and $L^2_\rho(\mathbb{R}^n)$ are defined with respect to the inner product (1.2), we may consider equation (3.12) as operator equation:

$$\Delta_0 u = \eta, \quad \Delta_0 : D(\Delta_0) \subseteq L^2_\rho(\mathbb{R}^n) \to L^2_\rho(\mathbb{R}^n), \quad \eta \in L^2_\rho(\mathbb{R}^n).$$

The relations (3.13) implies that the operators $\phi\Delta$ with domain of definition $D(\Delta_0) = C_0^\infty(\mathbb{R}^n)$ being symmetric. Let us note that the operator $\phi\Delta$ is not symmetric in the standard Lebesgue space $L^2(\mathbb{R}^n)$, because of the appearance of $\phi(x)$ (see [[53], pages 185-187]). From (1.5) and (3.13) we have

$$\|u\|_{L^2_\rho} \leq c(\Delta_0 u, u)_{L^2_\rho}, \quad \text{for all } u \in D(\Delta_0). \tag{3.14}$$

From (3.13) and (3.14) we conclude that Δ_0 is a symmetric, strongly monotone operator on $L^2_\rho(\mathbb{R}^n)$. The energy scalar product is given by:

$$(u,v)_E = \int_{\mathbb{R}^n} \nabla u \nabla v dx,$$

and the energy space is the completion of $D(\Delta_0)$ with respect to $(u,v)_E$. It is obvious that the energy space X_E is the homogeneous Sobolev space $D(\mathbb{R}^n)$. The energy extension Δ_E, namely

$$\phi\Delta : D(\mathbb{R}^n) \to D^{-1}(\mathbb{R}^n),$$

is defined to be the duality mapping of $D(\mathbb{R}^n)$. For every $\eta \in D^{-1}(\mathbb{R}^n)$ the equation (3.12), has a unique solution. Define $D(\Delta_1)$ to be the set of all solutions of the equation (3.12) for arbitrary $\eta \in L^2_\rho(\mathbb{R}^n)$. The operator extension Δ_1 of Δ_0, [see [58], Theorem 19.C] is the restriction of the energy extension Δ_E to the set $D(\Delta_1)$.

3.2. Function spaces and statements

The operator Δ_1 is self-adjoint and therefore graph-closed. Its domain is a Hilbert space with respect to the graph scalar product

$$(u,v)_{D(\Delta_1)} = (u,v)_{L^2_\rho} + (\Delta_1 u, \Delta_1 v)_{L^2_\rho}, \quad \text{for all } u,v \in D(\Delta_1).$$

The norm induced by the scalar product $(u,v)_{D(\Delta_1)}$ is

$$\|u\|_{D(\Delta_1)} = \left\{ \int_{\mathbb{R}^n} \rho|u|^2 dx + \int_{\mathbb{R}^n} \phi|\Delta u|^2 dx \right\}^{\frac{1}{2}}.$$

which is equivalent to the norm

$$\|\Delta_1 u\|_{L^2_\rho} = \left\{ \int_{\mathbb{R}^n} \phi|\Delta u|^2 dx \right\}^{\frac{1}{2}}.$$

So, we have established the *evolution quartet*

$$D(\Delta_1) \subset D(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n) \subset D^{-1}(\mathbb{R}^n), \tag{3.15}$$

where all the embedding are dense and compact. A consequence of the compactness of the embedding in (3.15) is that the eigenvalue problem

$$-\Delta u = \mu u, x \in \mathbb{R}^n, \tag{3.16}$$

has a complete system of eigenfunctions $\{w_n, \mu_n\}$ with the following properties:

$$\begin{cases} -\Delta w_j = \mu w_j, & j = 1,2\cdots, \quad w_j \in D(\mathbb{R}^n), \\ \\ 0 < \mu_1 \le \mu_2 \le \cdots, \quad \mu_j \to \infty, & as \quad j \to \infty. \end{cases} \tag{3.17}$$

It can be shown, as in [8], that every solution of (3.16) is such that

$$u(x) \longrightarrow 0, \quad as \quad |x| \longrightarrow \infty, \tag{3.18}$$

uniformly with respect to x. Finally, we give the definition of *weak solutions* for the problem (3.1).

Definition 3.1 A weak solution of (3.1) is (u_1, u_2) such that

- $(u_1,u_2) \in (L^2[0,T;D(\mathbb{R}^n)])^2$, $\quad (u_1', u_2') \in (L^2[0,T;L^l_\rho(\mathbb{R}^n)])^2$ and $(u_1'', u_2'') \in (L^2[0,T;D^{-1}(\mathbb{R}^n)])^2$,
- For all $(v,w) \in (C_0^\infty([0,T]\times\mathbb{R}^n))^2$, (u_1,u_2) satisfies the generalized formula:

$$\begin{cases} \int_0^T \left(\left(|u_1'|^{l-2}u_1'\right)', v\right)_{L^l_\rho} ds + \alpha \int_0^T (u_2, v)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} \nabla u_1 \nabla v dx ds \\ + \int_0^T \int_{\mathbb{R}^n} \nabla u_1' \nabla v dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_1(s-\tau)\nabla u_1(\tau) d\tau \nabla v(s) dx ds = 0, \\ \\ \int_0^T \left(\left(|u_2'|^{l-2}u_2'\right)', w\right)_{L^l_\rho} ds + \alpha \int_0^T (u_1, w)_{L^2_\rho} ds + \int_0^T \int_{\mathbb{R}^n} \nabla u_2 \nabla w dx ds \\ + \int_0^T \int_{\mathbb{R}^n} \nabla u_2' \nabla w dx ds - \int_0^T \int_{\mathbb{R}^n} \int_0^s g_2(s-\tau)\nabla u_2(\tau) d\tau \nabla w(s) dx ds = 0. \end{cases}$$

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- (u_1, u_2) satisfies the initial conditions
  $(u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \quad (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2.$

We are now ready to state and prove our existence results.

3.3 Well-posedness result for nonlinear case

This section is devoted to prove the existence and uniqueness of solutions to the system (3.1) taking account the nonlinear case in the terms responsible on the relation between tow equations, that is replacing $\alpha u_1, \alpha u_2$ by $f_1(u_1, u_2), f_2(u_1, u_2)$ introduced in the last section. First, we prove the existence of the unique solution of the restricted problem on B_R, the main ingredient used here is the Galerkin approximations introduced in [33].

Lemma 3.2 *Assume that (A1), (A2), (3.56)-(3.60) are satisfied. Suppose that the constants* $T > 0,\ R > 0$ *and the initial conditions*

$$(u_{10}, u_{20}) \in (D(B_R))^2, (u_{11}, u_{21}) \in (L^l_\rho(B_R))^2,$$

are given. Then there exists a unique solution for the problem (3.1) such that

$$u_i \in C[0, T; D(B_R)] \quad and \quad u_i' \in C[0, T; L^l_\rho(B_R)].$$

Proof. The existence is proved by using the Galerkin method, which consists in constructing approximations of the solution, then we obtain a priori estimates necessary to guarantee the convergence of these approximations. So, we take $\{w_i\}_{i=1}^\infty$ be the eigen-functions of the operator $-\Delta$. Then $\{w_i\}_{i=1}^\infty$ is an orthogonal basis of $D(B_R)$ which is orthonormal in $L^2_\rho(B_R)$.
Let

$$V_m = span\{w_1, w_2, \cdots, w_m\},$$

and the projection of the initial data on the finite dimensional subspace V_m is given by:

$$u_{10}^m = \sum_{j=0}^m a_j w_j, \quad u_{20}^m = \sum_{j=0}^m b_j w_j, \quad u_{11}^m = \sum_{j=0}^m c_j w_j, \quad u_{21}^m = \sum_{j=0}^m d_j w_j,$$

We search approximate solutions

$$u_1^m(x,t) := \sum_{j=0}^m h_j^m(t) w_j(x), \quad u_2^m(x,t) := \sum_{j=0}^m k_j^m(t) w_j(x),$$

of the approximate problem in V_m

$$\begin{cases} \int_{B_R} \left(\rho(x)\left(|u_1'^m|^{l-2}u_1'^m\right)' w - \int_0^t g_1(t-s)\nabla u_1^m(s,x)\nabla w ds\right) dx \\ + \int_{B_R} \left(\rho(x) f_1(u_1^m, u_2^m) w + \nabla u_1^m \nabla w + \nabla u_1'^m \nabla w\right) dx = 0, \\ \\ \int_{B_R} \left(\rho(x)\left(|u_2'^m|^{l-2}u_2'^m\right)' w - \int_0^t g_2(t-s)\nabla u_2^m(s,x)\nabla w ds\right) dx \\ + \int_{B_R} \left(\rho(x) f_2(u_1^m, u_2^m) w + \nabla u_2^m \nabla w + \nabla u_2'^m \nabla w\right) dx = 0, \\ \\ u_1^m(0) = u_{10}^m, u_1'^m(0) = u_{11}^m, u_2^m(0) = u_{20}^m, u_2'^m(0) = u_{21}^m. \end{cases} \tag{3.19}$$

Based on standard existence theory for differential equations, one can conclude the existence of solution (u_1^m, u_2^m) of (3.19) on a maximal time interval $[0, t_m)$, for each $m \in \mathbb{N}$.

• (A priori estimate 1): In (3.19), let $w = (u_1^m)'$ in the first equation and $w = (u_2^m)'$ in the second equation, add the resulting equations and integrate by parts to obtain

$$\frac{d}{dt}E^m(t) = \frac{1}{2}\sum_{i=1}^{2}(g_i' \circ \nabla u_i^m)(t) - \frac{1}{2}\sum_{i=1}^{2} g_i(t)\|\nabla u_i^m(t)\|_2^2 - \sum_{i=1}^{2}\|\nabla u_i'^m\|_2^2, \tag{3.20}$$

This means, using $(A1)$, that for some positive constant C independent of t and m, we have

$$E^m(t) \leq E^m(0) \leq C. \tag{3.21}$$

• (A priori estimate 2): In (3.19), let $w = -\Delta u_1'^m$ in the first equation and $w = -\Delta u_2'^m$ in the second equation, add the resulting equations, integrate by parts and use $(A1)$ to obtain

$$\begin{aligned} & \frac{d}{dt}\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\Delta u_i'^m\|_{L_\rho^l}^l + \frac{1}{2}\left(1 - \int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2 + \frac{1}{2}(g_i \circ \Delta u_i^m)\right) \\ = & \sum_{i=1}^{2}\left(\frac{1}{2}(g_i' \circ \Delta u_i^m) - \frac{1}{2}g_i(t)\|\Delta u_i^m\|_2^2 + \|\Delta u_i'^m\|_2^2\right) \\ - & \sum_{i=1}^{2}\int_{B_R}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i'^m dx \\ \leq & -\sum_{i=1}^{2}\int_{B_R}\rho(x) f_i(u_1^m, u_2^m)\Delta u_i'^m dx. \end{aligned} \tag{3.22}$$

Then, integrating over $(0,t)$ yields

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{l-1}{l}\|\Delta u_i'^m\|_{L_\rho^l}^l+\frac{1}{2}\left(1-\int_0^t g_i(s)ds\right)\|\Delta u_i^m\|_2^2+\frac{1}{2}(g_i\circ\Delta u_i^m)\right)\\
\leq\ &\sum_{i=1}^{2}\left(\|\Delta u_{i1}^m\|_{L_\rho^l}^l+\|\Delta u_{i0}^m\|_2^2-\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i^m dx\right)\\
+\ &\sum_{i=1}^{2}\int_{B_R}\rho(x)\left(f_i(u_{10}^m,u_{20}^m)\Delta u_{i0}^m\right)dx \qquad (3.23)\\
+\ &\int_0^t\int_{B_R}\rho(x)\left(\frac{\partial f_1}{\partial u_2}u_2'^m\Delta u_1^m+\frac{\partial f_2}{\partial u_1}u_1'^m\Delta u_2^m\right)dxds.
\end{aligned}$$

To estimate the terms on the right hand side of (3.24), we use (3.56)-(3.58), Young's inequality and (1.5) and take (3.21) into account to get

$$\begin{aligned}
&\int_{B_R}\rho(x)f_i(u_1^m,u_2^m)\Delta u_i^m\leq k\int_{B_R}\rho(x)\left(|u_1^m|+|u_2^m|+|u_1^m|^{\beta_{i1}}+|u_2^m|^{\beta_{i2}}\right)\Delta u_i^m,\\
&\leq\delta\|\Delta u_i^m\|_{L_\rho^2}^2+\frac{c}{\delta}\int_{B_R}\rho(x)\left(|u_1^m|^2+|u_2^m|^2+|u_1^m|^{2\beta_{i1}}+|u_2^m|^{2\beta_{i2}}\right),\\
&\leq\delta\|\Delta u_i^m\|_{L_\rho^2}^2+\frac{c}{\delta}\left(\|u_1^m\|_{L_\rho^2}^2+\|u_2^m\|_{L_\rho^2}^2+\|u_1^m\|_{L_\rho^2}^{2\beta_{i1}}+\|u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right),\\
&\leq\delta\|\Delta u_i^m\|_{L_\rho^2}^2+\frac{c}{\delta}\left(\|\nabla u_1^m\|_{L_\rho^2}^2+\|\nabla u_2^m\|_{L_\rho^2}^2+\|\nabla u_1^m\|_{L_\rho^2}^{2\beta_{i1}}+\|\nabla u_2^m\|_{L_\rho^2}^{2\beta_{i2}}\right),\\
&\leq\delta\|\Delta u_i^m\|_{L_\rho^2}^2+\frac{c}{\delta}E^m(0)E^m(t),\\
&\leq\delta\|\Delta u_i^m\|_{L_\rho^2}^2+\frac{c}{\delta}. \qquad (3.24)
\end{aligned}$$

Since $1\leq\beta_{ij}, i,j=1,2$. Now, we estimate

$$I:=\int_{B_R}\rho(x)\frac{\partial f_i}{\partial u_1}u_i'^m\Delta u_i^m.$$

First, we observe that

$$\frac{\beta_{1j}-1}{2\beta_{1j}}+\frac{1}{2\beta_{1j}}+\frac{1}{2}=1,$$

and use $(A2)$ and the generalized Hölder's inequality to infer

$$\begin{aligned}
|I|\ &\leq\ d\int_{B_R}\rho(x)\left(1+|u_1^m|^{\beta_{11}-1}+|u_2^m|^{\beta_{12}-1}\right)u_i'^m\Delta u_i^m,\\
&\leq\ d\left(\|u_i'^m\|_{L_\rho^2}+\|u_i'^m\|_{L_\rho^{2\beta_{11}}}\|u_1^m\|_{L_\rho^{2\beta_{11}}}^{\beta_{11}-1}+\|u_i'^m\|_{L_\rho^{2\beta_{12}}}\|u_2^m\|_{L_\rho^{2\beta_{12}}}^{\beta_{12}-1}\right)\|\Delta u_i^m\|_{L_\rho^2}.
\end{aligned}$$

Then, by (1.5), (3.21) and Young's inequality, we arrive at

$$\begin{aligned}
|I|\ &\leq\ c\left(1+\|\nabla u_1^m\|_2^{\beta_{11}-1}+\|\nabla u_2^m\|_2^{\beta_{12}-1}\right)\|\nabla u_i'^m\|_{L_\rho^2}\|\Delta u_i^m\|_{L_\rho^2},\\
&\leq\ c\left(\|\nabla u_i'^m\|_{L_\rho^2}\cdot\|\Delta u_i^m\|_{L_\rho^2}\right)\leq c\|\nabla u_i'^m\|_{L_\rho^2}^2+c\|\Delta u_i^m\|_{L_\rho^2}^2. \qquad (3.25)
\end{aligned}$$

Since the other terms in (3.24) can be similarly treated and the norms of the initial data are uniformly bounded, we combine (3.24), (3.25), use $(A1)$ and take δ small enough to end up with

$$\sum_{i=1}^{2}\left(\|\nabla u_i'^m\|_{L_\rho^l}^l+\|\Delta u_i^m\|_2^2\right) \;\leq\; c+c\sum_{i=1}^{2}\int_0^t\left(\|\nabla u_i'^m\|_{L_\rho^l}^l+\|\Delta u_i^m\|_2^2\right)ds.$$

Using Gronwall's inequality, this implies that

$$\sum_{i=1}^{2}\left(\|\nabla u_i'^m\|_{L_\rho^l}^l+\|\Delta u_i^m\|_2^2\right)\leq C,\quad \forall t\in[0,T] \text{ and } m\in\mathbb{N}. \tag{3.26}$$

• (A priori estimate 3): In (3.19), let $w=(u_1^m)''$ in the first equation and $w=(u_2^m)''$ in the second equation. Then, by exploiting the previous estimates and using similar arguments, we find

$$\sum_{i=1}^{2}\|u_i''^m\|_2^2\leq C,\quad \forall t\in[0,T] \text{ and } m\in\mathbb{N}. \tag{3.27}$$

From (3.21), (3.26) and (3.27), we conclude that

$$\begin{aligned}
u_i^m &\text{ are uniformly bounded in } L^\infty(0,T;D(B_R)),\\
u_i^{m'} &\text{ are uniformly bounded in } L^\infty(0,T;L_\rho^l(B_R)),\\
u_i^{m''} &\text{ are uniformly bounded in } L^2(0,T;D^{-1}(B_R)),
\end{aligned}$$

which implies that there exists subsequences of $\{u_i^m\}$, which we still denote in the same way, such that

$$\begin{aligned}
u_i^m &\overset{*}{\longrightarrow} \textit{weak } u_i \text{ in } L^\infty(0,T;D(B_R)),\\
u_i^{m'} &\overset{*}{\longrightarrow} \textit{weak } u_i' \text{ in } L^\infty(0,T;L_\rho^l(B_R)),\\
u_i^{m''} &\overset{*}{\longrightarrow} \textit{weak } u_i'' \text{ in } L^2(0,T;D^{-1}(B_R)).
\end{aligned} \tag{3.28}$$

In the sequel, we will deal with the nonlinear term. By Aubin's Lemma (see [33]), we find, up to a subsequence, that

$$u_i^m\rightarrow u_i \text{ strongly in } L^2(0,T;L_\rho^l(B_R)). \tag{3.29}$$

Then,

$$u_i^m\rightarrow u_i \text{ almost everywhere in } (0,T)\times B_R, \tag{3.30}$$

and therefore, from (3.59), (3.60),

$$f_i(u_1^m,u_2^m)\rightarrow f_i(u_1,u_2) \;\text{ almost everywhere in }\; (0,T)\times B_R, \text{ for } i=1,2. \tag{3.31}$$

Also, as u_i^m are bounded in $L^\infty(0,T;L^2_\rho(B_R))$, then the use of (3.56)-(3.60) gives that $f_i(u_1^m,u_2^m)$ is bounded in $L^\infty(0,T;L^2_\rho(B_R))$. From (3.31), we can deduce that

$$f_i(u_1^m,u_2^m) \rightharpoonup f_i(u_1,u_2) \text{ in } L^2(0,T;L^2_\rho(B_R)), \quad \text{for } i=1,2.$$

Combining the results obtained above, we can pass to the limit and conclude that (u_1,u_2) is the solution of system (3.1) restricted un B_R.

■ In the next result, we will extend our solutions to $\mathbb{R}^n$.

Theorem 3.1 *Assume that (A1), (A2), (3.56)-(3.60) are satisfied. Suppose that the initial conditions*

$$(u_{10},u_{11}) \in (C_0^\infty(B_R)^2, (u_{20},u_{21}) \in (C_0^\infty(B_R))^2,$$

are given. Then for the problem (3.1), there exists a unique solution such that

$$(u_1,u_2) \in (C[0,T;D(\mathbb{R}^n)])^2 \quad \textit{and} \quad (u_1',u_2') \in (C[0,T;L^l_\rho(\mathbb{R}^n)])^2.$$

Proof. (a) **Existence.** Let $R_0>0$ such that $supp(u_{10},u_{20}) \subset B_{R_0}$ and $supp(u_{11},u_{21}) \subset B_{R_0}$. Then, for $R \geq R_0$, $R \in \mathbb{N}$, we consider the approximating problem

$$\begin{cases} \left(|u_1'^R|^{l-2}u_1'^R\right)' + f_1(u_1^R,u_2^R) - \phi(x)\Delta\left(u_1^R + \int_0^t g_1(s)u_1^R(s-t,x)ds + u_1'^R\right) = 0, x \in B_R \times \mathbb{R}^+, \\ \\ \left(|u_2'^R|^{l-2}u_2'^R\right)' + f_2(u_1^R,u_2^R) - \phi(x)\Delta\left(u_2^R + \int_0^t g_2(s)u_2^R(s-t,x)ds + u_2'^R\right) = 0, x \in B_R \times \mathbb{R}^+, \\ \\ (u_1^R(0,x),u_2^R(0,x)) = (u_1^0(x),u_2^0(x)) \in \left(C_0^\infty(B_R)\right)^2, \\ (u_1'^R(0,x),u_2'^R(0,x)) = (u_1^1(x),u_2^2(x)) \in \left(C_0^\infty(B_R)\right)^2. \end{cases} \tag{3.32}$$

By Lemma 3.2, problem (3.32) has a unique solution u_i^R such that

$$(u_1^R,u_2^R) \in (C[0,T;D(B_R)])^2 \quad \text{and} \quad ((u_1^R)',(u_2^R)') \in (C[0,T;L^l_\rho(B_R)])^2.$$

We extend the solution of the problem (3.32) as

$$(\widetilde{u}_1^R,\widetilde{u}_2^R) =: \begin{cases} (u_1^R,u_2^R), & if\ |x| \leq R, \\ 0, & otherwise. \end{cases} \tag{3.33}$$

The solution (u_1^R,u_2^R) satisfies the estimates

$$\begin{array}{ll} \|\widetilde{u}_i^R\|_{L^\infty(0,T;D(\mathbb{R}^n))} \leq K, & \|f(\widetilde{u}_i^R)\|_{L^\infty(0,T;D(\mathbb{R}^n))} \leq K, \\ \|(\widetilde{u}_i^R)'\|_{L^\infty(0,T;L^l_\rho(\mathbb{R}^n))} \leq K, & \|(\widetilde{u}_i^R)''\|_{L^\infty(0,T;D^{-1}(\mathbb{R}^n))} \leq K, \end{array} \tag{3.34}$$

where the constant K is independent of R. The estimates (3.34) imply that

$$\widetilde{u}_i^R \text{ is relatively compact in } C([0,T];L^2_\rho(\mathbb{R}^n)). \tag{3.35}$$

Next using relations (3.34) and (3.35), the continuity of the embedding

$$C([0,T];L^2_\rho(\mathbb{R}^n)) \subset L^2([0,T];L^2_\rho(\mathbb{R}^n)),$$

and the continuity of f_i we may extract a subsequence of $\widetilde{u}_i^R$, denoted by $\widetilde{u}_i^{R_m}$, such that as $R_m \to \infty$ we get

$$\begin{aligned} \widetilde{u}_i^{R_m} &\overset{*}{\longrightarrow} \widetilde{u}_i \text{ in } L^\infty(0,T;D(B_R)),\\ (\widetilde{u}_i^{R_m})' &\overset{*}{\longrightarrow} u_i' \text{ in } L^\infty(0,T;L^l_\rho(B_R)),\\ (\widetilde{u}_i^{R_m})'' &\overset{*}{\longrightarrow} u_i'' \text{ in } L^\infty(0,T;D^{-1}(B_R)),\\ f(\widetilde{u}_i^{R_m}) &\overset{*}{\longrightarrow} f(\widetilde{u}_i) \text{ in } L^\infty(0,T;D(B_R)). \end{aligned} \tag{3.36}$$

For fixed $R = R_m$, let L_m denote the operator of restriction

$$L_m : [0,T] \times \mathbb{R}^n \to [0,T] \times B_R.$$

It is clear that the restricted subsequence $L_m\widetilde{u}_i^{R_m}$ satisfies the estimates obtained in Lemma 3.2. Therefore there exists a subsequence $\widetilde{u}_i^{R_{m_j}} = \widetilde{u}_i^j$ for which it can be shown by following the procedure of Lemma 3.2, that $L_m\widetilde{u}_i^j$ converges weakly to solution $\widetilde{u}_i^m$. We have

$$\begin{cases} \int_0^T \left(L_m\left(|\widetilde{u}_1^{j'}|^{l-2}\widetilde{u}_1^{j'}\right)', v\right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_1(L_m\widetilde{u}_1^j, L_m\widetilde{u}_2^j), v\right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} \nabla L_m\widetilde{u}_1^j \nabla v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{B_R} \nabla\widetilde{u}_1^j \nabla v dx ds \\ + \int_0^T \int_{B_R} \nabla L_m\widetilde{u}_1^{\prime j} \nabla v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_1^{\prime j}|^{l-2}\widetilde{u}_1^{j'}\right)', v\right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_1(\widetilde{u}_1^j, \widetilde{u}_2^j), v\right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} \nabla\widetilde{u}_1^j \nabla v dx ds - \int_0^T \int_0^t g_1(t-s) \int_{\mathbb{R}^n} \nabla\widetilde{u}_1^j \nabla v dx ds, \\ \\ \int_0^T \left(L_m\left(|\widetilde{u}_2^{j'}|^{l-2}\widetilde{u}_2^{j'}\right)', v\right)_{L^l_\rho(B_R)} ds + \int_0^T \left(f_2(L_m\widetilde{u}_1^j, L_m\widetilde{u}_2^j), v\right)_{L^2_\rho(B_R)} ds \\ + \int_0^T \int_{B_R} \nabla L_m\widetilde{u}_2^j \nabla v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{B_R} \nabla\widetilde{u}_2^j \nabla v dx ds \\ + \int_0^T \int_{B_R} \nabla L_m\widetilde{u}_2^{\prime j} \nabla v dx ds \\ = \int_0^T \left(\left(|\widetilde{u}_2^{\prime j}|^{l-2}\widetilde{u}_2^{j'}\right)', v\right)_{L^l_\rho(\mathbb{R}^n)} ds + \int_0^T \left(f_2(\widetilde{u}_1^j, \widetilde{u}_2^j), v\right)_{L^2_\rho(\mathbb{R}^n)} \\ + \int_0^T \int_{\mathbb{R}^n} \nabla\widetilde{u}_2^j \nabla v dx ds - \int_0^T \int_0^t g_2(t-s) \int_{\mathbb{R}^n} \nabla\widetilde{u}_2^j \nabla v dx ds, \end{cases} \tag{3.37}$$

for every $v \in C_0^\infty([0,T] \times B_R)$. Passing to the limit in (3.37) as $j \longrightarrow \infty$, we obtain that $L_m\widetilde{u}_i = \widetilde{u}_i^m$. The equalities (3.37) hold for any $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$ since the

radius R is arbitrarily chosen. Therefore $\widetilde{u}_i$ is a solution of the problem (3.32).
(b) **Uniqueness.** Let us assume that $(u_{11}, u_{21}), (u_{12}, u_{22})$ are two strong solutions of (3.1). Then, $(z_1, z_2) = (u_{11} - u_{12}, u_{21} - u_{22})$ satisfies, for all $w \in D(\mathbb{R}^n)$

$$\begin{cases} \int_{\mathbb{R}^n} \left(\rho(x)\left(|z_1'|^{l-2}z_1'\right)' w + \nabla z_1 \nabla w + \int_0^t g_1(s)\nabla z_1(s-t,x)\nabla w ds\right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_1(z_1, z_2) w dx + \nabla z_1' \nabla w = 0, \\ \int_{\mathbb{R}^n} \left(\rho(x)\left(|z_2'|^{l-2}z_2'\right)' w + \nabla z_2 \nabla w + \int_0^t g_2(s)\nabla z_2(s-t,x)\nabla w ds\right) dx \\ + \int_{\mathbb{R}^n} \rho(x) f_2(z_1, z_2) w dx + \nabla z_2' \nabla w = 0. \end{cases} \tag{3.38}$$

Substituting $w = z_1'$ in the first equation and $w = z_2'$ in the second equation, adding the resulting equations, integrating by parts and using $(A1)$, yield

$$\begin{aligned} &\frac{d}{dt}\sum_{i=1}^{2}\left(\frac{l-1}{l}\|z_i'\|_{L_\rho^l}^l + \frac{1}{2}\left(1 - \int_0^t g_i(s)ds\right)\|\nabla z_i\|_2^2 + \frac{1}{2}(g_i \circ \nabla z_i)\right) \\ \leq & \int_{\mathbb{R}^n} \left([f_1(u_{21}, u_{22}) + f_1(u_{11}, u_{12})]\, z_1' + [f_2(u_{21}, u_{22}) + f_2(u_{11}, u_{12})]\, z_2'\right) dx. \end{aligned}$$

Making use of (3.60) and following similar arguments that used to obtain (3.25), we find

$$\begin{aligned} &\int_{\mathbb{R}^n} \left([f_1(u_{21}, u_{22}) + f_1(u_{11}, u_{12})]\, z_1' + [f_2(u_{21}, u_{22}) + f_2(u_{11}, u_{12})]\, z_2'\right) dx \\ \leq\ & k\int_{\mathbb{R}^n} \left(1 + |u_{11}|^{\beta_{11}-1} + |u_{12}|^{\beta_{11}-1} + |u_{21}|^{\beta_{12}-1} + |u_{22}|^{\beta_{12}-1}\right)(|z_1| + |z_2|) z_1' dx \\ +\ & k\int_{B_R} \left(1 + |u_{11}|^{\beta_{21}-1} + |u_{12}|^{\beta_{21}-1} + |u_{21}|^{\beta_{22}-1} + |u_{22}|^{\beta_{22}-1}\right)(|z_1| + |z_2|) z_2' dx, \\ \leq\ & c\sum_{i=1}^{2}\left(\|z_i'\|_{L_\rho^l}^l + \|\nabla z_i\|_2^2.\right) \end{aligned} \tag{3.39}$$

Combining (3.38)- (3.39), integrating over $(0, t)$ and using Gronwall's Lemma, then we deduce that

$$\sum_{i=1}^{2}\left(\|z_i'\|_{L_\rho^l}^l + \|z_i\|_2^2\right) = 0, \tag{3.40}$$

which means that $(u_{11}, u_{21}) = (u_{12}, u_{22})$. This completes the proof. ■ We can now state and prove the asymptotic behavior of the solution of (3.1).

3.4 Decay rate for linear cases

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is similar to that of the memory terms, making some small perturbation in the associate energy, for this purpose, we introduce the functional

$$\psi(t)=\sum_{i=1}^{2}\int_{\mathbb{R}^n}\rho(x)u_i|u_i'|^{l-2}u_i'dx. \tag{3.41}$$

The following Lemma will be useful in the proof of our next result.

Lemma 3.3 *Under the assumptions (A1), (A2), the functional ψ satisfies, along the solution of (3.1),*

$$\psi'(t)\leq\sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l-(k-1-\delta+|\alpha|c)\sum_{i=1}^{2}\|\nabla u_i\|_2^2+c\sum_{i=1}^{2}(g_i\circ\nabla u_i), \tag{3.42}$$

for positive constants c.

Proof. From (3.41), integrate by parts over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi'(t) &= \int_{\mathbb{R}^n}\rho(x)u_1'^l dx+\int_{\mathbb{R}^n}\rho(x)u_1\left(|u_1'|^{l-2}u_1'\right)'dx\\
&+ \int_{\mathbb{R}^n}\rho(x)u_2'^l dx+\int_{\mathbb{R}^n}\rho(x)u_2\left(|u_2'|^{l-2}u_2'\right)'dx,\\
&= \int_{\mathbb{R}^n}\left(\rho(x)u_1'^l-u_1\Delta u_1-u_1\Delta u_1'-\alpha\rho(x)u_1u_2+u_1\int_0^t g_1(t-s)\Delta u_1(s,x)ds\right)dx\\
&+ \int_{\mathbb{R}^n}\left(\rho(x)u_2'^l-u_2\Delta^2 u_2-u_2\Delta u_2'-\alpha\rho(x)u_1u_2+u_2\int_0^t g_2(t-s)\Delta u_2(s,x)ds\right)dx,\\
&= \sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l-\left(1-\int_0^t g_i(s)ds\right)\sum_{i=1}^{2}\|\nabla u_i\|_2^2\\
&- \sum_{i=1}^{2}\|\nabla u_i'\|_2^2-2\alpha\int_{\mathbb{R}^n}\rho(x)u_1u_2dx\\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}^n}\nabla u_i\int_0^t g_i(t-s)(\nabla u_i(s)-\nabla u_i(t))dsdx.
\end{aligned}$$

Recalling that $\int_0^t g_i(s)ds\leq\int_0^\infty g_i(s)ds=1-k_i$, using Young's inequality, Lemma 1.3 and Lemma 3.1, we obtain

$$\begin{aligned}
\psi'(t) &\leq \sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l-\sum_{i=1}^{2}\|\nabla u_i'\|_2^2-(k_i-1+|\alpha|\|\rho\|_{L^s(\mathbb{R}^n)}^{-1})\sum_{i=1}^{2}\|\nabla u_i\|_2^2\\
&+ \delta\sum_{i=1}^{2}\|\nabla u_i\|_2^2+\frac{1}{4\delta}\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left(\int_0^t g_i(t-s)|\nabla u_i(s)-\nabla u_i(t)|ds\right)^2dx,\\
&\leq \sum_{i=1}^{2}\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l-\sum_{i=1}^{2}\|\nabla u_i'\|_2^2-(k-1-\delta+|\alpha|c)\sum_{i=1}^{2}\|\nabla u_i\|_2^2\\
&+ \frac{(1-k)}{4\delta}\sum_{i=1}^{2}(g_i\circ\nabla u_i).
\end{aligned}$$

For α small enough and $k = \min\{k_1, k_2\}$. ■ Our main result reads as follows.

Theorem 3.2 *Let $(u_{10}, u_{11}), (u_{20}, u_{21}) \in D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)$ and suppose that (A1), (A2) hold. Then there exist positive constants W, ω such that the energy of solution given by (3.1) satisfies,*

$$E(t) \leq WE(0)\exp\left(-\omega\int_0^t \xi(s)ds\right), \forall t \geq 0. \tag{3.43}$$

In order to prove this theorem, let us define

$$L(t) = N_1E(t) + \varepsilon\psi(t), \quad \forall\varepsilon > 0. \tag{3.44}$$

for $N_1 > 1$, we need the next lemma, which means that there is equivalence between the perturbed energy and energy functions.

Lemma 3.4 *For $N_1 > 1$, we have*

$$\beta_1 L(t) \leq E(t) \leq L(t)\beta_2, \quad \forall t \geq 0, \tag{3.45}$$

holds for some positive constants β_1 and β_2.

Proof. By (3.41) and (3.44), we have

$$\begin{aligned} |L(t) - N_1E(t)| &\leq \varepsilon|\psi_1(t)|, \\ &\leq \varepsilon\sum_{i=1}^{2}\int_{\mathbb{R}^n}\left|\rho(x)u_i|u_i'|^{l-2}u_i'\right| dx. \end{aligned}$$

Thanks to Hölder's and Young's inequalities with exponents $\frac{l}{l-1}$, l, since $\frac{2n}{n+2} \geq l \geq 2$, we have by using Lemma 1.3

$$\begin{aligned} \int_{\mathbb{R}^n}\left|\rho(x)u_i|u_i'|^{l-2}u_i'\right| dx &\leq \left(\int_{\mathbb{R}^n}\rho(x)|u_i|^l dx\right)^{1/l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right)^{(l-1)/l}, \\ &\leq \frac{1}{l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i|^l dx\right) + \frac{l-1}{l}\left(\int_{\mathbb{R}^n}\rho(x)|u_i'|^l dx\right), \\ &\leq c\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + c\|\rho\|^l_{L^s(\mathbb{R}^n)}\|\nabla u_i\|^l_2. \end{aligned} \tag{3.46}$$

Then, since $l \geq 2$, we have by using (3.4)

$$\begin{aligned} |L(t) - N_1E(t)| &\leq \varepsilon c\sum_{i=1}^{2}\left(\|u_i'\|^l_{L^l_\rho(\mathbb{R}^n)} + \|\nabla u_i\|^l_2\right), \\ &\leq \varepsilon c(E(t) + E^{l/2}(t)), \\ &\leq \varepsilon cE(t)(1 + E^{[(l/2)-1]}(t)), \\ &\leq \varepsilon cE(t)(1 + E^{[(l/2)-1]}(0)), \\ &\leq \varepsilon cE(t). \end{aligned}$$

Consequently, (3.45) follows. ■

Proof of Theorem 3.2 From (3.4), results of Lemma 3.3, we have

$$\begin{aligned} L'(t) &= N_1 E'(t) + \varepsilon \psi'(t), \\ &\le N_1 \Big(\frac{1}{2} \sum_{i=1}^{2} (g_i' \circ \nabla u_i)(t) - \sum_{i=1}^{2} \|\nabla u_i'\|_2^2 \Big) \\ &+ \varepsilon \sum_{i=1}^{2} \Big(\|u_i'\|_{L^l_\rho(\mathbb{R}^n)}^l - (k - 1 - \delta + |\alpha| c) \|\nabla u_i\|_2^2 + c(g_i \circ \nabla u_i) \Big), \end{aligned}$$

At this point, we choose N_1 large and ε so small such that

$$L'(t) \le M_0 \sum_{i=1}^{2} (g_i \circ \nabla u_i) - \varepsilon E(t), \quad \forall t \ge 0. \tag{3.47}$$

Multiplying (5.23) by $\xi(t)$ gives

$$\xi(t) L'(t) \le -\varepsilon \xi(t) E(t) + M_0 \xi(t) \sum_{i=1}^{2} (g_i \circ \nabla u_i). \tag{3.48}$$

The last term can be estimated, using $(A1)$ as follows

$$\begin{aligned} \xi(t) \sum_{i=1}^{2} (g_i \circ \nabla u_i) &\le \sum_{i=1}^{2} \xi_i(t) \int_{\mathbb{R}^n} \int_0^t g_i(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le \sum_{i=1}^{2} \int_{\mathbb{R}^n} \int_0^t \xi_i(t-s) g_i(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le -\sum_{i=1}^{2} \int_{\mathbb{R}^n} \int_0^t g_i'(t-s) |u_i(t) - u_i(s)|^2 ds dx, \\ &\le -\sum_{i=1}^{2} (g_i' \circ \nabla u_i) \le -E'(t). \end{aligned} \tag{3.49}$$

Thus, (5.23) becomes

$$\xi(t) L'(t) + M_0 E'(t) \le -\varepsilon \xi(t) E(t) \quad \forall t \ge 0. \tag{3.50}$$

Using the fact that ξ is a nonincreasing continuous function as ξ_1 and ξ_2 are nonincreasing and so ξ is differentiable, with $\xi'(t) \le 0$ for a.e t, then

$$(\xi(t) L(t) + M_0 E(t))' \le \xi(t) L'(t) + M_0 E'(t) \le -\varepsilon \xi(t) E(t) \quad \forall t \ge 0. \tag{3.51}$$

Since, using (3.45)

$$F = \xi L + M_0 E \sim E, \tag{3.52}$$

we obtain, for some positive constant ω

$$F'(t) \leq -\omega\xi(t)F(t) \quad \forall t \geq 0. \tag{3.53}$$

Integration over $(0,t)$ leads to, for some constant $\omega > 0$ such that

$$F(t) \leq WF(0)\exp\left(-\omega\int_0^t \xi(s)ds\right), \forall t \geq 0. \tag{3.54}$$

Recalling (3.52), estimate (3.54) yields the desired result (3.43).
This completes the proof of Theorem (3.2).

3.5 Concluding comments

1- One can easily obtain the same result in Theorem (3.2) in the nonlinear case

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1,u_2) - \phi(x)\Delta\left(u_1 + \int_0^t g_1(s)u_1(t-s,x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1,u_2) - \phi(x)\Delta\left(u_2 + \int_0^t g_2(s)u_2(t-s,x)ds + u_2'\right) = 0, \\ (u_1(0,x), u_2(0,x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0,x), u_2'(0,x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{3.55}$$

where our nonlinearity is given by the functions f_1, f_2 satisfying the next assumptions:
(***hyp1***) The functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ (for i=1,2) is of class C^1 and there exists a function F such that

$$f_1(x,y) = \frac{\partial F}{\partial x}, \qquad f_2(x,y) = \frac{\partial F}{\partial y}, \tag{3.56}$$

$$F \geq 0,\ xf_1(x,y) + yf_2(x,y) - F(x,y) \geq 0. \tag{3.57}$$

and

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}) \quad \forall (x,y) \in \mathbb{R}^2, \tag{3.58}$$

for some constant $d > 0$ and $1 \leq \beta_{ij} \leq \frac{n}{n-2}$ for $i,j = 1,2$.
(***hyp2***) There exists a positive constant k such that

$$|f_i(x,y)| \leq k(|x| + |y| + |x|^{\beta_{i1}} + |y|^{\beta_{i2}}), \tag{3.59}$$

and

$$\begin{aligned} &|f_i(x,y) - f_i(r,s)| \\ &\leq k(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1} + |r|^{\beta_{i1}-1} + |s|^{\beta_{i2}-1})(|x-r| + |y-s|), \end{aligned} \tag{3.60}$$

for all $(x, y), (r, s) \in \mathbb{R}^2$ and $i = 1, 2$. Noting that we follow the same steps in the linear cases with the same perturbed function and some calculations related with the presence of f_1, f_2.
2. Let us remark that, it is similar to study the question of existence and decay of solution of the same problem with the presence of weak-viscoelasticity in the form

$$\begin{cases} \left(|u_1'|^{l-2}u_1'\right)' + f_1(u_1, u_2) - \phi(x)\Delta\left(u_1 + \alpha_1(t)\int_0^t g_1(s)u_1(t-s, x)ds + u_1'\right) = 0, \\ \left(|u_2'|^{l-2}u_2'\right)' + f_2(u_1, u_2) - \phi(x)\Delta\left(u_2 + \alpha_2(t)\int_0^t g_2(s)u_2(t-s, x)ds + u_2'\right) = 0, \\ (u_1(0, x), u_2(0, x)) = (u_{10}(x), u_{20}(x)) \in (D(\mathbb{R}^n))^2, \\ (u_1'(0, x), u_2'(0, x)) = (u_{11}(x), u_{21}(x)) \in (L^l_\rho(\mathbb{R}^n))^2, \end{cases} \tag{3.61}$$

where we should need additional, conditions on α as follows

$$1 - \alpha_i(t)\int_0^t g_i(t)dt \geq k_i > 0, \int_0^\infty g_i(t)dt < +\infty, \alpha_i(t) > 0, \tag{3.62}$$

$$\lim_{t\to+\infty} \frac{-\alpha'(t)}{\alpha(t)\xi(t)} = 0 \tag{3.63}$$

where

$$\alpha(t) = \min\{\alpha_1(t), \alpha_2(t)\}, \quad \forall t \geq 0.$$

Under this additional conditions on α, by using the Lemma 1.2, the decay of energy associate with problem (3.61) is given in the next result

Theorem 3.3 *Let $(u_{i0}, u_{i1}) \in (D(\mathbb{R}^n) \times L^l_\rho(\mathbb{R}^n)), i = 1, 2$ and suppose that (A1), (A2), (3.56)-(3.60) hold. Then there exist positive constants W, ω such that the energy of solution given by (3.61) satisfies,*

$$E(t) \leq WE(t_0)\exp\left(-\omega\int_{t_0}^t \alpha(s)\xi(s)ds\right), \tag{3.64}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad \forall t \geq t_0 \geq 0.$

3.5. Concluding comments

Chapter 4

Transmission system in thermoelasticity with infinite memories

This Chapter describes a polynomial decay rate of solution for a transmission problem with $1-d$ mixed type I and type II thermoelastic system with infinite memories acting in the first and second parts. The main contribution here is to show that the t^{-1} is the sharp decay rate of our problem (4.1). That is to show that for this types of materials the dissipation produced by the infinite memories are not strong enough to produce an exponential decay of the solution.

4.1 Introduction and Previous Stability Results

In the present paper, we consider a transmission problem with $1-d$ mixed type I and type II thermoelastic system and memories terms for $t > 0$ in the following:

$$\begin{cases} \rho_1 u'' - a_1\Big(u_{xx} - \int_{-\infty}^{t} \mu_1(t-s)u_{xx}(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L,0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L,0), \\ \rho_2 v'' - a_2\Big(v_{xx} - \int_{-\infty}^{t} \mu_2(t-s)v_{xx}(s)ds\Big) + \beta_2 q_x = 0, & x \in (0,L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0,L), \\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \end{cases} \tag{4.1}$$

where u, v are the displacement of the system at time t in $(-L, 0)$ and $(0, L)$ and θ, q are respectively the temperature difference with respect to a fixed reference temperature, w_1, w_2 are the so-called thermal displacement, which satisfies

$$w_1(., t) = \int_0^t \theta(., s)ds + w_1(., 0)$$

and

$$w_2(., t) = \int_0^t q(., s)ds + w_2(., 0).$$

The parameters $a_1, a_2, \rho_1, \rho_2, \beta_1, \beta_2, c_1.c_2, k, l$ and $L < \infty$ are assumed to be positive constants.

The system (4.1) satisfies the Dirichlet boundary conditions:

$$\begin{cases} u(-L, t) = v(L, t) = 0, & t > 0, \\ \\ w_1(-L, t) = w_2(L, t) = 0, & t > 0, \end{cases} \tag{4.2}$$

and the following initial conditions:

$$\begin{cases} u(., 0) = u^0(x), u'(., 0) = u^1(x), w_1(., 0) = w_1^0(x), \theta(., 0) = \theta^0(x) \\ \\ v(., 0) = v^0(x), v'(., 0) = v^1(x), w_2(., 0) = w_2^0(x), q(., 0) = q^0(x). \end{cases} \tag{4.3}$$

We treat the infinite memories as Dafermos [15], [16] adding a new variables η_1, η_2 to the system which corresponds to the relative displacement history. Let us define the auxiliary variables

$$\eta_1 = \eta_1^t(x, s) = u(x, t) - u(x, t - s), \quad (x, s) \in (-L, 0) \times \mathbb{R}^+.$$

and

$$\eta_2 = \eta_2^t(x, s) = v(x, t) - v(x, t - s), \quad (x, s) \in (0, L) \times \mathbb{R}^+.$$

By differentiation we have

$$\frac{d}{dt}\eta_1^t(x, s) = -\frac{d}{ds}\eta_1^t(x, s) + \frac{d}{dt}u(x, t), \quad (x, s) \in (-L, 0) \times \mathbb{R}^+,$$

and

$$\frac{d}{dt}\eta_2^t(x, s) = -\frac{d}{ds}\eta_2^t(x, s) + \frac{d}{dt}v(x, t), \quad (x, s) \in (0, L) \times \mathbb{R}^+,$$

and we can take as initial condition $(t = 0)$

$$\eta_1^0(x, s) = u^0(x, 0) - u^0(x, -s), \quad (x, s) \in (-L, 0) \times \mathbb{R}^+.$$

and

$$\eta_2^0(x,s) = v^0(x,0) - v^0(x,-s), \quad (x,s) \in (0,L) \times \mathbb{R}^+.$$

Thus, the original memories terms can be rewritten as

$$\begin{array}{rcl} \int_{-\infty}^t \mu_1(t-s)u_{xx}(s)ds &=& \int_0^\infty \mu_1(s)u_{xx}(t-s)ds \\ &=& \left(\int_0^\infty \mu_1(t)dt\right) u_{xx} - \int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds. \end{array}$$

and

$$\begin{array}{rcl} \int_{-\infty}^t \mu_2(t-s)v_{xx}(s)ds &=& \int_0^\infty \mu_2(s)v_{xx}(t-s)ds \\ &=& \left(\int_0^\infty \mu_2(t)dt\right) v_{xx} - \int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds. \end{array}$$

The problem (4.1) is transformed into the system

$$\begin{cases} \rho_1 u'' - a_1\Big(\mu_{01}u_{xx} + \int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L,0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L,0), \\ \rho_2 v'' - a_2\Big(\mu_{02}v_{xx} + \int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds\Big) + \beta_2 q_x = 0, & x \in (0,L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0,L), \\ \frac{d}{dt}\eta_1^t(x,s) + \frac{d}{ds}\eta_1^t(x,s) - \frac{d}{dt}u(x,t) = 0, & x \in (-L,0), \\ \frac{d}{dt}\eta_2^t(x,s) + \frac{d}{ds}\eta_2^t(x,s) - \frac{d}{dt}v(x,t) = 0, & x \in (0,L), \\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \\ \eta_1^0(x,s) = u^0(x,0) - u^0(x,-s), s > 0 \\ \eta_2^0(x,s) = v^0(x,0) - v^0(x,-s), s > 0 \end{cases} \tag{4.4}$$

where $\mu_{0i} = 1 - \int_0^\infty \mu_i(t)dt, \qquad i = 1,2.$

The stability of various transmission problems on thermoelasticity have been considered [43], [13], [12] [19], [17], [42], [45], [46] and [52]. The transmission problem to hyperbolic equations was studied by Dautray and Lions [14] where the existence and regularity of solutions for the linear problem have been proved. In [45], the authors considered the transmission problem of viscoelastic waves

$$\begin{cases} \rho_1 u'' - \alpha_1 u_{xx} = 0, & x \in (0,L_0), \\ \rho_2 v'' - \alpha_2 v_{xx} + \int_0^t g(t-s)v_{xx}(s)ds = 0, & x \in (L_0,L), \end{cases} \tag{4.5}$$

satisfying boundary conditions and initial conditions. The authors studied the wave propagations over materials consisting of elastic and viscoelastic components. They showed that the viscoelastic part produce exponential decay of the solution. In [40], the authors investigated a 1D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero. Marzocchi et al [41] studied a multidimensional linear thermoelastic transmission problem. An existence and regularity result has been proved. When the solution is supposed to be spherically symmetric, the authors established an exponential decay result similar to [40]. Next, Rivera and *all* [46], considered a transmission problem in thermoelasticity with memory. As time goes to infinity, they showed the exponential decay of the solution in case of radially symmetric situations. We must mention the pioneer work by Rivera and *all* in [17] where a semilinear transmission problem for a coupling of an elastic and a thermoelastic material is considered. The heat conduction is modeled by Cattaneo's law removing the physical paradox of infinite propagation speed of signals. The damped, totally hyperbolic system is shown to be exponentially stable. In 2009, Mesaoudi and *all* [42] proposed and studied a 1D linear thermoelastic transmission problem, where the heat conduction is described by the theories of Green and Naghdi. By using the energy method, they proved that the thermal effect is strong enough to produce an exponential stability of the solution.

The earliest result in this direction was established by [56], where the dynamical behavior of the system is described by

$$\begin{cases} \rho_1 u_1'' - a_1 u_{1,xx} + \beta_1 \theta_{1,x} = 0, & x \in (-1,0), \\ c_1 \tau_1'' - b\theta_{1,xx} + \beta_1 u'_{1,x} = 0, & x \in (-1,0), \\ \rho_2 u_2'' - a_2 u_{2,xx} + \beta_2 \theta_{2,x} = 0, & x \in (0,1), \\ c_2 \tau_2'' - k\tau_{2,xx} + \beta_2 u'_{2,x} = 0, & x \in (0,1), \end{cases} \tag{4.6}$$

the system consists of two kinds of thermoelastic components, one is of type I, another one is of type II. Under certain transmission conditions, these two components are coupled at the interface. The authors proved that the system is lack of exponential decay rate and further obtain the sharp polynomial decay rate. Our paper is devoted to show that our system can achieve polynomial decay rate. That is, our main result here is to show that for this types of materials the dissipation produced by the viscoelastic part is not strong enough to produce an exponential decay of the solution despite that the infinite memory satisfies assumptions (4.7) and (4.8)

4.2 Preliminaries and the semi-group approach

For simplicity reason denote $u(x,t) = u, v(x,t) = v, w_i(x,t) = w_i, i = 1,2, q(x,t) = q$, when there is no confusion. Here $u' = du(t)/dt, v' = dv(t)/dt$ and $u'' = d^2u(t)/dt^2, v'' = d^2v(t)/dt^2, w_i'' = d^2w_i(t)/dt^2, i = 1,2$.
First we recall and make use the following assumptions on the functions μ_i as:
We assume that the function $\mu_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$1 - \int_0^\infty \mu_i(t)dt = \mu_{0i} > 0, i = 1,2 \quad \forall t \in \mathbb{R}^+ \tag{4.7}$$

and that there exists a constants $k_1 > 0$ such that

$$\mu_i'(t) + k_1\mu_i(t) \le 0 \quad \forall t \in \mathbb{R}^+, i = 1,2 \tag{4.8}$$

We denote by $\mathcal{A}$ the unbounded operator in an appropriate Hilbert state space:
Let

$$V^k(0,L) = \{h \in H^k(0,L); h(L) = 0\}$$

and

$$V^k(-L,0) = \{h \in H^k(-L,0); h(-L) = 0\}.$$

Let

$$\begin{aligned} \mathcal{H} &= V^1(-L,0) \times L^2(-L,0) \times L^2(-L,0) \times \\ &\times V^1(0,L) \times L^2(0,L) \times V^1(0,L) \times L^2(0,L), \end{aligned} \tag{4.9}$$

equipped, for $(u, u^1, \theta, v, v^1, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{w}_2, \tilde{q}) \in \mathcal{H}$ with an inner product

$$\Big\langle (u, u^1, \theta, v, v^1, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{w}_2, \tilde{q}) \Big\rangle_{\mathcal{H}}$$

$$=$$

$$\int_{-L}^0 \Big[a_1\Big(\mu_{01}u_x + \int_0^t \mu_2(s)\eta_{1,x}^t(s)ds\Big)\overline{\tilde{u}_x} + \rho_1 u^1\overline{\tilde{u^1}} + c_1\theta\overline{\tilde{\theta}}\Big]dx$$

$$+ \int_0^L \Big[a_2\Big(\mu_{02}v_x + \int_0^t \mu_2(s)\eta_{2,x}^t(s)ds\Big)\overline{\tilde{v}_x} + \rho_2 v^1\overline{\tilde{v^1}} + kw_{2,x}\overline{\tilde{w_{2,x}}} + c_2 q_x\overline{\tilde{q}_x}\Big]dx.$$

with domain

$$\mathcal{D}(\mathcal{A}) = (u, u^1, \theta, v, v^1, w_2, q) \in \mathcal{H} : \begin{cases} u, \theta \in H^2(-L,0), u^1 \in H^1(-L,0), \\ v \in H^2(0,L), v^1, q \in H^1(0,L), w_2 \in H^2(0,L), \\ \theta(-L) = q(L) = 0, l\theta_x(0) = kw_{2,x}(0) \\ a_1\mu_0 u_x(0) - \beta_1\theta(0) = a_2 v_x(0) - \beta_2 q(0) \\ u(0) = v(0), \theta(0) = q(0), \end{cases} \tag{4.10}$$

and

$$\mathcal{A}\begin{pmatrix} u \\ u^1 \\ \theta \\ v \\ v^1 \\ w_2 \\ q \end{pmatrix} = \begin{pmatrix} u^1 \\ \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds\Big) - \beta_1\theta_x\Big) \\ c_1^{-1}\Big(-\beta_1 u_x^1 + l\theta_{xx}\Big) \\ v^1 \\ \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds\Big) - \beta_2 q_x\Big) \\ q \\ c_2^{-1}\Big(-\beta_2 v_x^1 + kw_{2,xx}\Big) \end{pmatrix} \tag{4.11}$$

Lemma 4.1 *Let $\mathcal{A}$ and $\mathcal{H}$ be given in (4.9) and (4.11). Then, the operator $\mathcal{A}$ is dissipative in $\mathcal{H}$.*

Proof. Let $W = (u, u^1, \theta, v, v^1, w_2, q)^T$, then it is note hard to see that

$$\begin{aligned}
\mathcal{R}(\mathcal{A}W, W)_{\mathcal{H}} &= \mathcal{R}\Big(\int_{-L}^{0} \rho_1^{-1} a_1 \Big(\mu_{01} u_x^1 + \int_0^\infty \mu_1(s) \eta_{1,x}^t(s) ds\Big) \overline{u_x} dx \\
&+ \int_{-L}^{0} \rho_1^{-1} \Big(a_1 \Big(\mu_{01} u_x + \int_0^\infty \mu_1(s) \eta_{1,x}^t(s) ds\Big) - \beta_1 \theta\Big)_x \overline{u^1} dx \\
&+ \int_{-L}^{0} c_1^{-1} \Big(l \theta_{xx} - \beta_1 u_x'\Big) \overline{\theta} dx \\
&+ \int_0^L \rho_2^{-1} a_2 \Big(\mu_{02} v_x^1 + \int_0^\infty \mu_2(s) \eta_{2,x}^t(s) ds\Big) \overline{v_x} dx \\
&+ \int_0^L \rho_2^{-1} \Big(a_2 \Big(\mu_{02} v_x + \int_0^\infty \mu_2(s) \eta_{2,x}^t(s) ds\Big) - \beta_2 q\Big)_x \overline{v^1} dx \\
&+ \int_0^L \Big(k w_{2,xx} - \beta_2 v_x^1\Big) \overline{q_x} dx + \int_0^L k q_x \overline{w_{2,x}} dx\Big) \\
&= \mathcal{R}\Big(\int_{-L}^{0} \rho_1^{-1} a_1 \Big(\mu_{01} u_x^1 + \int_0^\infty \mu_1(s) \eta_{1,x}^t(s) ds\Big) \overline{u_x} dx \\
&- \int_{-L}^{0} \rho_1^{-1} \Big(a_1 \Big(\mu_{01} u_x + \int_0^\infty \mu_1(s) \eta_{1,x}^t(s) ds\Big)\Big) \overline{u_x^1} dx \\
&+ \int_{-L}^{0} c_1^{-1} l \theta_{xx} \overline{\theta} dx + \int_{-L}^{0} c_1^{-1} \beta_1 u^1 \overline{\theta_x} dx - \int_{-L}^{0} \rho_1^{-1} \beta_1 \theta_x \overline{u_x^1} dx \\
&+ \int_0^L \rho_2^{-1} a_2 \Big(\mu_{02} v_x^1 + \int_0^\infty \mu_2(s) \eta_{2,x}^t(s) ds\Big) \overline{v_x} dx \\
&- \int_0^L \rho_2^{-1} a_2 \Big(\mu_{02} v_x + \int_0^\infty \mu_2(s) \eta_{2,x}^t(s) ds\Big) \overline{v_x^1} dx \\
&+ \int_0^L \Big(k w_{2,xx} - \beta_2 v_x^1\Big) \overline{q_x} dx + \int_0^L k q_x \overline{w_{2,x}} dx - \int_0^L \rho_2^{-1} \beta_2 q_x \overline{v^1} dx\Big) \\
&= \mathcal{R}\Big(\int_{-L}^{0} c_1^{-1} l \theta_{xx} \overline{\theta} dx + \int_0^L k w_{2,xx} \overline{q_x} dx + \int_0^L k q_x \overline{w_{2,x}} dx\Big) \\
&= -c_1^{-1} l \int_{-L}^{0} |\theta_x|^2 dx \\
&\leq 0. \qquad (4.12)
\end{aligned}$$

Then, (4.12) means that the operator $\mathcal{A}$ is dissipative in $\mathcal{H}$. ■ In the next Theorem, we shall prove that the operator (4.11) generates a C_0 semigroup of contractions on

$\mathcal{H}$.

Theorem 4.1 *Let $\mathcal{A}$ and $\mathcal{H}$ be given in (4.9) and (4.11). Then, $\mathcal{A}$ generates a C_0 semi-group $S(t)$ of contractions on $\mathcal{H}$.*

Proof. For any

$$F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H},$$

the equation

$$\mathcal{A}W = F$$

has a unique solution $W = (u, u^1, \theta, v, v^1, w_2, q)^T \in \mathcal{D}(\mathcal{A})$ satisfying the transmission and boundary conditions. Then, using Lemma 4.1 and Sobolev embedding theorem, one obtains $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, the Lumer-Phillips theorem (see [50]) gives the result. This completes the proof. ■ For $\mathcal{U} = (u, u^1, \theta, v, v^1, w_2, q)^T$, the problem (4.4) can then be reformulated under the abstract from

$$\mathcal{U}' = \mathcal{A}\mathcal{U}, \tag{4.13}$$

where $\mathcal{U}(0) = (u^0, u^1, \theta^0, v^0, v^1, w_2^0, q^0)^T \in \mathcal{H}$ is given.
The following is the well-known Gearhart-Herbst-Pruss-Huang theorem for dissipative systems. We will use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [24] and Huang [22]

Theorem 4.2 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\zeta : \zeta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta|\to\infty}} \|(i\zeta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

4.3 Lack of Exponential Stability

Following the techniques in [4], it is easy to check that $(\mathcal{H}, \|.\|_{\mathcal{H}})$ is a Hilbert space. In this section we prove the lack of exponential decay using Theorem 4.2, that is we show that there exists a sequence of values h_m such that

$$\|(ih_m I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{4.14}$$

It is equivalent to prove that there exist a sequence of data $F_m \in \mathcal{H}$ and a sequence of real numbers $h_m \in \mathbb{R}$, with $\|F_m\|_{\mathcal{H}} \leq 1$ such that

$$\|(ih_m I - \mathcal{A})^{-1} F_m\|_{\mathcal{H}} = \|U_m\|^2_{\mathcal{H}} \to \infty. \tag{4.15}$$

Theorem 4.3 *Assume that the kernel satisfying hypothesis (4.7) and (4.8). The semi group $S(t)$ on $\mathcal{H}$ is not exponentially stable.*

Proof. We will find a sequence of bounded functions

$$F_m = (f_{1,m}, f_{2,m}, f_{3,m}, f_{4,m}, f_{5,m}, f_{6,m}, f_{7,m}, f_{8,m}, f_{9,m})^T \in \mathcal{H}, h \in \mathbb{R},$$

for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. We consider the spectral equation

$$ihU_m - \mathcal{A}U_m = F_m.$$

and show that the corresponding solution U_m is not bounded when F_m is bounded in $\mathcal{H}$. Rewriting the spectral equation in term of its components, we get

$$\begin{cases} ihu - u^1 = f_{1m} \\ ih\rho_1 u^1 - \Big(a_1\Big(\mu_{01}u_{xx} + \int_0^\infty \mu_1(s)\eta^t_{1,xx}(s)ds\Big) - \beta_1\theta_x\Big) = \rho_1 f_{2m} \\ ihc_1\theta - \Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = c_1 f_{3m} \\ ihv - v^1 = f_{4m} \\ ih\rho_2 v^1 - \Big(a_2\Big(\mu_{02}v_{xx} + \int_0^\infty \mu_2(s)\eta^t_{2,xx}(s)ds\Big) - \beta_2 q_x\Big) = \rho_2 f_{5m} \\ ihw_2 - q = f_{6m} \\ ihc_2 q - \Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = c_2 f_{7m} \\ ih\eta^t_1 - u^1 + \eta^t_{1,s} = f_{8m} \\ ih\eta^t_2 - v^1 + \eta^t_{2,s} = f_{9m} \end{cases} \tag{4.16}$$

We prove that there exists a sequence of real numbers h_m so that (4.16) verified Let us consider $f_{1m} = f_{4m} = f_{6m} = f_{8m} = f_{9m} = 0$ and using the equations to eliminate the terms u^1, v^1 and chose $f_{2m} = f_{3m} = f_{5m} = f_{6m} = \lambda_m$ to obtain $u^1 = ihu$, $v^1 = ihv$ and $q = ihw_2$. Then, system (4.16) becomes

$$\begin{cases} -h^2 u - \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \int_0^\infty \mu_1(s)\eta^t_{2,xx}(s)ds\Big) - \beta_1\theta_x\Big) = \lambda_m \\ ih\theta - c_1^{-1}\Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = \lambda_m \\ -h^2 v - \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \int_0^\infty \mu_2(s)\eta^t_{2,xx}(s)ds\Big) - \beta_2 ihw_{2,x}\Big) = \lambda_m \\ -h^2 w_2 - c_2^{-1}\Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = \lambda_m \\ ih\eta^t_1 - ihu + \eta^t_{1,s} = 0 \\ ih\eta^t_2 - ihv + \eta^t_{2,s} = 0 \end{cases} \tag{4.17}$$

We look for solutions of the form

$$u = a\lambda_m, v = b\lambda_m, \theta = c\lambda_m, w_2 = d\lambda_m, u^1 = e\lambda_m,$$

$$v^1 = f\lambda_m, \eta_1^t(x,s) = \gamma(s)_1\lambda_m, \eta_2^t(x,s) = \gamma_2(s)\lambda_m$$

with $a, b, c, d, e, f \in \mathbb{C}$ and $\gamma_1(s), \gamma_2(s)$ depend on h and will be determined explicitly in what follows. From (4.17), we get a, b, c, d, e and f satisfy

$$\begin{cases} -h^2 a - \rho_1^{-1}\Big(a_1 h_m\Big(\mu_{01}a + \int_0^\infty \mu_1(s)\gamma_1(s)ds\Big) - \beta_1 ch\Big) = 1, \\ ihc - c_1^{-1}\Big(-\beta_1 e + lh_m c\Big) = 1, \\ -h^2 b - \rho_2^{-1}\Big(a_2 h_m\Big(\mu_{02}b + \int_0^\infty \mu_2(s)\gamma_2(s)ds\Big) - \beta_2 ihd\Big) = 1, \\ ihd - c_2^{-1}\Big(-\beta_2 f + kh_m d\Big) = 1, \\ \gamma_{1,s} + ih\gamma_1 - iha = 0. \\ \gamma_{2,s} + ih\gamma_2 - ihb = 0. \end{cases} \tag{4.18}$$

From $(4.18)_5$ and $(4.18)_6$ we get

$$\gamma_1(s) = a - ae^{-ihs}, \tag{4.19}$$

and

$$\gamma_2(s) = b - be^{-ihs}. \tag{4.20}$$

Then, from (4.19), (4.20) we have

$$\begin{aligned} \int_0^\infty \mu_1(s)\gamma_1(s)\,ds &= \int_0^\infty \mu_1(s)(a - ae^{-ihs})ds \\ &= a\int_0^\infty \mu_1(s)ds - a\int_0^\infty \mu_1(s)ae^{-ihs}ds \\ &= a(1-\mu_{01}) - a\int_0^\infty \mu_1(s)e^{-ihs}\,ds. \end{aligned} \tag{4.21}$$

and

$$\int_0^\infty \mu_2(s)\gamma_2(s)\,ds = b(1-\mu_{02}) - b\int_0^\infty \mu_2(s)e^{-ihs}\,ds. \tag{4.22}$$

Now, we would like to find the parameters constants. To this end, choosing

$$c_1 ih = h_m l, \qquad c_2 ih = kh_m, \tag{4.23}$$

using equations $(4.18)_2$ and $(4.18)_4$, we obtain

$$e = \frac{c_1}{\beta_1}, \tag{4.24}$$

$$f = \frac{c_2}{\beta_2}. \tag{4.25}$$

By equations (4.18)$_1$ and (4.18)$_3$, we have

$$c = \frac{1}{(-h^2 - \rho_1^{-1} h_m a_1 \mu_{01})}\Big(1 + \rho_1^{-1} h_m a_1 \int_0^\infty \mu_1(s)\gamma_1(s)ds - \rho_1^{-1} h_m \beta_1 c\Big),$$

$$d = \frac{1}{(-h^2 - \rho_2^{-1} h_m a_2 \mu_{02})}\Big(1 + \rho_2^{-1} h_m a_2 \int_0^\infty \mu_2(s)\gamma_2(s)ds - \rho_1^{-1} h_m \beta_2 c\Big).$$

Recalling from (4.24), (4.25) that

$$\begin{aligned} u^1 + v^1 &= e\lambda_m + f\lambda_m \\ &= \frac{c_1}{\beta_1}\lambda_m + \frac{c_2}{\beta_2}\lambda_m, \end{aligned}$$

we get

$$\|u^1\|_2^2 + \|v^1\|_2^2 = \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2.$$

Therefore we have

$$\begin{aligned} \lim_{m\to\infty} \|U_m\|_{\mathcal{H}}^2 &\geq \lim_{m\to\infty} [\|u^1\|_2^2 + \|v^1\|_2^2] \\ &= \lim_{m\to\infty} \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2 \\ &= +\infty \end{aligned}$$

which completes the proof. ■

4.4 Polynomial Stability

Lemma 4.2 *[7], [38], [56] For some constant $C > 0$, a C_0 semigroup $S(t) = e^{t\mathcal{A}}$ of contractions on a Hilbert space satisfies*

$$\|S(t)W_0\| \leq Ct^{-1}\|W_0\|_{\mathcal{D}(\mathcal{A})}, \forall W_0 \in \mathcal{D}(\mathcal{A}), t \to \infty,$$

if and only if the following conditions hold

1. $\rho(\mathcal{A}) \supseteq i\mathbb{R}$
2. $\lim_{\zeta\to\infty} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty$

Our main result reads as follows.

Theorem 4.4 *Assume that (4.7) and (4.8) hold. Then t^{-1} is the sharp decay rate. Therefore, the decay rate of the system cannot be faster than t^{-1}.*

Proof. 1. Using proof by contradiction. For this purpose, we assume that there exists $\tilde{\lambda} = i\tilde{\xi} \in \delta(\mathcal{A}), \tilde{\delta} \in \mathbb{R}, \tilde{\delta} \neq 0$ on the imaginary axis and $\tilde{W} = (u, u^1, \theta, v, v^1, w_2, q) \in \mathcal{D}(\mathcal{A})$ is the eigenvector corresponding to $\tilde{\lambda}$. Then,

$$\tilde{\lambda}u = u^1, \tag{4.26}$$

$$\tilde{\lambda}u^1 = \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \int_0^\infty \mu_1(s)\eta^t_{1,xx}(s)ds\Big) - \beta_1\theta_x\Big), \tag{4.27}$$

$$\tilde{\lambda}\theta = c_1^{-1}\Big(- \beta_1 u^1_x + l\theta_{xx}\Big), \tag{4.28}$$

$$\tilde{\lambda}v = v^1, \tag{4.29}$$

$$\tilde{\lambda}v^1 = \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \int_0^\infty \mu_2(s)\eta^t_{2,xx}(s)ds\Big) - \beta_2 q_x\Big), \tag{4.30}$$

$$\tilde{\lambda}w_2 = q, \tag{4.31}$$

$$\tilde{\lambda}q = c_2^{-1}\Big(- \beta_2 v^1_x + kw_{2,xx}\Big). \tag{4.32}$$

Since $\mathcal{A}$ is dissipative by Lemma 4.1, we have $\mathcal{R}(\mathcal{A}\tilde{W}, \tilde{W}) = -c_1^{-1}l\int_{-L}^0 |\theta_x|^2dx = 0$, which yields $\theta_x = \theta_{xx} = 0$, then by (4.28), we have $u^1_x = 0$, then $u = u^1 = 0$. Hence $(u, u^1, w_2, q) = 0$ which contradicts the fact that $\tilde{W} = 0$ is an eigenvector. This completes the proof.

2. We would now show that

$$\lim_{\zeta\to\infty} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty \tag{4.33}$$

We prove that there at least exists a sequence

$$V_n = (u_n, u^1_n, \theta_n, v_n, v^1_n, w_{2,n}, q_n) \in \mathcal{D}(\mathcal{A}),$$

with $\|V_n\|_{\mathcal{H}} = 1$, and a sequence $\zeta_n \in \mathbb{R}$ with $\zeta_n \to \infty$ such that

$$\lim_{n\to\infty} \zeta_n\|(i\zeta_n I - \mathcal{A})V_n\|_{\mathcal{H}} = 0$$

or, for $\Omega_1 = (-L, 0), \Omega_2 = (0, L)$

$$\zeta_n(i\zeta_n u_n - u_n^1) \to 0, \text{ in } H^1(\Omega_1), \tag{4.34}$$

$$\zeta_n\Big(i\zeta_n u_n^1 - \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{n,xx} + \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds\Big) - \beta_1\theta_{n,x}\Big)\Big) \to 0, \text{ in } L^2(\Omega_1),$$

$$\zeta_n\Big(i\zeta_n\theta_n - c_1^{-1}\Big(-\beta_1 u^1_{n,x} + l\theta_{n,xx}\Big)\Big) \to 0, \text{ in } L^2(\Omega_1), \tag{4.35}$$

$$\zeta_n(i\zeta_n v_n - v_n^1) \to 0, \text{ in } H^1(\Omega_2), \tag{4.36}$$

$$\zeta_n\Big(i\zeta_n v_n^1 - \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{n,xx} + \int_0^\infty \mu_2(s)\eta^t_{2,n,xx}(s)ds\Big) - \beta_2 q_{n,x}\Big)\Big) \to 0, \text{ in } L^2(\Omega_2),$$

$$\zeta_n(i\zeta_n w_{2,n} - q_n) \to 0, \text{ in } H^1(\Omega_2), \tag{4.37}$$

$$\zeta_n\Big(i\zeta_n q_n - c_2^{-1}\Big(-\beta_2 v^1_{n,x} + kw_{2,n,xx}\Big)\Big) \to 0, \qquad in \quad L^2(\Omega_2), \tag{4.38}$$

$$ih\eta_1^t - u^1_{1,n} + \eta^t_{1,s} = 0 \tag{4.39}$$

$$ih\eta_2^t - v^1_{1,n} + \eta^t_{2,s} = 0 \tag{4.40}$$

Noting that

$$Re\langle\zeta_n(i\zeta_n - \mathcal{A})V_n, V_n\rangle_{\mathcal{H}} = \zeta_n\|\sqrt{l} \quad \theta_{n,x}\|^2_{L^2} \to 0.$$

Then

$$\sqrt{\zeta_n} \quad \theta_{n,x} \to 0, \quad in \quad L^2(-L, 0). \tag{4.41}$$

By Poincaré's inequality, we get

$$\sqrt{\zeta_n} \quad \theta_n \to 0, \quad in \quad L^2(-L, 0). \tag{4.42}$$

Thanks to the Gagliardo-Nirenberg inequality, we have

$$\|\sqrt{\zeta_n} \quad \theta_n\|_{L^\infty} \le C_1\sqrt{\|\sqrt{\zeta_n} \quad \theta_{n,x}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n} \quad \theta_n\|_{L^2}} + C_2\|\sqrt{\zeta_n} \quad \theta_n\|_{L^2}. \tag{4.43}$$

Thus,

$$\sqrt{\zeta_n} \quad \theta_n(0) \to 0. \tag{4.44}$$

From (4.34), we have $\beta_1(i\zeta_n)^{-1}u^1_{n,x}$ is bounded in $L^2(-L,0)$. By (4.35) we have the boundedness of $(i\zeta_n)^{-1}\theta_{n,xx}$ in $L^2(-L,0)$.
Using again the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}\|_{L^\infty} &\leq d_1\sqrt{\|(\zeta_n)^{-1}\quad \theta_{n,xx}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n}\quad \theta_{n,x}\|_{L^2}} + d_2\|\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}\|_{L^2}\\ &\to 0.\end{aligned}$$

which gives

$$\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(-L)\to 0, \qquad \left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(0)\to 0. \tag{4.45}$$

Multiplying equation after (4.34) by $p(x)u_{n,x}$ in L^2-norm for $p(x)\in C^1[-L,0]$, to get

$$\begin{aligned}&-\zeta_n^2\langle u_n, p(x)u_{n,x}\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle \mu_{01}u_{n,xx}, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\ &-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\ &+\rho_1^{-1}\beta_1\langle\theta_{n,x}, p(x)u_{n,x}\rangle_{L^2(-L,0)}\to 0.\end{aligned} \tag{4.46}$$

Integration by parts gives

$$-\zeta_n^2\langle u_n, p(x)u_{n,x}\rangle_{L^2(-L,0)} = \zeta_n^2 p(-L)|u_n(-L)|^2 - \zeta_n^2 p(0)|u_n(0)|^2 + \zeta_n^2\Big\langle p_x(x)u_n, u_n\Big\rangle_{L^2(-L,0)}$$

$$\begin{aligned}-\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)} &= -\rho_1^{-1}a_1\mu_{01}p(0)|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01}p(-L)|u_{n,x}(-L)|^2\\ &+ \rho_1^{-1}a_1\mu_{01}\Big\langle p_x(x)u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}\end{aligned}$$

and

$$\begin{aligned}&-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\ &= -\rho_1^{-1}a_1p(0)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds u_{n,x}(0)\\ &+\rho_1^{-1}a_1p(-L)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds u_{n,x}(-L)\\ &+\rho_1^{-1}a_1\Big\langle p_x(x)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}.\end{aligned} \tag{4.47}$$

Since

$$\rho_1^{-1}\beta_1\langle\theta_{n,x},p(x)u_{n,x}\rangle_{L^2(-L,0)}\to 0,$$

then by the above integrations, for $p(x)=x\in C^1[-L,0]$, Eq.(4.46) takes the form

$$\begin{aligned}
&-\zeta_n^2|u_n(-L)|^2+\zeta_n^2\Big\langle u_n,u_n\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\mu_{01}|u_{n,x}(-L)|^2+\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds u_{n,x}(-L)\\
+\ &\rho_1^{-1}a_1\Big\langle\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0,
\end{aligned}\tag{4.48}$$

and hence $u_{n,x}(-L)$ and $\zeta_n u_n(-L)$ are bounded.
Similarly, taking $p(x)=x+L\in C^1[-L,0]$, Eq.(4.46) takes the form

$$\begin{aligned}
&-\zeta_n^2|u_n(0)|^2+\zeta_n^2\Big\langle u_n,u_n\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\mu_{01}|u_{n,x}(0)|^2+\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds u_{n,x}(0)\\
+\ &\rho_1^{-1}a_1\Big\langle\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.
\end{aligned}\tag{4.49}$$

Then, we get boundedness of $\zeta_n u_n(0)$ and $u_{n,x}(0)$.
Multiplying (4.35) by $u_{n,x}$ and taking the integration to get since $\zeta_n>0$,

$$i\zeta_n\Big\langle\theta_n,u_{n,x}\Big\rangle_{L^2(-L,0)}+c_1^{-1}\beta_1\Big\langle u_{1,n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}-c_1^{-1}l\Big\langle\theta_{n,xx},u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.$$

By (4.42), we have after dividing by $i\sqrt{\zeta_n}$

$$i\zeta_n\Big\langle\theta_n,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0$$

Integrating by part to get

$$\begin{aligned}
&l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)}-\theta_{n,x}(0)\overline{u_{n,x}(0)}\Big)+l\Big\langle\sqrt{\zeta_n}\theta_{n,x},(i\zeta_n)^{-1}u_{n,xx}\Big\rangle_{L^2(-L,0)}\\
&+\beta_1\sqrt{\zeta_n}\Big\langle u_{1,n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0
\end{aligned}\tag{4.50}$$

By(4.45)and the boundedness of $u_{n,x}(-L)$ and $u_{n,x}(0)$, we have

$$l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)}-\theta_{n,x}(0)\overline{u_{n,x}(0)}\Big)\to 0$$

Moreover, from equation after (4.34), we obtain that $(i\zeta_n)^{-1}u_{n,xx}$ is bounded in $L^2(-L,0)$, thus

$$l(\sqrt{\zeta_n}\theta_{n,x}, (i\zeta_n)^{-1}u_{n,xx}) \to 0$$

Hence by(4.50),we get

$$\sqrt{\sqrt{\zeta_n}} \quad u_{n,x} \to 0, \qquad in \qquad L^2(-L,0) \tag{4.51}$$

thanks to the Poincaré inequality, we have

$$\sqrt{\sqrt{\zeta_n}} \quad u_n \to 0, \quad in \qquad L^2(-L,0) \tag{4.52}$$

By (4.51),(4.52) and Galiardo-Nirenberg inequality, we get

$$\sqrt{\sqrt{\zeta_n}} \quad u_n(0) \to 0 \tag{4.53}$$

From equation after (4.34) and (4.41) and since $\zeta_n > 0$, we have

$$i\zeta_n u_{1,n} - \rho_1^{-1}a_1\Big(\mu_{01}u_{n,xx} + \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big) \to 0, \qquad in \quad L^2(-L,0), \tag{4.54}$$

Multiplying the above by u_n, we get

$$i\zeta_n\Big\langle u_{1,n}, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle\Big(\mu_{01}u_{n,xx} + \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big), u_n\Big\rangle_{L^2(-L,0)} \to 0.$$

Integrating by part, we get

$$\begin{aligned}
&-\Big\langle u_{1,n}, u_{1,n}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\mu_{01}u_{n,x}(0)\overline{u_n(0)} + \rho_1^{-1}a_1\mu_{01}u_{n,x}(-L)\overline{u_n(-L)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&+\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds\overline{u_n(0)} - \rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds\overline{u_n(-L)} \\
&+\rho_1^{-1}a_1\Big\langle\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned}$$

Since $u_{n,x}(0), u_{n,x}(-L)$ are bounded, by (4.51) and $u_n(-L) \to 0, u_n(0) \to 0$, we have

$$u_{1,n}, \zeta_n u_n \to 0, \quad in \quad L^2(-L,0). \tag{4.55}$$

Multiplying equation after (4.34) by $(x+L)u_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re\Big[-\Big\langle\zeta_n^2 u_{1,n}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\Big\langle\Big(\mu_{01}u_{n,xx} + \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big), (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \\
&= -\zeta_n^2|u_n(0)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds u_{n,x}(0) + \rho_1^{-1}a_1\Big\langle\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0,
\end{aligned}$$

Hence by (4.51)and (4.55)

$$\zeta_n u_n(0), u_{n,x}(0) \to 0 \tag{4.56}$$

Now, multiplying equation after (4.34) by $xu_{n,x}$, we get the real part as follows

$$2\Re\Big[-\Big\langle \zeta_n^2 u_n^1, xu_{n,x} \Big\rangle_{L^2(-L,0)} \tag{4.57}$$

$$-\rho_1^{-1} a_1 \Big\langle \Big(\mu_{01} u_{n,xx} + \int_0^\infty \mu_1(s) \eta^t_{1,n,xx}(s) ds \Big), xu_{n,x} \Big\rangle_{L^2(-L,0)} \Big]$$

$$= -\zeta_n^2 |u_n(-L)|^2 + \zeta_n^2 \Big\langle u_n, u_n \Big\rangle_{L^2(-L,0)} - \rho_1^{-1} a_1 \mu_{01} |u_{n,x}(-L)|^2 + \rho_1^{-1} a_1 \mu_{01} \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1} a_1 \int_0^\infty \mu_1(s) \eta^t_{1,n,x}(-L, s) ds u_{n,x}(-L) + \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu_1(s) \eta^t_{1,n,x}(s) ds, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0,$$

Then

$$\zeta_n u_n(-L), u_{n,x}(-L) \to 0. \tag{4.58}$$

Taking again equation after (4.34), multiplying by u_n, we have

$$\sqrt{\zeta_n} \Big\langle i\zeta_n u_n^1, u_n \Big\rangle_{L^2(-L,0)} + \rho_1^{-1} \sqrt{\zeta_n} \beta_1 \Big\langle \theta_{n,x}, u_n \Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1} \sqrt{\zeta_n} a_1 \mu_{01} \Big\langle u_{n,xx}, u_n \Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1} \sqrt{\zeta_n} a_1 \Big\langle \int_0^\infty \mu_1(s) \eta^t_{1,n,xx}(s) ds, u_n \Big\rangle_{L^2(-L,0)} \to 0, \tag{4.59}$$

By (4.51) and (4.56), we have

$$-\rho_1^{-1} \sqrt{\zeta_n} a_1 \mu_{01} \Big\langle u_{n,xx}, u_n \Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1} a_1 \mu_{01} \sqrt{\zeta_n} u_{n,x}(0) \overline{u_n(0)} + \rho_1^{-1} a_1 \mu_{01} \sqrt{\zeta_n} u_{n,x}(-L) \overline{u_n(-L)}$$

$$+\rho_1^{-1} a_1 \mu_{01} \sqrt{\zeta_n} \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0$$

and

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_n\Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds\overline{u_n(0)}$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds\overline{u_n(-L)}$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.60}$$

Thus by(4.60) and (4.41), we have

$$\sqrt{\sqrt{\zeta_n}}u_n^1 \to 0, \quad in \qquad L^2(-L,0). \tag{4.61}$$

Multiplying equation after (4.34) by $(x+L)u_{n,x}$, we have

$$\Big\langle i\sqrt{\zeta_n}\zeta_n u_n^1, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_{01}\Big\langle u_{n,xx}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \tag{4.62}$$

Integrating by parts and using (4.41)and the boundedness of $u_{n,x}$ in $L^2(-L,0)$, we get

$$-\sqrt{\zeta_n}|u_n^1(0)|^2 + \sqrt{\zeta_n}\Big\langle u_n^1, u_n^1\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}|u_{n,x}(0)|^2$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds u_{n,x}(0)$$

$$-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_{-L}^0 \Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.63}$$

Thus by (4.51)and (4.61)

$$\sqrt{\sqrt{\zeta_n}}u_n^1(0), \sqrt{\sqrt{\zeta_n}}u_{n,x}(0) \to 0 \tag{4.64}$$

Multiplication of (4.54) by $u_{n,x}$ yields

$$i\zeta_n\Big\langle u_n^1, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \tag{4.65}$$

Due to (4.56)and (4.58), we get

$$-\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}$$
$$= \frac{1}{2}(-\rho_1^{-1}a_1\mu_{01})|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01})|u_{n,x}(-L)|^2$$
$$-\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)dsu_{n,x}(0) + \rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)dsu_{n,x}(-L)$$
$$+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.66}$$

Thus, it follows from (4.65)that

$$(i\zeta_n u_n^1, u_{n,x}) \to 0 \tag{4.67}$$

Taking the product of (4.54) with θ_n, yields

$$i\zeta_n\Big\langle u_{1,n}, \theta_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, \theta_n\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, \theta_n\Big\rangle_{L^2(-L,0)} \to 0, \qquad in \quad L^2(-L,0), \tag{4.68}$$

Due to (4.41),(4.44)and (4.56)

$$-\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, \theta_n\Big\rangle_{L^2(-L,0)}$$
$$= -\rho_1^{-1}a_1\mu_{01}u_{n,x}(0)\overline{\theta_n(0)} + \rho_1^{-1}a_1\mu_{01}u_{n,x}(-L)\overline{\theta_n(-L)}$$
$$+\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, \theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.69}$$

and

$$-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, \theta_n\Big\rangle_{L^2(-L,0)}$$
$$= -\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds\overline{\theta_n(0)}$$
$$+\rho_1^{-1}a_1\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds\overline{\theta_n(-L)}$$
$$+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, \theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{4.70}$$

Then from (4.68) that

$$i\zeta_n\Big\langle u_n^1, \theta_n\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.71}$$

Multiplying (4.35)by u_n^1, we have

$$\Big\langle i\zeta_n\theta_n, u_n^1\Big\rangle_{L^2(-L,0)} - c_1^{-1}l\Big\langle \theta_{n,xx}, u_n^1\Big\rangle_{L^2(-L,0)} + c_1^{-1}\beta_1\Big\langle u_{n,x}^1, u_n^1\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.72}$$

by (4.67), (4.71) we have

$$\left\langle \theta_{n,xx}, u_n^1 \right\rangle_{L^2(-L,0)} \to 0 \tag{4.73}$$

Integrating by part

$$\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} - \left\langle \theta_{n,x}, u_{n,x}^1 \right\rangle_{L^2(-L,0)} \to 0 \tag{4.74}$$

Due to (4.45)and (4.64), we get

$$\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} \to 0. \tag{4.75}$$

From (4.74) we have

$$\left\langle \theta_{n,x}, u_{n,x}^1 \right\rangle_{L^2(-L,0)} \to 0. \tag{4.76}$$

Multiplying (4.35) by $(x+L)\theta_{n,x}$, integrating to get

$$\Re\Big[\left\langle i\zeta_n\theta_n, (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)} - c_1^{-1}\left\langle (l\theta_{n,xx} - \beta_1 u_{n,x}^1), (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)}\Big] \to 0 \tag{4.77}$$

By (4.41) and (4.42), we get

$$\left\langle i\zeta_n\theta_n, (x+L)\theta_{n,x} \right\rangle_{L^2(-L,0)} \to 0 \tag{4.78}$$

Thus by (4.77) and (4.41), we have

$$-c_1^{-1}l\theta_{n,x}(0)\overline{\theta_{n,x}(0)} + 2\Re[c_1^{-1}\beta_1(u_{n,x}^1, (x+L)\theta_{n,x})] \to 0 \tag{4.79}$$

Then by (4.76), we get

$$\theta_{n,x}(0) \to 0 \tag{4.80}$$

Hence, by (4.66),(4.56),(4.44)and (4.80), we have

$$u_{n,x}(0), u_n(0), \theta_n(0), \theta_{n,x}(0) \to 0 \tag{4.81}$$

On the hand taking the product of (4.38) with $(x-L)w_{2,n,x}$, yields

$$\begin{aligned}&\Re\Big[i\zeta_n\left\langle q_n, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)} + c_2^{-1}\beta_2\left\langle v_{n,x}^1, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)}\\ &-c_2^{-1}k\left\langle w_{2,n,xx}, (x-L)w_{2,n,x} \right\rangle_{L^2(0,L)}\Big] \to 0,\end{aligned} \tag{4.82}$$

Using the transmission conditions in (4.1), we get

$$(q_n, q_n) + c_2^{-1}k(w_{2,n,x}, w_{2,n,x}) - 2\Re\Big[c_2^{-1}\beta_2\left\langle v_{n,x}, (x-L)q_{n,x} \right\rangle_{L^2(0,L)}\Big] \to 0. \tag{4.83}$$

Taking the product of equation after (4.36) with $(x-L)v_{n,x}$ to obtain

$$i\zeta_n\Big\langle v_n^1,(x-L)v_{n,x}\Big\rangle_{L^2(0,L)} - \rho_2^{-1}a_2\Big\langle \mu_{02}v_{n,xx} + \int_0^\infty \mu_2(s)\eta_{2,n,xx}^t(s)ds,(x-L)v_{n,x}\Big\rangle_{L^2(0,L)}$$
$$+\rho_2^{-1}\beta_2\Big\langle q_{n,x},(x-L)v_{n,x}\Big\rangle_{L^2(0,L)} \to 0, \quad (4.84)$$

Integrating (4.84)by parts we have

$$\Big\langle v_n^1,v_n^1\Big\rangle_{L^2(0,L)} + \rho_2^{-1}a_2\Big\langle \mu_{02}v_{n,x} + \int_0^\infty \mu_2(s)\eta_{2,n,x}^t(s)ds, v_{n,x}\Big\rangle_{L^2(0,L)}$$
$$+2\Re\Big[\rho_2^{-1}\beta_2\Big\langle q_{n,x},(x-L)q_{n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0 \quad (4.85)$$

Thus by (4.83)and (4.85)

$$a_2\Big\langle \mu_{02}v_{n,x} + \int_0^\infty \mu_2(s)\eta_{2,n,x}^t(s)ds, v_{n,x}\Big\rangle_{L^2(0,L)} + \Big\langle \rho_2 v_n^1, v_n^1\Big\rangle_{L^2(0,L)}$$
$$+c_2\Big\langle q_n,q_n\Big\rangle_{L^2(0,L)} + k\Big\langle w_{2,n,x},w_{2,n,x}\Big\rangle_{L^2(0,L)} \to 0 \quad (4.86)$$

Then,

$$v_{n,x}, v_n^1, w_{2,n,x}, q_n \to 0, \qquad in \qquad L^2(0,L) \quad (4.87)$$

Thus (4.87) together with (4.42), (4.55) and (4.87), we obtain

$$V_n = (u_n, u_n^1, \theta_n, v_n, v_n^1, w_{2,n}, q_n)^T \to 0 \quad (4.88)$$

which contradicts $\|V_n\| = 1$ therefore, (4.33) holds.

■

Chapter 5

Blow up of solutions for coupled system Love-equations with infinite memories

A coupled system of nonlinear Love equations with infinite memories is considered. The nonexistence of weak solution is proved

5.1 Introduction

Let $u = u(x,t), v = v(x,t), x \in \Omega$ be a bounded domain of $\mathbb{R}$ with smooth boundary $\partial\Omega, t > 0$. We consider a system of nonlinear Love-equation for $t > 0$ in the form

$$\begin{cases} \left(|u'|^{l-2}u'\right)' - \left(\lambda_0 u_x + \lambda_1 u'_x + \left(|u'_x|^{l-2}u'_x\right)'\right)_x + \alpha \int_{-\infty}^{t} \mu_1(t-s)u_{xx}(s)ds = f_1(u,v), \\ \left(|v'|^{l-2}v'\right)' - \left(\lambda_0 v_x + \tilde{\lambda}_1 v'_x + \left(|v'_x|^{l-2}v'_x\right)'\right)_x + \alpha \int_{-\infty}^{t} \mu_2(t-s)u_{xx}(s)ds = f_2(u,v), \end{cases} \tag{5.1}$$

where $\lambda_0, \lambda_1, \tilde{\lambda}_1, \alpha > 0, l > 2$, are constants. The functions $\mu_1, \mu_2, f_1(u,v), f_2(u,v)$ are specified later. Equation (5.1) satisfies the homogeneous Dirichlet boundary conditions:

$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \qquad t > 0, \tag{5.2}$$

and the following initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), \quad u'(x,0) = u_1(x) \quad in \quad H_0^1 \cap H^2, \\ v(x,-t) = v_0(x,t), \quad v'(x,0) = v_1(x) \quad in \quad H_0^1 \cap H^2. \end{cases} \tag{5.3}$$

To deal with a wave equations with infinite histories, we assume that the kernel functions μ_1, μ_2 satisfy the following hypothesis:
(Hyp1:) $\mu_1, \mu_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are a nonincreasing C^1 functions such that

$$\lambda_0 - \alpha \int_0^\infty \mu_1(s)ds = l > 0, \quad \mu_1(0) > 0. \tag{5.4}$$

$$\lambda_0 - \alpha \int_0^\infty \mu_2(s)ds = k > 0, \quad \mu_2(0) > 0. \tag{5.5}$$

The famous and widely used technical lemma 1.2 will play an important role in the sequel.

5.2 Blow up result

We prove that if (5.11) hold, then the blow up of solutions of (5.1) for all time occurs when the initial energy is negative.
Let us assume that there exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants $p > l$, such that

$$\begin{aligned}
&\frac{\partial \mathcal{F}}{\partial u}(u, v) = f_1(u, v), \qquad \frac{\partial \mathcal{F}}{\partial v}(u, v) = f_2(u, v), \\
&u f_1(u, v) + v f_2(u, v) = p\mathcal{F}(u, v), \quad \forall (u, v) \in \mathbb{R}^2, \\
&c_0(|v|^p + |u|^p) \leq p\mathcal{F}(u, v) \leq c_1(|v|^p + |u|^p), \quad \forall (u, v) \in \mathbb{R}^2, c_0, c_1 \in \mathbb{R}_*^+,
\end{aligned} \tag{5.6}$$

We introduce the energy functional $E(t)$ associated with system (5.1)-(5.3)

$$2E(t) = \frac{2(l-1)}{l}\Big[\|u'\|_l^l + \|u_x'\|_l^l + \|v'\|_l^l + \|v_x'\|_l^l\Big] + J(t) - 2\int_\Omega \mathcal{F}(u, u_x)dx. \tag{5.7}$$

where

$$\begin{aligned}
J(t) &= \Big(\lambda_0 - \alpha \int_0^\infty \mu_1(s)ds\Big)\|u_x\|_2^2 + \Big(\lambda_0 - \alpha \int_0^\infty \mu_2(s)ds\Big)\|v_x\|_2^2 \\
&+ \alpha \int_0^\infty \mu_1(s)\|u_x(t) - u_x(t-s)\|_2^2 ds + \alpha \int_0^\infty \mu_2(s)\|v_x(t) -_x (t-s)\|_2^2 ds.
\end{aligned}$$

Remark 5.1 Note that the integral $\int_\Omega \mathcal{F}(u, v)dx$ in (5.7) makes sense because $H_0^1(\Omega) \cap L^p(\Omega)$ for $p > 2$.

Example 5.1 The functions $f_1(u,v), f_2(u,v)$ can be given by the formulas

$$\begin{cases} f_1(u,v) = a_1|u+v|^{p-2}(u+v) + b_1|u|^{(p-1)/2}u|v|^{p/2}, \\ \\ f_1(u,v) = a_1|u+v|^{p-2}(u+v) + b_1|v|^{(p-1)/2}v|u|^{p/2}, a_1, b_1 > 0, a_1, b_1 > 0, p > 2. \end{cases}$$

It is note hard to see this Lemma (Using Lemma 1.2).

Lemma 5.1 *Supose that (Hyp1) holds. Let u be solution of system (5.1)-(5.3). Then the energy functional (5.7) is a nonincreasing function, i.e., for all $t \geq 0$,*

$$\begin{aligned} \frac{d}{dt}E(t) &= -\lambda_1\|u'_x\|_2^2 + \frac{\alpha}{2}\int_0^\infty \mu'_1(s)\|u_x(t) - u_x(t-s)\|_2^2 ds \\ &- \lambda_1\|v'_x\|_2^2 + \frac{\alpha}{2}\int_0^\infty \mu'_2(s)\|v_x(t) - v_x(t-s)\|_2^2 ds. \end{aligned}$$

For reader, we state this Lemma with its proof.

Lemma 5.2 *Let $\nu > 0$ be a real positive number and let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t), \tag{5.8}$$

defined in $[0,\infty)$. If $L(0) > 0$, then the solution does not exist for $t \geq L(0)^{-\nu}\xi^{-\nu}\nu^{-1}$.

Proof. The direct integration of (5.8) gives

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi\nu t$$

Thus, we get the following estimate:

$$L^{\nu}(t) \geq \Big[L^{-\nu}(0) - \xi\nu t\Big]^{-1}. \tag{5.9}$$

It is clear that the right-hand side of (5.9) is unbounded for

$$\xi\nu t = L^{-\nu}(0).$$

Lemma 5.2 is proved. ■ Our goal is to prove that when the initial energy is positive, the solution of system (5.1) blows up for all time under the (5.6), (5.11). We introduce the following notation for some constant γ,

$$\gamma = \eta^{1/(2-p)}, \quad E_1 = \Big(\frac{1}{2} - \frac{1}{p}\Big)\eta^{2/(1-p)}, \quad E_2 = \Big(\frac{1}{q} - \frac{1}{p}\Big)\eta^{2/(1-p)}, \tag{5.10}$$

wher η, q are a constants will be specified later.

5.2. Blow up result

Lemma 5.3 *(see [59], Lemma 3.3) Suppose (Hpp1), (5.6) hold. Let* (u,v) *be solution of (5.1)-(5.3). Assume that* $p>2$, $\|u_{0,x}(0)\|_2^2+\|v_{0,x}(0)\|_2^2>\eta^{2/(1-p)}$ *and* $E(0)<E_2$*. Then, there exists a constant* $\gamma_0>\gamma$ *such that* $J(t)>\gamma_0^2$ *and*

$$p\int_\Omega \mathcal{F}(u,v)dx\geq \eta\gamma_0^p,\qquad \forall t\geq 0.$$

The result here reads as follows.

Theorem 5.1 *Assume that (5.6) hold and* $p>l$*. For any* $(u_0(0),v_0(0))\in H_0^1\cap H^2$ *such that*

$$\|u_{0,x}(0)\|_2^2+\|v_{0,x}(0)\|_2^2>\gamma^2,\qquad E(0)<E_2.$$

There exist a number $q, 2<q<p$*, such that*

$$\max\Big(\int_0^\infty \mu_1(s)ds,\int_0^\infty \mu_2(s)ds\Big)<\frac{\lambda_0(1+q/2)}{\lambda\Big(1/2q+1+q/2\Big)}. \tag{5.11}$$

Then, any solutions (u,v) *of (5.1)-(5.2) satisfies* $\forall t>0$,

$$\|u\|_p^p+\|v\|_p^p\to\infty.$$

Proof. Let

$$H(t)=E_2-E(t),\qquad \forall t>0. \tag{5.12}$$

By multiplying the first equation in (5.1) by $-u'$ and the second by $-v'$, integrating over Ω and using (5.7), Lemma 1.2, Lemma 5.1, we obtain

$$\begin{aligned}
H'(t) &= -E'(t)\\
&= \lambda_1\|u'_x\|_2^2-\frac{1}{2}\alpha\int_0^\infty \mu_1'(s)\|u_x(t)-u_x(t-s)\|_2^2ds\\
&+ \tilde{\lambda}_1\|v'_x\|_2^2-\frac{1}{2}\alpha\int_0^\infty \mu_2'(s)\|v_x(t)-v_x(t-s)\|_2^2ds\\
&\geq 0,\qquad \forall t>0.
\end{aligned} \tag{5.13}$$

Consequently, since E' is absolutely continuous, we have

$$H(0)=E_2-E(0)>0, \tag{5.14}$$

which implies

$$0<H(0)\leq H(t)=E_2-\frac{l-1}{l}\Big[\|u'\|_l^l+\|u'_x\|_l^l+\|v'\|_l^l+\|v'_x\|_l^l\Big]-\frac{1}{2}J(t)+\int_\Omega \mathcal{F}(u,v)dx,\forall t>0.$$

From (Hp1), (5.10) and result in Lemma 5.3, we obtain

$$\begin{aligned}
&E_2 - \frac{l-1}{l}\Big[\|u'\|_l^l + \|u'_x\|_l^l + \|v'\|_l^l + \|v'_x\|_l^l\Big] - \frac{1}{2}J(t) + \int_\Omega \mathcal{F}(u,v)dx \\
&< E_2 - \frac{1}{2}\gamma_0^2 + \int_\Omega \mathcal{F}(u,v)dx \\
&< E_2 - \frac{1}{2}\gamma^2 + \int_\Omega \mathcal{F}(u,v)dx \\
&< E_1 - \frac{1}{2}\gamma^2 + \int_\Omega \mathcal{F}(u,v)dx \\
&< \int_\Omega \mathcal{F}(u,v)dx \qquad (5.15)
\end{aligned}$$

One gets

$$0 < H(0) \leq H(t) \leq \int_\Omega \mathcal{F}(u,v)dx. \qquad (5.16)$$

Then, we define a functionals

$$\begin{aligned}
M(t) &= \frac{1}{2}\int_\Omega \Big(|u'|^{l-1}u'u + |v'|^{l-1}v'v\Big)dx, \\
N(t) &= \lambda_1 \frac{1}{2}\int_\Omega \Big(|u_x|^2 + |v_x|^2\Big)dx + \int_\Omega \Big(|u'_x|^{l-1}u'_x u + |v'_x|^{l-1}v'_x v\Big)dx,
\end{aligned}$$

and introduce

$$L(t) = H^{1-\sigma}(t) + \varepsilon M(t) + \varepsilon N(t), \qquad (5.17)$$

for ε small enough and

$$0 < \sigma < 1, 2/(1-2\sigma) \leq p. \qquad (5.18)$$

We now show that $L(t)$ satisfies the differential inequality in Lemma 5.2. By taking the derivative of (5.17) and using (5.6) and Lemma 1.2, we obtain

$$\begin{aligned}
L'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{l-1}{l}\Big[\|u'\|_l^l + \|u'_x\|_l^l + \|v'\|_l^l + \|v'_x\|_l^l\Big] \\
&- \varepsilon\Big(\lambda_0 - \alpha\int_0^\infty \mu_1(s)ds\Big)\|u_x\|_2^2 - \varepsilon\Big(\lambda_0 - \alpha\int_0^\infty \mu_2(s)ds\Big)\|v_x\|_2^2 \\
&+ \varepsilon\alpha\int_0^\infty \mu_1(s)\int_\Omega [u_x(t-s) - u_x(s)]u_x dxds \\
&+ \varepsilon\alpha\int_0^\infty \mu_2(s)\int_\Omega [v_x(t-s) - v_x(s)]v_x dxds \\
&+ \varepsilon\int_\Omega f_1(u,v)udx + \varepsilon\int_\Omega f_2(u,v)vdx. \qquad (5.19)
\end{aligned}$$

By the Cauchy-Schwarz and Young inequalities, we find

$$\begin{aligned}
&\int_0^\infty \mu_1(s)\int_\Omega [u_x(t-s)-u_x(s)]u_x dxds\\
&\le \int_0^\infty \mu_1(s)\|u_x(t-s)-u_x(s)\|_2 ds\|u_x\|_2\\
&\le \gamma\int_0^\infty \mu_1(s)\|u_x(t-s)-u_x(s)\|_2^2 ds+\frac{\int_0^\infty \mu_1(s)ds}{4\gamma}\|u_x\|_2^2, \gamma>0.
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty \mu_2(s)\int_\Omega [v_x(t-s)-v_x(s)]v_x dxds\\
&\le \int_0^\infty \mu_2(s)\|v_x(t-s)-v_x(s)\|_2 ds\|v_x\|_2\\
&\le \gamma\int_0^\infty \mu_2(s)\|v_x(t-s)-v_x(s)\|_2^2 ds+\frac{\int_0^\infty \mu_2(s)ds}{4\gamma}\|v_x\|_2^2, \gamma>0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
L'(t) \;\ge\;& (1-\sigma)H^{-\sigma}(t)H'(t)+\frac{l-1}{l}\Big[\|u'\|_l^l+\|u'_x\|_l^l+\|v'\|_l^l+\|v'_x\|_l^l\Big]\\
-\;& \varepsilon\Big(\lambda_0-\alpha\int_0^\infty \mu_1(s)ds-\alpha\frac{\int_0^\infty \mu_1(s)ds}{4\gamma}\Big)\|u_x\|_2^2\\
-\;& \varepsilon\Big(\lambda_0-\alpha\int_0^\infty \mu_2(s)ds-\alpha\frac{\int_0^\infty \mu_2(s)ds}{4\gamma}\Big)\|v_x\|_2^2\\
+\;& \varepsilon\alpha\gamma\int_0^\infty \mu_1(s)\|u_x(t-s)-u_x(s)\|_2^2 ds\\
+\;& \varepsilon\alpha\gamma\int_0^\infty \mu_2(s)\|v_x(t-s)-v_x(s)\|_2^2 ds\\
+\;& \varepsilon\int_\Omega f_1(u,v)udx+\varepsilon\int_\Omega f_2(u,v)vdx. \qquad (5.20)
\end{aligned}$$

By (5.6), we obtain

$$\begin{aligned}
&\int_\Omega f_1(u,u_x)udx+\int_\Omega f_2(u,u_x)u_x dx\\
&\le c_1\Big(\|u\|_p^p+\|v\|_p^p\Big). \qquad (5.21)
\end{aligned}$$

Then, it follows since (5.18), $\sigma<1$ and due to the inequality

$$(1-\sigma)H^{-\sigma}(t)H'(t)>0, \forall t>0,$$

that

$$\begin{aligned} L'(t) &\geq \varepsilon\frac{l-1}{l}\Big[\|u'\|_l^l+\|u_x'\|_l^l+\|v'\|_l^l+\|v_x'\|_l^l\Big] \\ &- \varepsilon\Big(\lambda_0-\alpha\int_0^\infty\mu_1(s)ds+\alpha\frac{\int_0^\infty\mu_1(s)ds}{4\gamma}\Big)\|u_x\|_2^2 \\ &- \varepsilon\Big(\lambda_0-\alpha\int_0^\infty\mu_2(s)ds+\alpha\frac{\int_0^\infty\mu_2(s)ds}{4\gamma}\Big)\|v_x\|_2^2 \\ &+ \varepsilon\alpha\gamma\int_0^\infty\mu_1(s)\|u_x(t-s)-u_x(s)\|_2^2ds \\ &+ \varepsilon\alpha\gamma\int_0^\infty\mu_2(s)\|v_x(t-s)-v_x(s)\|_2^2ds \\ &+ \varepsilon c_1\Big(\|v\|_p^p+\|u\|_p^p\Big). \end{aligned} \tag{5.22}$$

Adding and substituting $qE(t), 2<q<p$, by using the definition of $H(t), E_2$, we get

$$\begin{aligned} L'(t) &\geq \varepsilon(1+\frac{q}{2})\frac{l-1}{l}\Big[\|u'\|_l^l+\|u_x'\|_l^l+\|v'\|_l^l+\|v_x'\|_l^l\Big]+\varepsilon qH(t)-q\varepsilon E_2 \\ &+ \frac{q}{2}\Big(\lambda_0-\alpha\int_0^\infty\mu_1(s)ds\Big)\|u_x\|_2^2 \\ &+ \frac{q}{2}\Big(\lambda_0-\alpha\int_0^\infty\mu_2(s)ds\Big)\|v_x\|_2^2 \\ &+ \varepsilon\alpha(\frac{q}{2}-\gamma)\int_0^\infty\mu_1(s)\|u_x(t-s)-u_x(s)\|_2^2ds \\ &+ \varepsilon\alpha(\frac{q}{2}-\gamma)\int_0^\infty\mu_2(s)\|v_x(t-s)-v_x(s)\|_2^2ds \\ &+ \varepsilon(c_1-q)\Big(\|u\|_p^p+\|v\|_p^p\Big) \end{aligned} \tag{5.23}$$

for some

$$a_1=\frac{q}{2}-\gamma>0, \quad i.e.\gamma<q/2,$$

$$a_2=\lambda_0[-1+\frac{q}{2}]-\alpha\Big[\frac{1}{4\gamma}-1+\frac{q}{2}\Big]\int_0^\infty\mu_1(s)ds>0.$$

$$a_4=\lambda_0[-1+\frac{q}{2}]-\alpha\Big[\frac{1}{4\gamma}-1+\frac{q}{2}\Big]\int_0^\infty\mu_2(s)ds>0.$$

$$a_3=c_1-q>0.$$

Then, estimate (5.23) becomes

$$\begin{aligned}
L'(t) &\geq \varepsilon(1+\frac{q}{2})\frac{l-1}{l}\Big[\|u'\|_l^l + \|u'_x\|_l^l + \|v'\|_l^l + \|v'_x\|_l^l\Big] + \varepsilon q H(t) - q\varepsilon E_2 \\
&+ \varepsilon\alpha a_2\|u_x\|_2^2 + \varepsilon\alpha a_1\int_0^\infty \mu_1(s)\|u_x(t)-u_x(t-s)\|_2^2 ds \\
&+ \varepsilon\alpha a_4\|v_x\|_2^2 + \varepsilon\alpha a_1\int_0^\infty \mu_2(s)\|v_x(t)-v_x(t-s)\|_2^2 ds \\
&+ \varepsilon a_3\Big(\|u\|_p^p + \|v\|_p^p\Big). \qquad (5.24)
\end{aligned}$$

Since $\gamma_0 > \eta^{(2-p)/p}$, we take $a_5 = c_1 - q - 2E_2(\eta^{-1}\gamma_0^{-p}) > 0$.
We can now find a positive constants $\nu > 0$ such that

$$\begin{aligned}
&L'(t) \geq \varepsilon\nu\Big[H(t) + \frac{l-1}{l}\Big[\|u'\|_l^l + \|u'_x\|_l^l + \|v'\|_l^l + \|v'_x\|_l^l + \|u_x\|_2^2 + \|v_x\|_2^2 \\
&+ \int_0^\infty \mu_1(s)\|u_x(t)-u_x(t-s)\|_2^2 ds + \int_0^\infty \mu_2(s)\|v_x(t)-v_x(t-s)\|_2^2 ds \\
&+\|u\|^p + \|v\|^p\Big].
\end{aligned}$$

Thus, we can choose $\varepsilon > 0$ small enough such that

$$L(t) \geq L(0) > 0, \quad \forall t > 0. \qquad (5.25)$$

Other hand, we have by (5.18), Hölder's and Young's inequalities,

$$\begin{aligned}
\int_\Omega \Big(|u'|^{l-2}u'udx\Big)^{1/(1-\sigma)} &\leq \Big|\int_\Omega |u'|^{l-1}udx\Big)\Big|^{1/(1-\sigma)} \leq \Big|\|u'\|_l^{l-1}\|u\|_l\Big)\Big|^{1/(1-\sigma)} \\
&\leq \Big|c|\Omega|^{\frac{1}{l}-\frac{1}{p}}\Big(\|u'\|_l^l + \|u\|_p^l\Big)\Big|^{1/(1-\sigma)} \\
&\leq c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big|\|u'\|_l^l + \|u\|_p^p\Big|^{1/(1-\sigma)} \\
&\leq c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big(\|u\|_p^{\tau/(1-\sigma)} + \|u'\|_2^{s/(1-\sigma)}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Big(|v'|^{l-2}v'vdx\Big)^{1/(1-\sigma)} &\leq \Big|\int_\Omega |v'|^{l-1}vdx\Big|^{1/(1-\sigma)} \leq \Big|\|v'\|_l^{l-1}\|v\|_l\Big)\Big|^{1/(1-\sigma)} \\
&\leq \Big|c|\Omega|^{\frac{1}{l}-\frac{1}{p}}\Big(\|v'\|_l^l + \|v\|_p^l\Big)\Big|^{1/(1-\sigma)} \\
&\leq c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big|\|v'\|_l^l + \|v\|_p^p\Big|^{1/(1-\sigma)} \\
&\leq c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big(\|v\|_p^{\tau/(1-\sigma)} + \|v'\|_2^{s/(1-\sigma)}\Big),
\end{aligned}$$

similarly

$$\begin{aligned}
\int_\Omega \Big(|u'_x|^{l-2}u'_x u dx\Big)^{1/(1-\sigma)} &\le \Big|\int_\Omega |u'_x|^{l-1}u dx\Big)\Big|^{1/(1-\sigma)} \le \Big|\|u'_x\|_l^{l-1}\|u\|_l\Big)\Big|^{1/(1-\sigma)} \\
&\le \Big|c|\Omega|^{\frac{1}{l}-\frac{1}{p}}\Big(\|u'_x\|_l^l+\|u\|_p^l\Big)\Big|^{1/(1-\sigma)} \\
&\le c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big|\|u'_x\|_l^l+\|u\|_p^p\Big|^{1/(1-\sigma)} \\
&\le c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big(\|u\|_p^{\tau/(1-\sigma)}+\|u'_x\|_2^{s/(1-\sigma)}\Big),
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Big(|v'_x|^{l-2}v'_x v dx\Big)^{1/(1-\sigma)} &\le \Big|\int_\Omega |v'_x|^{l-1}v dx\Big|^{1/(1-\sigma)} \le \Big|\|v'_x\|_l^{l-1}\|v\|_l\Big)\Big|^{1/(1-\sigma)} \\
&\le \Big|c|\Omega|^{\frac{1}{l}-\frac{1}{p}}\Big(\|v'_x\|_l^l+\|v\|_p^l\Big)\Big|^{1/(1-\sigma)} \\
&\le c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big|\|v'_x\|_l^l+\|v\|_p^p\Big|^{1/(1-\sigma)} \\
&\le c|\Omega|^{(\frac{1}{l}-\frac{1}{p})/(1-\sigma)}\Big(\|v\|_p^{\tau/(1-\sigma)}+\|v'_x\|_l^{s/(1-\sigma)}\Big).
\end{aligned}$$

for $\frac{1}{\tau}+\frac{1}{s}=1$. We take $s=l(1-\sigma)$ to get

$$\frac{\tau}{1-\sigma}+\frac{l}{1-2\sigma}=1.$$

By using the algebraic inequality

$$z^\nu \le (z+1) \le \Big(1+\frac{1}{a}\Big)(z+a), \qquad \forall z\ge 0, \quad 0<\nu\le 1, a\ge 0, \tag{5.26}$$

we find

$$\|u\|_p^{2/(1-2\sigma)} \le c\Big(\|u\|_p^p+H(t)\Big), \quad \forall t\in[0,T_*). \tag{5.27}$$

Then

$$\Big(\int_\Omega |u'|^{l-1}u'u+|v'|^{l-1}v'v dx\Big)^{1/(1-\sigma)} \le c\Big(\|u\|_p^p+\|u'\|_l^l+\|v\|_p^p+\|v'\|_l^l+H(t)\Big).$$

Similarly, we obtain

$$\Big(\int_\Omega |u'_x|^{l-1}u'_x u+|v'_x|^{l-1}v'_x v dx\Big)^{1/(1-\sigma)} \le c\Big(\|u_x\|_p^p+\|u'_x\|_l^l+\|v_x\|_p^p+\|v'_x\|_l^l+H(t)\Big).$$

Thus

$$\begin{aligned}
L(t) &= H^{1-\sigma}(t)+\varepsilon\frac{1}{2}\int_\Omega\Big(|u'|^{l-1}u'u+|v'|^{l-1}v'v\Big)dx \\
&+ \varepsilon\lambda_1\frac{1}{2}\int_\Omega\Big(|u_x|^2+|v_x|^2\Big)dx+\varepsilon\int_\Omega\Big(|u'_x|^{l-1}u'_x u+|v'_x|^{l-1}v'_x v\Big)dx.
\end{aligned}$$

Then

$$L^{1-\sigma}(t) \leq c\Big[H(t) + \|u\|_p^p + \|v\|_p^p + \|u'\|_l^l + \|v'\|_l^l + \|u'_x\|_l^l + \|v'_x\|_l^l\Big].$$

This yields

$$L'(t) \geq a_0 L^{1/(1-\sigma)(t)}, \qquad \forall t \geq 0. \tag{5.28}$$

Finally, as a result of simple integration of (5.28), completes the proof of Theorem 5.1. ■

Bibliography

[1] F. Alabau-Boussouira and P. Cannarsa, *A general method for proving sharp energy decay rates for memory-dissipative evolution equations*, C. R. Math. Acad. Sci. Paris, Ser. I 347, (2009), 867-872.

[2] D. Andrade, M. A. Jorge Silva and T. F. Ma, *Exponential stability for a plate equation with p-Laplacian and memory terms*, Math. Meth. Appl. Sci. 35, 417-426, 2012.

[3] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.

[4] A. Beniani, A. Benaissa and Kh. Zennir, *Polynomial decay of solutions to the Cauchy problem for a Petrowsky-Petrowsky system in* $\mathbb{R}^n$, Acta. Appl. Math., Vo 146, (1) (2016) pp 67-79.

[5] K. Bouhali and F. Ellaggoune, *Viscoelastic wave equation with logarithmic nonlinearities in* $\mathbb{R}^n$, J. Part. Diff. Eq., Vol. 30, No. 1, (2017) pp. 47-63.

[6] K. Bouhali and F. Ellaggoune, *Existence and decay of solution to coupled system of viscoelastic wave equations with strong damping in* $\mathbb{R}^n$, Bol. Soc. Paran. Mat, Vol. 00, No. 0, (0000) pp. 00-00. doi:10.5269/bspm.41175.

[7] A. Borichev, Y. Tomilov, *Optimal polynomial decay of functions and operator semigroups*, Math. Ann. 347, (2010) 455-478.

[8] K.J. Brown, N. M Stavrakakis, *Global bifurcation results for semilinear elliptic equations on all of* $\mathbb{R}^n$, Duke Math. J. 85 (1996), 77-94.

[9] M. M. Cavalcanti, H.P. Oquendo, *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim. 42(4)(2003)1310–1324.

[10] T. Cazenave and A. Haraux *Equations d'evolution avec nonlinearite logarithmique*, Ann. Fac. Sci. Toulouse Math. (5) 2 (1980), no. 1, 21-51.

[11] H. Chen, P. Luo and G. Liu *Global solution and blow6up of a semilinear heat equation with logarithmic nonlinearity*, J. Math. Anal. Appl. 422 (2015) 84-98.

[12] G. Claudio; N. M. Grazia; P. Vittorino, *Exponential stability in linear heat conduction with memory: a semigroup approach*, Commun. Appl. Anal. 5 (2001), no. 1, 121-133.

[13] G. Claudio; P. Vittorino, *Stability of abstract linear thermoelastic systems with memory*, Math. Models Methods Appl. Sci. 11 (2001), no. 4, 627-644.

[14] R. Dautray and J. L. Lions, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Vol. 1, Masson, Paris, 1984.

[15] C. M. Dafermos, *On the existence and the asymptotic stability of solution to the equations of linear thermoelasticity*, Arch. Ration. Mech. Anal., 29, (1968) 241-271.

[16] C. M. Dafermos, H.P. Oquendo. *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. 37(1970), 297-308.

[17] Hugo D. Fernandez Sare, Jaime E. Munoz Rivera and Reinhard Racke. Stability for a Transmission Problem in Thermoelasticity with Second Sound. Journal of Thermal Stresses Vol. 31, (2008), Iss. 12.

[18] M. A. Jorge Silva and T. F. Ma, *On a viscoelastic plate equation with history setting and perturbation of p-Laplacian type*, IMA Journal of Applied Mathematics, 2012 1-17.

[19] C. Juan Vila Bravo, E. Jaime Munoz Rivera, *The transmission problem to thermoelastic plate of hyperbolic type*, IMA Journal of Applied Mathematics, Volume 74, Issue 6, 1 (2009), pp 950-962.

[20] X. S. Han, *Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics*, Bull. Korean Math. Soc. 50(1) (2013), 275-283.

[21] Z. Hongwei, L. Gongwei and H. Qingying, *Exponential Decay of Energy for a Logarithmic Wave Equation.* J. Part. Diff. Eq., Vol. 28, No. 3(2015), pp. 269-277.

[22] F. Huang, *Characteristic condition for exponential stability of linear dynamical systems in Hilbert space*, Ann. Differential. Equations, 1 (1985)(1) 43-56.

[23] J. Ignaczak, M. Ostoja-Starzewski, *Thermoelasticity with Finite Wave Speeds*, Oxford mathematical Monographs. Oxford University Press, New York (2010).

[24] L. Gearhart, *Spectral theory for contraction semigroups on Hilbert spaces*, Trans. AMS 236 (1978) 385-394.

[25] P. Gorka, *Logarithmic Klein-Gordon equation*, Acta Physica polonica B. 40 (2009) 59-66.

[26] Green and P. M. Naghdi, *Thermoelasticity without energy dissipation*, J. Elast., 31 (1993), 189-208.

[27] M. Kafini, *uniforme decay of solutions to Cauchy viscoelastic problems with density*, Elecron. J. Differential Equations Vol.2011 (2011) No. 93, pp. 1-9.

[28] M. Kafini and S. A. Messaoudi, *On the uniform decay in viscoelastic problem in* $\mathbb{R}^n$, Appl. Math. Comput 215 (2009) 1161-1169.

[29] M. Kafini, S. A. Messaoudi and Nasser-eddine Tatar, *Decay rate of solutions for a Cauchy viscoelastic evolution equation*, Indag. Math., 22 (2011) 103-115.

[30] N.I karachalios, N.M, Stavrakakis, *Existence of global attractor for semilinear dissipative wave equations on* $\mathbb{R}^n$, J. Differential Equations, 157 (1999) 183-205.

[31] N.I karachalios, N.M, Stavrakakis,*Global existence and blow-up results for some non-linear wave equations on* $\mathbb{R}^n$, Adv. Differential Equations, 6(2) (2001) 155-174.

[32] M.C. Leseduarte, A. Magana, *Quintanilla, R.: On the time decay of solutions in porous- thermo-elasticity of type II* Discrete Contin. Dyn. Syst. Ser. B13 (2010), 375-391.

[33] J. L. Lions, *Quelques méthodes de résolution des problemes aux limites non lineaires*, Dunod, Paris, 1969.

[34] H. W. Lord, Y. Shulman, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids 15 (1967), 299-309.

[35] W. Liu, G. Li and L. Hong, *Decay of solutions for a plate equation with p-Laplacian and memory term*, Electronic J. Differential Equations 129, (2012), 1-5.

[36] Y. Liu, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal. 64 (2006) 2665-2687.

[37] Z.Y. Liu, R. Quintanilla, *Energy decay rate of a mixed type II and type III thermoelastic system*, Discrete Contin. Dyn. Syst. Ser. B14, (2010) 1433-1444.

[38] Z. Liu, B. Rao, *Characterization of polynomial decay rate for the solution of linear evolution equation*, Z. Angew. Math. Phys. 56 (2005) 630-644.

[39] Z. Liu, R. Quintanilla, *Analyticity of solutions in type III thermoelastic plates*, IMA J. Appl. Math.75 (2010), 356-365.

[40] A. Marzocchi, J. E. Munos Rivera and M. G. Naso, *Asymptotic behavior and exponential stability for a transmission problem in thermoelasticity* Math. Methods Appl. Sci., 25(2002), 955-980.

[41] A. Marzocchi, J. E. Munoz Rivera and M. G. Naso, *Transmission problem in thermoelasticity with symmetry*, IMA J. Appl. Math., 68 (2002) 23-46.

[42] S. A. Mesaoudi, B. Said-Houari, Energy decay in a transmission problem in thermoelasticity of type III, IMA Journal of Applied Mathematics, 74 (2009) 344-360.

[43] A. Moncef, *On uniform decay of a nonsimple thermoelastic bar with memory*, NJ. Math. Anal. Appl. 402 (2013), no. 2, 745-757.

[44] J. E. Munoz Rivera and M.G. Naso, *On the decay of the energy for systems with memory and indefinite dissipation*, Asymptotic. Anal. 49 (3), (2006), 189-204.

[45] J. E. Munoz Rivera and H.P. Oquendo, *The Transmission Problem of Viscoelastic Waves*, Acta Appl Math, (2000) 62: 1, pp 1-21.

[46] J. E. Munoz Rivera and M. G. Naso, *About Asymptotic Behavior for a Transmission Problem in Hyperbolic Thermoelasticity*, Acta Appl Math, (2007) 99: 1, pp 1-21.

[47] Muhammad I. Mustafa and S. A. Messaoudi, *General stability result for viscoelastic wave equations*, J. Math. Phys. 53 (2012), 053702.

[48] D. Ouchenane, Kh. Zennir and M. Bayoud, *Global nonexistence of solutions for a system of nonlinear viscoelastic wave equations with degenerate damping and source terms*, Ukrainian Mathematical Journal 65, No. 7, 654-669, 2013.

[49] Papadopulos, P.G. Stavrakakies, *Global existence and blow-up results for an equations of Kirchhoff type on* $\mathbb{R}^n$, Topol. Methods Nolinear Anal. 17, (2001), 91-109.

[50] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin (1983).

[51] Perikles G. Papadopoulos, M. Nikos Stavrakakis, *Central manifold theory for the generalized equation of Kirchhoff strings on* $\mathbb{R}^n$, Nonlinear Analysis 61, (2005), 1343-1362.

[52] C. A. Raposo, W. D. Bastosand J. A. J. Avila, *A Transmission Problem for Euler-Bernoulli beam with Kelvin-Voigt Damping*, Applied Mathematics and Information Sciences, An International Journal, 5(1)(2011) 17-28.

[53] M. Reed and B. Simon, *Methods of Mathematical Physics III: Scattering Theory*, Academic Press, New York, 1979.

[54] F. Sun and M. Wang, *Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping*, Nonlinear Analysis, (2007) 66, 2889-2910.

[55] R. Torrejon and J. M. Yong, *On a quasilinear wave equation with memory*, Nonlinear Anal.16 (1991), no. 1, 61-78.

[56] J. Wang, J. H.Zhomg and G. Q. Xu, *Energy decay rate of transmission problem between thermoelasticity of type I and type II*, Z. Angew. Math. Phys., (2017) 68:65.

[57] Z. Yong, *A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in* $\mathbb{R}^n$, Appl. Math. Lett. 18 (2005), 281-286.

[58] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. II, Monotone Operators*, Springer-Verlag, Berlin, 1990.

[59] Kh. zennir, *Growth of Solutions with Positive Initial Energy to System of Degeneratly Damped Wave Equations with Memory*, Lobachevskii Journal of Mathematics, Vol. 35, No. 2,(2014) pp. 147-156.

[60] X. Zhang, E. Zuazua, *Decay of solutions of the system of thermoelasticity of type III*, Commun. Contemp. Math. 5 (2003), 25-83.

ملخص

في هذه الأطروحة، ندرس مسألة وجود وعدم وجود الحلول الضعيفة مع تطور الزمن وكذلك السلوكياتالتقاربية لبعض المسائل الاهتزازية الزائدية اللزجة في وجود تخميدات مختلفة وقوى خارجيةمختلفة في المصادر. في هذا الصدد، أثبتنا عدة نتائج تبين كيفيات الاضمحلال تحت اعتبارات ملائمة على دالة الذاكرة وبعض معاييرالمعادلات. نستخدم طرق متعددة وجديدة مثل طريقة عمق البئر لتحديد نتائج الاستقرار المطلوبة للمسائل. نتائجنا تعمم العديد من النتائج الموجودة في أدبيات الاختصاص الدقيق.

الكلمات المفتاحية:

معادلة الموجة،اللزوجة،التخميد، النظام المزدوج،الوجود، معدل الاضمحلال، نظام النقل، الذاكرة النشيطة ، الانفجار.

Abstract:

In this thesis, we study the existence/nonexistence in time as well as the asymptotic behavior of some viscoelastic problems in the presence of different damping and different nonlinearities in the sources. In this regard, we prove several decay results under appropriate assumptions on the kernels and the structural parameters of the equations. We use the multiplier method, the well depth method to establish the desired stability results of the problems. Our results generalize many results existing in the literature.

Keywords: Wave equation, Viscoelastic, Damping, Coupled system, Existence, Decay rate, Thermo-elasticity, Transmission system, Infinite memory, Blow up.

Résumé

Dans cette thèse, nous étudions l'existence/non-existence dans le temps ainsi que le comportement asymptotique de certains problèmes viscoélastiques en présence d'amortissement différent et de non-linéarités différentes dans les sources. À cet regard, nous prouvons plusieurs résultats sous des hypothèses appropriées sur les noyaux et les paramètres structurels des équations. Nous utilisons la méthode du multiplicateur, la méthode de la profondeur du puits, pour établir les résultats souhaités en termes de stabilité des problèmes. Nos résultats généralisent de nombreux résultats existants dans la littérature.

Mots-clés: Equation des ondes, Viscoélastique, Amortissement, Système couplé, Existence, Taux de la décroissance, Thermoélasticité, Système de transmission, Mémoire infinie, Explosion en temps fini.