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Par: Salah BOULARES

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THÈME

Quelques Inégalités Intégrales Appliquées à Certaines Classes Des Equations aux Dérivées Partielles.

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Devant le jury :

Denche Mohamed	Prof.	Université des Frères Mentouri	Président
Khellaf Hassane	Prof.	Université des Frères Mentouri	Rapporteur
Ellaggoune Fateh	Prof.	Université de Guelma	Examinateur
Berkane Abdelhak	Prof.	Université des Frères Mentouri	Examinateur
Akrout Kamel	Prof.	Université Larbi Tébessi-Tébessa	Examinateur
Memou Ameur	MCA.	Université de M'sila	Examinateur

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Introduction

The integral inequalities play a fundamental role in the theory of differential and integral equations. There are a necessary tool in the study of various classes of equations. In the past few years, many authors have established several linear integral inequalities, non linear integral inequalities, integrodifferential inequalities and delay integral inequalities and presented some of its applications to the qualitative study of differential equations, we refer the reader to literatures [4, 29, 34, 40].

With the development of science and technology, an important type of inequalities, such as the Gronwall Bellman type inequality, the Gronwall type inequality, the Henry-Bihari type inequality, the Henry-Gronwall type inequality, occupies a great place in the research works of the modelling of engineering and science problems as well as the qualitative analysis of the solutions to differential equations. Nowadays, after the development that have seen the fractional differential equations, integral inequalities with weakly singular kernels has become greater [16, 28, 32, 33, 48, 50]. Henry [16] suggested a recent method to search solutions and to prove some results about linear integral inequalities with a weakly singular kernel. Ye et al. in [48] have worked with a generalized inequality to investigate the dependence of the solution for a fractional differential equation.

The inequalities are inadequate and it is necessary to seek some new integral inequalities, delay integral inequalities, in the case of the functions with one and several variables for used as tools to study the existence, uniqueness, stability, Ulam stability and continuous dependence of the solution for some classes of partial differential equations, delay partial differential equations, integral equations. Let us give the review of each chapter of the thesis.

In chapter 1 we present a number of classical facts in the domain of Gronwall inequalities and some non-linear inequalities in one variable and in several variables, in the last section we present some linear and non-linear fractional integral inequalities.

In the chapter 2 We will study the same non-linear integral inequalities for two-variable functions, which are studied by [12] with the term delay.

in chapter 3 we establish some non-linear retarded integral inequalities for functions of n independent variables which can be used as handy tools in the theory of partial differential and integral equations. These new inequalities represent a generalization of the results obtained in [17]. Some applications of our results are also given.

In the first section of chapter 4 we give some necessary concepts on the generalized fractional and conformable fractional calculus. In the second section of chapter 4, the main contribution, using the method introduced by Zhu [50] novel weakly singular integral inequalities are established. In the third section of of chapter 4, we study the following inequalities type

$$u(t) \leq a(t) + b(t) \int_a^t f(s) u(s) d_\alpha s + \int_a^t f(t) W\left(\int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau \right) d_\alpha s,$$

$$u(t) \leq a(t) + b(t) \int_a^t f(s) g(u(s)) d_\alpha s + \int_a^t f(t) W\left(\int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau \right) d_\alpha s.$$

Where $a(.), b(.), f(.), W(.), \Phi(.)$ and $k(.,.)$ are given functions satisfied some conditions supposed later. This section is based on Rui A. C. Ferreira and Delfim F. M. Torres [14], we generalised the results in conformable fractional version integral inequalities with the help of the Katugampola conformable fractional calculus. In the fourth section of of chapter 4, we give an appliction for the second and third section to illustrate the usefulness of our results, such that we present the existence, uniqueness and Ulam stability for the solution of the following problem

$$\begin{cases} {}^C D_{0^+}^{\beta,\chi} x(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{1}$$

where ${}^C D_{a^+}^{\beta,\chi}$ is the Caputo derivatives with respect to χ, $\beta \in (0,1)$ and the continuous function $f : J \times R \to R$ satisfying some conditions will be specified later for the second section, and we gives a bound on the solution of the following integral equation

$$u(t) = k + \int_0^{\lambda(t)} F\left(s, u(s), \int_0^s K(\tau, u(\tau)) d_\alpha \tau \right) d_\alpha s, \quad t \in [0,b],$$

for the third section.

Chapter 1

Classical Gronwall Inequalities

This chapter is presenting a number of classical facts in the domain of Gronwall inequalities, some non-linear inequalities in one variable and in several variables, in the last section we present some linear and non-linear fractional integral inequalities.

1.1 Some Linear Gronwall Inequalities

In 1919, Gronwall in [15] proved the following linear Gronwall's inequality.

Theorem 1.1. *Let $u(t)$, $a(t)$ and $k(t)$ be real continuous functions defined in $[a,b]$, $a(t) \geq 0$, for $t \in [a,b]$. Assume that*

$$u(t) \leq a(t) + \int_a^t k(s)\, u(s)\, ds,\ \ t \in [a,b]. \tag{1.1}$$

Then

$$u(t) \leq a(t) + \int_a^t a(s)\, k(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds,\ \ t \in [a,b]. \tag{1.2}$$

Proof. Define the function

$$y(t) = \int_a^t k(s)\, u(s)\, ds$$

for $t \in [a,b]$. Then we have $y(a) = 0$, and

$$\begin{aligned} y'(t) &= k(t)\, u(t) \\ &\leq a(t)\, k(t) + y(t)\, k(t), \quad t \in [a,b]. \end{aligned}$$

By multiplication with $\exp\left(-\int_a^t k(\tau)\, d\tau\right)$, we obtain

$$\frac{d}{dt}\left(y(t) \exp\left(-\int_a^t k(\tau)\, d\tau\right)\right) \leq a(t)\, k(t) \exp\left(-\int_a^t k(\tau)\, d\tau\right),$$

By integration on $[a,t]$, one gets

$$y(t) \exp\left(-\int_a^t k(\tau)\, d\tau\right) \leq \int_a^t a(s)\, k(s) \exp\left(-\int_a^s k(\tau)\, d\tau\right) ds,$$

then

$$y(t) \leq \int_a^t a(s)\, k(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds.$$

Since

$$u(t) \leq a(t) + y(t),$$

we find

$$u(t) \leq a(t) + \int_a^t a(s)\, k(s) \exp\left(\int_s^t k(\tau)\, d\tau\right) ds.$$

□

Corollary 1.1. *If* a *is a constant in (1.1) then (1.2) become*

$$u(t) \leq a \exp\left(\int_a^t b(s)\,ds\right).$$

Giuliano, Kharlamov,Willet, and Beesack in [7] investigated the following integral inequalities

Theorem 1.2. *Let* $u(t)$ *and* $k(t)$ *be continuous functions in* $[a,b]$, *and let* $a(t)$ *and* $b(t)$ *be Riemann integrable functions in* $[a,b]$, *with* $k(t)$ *and* $b(t)$ *are non-negative in* $[a,b]$.
i) If

$$u(t) \leq a(t) + b(t)\int_a^t k(s)\,u(s)\,ds,\ \ t \in [a,b]. \tag{1.3}$$

Then

$$u(t) \leq a(t) + b(t)\int_a^t a(s)\,k(s)\exp\left(\int_s^t b(\tau)\,k(\tau)\,d\tau\right)ds,\ \ t \in [a,b]. \tag{1.4}$$

ii) If $\leq$ *is replaced by* $\geq$ *in both (1.3) and (1.4), the result remain valid.*
iii) Both i) and ii) remain valid if $\int_a^t$ *is replaced by* $\int_t^b$ *and* $\int_s^t$ *by* $\int_t^s$ *throughout.*

Theorem 1.2 can be generalized as follows

Corollary 1.2. *[5] Let* $u(t)$ *and* $k_i(t)$ $(i = 1,2,...,n)$ *be continuous functions in* $[a,b]$, *and let* $a(t)$ *and* $b_i(t)$ *be Riemann integrable functions in* $[a,b]$, *with* $k_i(t)$ *and* $b_i(t)$ $(i = 1,2,...,n)$ *be non-negative in* $[a,b]$. *Assume that*

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t)\int_a^t k_i(s)\,u(s)\,ds,\ \ t \in [a,b].$$

Then

$$u(t) \leq a(t) + b(t)\int_a^t a(s)\sum_{i=1}^{n} k_i(s)\exp\left(\int_s^t b(\tau)\sum_{i=1}^{n} k_i(\tau)\,d\tau\right)ds,\ \ t \in [a,b].$$

1.2 Some non-linear integral inequalities in one variable

We present the definition of subadditive and submultiplicative functions:

Definition 1.1. *A function* $f : \mathbb{R}_+ \to \mathbb{R}_+$, *is said to be*
i) subadditive if $f(x+y) \leq f(x) + f(y)$, *for* $x, y \in \mathbb{R}_+$.
ii) submiltiplicative if $f(xy) \leq f(x)\,f(y)$.

Pachpatte in [41] have presented the following integral inequalities

Theorem 1.3. *Let $u(t), f(t), g(t)$ and $h(t)$ be non-negative continuous functions defined on $\mathbb{R}_+$. Let $w(u)$ be a continuous non-decreasing and submultiplicative function defined on $\mathbb{R}_+$ and $w(u) > 0$ on $\mathbb{R}_+$. If*

$$u(t) \leq u_0 + g(t) \int_0^t f(s) u(s) \, ds + \int_0^t h(s) w(u(s)) \, ds,$$

for all $t \in \mathbb{R}_+$, where u_0 is a positive constant, then for $0 \leq t \leq t_1$, we have

$$u(t) \leq a(t) G^{-1} \left[G(u_0) + \int_0^t h(s) w(u(s)) \, ds \right],$$

where

$$a(t) = 1 + g(t) \int_0^t f(s) \exp\left(\int_0^t g(\tau) f(\tau) \, d\tau \right) ds, \tag{1.5}$$

for $t \in \mathbb{R}_+$, and

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)}, r > 0, r_0 > 0,$$

and G^{-1} is the inverse function of G, and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G(u_0) + \int_0^t h(s) w(u(s)) \, ds \in Don\left(G^{-1}\right),$$

for all $t \in \mathbb{R}_+$ lying in the interval $[0, t_1]$.

Pachpatte in [41] also proved the following new generalization of the past theorem.

Theorem 1.4. *Let $u(t), f(t), g(t)$ and $h(t)$ be non-negative continuous functions defined on $\mathbb{R}_+$. Let $w(u)$ be a continuous non-decreasing and submultiplicative function defined on $\mathbb{R}_+$ and $w(u) > 0$ on $\mathbb{R}_+$. Let $p(t) > 0, \phi(t) \geq 0$ be continuous and non-decreasing functions defined on $\mathbb{R}_+$, and $\phi(0) = 0$. If*

$$u(t) \leq p(t) + g(t) \int_0^t f(s) u(s) \, ds + \phi\left(\int_0^t h(s) w(u(s)) \, ds \right),$$

for all $t \in \mathbb{R}_+$, then for $0 \leq t \leq t_2$,

$$u(t) \leq a(t) \left[p(t) + \phi^{-1}\left(F^{-1}\left[F(A(t)) + \int_0^t h(s) w(u(s)) \, ds \right] \right) \right],$$

where $a(t)$ is defined by (1.5) and

$$A(t) = \int_0^t h(s) w(a(s) p(s)) \, ds,$$

for $t \in \mathbb{R}_+$, and

$$F(r) = \int_{r_0}^{r} \frac{ds}{w(\phi(s))}, r > 0, r_0 > 0,$$

and F^{-1} is the inverse function of F, and $t_2 \in \mathbb{R}_+$ is chosen so that -1

$$F(A(t)) + \int_0^t h(s)\, w(u(s))\, ds \in Don\left(F^{-1}\right),$$

for all $t \in \mathbb{R}_+$ lying in the interval $[0, t_2]$.

Remark 1.1. *Theorem 1.4, become Theorem 1.3, for $p(t)$ is a constant and ϕ is the identity function.*

1.3 Some non-linear integral inequalities in Several Variables

During the past few years, the discovery and the application of the new generalizations of the Gronwall-Bellman inequality in more than one independent variables have attracted the interest of many authors. In [37], Pachpatte considered the finite difference inequality in two independent variables. Integral inequalities of the Gollwitzer type in n independent variables are established by Yang in [13] .

Throughout this section, we assume that $I =]a, b[$ in any bounded open set in the dimensional Euclidean space and that our integrals are on $\mathbb{R}^n\,(n \geq 1)$, where $a = (a_1, a_2, ..., a_n)\,, b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n_+$. For $x = (x_1, x_2, ..., x_n)\,, t = (t_1, t_2, ..., t_n) \in I$, we shall denote the integral

$$\int_a^x = \int_{a_1}^{x_1} \int_{a_2}^{x_2} .... \int_{a_n}^{x_n} ...dt_n...dt_1$$

Furthermore, for $x, t \in \mathbb{R}^n$, we shall write $t \leq x$ whenever $t_i \leq x_i, i = 1, 2, ..., n$ and $0 \leq a \leq x \leq b$, for $x \in I$ and $D = D_1 D_2 ... D_n$,where $D_1 = \frac{\partial}{\partial x_i}$, for $i = 1, 2, ..., n$ Let $C(I, \mathbb{R}_+)$ denote the class of continuous functions from I to $\mathbb{R}_+$. The following theorem deals with n-independent variables versions of the inequalities established in Pachpatte [36]

Theorem 1.5. *Let $u(x), f(x), a(x) \in C(I, \mathbb{R}_+)$ and let $K(x, t), D_i K(x, t)$ be in $C(I \times I, \mathbb{R}_+)$ for all $i = 1, 2, ..., n$ and c be a non-negative constant.*
1) If

$$u(t) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(s, \tau)\, u(\tau)\, d\tau\right] ds, \tag{1.6}$$

For $x \in I$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq c \left[1 + \int_a^x f(t) \exp\left(\int_a^t f(s) + k(b, s)\, ds\right) dt\right]. \tag{1.7}$$

2) If

$$u(t) \leq a(x) + \int_a^x f(s)\left[u(s) + \int_a^s k(s,\tau)\, u(\tau)\, d\tau\right] ds, \tag{1.8}$$

For $x \in I$ and $a \leq \tau \leq s \leq b$, then

$$u(x) \leq a(x) + e(x)\left[1 + \int_a^x f(t) \exp\left(\int_a^t f(s) + k(b,s)\, ds\right) dt\right], \tag{1.9}$$

Where

$$e(x) = \int_a^x f(s)\left[a(s) + \int_a^s k(s,\tau)\, a(\tau)\, d\tau\right] ds. \tag{1.10}$$

Theorem 1.6. *Let $u(x), f(x)\, a(x)$ and $k(x,t)$ be as defined in Theorem 1.5. Let $\Phi(u(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(x) \geq 0$ and let $W(u(x))$ be real-valued, positive, continuous,and non-decreasing function defined for $x \in I$. Assume that $a(x)$ is positive continuous function and non-decreasing for $x \in I$.*
If

$$u(x) \leq a(x) + \int_a^x f(t)\, g(u(t))\, dt + \int_a^x f(t)\, W\left(\int_a^t k(t,s)\, \Phi(u(s))\, ds\right) dt,$$

For $a \leq s \leq t \leq x \leq b$, then for $a \leq x \leq x^*$,

$$u(x) \leq \beta(x)\left\{a(x) + \int_a^x f(t)\, W\left[\Psi^{-1}\left(\Psi(\eta) + \int_a^t k(b,s)\, \Phi\left[\beta(s) \int_a^s f(\tau)\, d\tau\right] ds\right)\right] dt\right\},$$

Where

$$\beta(x) = G^{-1}\left(G(1) + \int_a^x f(s)\, ds\right),$$

$$\eta = \int_a^b k(b,s)\, \Phi(\beta(s)\, a(s))\, ds,$$

$$G(u) = \int_{u_0}^u \frac{1}{g(z)} dz, \quad u > 0\, (u_0 > 0),$$

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \geq x_0 > 0.$$

where G^{-1} is the inverse function of G, and Ψ is the inverse function of Ψ^{-1}, x^* is chosen so that $G(1) + \int_a^x ft(s)\, ds$ is in the domain of G^{-1}, and

$$\Psi(\eta) + \int_a^t k(b,s)\, \Phi\left[\beta(s) \int_a^s f(\tau)\, d\tau\right] ds,$$

is in the domain of Ψ^{-1}.

Remark 1.2. *Theorem 1.6, become Theorem 1.5, for a is a constant, $f \equiv 1$ and g, W and Φ are the identity functions*

1.4 Some fractional integral inequalities

1.4.1 Some definitions

We introduce some important functions and some necessary concepts on the fractional calculus, namely the Gamma function, the Riemann-Liouville integral and derivative, the Caputo derivative.

Definition 1.2. *[20] The Gamma function* $\Gamma(.)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

which converges in the right half of the complex plane, that is, $\mathrm{Re}(z) > 0$.
The Gamma function satisfies

$$\Gamma(z+1) = z\Gamma(z), \ \mathrm{Re}(z) > 0$$

and for any integer $n \geq 0$, *we have*

$$\Gamma(n+1) = n!.$$

Definition 1.3. *[20] The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $f : (a, +\infty) \to \mathbb{R}$ *is given by*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds$$

provided that the right side is pointwise defined on $(a, +\infty)$.

Definition 1.4. *[20] The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *of a function* $f : (a, +\infty) \to \mathbb{R}$ *is given by*

$$D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{ds}\right)^n \int_a^x \frac{f(s)}{(t-s)^{\alpha-n+1}} ds = \left(\frac{d}{ds}\right)^n I_{a^+}^{n-\alpha} f(s),$$

provided that the right side is pointwise defined on $(a, +\infty)$, *where* $n = [\alpha] + 1$, $[\alpha]$ *denotes the integer part of* α.

Definition 1.5. *[20] Let* $\alpha > 0$ *and* $n = [\alpha] + 1$, *for a function* $f \in AC^n([a, b], \mathbb{R})$ *the Caputo fractional derivative of order* α *of* f *is defined by*

$$\begin{aligned} \left({}^C D_{a^+}^{\alpha} f\right)(t) &= I^{n-\alpha} D^{(n)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds. \end{aligned}$$

where $D = \frac{d}{dt}$ *denotes the classical derivative and* $AC^n[a, b] = \{f \in C^{n-1}[a, b], f^{(n-1)}$ *absolutely continuous function*$\}$.

Lemma 1.1. *[20] Let $\alpha > 0$, $n = [\alpha] + 1$ and $f : [a, b] \to \mathbb{R}$ be a given fonction. Assume that $D_{a^+}^{\alpha} f$ and ${}^C D_{a^+}^{\alpha} f$ exist. Then*

$$ {}^C D_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha} . $$

Lemma 1.2. *[20] Let $\alpha > 0$, $n = [\alpha] + 1$. If $f \in L^1[a, b]$ and $f_{n-\alpha} \in AC^n[a, b]$, then the equality*

$$ \left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f\right)(t) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)} (t-a)^{\alpha-j} . $$

holds almost everywhere on $[a, b]$. In particular, if $0 < \alpha < 1$, then

$$ \left(I_{a^+}^{\alpha} D_{a^+}^{\alpha} f\right)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1} , $$

where $f_{n-\alpha} = I_{a^+}^{n-\alpha} f$ and $f_{1-\alpha} = I_{a^+}^{1-\alpha} f$.

Let $\alpha > 0$, then we have

$$ \left(I_{a^+}^{\alpha C} D_{a^+}^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^{k} . $$

1.4.2 Some Classical fractional integral inequalities

The following theorem given the generalized singular Gronwall inequality (see [48]).

Theorem 1.7. *Suppose $\beta > 0$, $f(t)$ is a non-negative function locally integrable on $[a, b)$, $b \leq \infty$ and $g(t)$ is a non-negative, non-decreasing continuous function defined on $g(t) \leq M$, $t \in [a, b)$, and suppose $u(t)$ is non-negative and locally integrable on $[a, b)$ with*

$$ u(t) \leq f(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds, \qquad t \in [a, b) . $$

Then

$$ u(t) \leq f(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} f(s)\, ds \right], \qquad t \in [a, b) . $$

If $f(t) = 0$ for all $t \in [a, b)$we find $u(t) = 0$ for all $t \in [a, b)$.

Proof. Let $B\phi(t) = g(t) \int_0^t (t-s)^{\beta-1} \phi(s)\, ds$, $t \geq 0$ for locally integrable functions ϕ. Then

$$ u(t) \leq f(t) + Bu(t) $$

Implies

$$ u(t) \leq \sum_{k=0}^{n-1} B^k f(t) + B^n u(t) $$

Let us prove that

$$B^n u(t) \leq \int_0^t \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1} u(s)\,ds \tag{1.11}$$

And $B^n u(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for each t in $0 \leq t \leq T$.
We know this relation (1.11) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k+1$, then the induction hephotessis implies

$$B^{k+1}u(t) = B\left(B^k u(t)\right) \leq g(t)\int_0^t (t-s)^{\beta-1}\left(\int_0^s \frac{(g(s)\Gamma(\beta))^k}{\Gamma(k\beta)}(s-\tau)^{k\beta-1} u(\tau)\,d\tau\right) ds.$$

Since $g(t)$ is nondecreasing, it follows that

$$B^{k+1}u(t) \leq (g(t))^{k+1}\int_0^t (t-s)^{\beta-1}\left(\int_0^s \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}(s-\tau)^{k\beta-1} u(\tau)\,d\tau\right) ds.$$

this implies that

$$\begin{aligned} B^{k+1}u(t) &\leq (g(t))^{k+1}\int_0^t \left(\int_\tau^t \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)}(t-s)^{\beta-1}(s-\tau)^{k\beta-1}\,ds\right) u(\tau)\,d\tau \\ &= \int_0^t \frac{(g(t)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)}(t-s)^{(k+1)\beta-1} u(s)\,ds, \end{aligned}$$

where the integral

$$\begin{aligned} \int_\tau^t (t-s)^{\beta-1}(s-\tau)^{k\beta-1}\,ds &= (t-\tau)^{k\beta+\beta-1}\int_0^1 (1-z)^{\beta-1} z^{k\beta-1}dz \\ &= (t-\tau)^{(k+1)\beta-1} B(k\beta,\beta) \\ &= \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}(t-\tau)^{(k+1)\beta-1}, \end{aligned}$$

is evaluated with the help of the substitution $s = \tau + z(t-\tau)$ and the definition of the beta function see [42]. The relation (1.11) is proved.
Since $B^n u(T) \leq \int_0^t \frac{(M\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1} u(s)\,ds \longrightarrow 0$ as $n \longrightarrow +\infty$ for $t \in [0,T)$, the Theorem is proved. □

Corollary 1.3. *Under the hypothesis of Theorem 1.7, let $f(t)$ be a non-decreasing function on $[a,b)$. Then we have*

$$u(t) \leq f(t)\,E_\beta\left(g(t)\Gamma(\beta)t^\beta\right),$$

where $E_\beta(t)$ is the Mittag-Leffler function defined by $E_\beta(t) = \sum_{n=1}^{\infty}\frac{t^n}{\Gamma(n\beta+1)}$ for $t > 0$.

The following new type of Gronwall-Bellman fractional integral inequality is proved by Wu, Qiong in [43].

Theorem 1.8. *Suppose* $0 < \beta < 1$*,and consider the interval* $I = [0, b)$ *where* $b \leq \infty$*. Suppose* $f(t)$ *is a non-negative function, which is locally integrable on* I *and* $h(t)$ *and* $g(t)$ *are non-negative, non-decreasing continuous function defined on* I*, with both bounded by a positive constant* M*. If* $u(t)$ *is non-negative, and locally integrable on* I *and satisfies*

$$u(t) \leq f(t) + h(t) \int_0^t u(s)\, ds + g(t) \int_0^t (t-s)^{\beta-1} u(s)\, ds, \qquad t \in I.$$

Then

$$u(t) \leq f(t) + \sum_{n=1}^{\infty} \sum_{i=1}^{n} \left[C_i^n h^{n-i}(t)\, g^i(t) \frac{(\Gamma(\beta))^n}{\Gamma(i\beta + n - i)} \int_0^t (t-s)^{i\beta-(i+1-n)} f(s)\, ds \right], \qquad t \in I.$$

Proof. Let ϕ be a locally integrable function and define an operator B on ϕ as follows

$$B\phi(t) = B(t) \int_0^t B(s)\, ds + g(t) \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \ t \geq 0 \tag{1.12}$$

From the inequality, 1.12 we obtain

$$u(t) \leq a(t) + Bu(t).$$

This implies

$$u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t). \tag{1.13}$$

As a similar proof of Theorem 1.7 we find

$$B^n u(t) \leq \sum_{i=0}^{n} C_i^n b^{n-i}(t)\, g^i(t) \frac{(\Gamma(\alpha))^n}{\Gamma(i\alpha + n - i)} \int_0^t (t-s)^{(i\alpha-(i+1-n))} a(s)\, ds,$$

and $B^n u(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for each t in $0 \leq t \leq T$, thus we get the desired inequality. □

Remark 1.3. *Theorem 1.8, become Theorem 1.7, for* $h \equiv 0$ *and* $0 < \beta < 1$*.*

Corollary 1.4. *Suppose the conditions in Theorem 1.8 are satisfied and furthermore,* $f(t)$ *is non-decreasing on* $0 \leq t < T$*.*

$$u(t) \leq f(t)\, E_\beta \left(g(t)\, \Gamma(\beta)\, t^\beta \right) \exp\left(\frac{1}{\beta} h(t)\, t \right).$$

Proof. From the proof of Theorem 1.8,

$$u(t) \leq a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} C_i^n b^{n-i}(t)\, g^i(t) \frac{(\Gamma(\alpha))\, i}{\Gamma(i\alpha + n - i)} \int_0^t (t-s)^{(i\alpha-(i+1-n))} a(s)\, ds$$

Since $a(t)$ is nondecreasing,

$$
\begin{aligned}
u(t) &\leq a(t)\sum_{n=1}^{\infty}\sum_{i=0}^{n} C_i^n b^{n-i}(t)\, g^i(t)\, \frac{(\Gamma(\alpha))\, i}{\Gamma(i\alpha+n-i)} \int_0^t (t-s)^{(i\alpha-(i+1-n))}\, a(s)\, ds \\
&\leq a(t) + \sum_{n=1}^{\infty}\sum_{i=0}^{n} C_i^n b^{n-i}(t)\, g^i(t)\, \frac{(\Gamma(\alpha))\, i}{\Gamma(i\alpha+n-i+1)} t^{i\alpha+n-i} \\
&\leq a(t)\, E_\alpha\left(g(t)\,\Gamma(\alpha)\, t^\alpha\right) \exp\left(\frac{1}{\alpha} b(t)\, t\right).
\end{aligned}
$$

This completes the proof. □

Medved, in [33] investigated the following fractional Integral Inequalities of Henry type

Definition 1.6. *Let $q > 0$, be a real number and $0 < T \leq \infty$, we say that a function $w : \mathbb{R}_+ \to \mathbb{R}$ satisfies a condition (q), if*

$$
\exp(-qt)\,[w(u)]^q \leq R(t)\, w\left(\exp(-qt)\, u^q\right), \quad \text{for all } u \in \mathbb{R}_+, t \in [0,T). \tag{1.14}
$$

where $R(t)$ is a continuous, non-negative function.

Example 1.1. *If $w(u) = u^m$, $m > 0$ then $e^{-qt}(w(u))^q = e^{(m-1)qt} w\left(e^{-qt}u^q\right)$ for any $q > 1$, i.e., the condition 1.14 is satisfied with $R(t) = e^{(m-1)qt}$.*

Theorem 1.9. *Let $f(t)$ be a non-decreasing, non-negative C^1-function on $[0,T)$, $g(t)$ be a continuous, non-negative function on $[0,T)$, $w : \mathbb{R}_+ \to \mathbb{R}$ be a continuous, non-decreasing function, $w(0) = 0$, $w(u) > 0$, on $[0,T)$ and $u(t)$ be a continuous, non-negative function on $[0,T)$, with*

$$
u(t) \leq f(t) + \int_0^t (t-s)^{\beta-1}\, g(s)\, w(u(s))\, ds, \quad t \in [0,T),
$$

where $\beta > 0$, Then the following assertions hold:
i) Suppose $\beta > \frac{1}{2}$ and w satisfies the condition (1.14) with $q = 2$. Then

$$
u(t) \leq \exp(t)\left\{\Omega^{-1}\left[\Omega\left(2f(t)^2\right) + g_1(t)\right]\right\}^{\frac{1}{2}}, \quad t \in (0, T_1),
$$

where

$$
g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t R(s)\, g(s)^2\, ds,
$$

$\Omega(r) = \int_{r_0}^{r} \frac{ds}{w(s)}$, $r_0 > 0$, Ω^{-1} is the inverse of Ω^{-1} and $T_1 \in \mathbb{R}_+$ such that $\Omega\left(2f(t)^2\right) + g_1(t) \in Dom\left(\Omega^{-1}\right)$ for all $t \in (0, T_1)$.
ii) Let $\beta \in \left(0, \frac{1}{2}\right)$, and w satisfies the condition (1.14) with $q = z + 2$, where $z = \frac{1-\beta}{\beta}$. Then

$$
u(t) \leq \exp(t)\left\{\Omega^{-1}\left[\Omega\left(2^{q-1} f(t)^q\right) + g_2(t)\right]\right\}^{\frac{1}{q}}, \quad t \in (0, T_2),
$$

where

$$g_2(t) = 2^{q-1} K_z^q \int_0^t R(s)\, g(s)^q \, ds,$$

$$K_z = \frac{\Gamma(1-\alpha q)}{1-\alpha q}, \quad \alpha = \frac{z}{z+1}, \quad q = \frac{z+2}{z+1}. \tag{1.15}$$

$T_2 \in \mathbb{R}_+$ *such that* $\Omega\left(2^{q-1} f(t)^q\right) + g_2(t) \in Dom\left(\Omega^{-1}\right)$ *for all* $t \in (0, T_2)$.

Theorem 1.10. *Suppose* $f(t), h(t)$ *are non-negative, integrable functions on* $[0,T)$ *and* $g(t), u(t)$ *are integrable, non-negative functions on* $[0,T)$ *with*

$$u(t) \leq f(t) + h(t) \int_0^t (t-s)^{\beta-1} g(s)\, u(s)\, ds, \qquad a.e\ .on\ [0,T)\ .$$

then the following assertions hold :
i) If $\beta \geq \frac{1}{2}$ *then*

$$u(t) \leq \exp(t)\, \Phi(t)^{\frac{1}{2}}, \quad a.e\ .on\ [0,T)$$

Where

$$\Phi(t) = 2f(t)^2 + 2Kh(t)^2 \int_0^t f(s)^2 g(s)^2 \exp\left[K \int_s^t h(r)^2 g(r)^2 \, dr\right] ds,$$

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}$$

ii) $\beta = \frac{1}{z+1}$ *for some* $z \geq 1$ *then*

$$u(t) \leq \exp(t)\, \Psi(t)^{\frac{1}{q}}, \quad a.e\ .on\ [0,T)\ ,$$

where

$$\Psi(t) = 2^{q-1} f(t)^q + 2^{q-1} K_z^q h(t)^q \int_0^t f(s)^q g(s)^q \exp\left[2^{q-1} K_z^q \int_s^t h(r)^q g(r)^q \, dr\right] ds,$$

$q = z+2$, *and* K_z *is defined by (1.15).*

1.5 Some applications

We present three examples of application to study respectively the boundless, uniqueness and Ulam stability of the solution of the following fractional Cauchy problem

$$\begin{cases} {}^C D^{\alpha} u(t) + \lambda u(t) = f(t, u(t)), \ t \in J = [0,T], \\ u(0) = u_0, \end{cases} \tag{1.16}$$

where ${}^C D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continous function. Next, we introduce the following assumptions

H1) there exists $\varphi, \psi \in C(J, \mathbb{R}_+)$, such that

$$|f(t,u)| \leq \varphi(t)|u| + \psi(t) \text{ for all} t \in J, \text{ and all } u \in \mathbb{R}.$$

H2) There exists $L_f > 0$, such that

$$|f(t,u_1) - f(t,u_2)| \leq L_f |u_1 - u_2| \text{ for all} t \in J \text{ and all } u_1, u_2 \in \mathbb{R}.$$

Clearly that if H1) is satisfied. Then there exist at least one solution of (1.16) note $u(t)$, such that $u(t)$ is a solution of the following integrel equation

$$u(t) = u_0 E_\alpha(-\lambda t^\alpha) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s, u(s))\, ds.$$

We consider the following inequality

$$\left|{}^C D^\alpha u(t) + \lambda u(t) - f(t, u(t))\right| < \epsilon, \text{ for } \epsilon > 0. \tag{1.17}$$

Theorem 1.11. *Suppose that H1) is satisfied. Then*

$$|u(t)| \leq \left(|u_0| + \frac{\psi^*}{\Gamma(\alpha+1)} T^\alpha\right) E_\alpha(\varphi^* t^\alpha).$$

where $\varphi^* = \sup_{t \in J} \varphi(t)$, $\psi^* = \sup_{t \in J} \psi(t)$.

Proof. We have

$$u(t) = u_0 E_\alpha(-\lambda t^\alpha) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s, u(s))\, ds,$$

From H1) and by the fact of $E_\alpha(-\lambda t^\alpha) \leq 1$, $E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$, for any $\lambda > 0$, and $t \in J$. we find

$$\begin{aligned} |u(t)| &= |u_0 E_\alpha(-\lambda t^\alpha)| + \int_0^t (t-s)^{\alpha-1} |E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)| |f(s,u(s))| ds \\ &\leq |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) |u(s)|\, ds \\ &\leq |u_0| + \frac{\psi^*}{\Gamma(\alpha+1)} T^\alpha + \frac{\varphi^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)|\, ds. \end{aligned}$$

Using Corollary 1.3, we get

$$|u(t)| \leq \left(|u_0| + \frac{\psi^*}{\Gamma(\alpha+1)} T^\alpha\right) E_\alpha(\varphi^* t^\alpha).$$

□

Theorem 1.12. *Suppose that H2) is satisfied. Then Eq. (1.16) has a unique solution on J.*

Proof. we suppose that $u_1(t), u_2(t)$ are two solutions of Eq (1.1). Then

$$\begin{aligned} |u_1(t) - u_2(t)| &= \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^{\alpha}) (f(s, u_1(s)) - f(s, u_2(s))) ds \right| \\ &\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u_1(s) - u_2(s)| ds. \end{aligned}$$

Using Theorem 1.7, we obtain $u_1 = u_2$. □

Definition 1.7. *Eq. (1.16) is Ulam-Hyers stable if there exists* $c > 0$, *such that for every* $\epsilon > 0$, *and for every solution* v *of (1.17) there is a solution* u *of Eq. (1.16) with*

$$|u(t) - v(t)| \leq \epsilon E_{\alpha}(L_f t^{\alpha}) c, \qquad \text{for all} \quad t \in J.$$

Remark 1.4. *If* v *is a solution of (1.17) then* v *is a solution of*

$$\left| v(t) - v(0) E_{\alpha}(-\lambda t^{\alpha}) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^{\alpha}) f(s, v(s)) ds \right| \leq \frac{\epsilon}{\Gamma(\alpha+1)} T^{\alpha}.$$

Theorem 1.13. *[18] Suppose that H2) is satisfied. Then,the solution of (1.16) is Ulam-Hyers stable.*

Proof. Let v be a solution of (1.17) and u the unique solution of the following problem

$$\begin{cases} {}^{C}D^{\alpha} u(t) + \lambda u(t) = f(t, u(t)), \ \alpha \in (0,1), t \in J, \\ u(0) = v(0), \end{cases}$$

then

$$u(t) = v(0) E_{\alpha}(-\lambda t^{\alpha}) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^{\alpha}) f(s, u(s)) ds.$$

From Remark 1.4, we find

$$\begin{aligned}
& |v(t) - u(t)| \\
\leq & \left| v(t) - v(0) E_\alpha(-\lambda t^\alpha) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s, u(s)) \, ds \right| \\
\leq & \left| y(t) - v(0) E_\alpha(-\lambda t^\alpha) - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s, v(s)) \, ds \right. \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s, v(s)) \, ds \\
& \left. - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s, u(s)) \, ds \right| \\
\leq & \frac{\epsilon}{\Gamma(\alpha+1)} T^\alpha + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, v(s)) - f(s, u(s))| \, ds \\
\leq & \frac{\epsilon}{\Gamma(\alpha+1)} T^\alpha + \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s) - u(s)| \, .
\end{aligned}$$

Using Corollary 1.3, we get

$$|v(t) - u(t)| \leq \frac{\epsilon}{\Gamma(\alpha+1)} T^\alpha E_\alpha(L_f t^\alpha) \, .$$

Thus, Eq. (1.16) is Ulam-Hyers stable. □

Chapter 2

On Some Non-Linear Integral Inequalities For Two Variable Functions With A Term Of Delay

Dragomir and Kim ([12], Theorem 2.3) have presented the following type of integral inequalities for functions with two variables without term of delay

$$\begin{aligned} u(x,y) &\leq a(x,y) + f(x,y)\, H\left(\int_0^x \int_y^\infty d(s,t)\, W(u(s,t))\, dtds\right) \\ &+ b(x,y) \int_\alpha^x c(s,y)\, u(s,y)\, ds. \end{aligned}$$

In this chapter we present some non-linear delay Integral Inequalities for two variable functions with a term of delay, secondly we establish some further delay Integral inequalities. Finally, applications to typical are presented to show the efficiency of the proposed approach. we present three examples of application to study respectively the boundless, uniqueness and stability of the solution for a problem of differential equations with delay. The preset results in press are a generalisation of some inequalities proved in [12].

2.1 Certain Non-Linear Integral Inequalities with a term of delay

We start by proving some lemmas, which we use in this chapter

Lemma 2.1. *Let $u(t)$, $k(t)$, $a(t)$ and $b(t)$ be Riemann integrable functions on $[0,\infty)$ with $u(t), k(t)$ and $b(t)$ non-negative on $[0,\infty)$, and $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ are non-decreasing functions, with $\alpha(t) \leq t$, $\beta(t) \geq t$ for $t > 0$,*
1. If

$$u(t) \leq a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} k(s)\, u(s)\, ds, \text{for } t_0 \leq t,\ t_0, t \in [0,\infty),$$

then

$$u(t) \leq a(t) + b(t) \int_{\alpha(t_0)}^{\alpha(t)} a(s)\, k(s) \exp\left(\int_{\alpha(s)}^{\alpha(t)} b(r)\, k(r)\, dr\right) ds, t \in [0,\infty). \tag{2.1}$$

2. If

$$u(t) \leq a(t) + b(t) \int_{\beta(t)}^{\infty} k(s)\, u(s)\, ds, \text{for } t \in [0,\infty),$$

then

$$u(t) \leq a(t) + b(t) \int_{\beta(t)}^{\infty} a(s)\, k(s) \exp\left(\int_{\beta(t)}^{\beta(s)} b(r)\, k(r)\, dr\right) ds,\ t \in [0,\infty). \tag{2.2}$$

Proof. 1. Define a function $v(t)$ by

$$v(t) = \int_{\alpha(t_0)}^{\alpha(t)} k(s)\, u(s)\, ds, \tag{2.3}$$

then $v(t_0) = 0$ and

$$u(t) \le a(t) + b(t)\, v(t), \tag{2.4}$$

by integrating (2.3) and using (2.4), we get

$$v'(t) \le a(\alpha(t))\, k(\alpha(t))\, \alpha'(t) + b(\alpha(t))\, k(\alpha(t))\, v(\alpha(t))\, \alpha'(t),$$

we multiply the last inquality by the integrating factor $\exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right)$, we get

$$\begin{aligned} &[v'(t) - b(\alpha(t))\, k(\alpha(t))\, v(t)\, \alpha'(t)] \exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right) \\ &\quad \le a(\alpha(t))\, k(\alpha(t))\, \alpha'(t) \exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right). \end{aligned}$$

It follows that

$$\frac{d}{dt}\left[v(t) \exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right)\right] \le a(\alpha(t))\, k(\alpha(t))\, \alpha'(t) \exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right).$$

By integrating the lasting inequality from t_0 to t, with the chang of variable we obtain

$$v(t) \exp\left(-\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right) \le \int_{\alpha(t_0)}^{\alpha(t)} a(s)\, k(s) \exp\left(-\int_{\alpha(t_0)}^{s} b(r)\, k(r)\, dr\right) ds.$$

Thus

$$v(t) \le \int_{\alpha(t_0)}^{\alpha(t)} a(s)\, k(s) \exp\left(-\int_{\alpha(t_0)}^{s} b(r)\, k(r)\, dr\right) \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} b(r)\, k(r)\, dr\right) ds.$$

i.e.

$$v(t) \le \int_{\alpha(t_0)}^{\alpha(t)} a(s)\, k(s) \exp\left(\int_{s}^{\alpha(t)} b(r)\, k(r)\, dr\right) ds.$$

Since $\alpha(t) \le t$, we get

$$v(t) \le \int_{\alpha(t_0)}^{\alpha(t)} a(s)\, k(s) \exp\left(\int_{\alpha(s)}^{\alpha(t)} b(r)\, k(r)\, dr\right) ds.$$

Using the bound of $v(t)$ in $u(t) \le a(t) + b(t)\, v(t)$, we get the required inequality in (2.1).

2. Dfine a function $v(t)$ by

$$v(t) = \int_{\beta(t)}^{\infty} k(s)\, u(s)\, ds, \tag{2.5}$$

then $v(\infty) = 0$ and

$$u(t) \leq a(t) + b(t)\, v(t). \tag{2.6}$$

By integrating (2.5) and using (2.6), we get

$$v'(t) = -\beta'(t)\, k(\beta(t))\, u(\beta(t)),$$

and

$$\begin{aligned} v'(t) &\geq -k(\beta(t))\,[a(\beta(t)) + b(\beta(t))\, v(\beta(t))]\,\beta'(t) \\ &\geq -b(\beta(t))\, k(\beta(t))\, v(t)\,\beta'(t) - a(\beta(t))\, k(\beta(t))\,\beta'(t), \end{aligned}$$

i.e.

$$v'(t) + b(\beta(t))\, k(\beta(t))\, v(t)\,\beta'(t) \geq -a(\beta(t))\, k(\beta(t))\,\beta'(t),$$

we multiply the last inquality by the integrating factor $\exp\left(-\int_{\beta(t)}^{\infty} b(r)\, k(r)\, dr\right)$, we have

$$\begin{aligned} &[v'(t) + b(\beta(t))\, k(\beta(t))\, v(t)\,\beta'(t)] \exp\left(-\int_{\beta(t)}^{\infty} b(r)\, k(r)\, dr\right) \\ \geq\; & -a(\beta(t))\, k(\beta(t))\,\beta'(t) \exp\left(-\int_{\beta(t)}^{\infty} b(r)\, k(r)\, dr\right). \end{aligned}$$

Thus

$$\frac{d}{dt}\left[v(t) \exp\left(-\int_{\beta(t)}^{\infty} b(r)\, k(r)\, dr\right)\right] \geq -a(\beta(t))\, k(\beta(t))\,\beta'(t) \exp\left(-\int_{\beta(t)}^{\infty} b(r)\, k(r)\, dr\right).$$

By integrating the lasting inequality from t to ∞, with the chang of variable we obtain

$$v(t) \leq \int_{\beta(t)}^{\infty} a(s)\, k(s) \exp\left(\int_{\beta(t)}^{s} b(r)\, k(r)\, dr\right) ds.$$

Since $\beta(t) \geq t$, we get

$$v(t) \leq \int_{\beta(t)}^{\infty} a(s)\, k(s) \exp\left(\int_{\beta(t)}^{\beta(s)} b(r)\, k(r)\, dr\right) ds,$$

Using the bound of $v(t)$ in $u(t) \leq a(t) + b(t)\, v(t)$, we get the required inequality in (2.2). □

Lemma 2.2. *Let $u(x,y), a(x,y), b(x,y)$ be non-negative continuous functions defined for $x, y \in \mathbb{R}_+$ and $\alpha, \beta \in C^1([0,\infty), [0,\infty))$ are non-decreasing functions.*

1. *Assume that $a(x,y)$ is non-decreasing in x and non-increasing in y and $\alpha(t) \leq t$, $\beta(t) \geq t$*

for $t \geq 0,$ *and* $\alpha(0) = 0.$ *If*

$$u(x,y) \leq a(x,y) + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} b(s,t)\, u(s,t)\, dtds \text{ for all } x,y \in \mathbb{R}_+,$$

then

$$u(x,y) \leq a(x,y) \exp\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} b(s,t)\, dtds \right). \tag{2.7}$$

2. Assume that $a(x,y)$ *is non-increasing in each of the variables* $x,y \in \mathbb{R}_+$ *and* $\alpha(t) \geq t,$ $\beta(t) \geq t$ *for* $t \geq 0,$ *and* $\alpha(\infty) = \beta(\infty) = \infty.$ *If*

$$u(x,y) \leq a(x,y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s,t)\, u(s,t)\, dtds \text{ for all } x,y \in \mathbb{R}_+,$$

then

$$u(x,y) \leq a(x,y) \exp\left(\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s,t)\, dtds \right). \tag{2.8}$$

Proof. Fix any $X, Y \in \mathbb{R}_+$. Then for $0 \leq x \leq X$ and $Y \leq y$ we have

$$u(x,y) \leq v(x,y),$$

where

$$v(x,y) = a(X,Y) + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} b(s,t)\, u(s,t)\, dtds.$$

Clearly, $v(x,y)$ is non-decreasing in x and non-increasing in y and

$$v(0,y) = a(X,Y), \tag{2.9}$$

$$\begin{aligned} \frac{\partial}{\partial x} v(x,y) &= \alpha'(x) \int_{\beta(y)}^{\infty} b(\alpha(x),t)\, u(\alpha(x),t)\, dt \\ &\leq \alpha'(x)\, v(x,y) \int_{\beta(y)}^{\infty} b(\alpha(x),t)\, dt. \end{aligned}$$

i.e.

$$\frac{\frac{\partial}{\partial x} v(x,y)}{v(x,y)} \leq \alpha'(x) \int_{\beta(y)}^{\infty} b(\alpha(x),t)\, dt.$$

By keeping y fixed in the above inequality, setting $x = s$ and integrating from 0 to x, and the change of variable, and using

$$v(x,y) \leq a(X,Y) \exp\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} b(s,t)\, dtds \right).$$

By setting $x = X$ and $y = Y$ and using the fact that $u(x,y) \leq v(x,y)$ we get the inequality in

(2.7).

2.Fix any X, Y. Then, for $X \leq x$ $Y \leq y$ we have

$$u(x,y) \leq v(x,y),$$

where

$$v(x,y) = a(X,Y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s,t)\, u(s,t)\, dtds.$$

Clearly, $v(x,y)$ is non-increasing in each variable $x,\ y \in \mathbb{R}_+$ and

$$v(\infty, y) = v(x, \infty) = a(X,Y), \frac{\partial}{\partial x} v(x,\infty) = \frac{\partial}{\partial y} v(\infty, y) = 0, \tag{2.10}$$

and

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\partial}{\partial y} v(x,y) &= \alpha'(x)\, \beta'(y)\, b(\alpha(x), \beta(x))\, u(\alpha(x), \beta(x)) \\ &\leq \alpha'(x)\, \beta'(y)\, b(\alpha(x), \beta(x))\, v(\alpha(x), \beta(x)). \end{aligned}$$

Since $\alpha(x) \geq x,\ \beta(y) \geq y$ and $v(x,y)$ is non-increasing in each variable $x,\ y \in \mathbb{R}_+$ we have

$$\frac{\partial}{\partial y}\left[\frac{\frac{\partial}{\partial x} v(x,y)}{v(x,x)}\right] \leq \alpha'(x)\, \beta'(y)\, b(\alpha(x), \beta(x)).$$

By keeping x fixed in above inequality, setting $y = t$ integrating from y to ∞ and using (2.10), and again by keeping y fixed, setting $x = s$, integrating from x to ∞ in the resulting inequality and using (2.10) with the change of variable we obtain

$$v(x,y) \leq a(X,Y) \exp\left(\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} b(s,t)\, dtds\right).$$

By setting $x = X$ and $y = Y$ and using the fact that $u(x,y) \leq v(x,y)$ we get the inequality in (2.8). □

The following theorems deals some non-linear integral inequalities for two variable functions with a term of delay, which are important in the qualitative theory of differential equations.

Theorem 2.1. *Let $u(x,y), a(x,y), b(x,y), c(x,y), d(x,y), f(x,y)$ be real valued non-negative continuous functions defined for $x, y \in \mathbb{R}_+$, and $\alpha, \beta \in C^1([0,\infty), [0,\infty))$ are non-decreasing functions with $\alpha(x) \leq x,\ \beta(y) \geq y$ on $[0,\infty)$ and $\alpha(0) = 0$. Let $W(u(x,y))$ be real valued, positive, continuous, strictly non-decreasing, subdditive, and submultiplicative function for $u(x,y) \geq 0$ and let $H(u(x,y))$ be a real valued, continuous, positive, and non-decreasing function defined for $x, y \in \mathbb{R}_+$. Assume that $a(x,y)$ and $f(x,y)$ are non-decreasing in x for $x \in \mathbb{R}^+$. If*

$$\begin{aligned} u(x,y) &\leq a(x,y) + f(x,y) H\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t) W(u(s,t)) \, dtds\right) \\ &\quad + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} c(s,y) u(s,y) \, ds, \end{aligned} \tag{2.11}$$

for $x, y \in \mathbb{R}^+$ then

$$\begin{aligned} & u(x,y) \\ \leq\ & p(x,y)\{a(x,y) + \\ & f(x,y) H\left[G^{-1}\left(G(A) + \left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t) W(p(s,t) f(s,t)) \, dtds\right)\right)\right]\right\}, \end{aligned} \tag{2.12}$$

for $x, y \in \mathbb{R}^+$ where

$$p(x,y) = 1 + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} c(s,y) \exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y) c(r,y) \, dr\right) ds, \tag{2.13}$$

$$A = \int_0^{\infty} \int_0^{\infty} d(s,t) W(p(s,y) a(s,y)) \, dtds, \tag{2.14}$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{W(H(s))} \quad r \geq r_0 > 0. \tag{2.15}$$

Proof. Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + f(x,y) H\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t) W(u(s,t)) \, dtds\right), \tag{2.16}$$

from (2.11) we get

$$u(x,y) \leq z(x,y) + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} c(s,y) u(s,y) \, ds, \tag{2.17}$$

Clearly, $z(x,y)$ is a non-negative and continuous in x, setting y fixed in (2.17) and using Lemma 2.1, we obtain

$$u(x,y) \leq z(x,y) + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} z(s,y) c(s,y) \exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y) c(r,y) \, dr\right) ds.$$

Moreover, the non-decreasing of the function $z(x,y)$ yields

$$u(x,y) \leq z(x,y) p(x,y), \tag{2.18}$$

where $p(x,y)$ is defined by (2.13). From (2.16) we find

$$u(x,y) \le p(x,y)\left(a(x,y) + f(x,y)\,H(v(x,y))\right), \tag{2.19}$$

where $v(x,y)$ is defined by

$$v(x,y) = \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t)\,W(u(s,t))\,dtds.$$

Using (2.19) we obtain

$$\begin{aligned} v(x,y) \;\le\; & \int_0^{\infty}\int_0^{\infty} d(s,t)\,W(p(s,t)\,a(s,t))\,dtds \\ & + \int_0^{\alpha(x)}\int_{\beta(y)}^{\infty} d(s,t)\,W(p(s,t)\,f(s,t))\,W(H(v(s,t)))\,dtds, \end{aligned} \tag{2.20}$$

since W is subadutive and submultiplicative. Define $r(x,y)$ as the right side of the last above inequality, then

$$r(0,y) = \int_0^{\infty}\int_0^{\infty} d(s,t)\,W(p(s,y)\,a(s,y))\,dtds = A, \tag{2.21}$$

$$v(x,y) \le r(x,y),$$

$r(x,y)$ is non-decreasing in x and non-increasing in y and

$$\begin{aligned} r_x(x,y) \;=\; & \alpha'(x)\int_{\beta(y)}^{\infty} d(\alpha(x),t)\,W(p(\alpha(x),t)\,f(\alpha(x),t))\,W(H(v(\alpha(x),t)))\,dt \\ \le\; & \alpha'(x)\int_{\beta(y)}^{\infty} d(\alpha(x),t)\,W(p(\alpha(x),t)\,f(\alpha(x),t))\,W(H(r(x,t)))\,dt \\ \le\; & W(H(r(x,y)))\,\alpha'(x)\int_{\beta(y)}^{\infty} d(\alpha(x),t)\,W(p(\alpha(x),t)\,f(\alpha(x),t))\,dt. \end{aligned} \tag{2.22}$$

Dividing both sides of (2.22) by $W(H(r(x,y)))$ we obtain

$$\frac{r_x(x,y)}{W(H(r(x,y)))} \le \alpha'(x)\int_{\beta(y)}^{\infty} d(\alpha(x),t)\,W(p(\alpha(x),t)\,f(\alpha(x),t))\,dt. \tag{2.23}$$

Using (2.15) and (2.23) we find

$$G_x(r(x,y)) \le \alpha'(x)\int_{\beta(y)}^{\infty} d(\alpha(x),t)\,W(p(\alpha(x),t)\,f(\alpha(x),t))\,dt. \tag{2.24}$$

Now setting $x = \sigma$ in (2.24) and the integrating with respect to σ from 0 to x, and making the change of variable $s = \alpha(\sigma)$ we obtain

$$\begin{aligned} & G(r(x,y)) \\ \leq \; & G(r(0,y)) + \int_0^x \left(\int_{\beta(y)}^{\infty} \alpha'(s)\, d(\alpha(s), t)\, W(p(\alpha(s), t)\, f(\alpha(s), t))\, dt \right) ds \\ : \; & = G(r(0,y)) + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t)\, W(p(s,t)\, f(s,t))\, dtds, \end{aligned} \tag{2.25}$$

the last inequality imply that

$$r(x,y) \leq G^{-1}\left(G(r(0,y)) + \left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t)\, W(p(s,t)\, f(s,t))\, dtds \right) \right). \tag{2.26}$$

In view of (2.19), (2.21) and (2.26) and by the fact of $v(x,y) \leq r(x,y)$, we obtain the inequality (2.12). $\square$

Next, we shall present some important remark and corollaries resulting from the above Theorem.

Remark 2.1. *If $\alpha(x) = x$ and $\beta(y) = y$ in theorem 2.1 we get theorem 2.3 in [12].*

Corollary 2.1. *If $W(x) = H(x) = x$, and $r_0 = 1$ in Theorem 2.1 we get*

$$r(0,y) = \int_0^{\infty} \int_0^{\infty} d(s,t)\, p(s,y)\, a(s,y)\, dtds = A,$$

and

$$\begin{aligned} u(x,y) \; \leq \; & p(x,y) \{ a(x,y) + \\ & A f(x,y) \left[\exp\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t)\, p(s,t)\, f(s,t)\, dtds \right) \right] \right\}. \end{aligned}$$

Corollary 2.2. *If $W(x) = x^p$ with $0 < p < 1$, $H(s) = s$, and $r_0 = 0$ in Theorem 2.1 we get*

$$r(0,y) = \int_0^{\infty} \int_0^{\infty} d(s,t)\, p(s,y)\, a(s,y)\, dtds = A,$$

and

$$\begin{aligned} u(x,y) \; \leq \; & p(x,y) \{ a(x,y) + \\ & (1-p)^{\frac{1}{1-p}} f(x,y) \left(\frac{A^{1-p}}{1-p} + \left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t)\, p(s,t)\, f(s,t)\, dtds \right) \right)^{\frac{1}{1-p}} \right\}. \end{aligned}$$

By the same proof of Theorem 2.1, with using (2) of Lemma 2.1, we obtain the following theorem

Theorem 2.2. *Let* $u(x,y), a(x,y), b(x,y), c(x,y), d(x,y), f(x,y), W(u(x,y))$ *and* $H(u(x,y))$ *be as defined in theorem 2.1 and* $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ *be non-decreasing function with* $\alpha(x) \geq x$, $\beta(y) \geq y$ *on* $[0,\infty)$ *and* $\beta(\infty) = \infty$. *Assume that* $a(x,y), b(x,y)$, *and* $f(x,y)$ *are non-increasing in* x *for* $x \in \mathbb{R}_+$. *If*

$$\begin{aligned} u(x,y) &\leq a(x,y) + f(x,y) H\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} d(s,t) W(u(s,t))\, dtds\right) \\ &\quad + b(x,y)\int_{\beta(x)}^{\infty} c(s,y) u(s,y)\, ds, \end{aligned}$$

for $x, y \in \mathbb{R}_+$ *then*

$$\begin{aligned} u(x,y) &\leq p(x,y)\{a(x,y) + \\ &\quad f(x,y) H\left[G^{-1}\left(G(A) + \left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} d(s,t) W(p(s,t) f(s,t))\, dtds\right)\right)\right]\right\}, \end{aligned}$$

for $x, y \in \mathbb{R}^+$ *where*

$$p(x,y) = 1 + b(x,y)\int_{\beta(x)}^{\infty} c(s,y) \exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y) c(r,y)\, dr\right) ds,$$

$$A = \int_0^{\infty}\int_0^{\infty} d(s,t) W(p(s,y) a(s,y))\, dtds,$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{W(H(s))} \quad r \geq r_0 > 0.$$

Theorem 2.3. *Let* $u(x,y), a(x,y), b(x,y), c(x,y), f(x,y)$ *be real valuer non-nagative continuous functions defined for* $x, y \in \mathbb{R}_+$ *and* $L : \mathbb{R}_+^3 \to \mathbb{R}$ *be a continuous function and* $L(x,y,u)$ *is non-decreasing in* u *and satisfies the condition*

$$0 \leq L(x,y,u) - L(x,y,v) \leq M(x,y,v)\, \phi^{-1}(u-v),$$

for $u \geq v \geq 0$, *where* $M(x,y,v)$ *is a real valued non-negative continuous function defined for* $x, y, v \in \mathbb{R}_+$, *and* $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ *be non-decreasing function with* $\alpha(x) \leq x$, $\beta(y) \geq y$ *on* $[0,\infty)$. *Assume that* $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a continuous and strictly increasing function with* $\phi(0) = 0$, ϕ^{-1} *is the inverse function of* ϕ *and*

$$\phi^{-1}(uv) \leq \phi^{-1}(u)\, \phi^{-1}(v),$$

for $u, v \in \mathbb{R}_+$. Assume that $a(x,y), f(x,y)$ are non-decreasing in x. If

$$\begin{aligned} u(x,y) \leq\ & a(x,y) + f(x,y)\,\phi\left(\int_0^{\alpha(x)}\int_{\beta(y)}^{\infty} L(s,t,u(s,t))\,dtds\right) \\ & + b(x,y)\int_{\alpha(x_0)}^{\alpha(x)} c(s,y)\,u(s,y)\,ds, \end{aligned} \tag{2.27}$$

for $x, y \in \mathbb{R}_+$, then

$$\begin{aligned} & u(x,y) \leq p(x,y)\{a(x,y) + f(x,y)\,\phi[e(x,y) \\ & \times \exp\left(\int_0^{\alpha(x)}\int_{\beta(y)}^{\infty} M(s,t,p(s,t)\,a(s,t))\,\phi^{-1}(p(s,t)\,f(s,t))\,dtds\right)\Big]\Big\}, \end{aligned} \tag{2.28}$$

for $x, y \in \mathbb{R}_+$, where

$$p(x,y) = 1 + b(x,y)\int_{\infty}^{\alpha(x)} c(s,y)\exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y)\,c(r,y)\,dr\right)ds, \tag{2.29}$$

$$e(x,y) = \int_0^{\alpha(x)}\int_{\beta(y)}^{\infty} L(s,t,p(s,t)\,a(s,t))\,dtds. \tag{2.30}$$

Proof. Define a function $z(x,y)$ by

$$z(x,y) = a(x,y) + f(x,y)\,\phi\left(\int_0^{\alpha(x)}\int_{\beta(y)}^{\infty} L(s,t,u(s,t))\,dtds\right), \tag{2.31}$$

then from (2.27) we find

$$u(x,y) \leq z(x,y) + b(x,y)\int_{\alpha(x_0)}^{\alpha(x)} c(s,y)\,u(s,y)\,ds. \tag{2.32}$$

Clearly that, $z(x,y)$ is a non-negative and continuous in x. Setting y fixed in (2.32) and using (1) of Lemma 2.1 to (2.32), we get

$$u(x,y) \leq z(x,y) + b(x,y)\int_{\alpha(x_0)}^{\alpha(x)} z(s,y)\,c(s,y)\exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y)\,c(r,y)\,dr\right)ds.$$

Moreover, the non-decreasing of the function $z(x,y)$ yields

$$u(x,y) \leq z(x,y)\,p(x,y), \tag{2.33}$$

where $p(x,y)$ is defined by (2.29). From (2.31) and (2.33) we have

$$u(x,y) \leq p(x,y)\left(a(x,y) + f(x,y)\,\phi(v(x,y))\right), \tag{2.34}$$

where $v(s,y)$ is defined by

$$v(x,y) = \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} L(s,t,u(s,t))\,dtds.$$

The hypotheses on L and ϕ, and (2.34) yields

$$\begin{aligned} v(x,y) &\leq \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} \left(L\left(s,t,p(s,t)\left[a(s,t) + f(s,t)\,\phi(v(s,t))\right]\right)\right. \\ &\quad \left. -L(s,t,p(s,t)\,a(s,t)) + L(s,t,p(s,t)\,a(s,t))\right) dtds \\ &\leq \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} L(s,t,p(s,t)\,a(s,t))\,dtds \\ &\quad + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} M(s,t,p(s,t)\,a(s,t))\,\phi^{-1}(p(s,t)\,f(s,t)\,\phi(v(s,t)))\,dtds \end{aligned}$$

$$\leq e(x,y) + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} M(s,t,p(s,t)\,a(s,t))\,\phi^{-1}(p(s,t)\,f(s,t))\,v(s,t)\,dtds, \tag{2.35}$$

where $e(x,y)$ is defined by (2.30). Clearly that the function $e(x,y)$ is non-negative, continuous non-decreasing in x and non-increasing in y. Using (1) of Lemma 2.2 we find

$$v(s,y) \leq e(x,y) \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} M(s,t,p(s,t)\,a(s,t))\,\phi^{-1}(p(s,t)\,f(s,t))\,dtds. \tag{2.36}$$

In view of (2.34) and (2.36) we conclude the inequality (2.28). □

Corollary 2.3. *If $L(s,t,u(s,t)) = u(s,t)$ and $\Phi(x) = x$ in Theorem 2.3 we get*

$$e(x,y) = \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} p(s,t)\,a(s,t)\,dtds$$

$$M(x,y,v) = 1$$

and

$$\begin{aligned} u(x,y) &\leq p(x,y)\{a(x,y) + \\ &\quad f(x,y)\left[e(x,y) \times \exp\left(\int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} p(s,t)\,f(s,t)\,dtds\right)\right]\right\}. \end{aligned}$$

By the same proof of Theorem 2.3, with using (2) of Lemma 2.2 we obtain the following theorem

Theorem 2.4. *Let* $u(x,y), a(x,y), b(x,y), c(x,y), f(x,y), L, M, \Phi$ *and* Φ^{-1}*be as defined in Theorem 2.3. and* $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ *be non-decreasing function with* $\alpha(x) \geq x$, $\beta(y) \geq y$ *on* $[0,\infty)$ *and* $\alpha(\infty) = \beta(\infty) = \infty$. *Assume that* $a(x,y), f(x,y)$ *are non-increasing in* x *for* $x \in \mathbb{R}_+$ *If*

$$\begin{aligned} u(x,y) &\leq a(x,y) + f(x,y)\,\Phi\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} L(s,t,u(s,t))\,dtds\right) \\ &\quad + b(x,y)\int_{\alpha(x)}^{\infty} c(s,y)\,u(s,y)\,ds \end{aligned}$$

for $\beta, x, y \in \mathbb{R}_+$, *then*

$$\begin{aligned} u(x,y) &\leq p(x,y)\{a(x,y) + f(x,y)\,\Phi[e(x,y) \\ &\quad \times \exp\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} M(s,t,p(s,t)\,a(s,t))\,\Phi^{-1}(p(s,t)\,f(s,t)\,dtds)\,dtds\right)\Big]\Big\} \end{aligned}$$

for $x, y \in \mathbb{R}_+$ *where*

$$p(x,y) = 1 + b(x,y)\int_{\alpha(x)}^{\infty} c(s,y)\exp\left(\int_{\alpha(s)}^{\alpha(x)} b(r,y)\,c(r,y)\,dr\right)ds$$

$$e(x,y) = \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} L(s,t,p(s,t)\,a(s,t))\,dtds \tag{2.37}$$

Remark 2.2. *If* $\alpha(x) = x$ *and* $\beta(y) = y$ *in theorem 2.4 we get theorem 2.5 in [12]*

2.2 Further delay inequalities

In this section we use the following class of function (see [7]).

Definition 2.1. *A function* $g : [0,\infty) \to [0,\infty)$ *is said to belong to the class* S *if*
(i) $g(u)$ *is positive, non-decreasing and continuous for* $u \geq 0$,
(ii) $\frac{1}{v}g(u) \leq g\left(\frac{u}{v}\right)$, $u > 0, v \geq 1$.

Example 2.1. *If* $g(u) = u^m$, $0 < m < 1$, *then* $\frac{1}{v}u^m \leq \left(\frac{u}{v}\right)^m$, *for any* $u > 0, v \geq 1$.

Theorem 2.5. *Let* $u(x,y), a(x,y), b(x,y), c(x,y), f(x,y)$ *be real valued non-negative continuous function defined for* $x, y \in \mathbb{R}_+$ *and let* $g \in S$. *and* $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ *be non-decreasing function with* $\alpha(x) \leq x$, $\beta(y) \geq y$ *on* $[0,\infty)$ *and* $\alpha(0) = 0$. *Also let* $W(u(x,y))$ *be real valued, positive, continuous, strictly non-decreasing subadditive and submultiplicative function for* $u(x,y) \geq 0$ *and let* $H(u(x,y))$ *be a real valued, continuous, positive, and non-decreasing function defined for* $x, y \in \mathbb{R}_+$. *Assume that a function* $m(x,y)$ *is a non-decreasing in* x *and* $m(x,y) \geq 1$, *which is defined by*

$$m\left(x,y\right)=a\left(x,y\right)+f\left(x,y\right)H\left(\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}c\left(s,t\right)W\left(u\left(s,t\right)\right)dtds\right)\text{ for }x,y\in\mathbb{R}_{+}.$$

If

$$u\left(x,y\right)\leq m\left(x,y\right)+\int_{\alpha(x_0)}^{\alpha(x)}b\left(s,y\right)g\left(u\left(s,y\right)\right)ds, \tag{2.38}$$

for $x,y\in\mathbb{R}_{+}$ *then*

$$\begin{aligned} u\left(x,y\right) &\leq F\left(x,y\right)\{a\left(x,y\right)+f\left(x,y\right)\times \\ & H\left[G^{-1}\left(G\left(B\right)+\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}c\left(s,t\right)W\left(F\left(s,t\right)f\left(s,t\right)\right)dtds\right)\right]\right\}, \end{aligned} \tag{2.39}$$

where

$$F\left(x,y\right)=\Omega^{-1}\left(\Omega\left(1\right)+\int_{\alpha(x_0)}^{\alpha(x)}b\left(s,y\right)ds\right), \tag{2.40}$$

$$B=\int_{0}^{\infty}\int_{0}^{\infty}c\left(s,t\right)W\left(F\left(s,t\right)a\left(s,t\right)\right)dtds, \tag{2.41}$$

$$\Omega\left(u\right)=\int_{u_0}^{u}\frac{ds}{g\left(s\right)}\quad u\geq u_0>0, \tag{2.42}$$

where Ω^{-1} *is the inverse function of* Ω; G,G^{-1} *are defined in theorem 2.1,* $\Omega\left(1\right)+\int_{\alpha(x_0)}^{\alpha(x)}b\left(s,y\right)ds$ *is in the domain of* Ω^{-1} *and*

$$G\left(B\right)+\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}b\left(s,t\right)W\left(F\left(s,t\right)f\left(s,t\right)\right)dtds,$$

is in the domain of G^{-1} *for* $x,y\in\mathbb{R}_{+}$

Proof. Clearly that, $m\left(x,y\right)$ be a positive, continuous, non-decreasing.In view of (2.38) it yields

$$\frac{u\left(x,y\right)}{m\left(x,y\right)}\leq 1+\int_{\alpha(x_0)}^{\alpha(x)}b\left(s,y\right)g\left(\frac{u\left(s,y\right)}{m\left(s,y\right)}\right)ds, \tag{2.43}$$

since $g\in S$. The inequality (2.43) may be treated as a one dimensionel Bihari inequality [5], for any fixed $y\in\mathbb{R}_{+}$, it implies that

$$u\left(x,y\right)\leq m\left(x,y\right)F\left(x,y\right),$$

where $F\left(x,y\right)$ is defined by (2.40). Now by the same proof of Theorem 2.1 we obtain the inequality (2.39). □

Now, we can give the following remark and corollary that are obvious consequences of the above theorem.

Remark 2.3. *If $\alpha(x) = x$ and $\beta(y) = y$ in Theorem 2.1 we get Theorem 3.1 in [11]*

Corollary 2.4. *If $W(x) = H(x) = g(x) = \alpha(x) = x$, $x_0 = 0$, $u = 1$ and $b(x, y) = y$ in Theorem 2.5 we get*

$$F(x, y) = \exp(xy),$$

$$B = \int_0^\infty \int_0^\infty \exp(st)\, c(s, t)\, a(s, t)\, dtds,$$

and

$$\begin{aligned} u(x, y) \leq\ & \exp(xy) \{a(x, y) + Bf(x, y) \times \\ & \left(\exp \int_0^{\alpha(x)} \int_{\beta(y)}^\infty \exp(st)\, c(s, t)\, f(s, t)\, dtds\right)\right\}. \end{aligned}$$

Theorem 2.6. *Let $u(x, y), a(x, y), b(x, y), c(x, y), f(x, y), W(u(x, y))$, and $H(u(x, y))$ be as defined in Theorem 2.5 and $g \in S$, and $\alpha, \beta \in C^1([0, \infty), [0, \infty))$ be non-decreasing function with $\alpha(x) \geq x$, $\beta(y) \geq y$ on $[0, \infty)$ and $\alpha(\infty) = \infty$. Assume that a function $m(x, y)$ is non-increasing in x and $m(x, y) \geq 1$,which is defined by*

$$m(x, y) = a(x, y) + f(x, y)\, H\left(\int_{\alpha(x)}^\infty \int_{\beta(y)}^\infty c(s, y)\, W(u(s, t))\, dtds\right),$$

for $x, y \in \mathbb{R}_+$. If

$$u(x, y) \leq m(x, y) + \int_{\alpha(x)}^\infty b(s, t)\, g(u(s, y))\, ds, \tag{2.44}$$

for $x, y \in \mathbb{R}_+$ then

$$\begin{aligned} u(x, y) \leq\ & F(x, y) [a(x, y) + f(x, y) \times \\ & H\left[G^{-1}\left(G(B) + \int_{\alpha(x)}^\infty \int_{\beta(y)}^\infty c(s, t)\, W(F(s, t)\, f(s, t))\, dtds\right)\right], \end{aligned} \tag{2.45}$$

for $x, y \in \mathbb{R}_+$, where

$$F(x, y) = \Omega^{-1}\left(\Omega(1) + \int_{\alpha(x)}^\infty b(s, y)\, ds\right), \tag{2.46}$$

Where B is defined in (2.41), and Ω is defined in (2.42), Ω^{-1} is the inverse function of Ω; G, G^{-1} are defined in Theorem 2.1, $\Omega(1) + \int_{\alpha(x_0)}^{\alpha(x)} b(s, y)\, ds$ is in the domain of Ω^{-1}, and

$$G(B) + \int_{\alpha(x)}^\infty \int_{\beta(y)}^\infty b(s, t)\, W(F(s, t)\, f(s, t))\, dtds,$$

is in the domain of G^{-1}, for $x, y \in \mathbb{R}_+$.

Proof. Clearly that, $m(x,y)$ is a positive, continuous, non-decreasing in x. In view of 2.44 it yields

$$\frac{u(x,y)}{m(x,y)} \leq 1 + \int_{\alpha}^{x} b(s,y)\, g\left(\frac{u(s,y)}{m(s,y)}\right) ds,$$

since $g \in S$. The inequality (2.44) may be treated as a onedimensional Bihari inequality [5] for any fixed $y, y \in \mathbb{R}_+$, which implies that

$$u(x,y) \leq F(x,y)\, m(x,y),$$

where $F(x,y)$ is defined by (2.46). Now, by the same proof of Theorem 2.2, we obtain the inequality (2.45). □

Remark 2.4. *If $\alpha(x) = x$ and $\beta(y) = y$ in Theorem 2.1 we get Theorem 3.2 in [12].*

Theorem 2.7. *Let $u(x,y), a(x,y), b(x,y), f(x,y), L, M, \Phi,$ and Φ^{-1} be as defined in Theorem 2.3, and let $g \in S$, and $\alpha, \beta \in C^1([0,\infty), [0,\infty))$ be non-decreasing function with $\alpha(x) \leq x$, $\beta(y) \geq y$ on $[0,\infty)$ and $\alpha(0) = 0$. Assume that a function $n(x,y)$ is non-decreasing in x and $n(x,y) \geq 1$, which is defined by*

$$n(x,y) = a(x,y) + f(x,y)\, \Phi\left(\int_{0}^{\alpha(x)} \int_{\beta(y)}^{\infty} L(s,t,u(s,t))\, dt ds\right) \text{ for } x, y \in \mathbb{R}_+.$$

If

$$u(x,y) \leq n(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} b(s,y)\, g(u(s,y))\, ds \text{ for } x, y \in \mathbb{R}_+, \tag{2.47}$$

then

$$\begin{aligned} u(x,y) \leq\; & F(x,y)\{a(x,y) + f(x,y)\, \Phi[e(x,y) \times \\ & \exp\left(\int_{0}^{\alpha(x)} \int_{\beta(y)}^{\infty} M(s,t,F(s,t)\, a(s,t))\, \Phi^{-1}(F(s,t)\, f(s,t))\, dt ds\right)]\}. \end{aligned} \tag{2.48}$$

for $x, y \in \mathbb{R}_+$, where F is defined in (2.40), $e(x,y)$ is defined in (2.30), Ω is defined in (2.42), Ω^{-1} is the inverse function of Ω, and $\Omega(1) + \int_{\alpha(x_0)}^{\alpha(x)} b(s,y)\, ds$ is in the domain of Ω^{-1}

Proof. Clearly that, $n(x,y)$ is a positive, continuous, non-decreasing in x. In view of (2.47) it yields

$$\frac{u(x,y)}{n(x,y)} \leq 1 + \int_{\alpha(x_0)}^{\alpha(x)} b(s,y)\, g\left(\frac{u(s,y)}{n(s,y)}\right) ds, \tag{2.49}$$

since $g \in S$. The inequality (2.49) may be treated as a one-dimensional Bihari inequality [5] for any fixed $y, y \in \mathbb{R}_+$, it implies that

$$u(x,y) \leq F(x,y)\, n(x,y).$$

New , by the same proof of Theorem 2.3, we obtain the inequality (2.48). □

Remark 2.5. *If $\alpha(x) = x$ and $\beta(y) = y$ in Theorem 2.1 we get Theorem 3.3 in [11]*

Theorem 2.8. *Let $u(x,y), a(x,y), b(x,y), f(x,y), L, M, \Phi$ and Φ^{-1} be as defined in Theorem (2.3), and let $g \in S$, and $\alpha, \beta \in C^1([0,\infty),[0,\infty))$ be non-decreasing function with $\alpha(x) \geq x$, $\beta(y) \geq y$ on $[0,\infty)$. Assume that a functions $n(x,y)$ is nonincreasing in x and $n(x,y) \geq 1$, which is defined by*

$$n(x,y) = a(x,y) + f(x,y)\Phi\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} L(s,t,u(s,t))\,dtds\right),$$

for $x, y \in \mathbb{R}_+$. If

$$u(x,y) \leq n(x,y) + \int_{\alpha(x)}^{\infty} b(s,y)\,b(u(s,y))\,ds,$$

for $x, y \in \mathbb{R}_+$ then

$$\begin{aligned} u(x,y) \quad &\leq \quad F(x,y)\{a(x,y) + f(x,y)\Phi[e(x,y) \\ &\times \exp\left(\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} M(s,t,F(s,t)\,a(s,t))\,\Phi^{-1}(F(s,t)\,f(s,t))\,dtds\right)\Big]\Big\}, \end{aligned}$$

for $x, y \in \mathbb{R}_+$, where F is defined in (2.46), $e(x,y)$ is defined in (2.37), Ω is defined in (2.42), Ω^{-1} is the inverse function of Ω, and $\Omega(1) + \int_{\alpha(x)}^{\infty} b(x,y)\,ds$ is the domain of Ω^{-1}. The proof of this theorem follow by an argument similar to that in Theorem (2.7) with suitable changes. We omit the details

Remark 2.6. *If $\alpha(x) = x$ and $\beta(y) = y$ in Theorem 2.1 we get Theorem 3.4 in [11]*

2.3 Some Applications

Using [12] and [29] we study certain properties of solutions of the following terminal-value problem for the partial differential equation

$$u_{xy}(x,y) = \alpha'(x)\,\beta'(y)\,h(\alpha(x),\beta(y),u(\alpha(x),\beta(y))) + r(\alpha(x),\beta(y)), \tag{2.50}$$

$$u(x,\infty) = \sigma_{\infty}(x), u(0,y) = \tau(y), u(0,\infty) = k, \tag{2.51}$$

where $h : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R}_+^2 \to \mathbb{R}$, $\sigma_{\infty}, \tau : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and k is a real constant.

We present three examples of application to study respectively the boundless, uniqueness and stability of the solution of (2.50)-(2.51).

Example 01: gives the bound of the solution of (2.50),(2.51).

Assume that all functions of problem (2.50)- (2.51) are defined and continuous on their respective domains of definitions,

$$|h(x,y,u)| \leq d(x,y)\,W(|u|), \tag{2.52}$$

and

$$\left|\sigma_{\infty}(x)+\tau(y)-k-\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}r(s,t)\,dtds\right|\leq a(x,y)+b(x,y)\int_{\alpha(x_0)}^{\alpha(x)}c(s,y)\,u(s,y)\,ds, \quad (2.53)$$

where $\alpha(x), \beta(y), a(x,y), b(x,y), c(x,y)$ and $W(u)$ are as defined in Theorem 2.1. If $u(x,y)$ is a solution of (2.50)-(2.51), then

$$\begin{aligned} u(x,y) &= \sigma_{\infty}(x)+\tau(y)-k \\ &\quad -\int_{0}^{x}\int_{y}^{\infty}\alpha'(s)\,\beta'(t)\,[h(\alpha(s),\beta(t),u(\alpha(s),\beta(t)))+r(\alpha(s),\beta(t))]\,dtds \\ &= \sigma_{\infty}(x)+\tau(y)-k-\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}h(s,t,u(s,t))+r(s,t)\,dtds, \end{aligned}$$

for $x,y\in\mathbb{R}_+$. From (2.52),(2.53) we get

$$|u(x,y)|\leq a(x,y)+\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}d(s,t)\,W(|u(s,t)|)+b(x,y)\int_{\alpha(x_0)}^{\alpha(x)}c(s,y)\,u(s,y)\,ds, \quad (2.54)$$

Now, a suitable application of Theorem 2.1 with $f(x,y)=1$ and $H(u)=u$ to (2.54) we get

$$|u(x,y)|\leq p(x,y)\left\{a(x,y)+G^{-1}\left(G(A)+\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}d(s,t)\,W(p(s,t))\,dtds\right)\right\},$$

for $x,y\in\mathbb{R}_+$, where $p(x,y), G$, and G^{-1} are defined in theorem 2.1.

example 02: gives the uniqueness of the solution of (2.50)-(2.51).

Let $u(x,y)$ and $v(x,y)$ tow solutions of problem (2.50)-(2.51). Such that

$$|h(x,y,u)-h(x,y,v)|\leq \epsilon d(x,y)\,W(|u-v|)\quad (0<\epsilon<1), \quad (2.55)$$

where $d(x,y)$ and $W(u)$ are as defined in theorem 2.1. Then

$$\begin{aligned} &u(x,y)-v(x,y) \\ =\ & -\int_{0}^{x}\int_{y}^{\infty}\alpha'(s)\,\beta'(t)\,[h(\alpha(s),\beta(t),u(\alpha(s),\beta(t)))+r(\alpha(s),\beta(t))]\,dtds \\ &+\int_{0}^{x}\int_{y}^{\infty}\alpha'(s)\,\beta'(t)\,[h(\alpha(s),\beta(t),v(\alpha(s),\beta(t)))+r(\alpha(s),\beta(t))]\,dtds \\ =\ & -\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}(h(s,t,u(s,t))-h(s,t,v(s,t)))\,dtds, \end{aligned}$$

for $x,y\in\mathbb{R}_+$.From (2.55) we get

$$|u(x,y)-v(x,y)|\leq \epsilon\int_{0}^{\alpha(x)}\int_{\beta(y)}^{\infty}d(s,t)\,W(|u(s,t)-v(s,t)|)\,dtds. \quad (2.56)$$

Now, a suitable application of Theorem 2.1 with $f(x,y) = \epsilon$, $H(u) = u$, and $a(x,y) = b(x,y) = 0$ to (2.56) we get

$$|u(x,y) - v(x,y)| \leq \epsilon G^{-1}\left(G(A) + \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t) W(\epsilon)\, dtds\right)$$

For $\epsilon \to 0$ we obtain $u(x,y) = v(x,y)$.

example 03: gives the stability of the solution of (2.50)-(2.51).

Let $u(x,y)$ and $v(x,y)$ tow solutions of (2.50) with the given initial boundary data

$$u(x,\infty) = \sigma_\infty(x), u(0,y) = \tau(y), u(0,\infty) = k, \tag{2.57}$$

$$v(x,\infty) = \sigma'_\infty(x), v(0,y) = \tau'(y), v(0,\infty) = k'. \tag{2.58}$$

such that the condition (2.55) is holds, and

$$\begin{aligned} & |\sigma_\infty(x) - \sigma'_\infty(x) + \tau(y) - \tau(y) - k + k'| \\ \leq\ & \epsilon a(x,y) + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} c(s,y)(u(s,y) - v(s,y))\, ds, (0 < \epsilon < 1), \end{aligned} \tag{2.59}$$

where $\alpha(x), \beta(y), a(x,y), b(x,y), c(x,y)$ and $W(u)$ are as defined in Theorem 2.1, and $\alpha(0) = 0$. Then

$$\begin{aligned} u(x,y) - v(x,y) &= \sigma_\infty(x) - \sigma'_\infty(x) + \tau(y) - \tau(y) - k + k' \\ &\quad - \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} (h(s,t,u(s,t)) - h(s,t,v(s,t)))\, dtds, \end{aligned}$$

for $x, y \in \mathbb{R}_+$. From (2.55),(2.59) we get

$$\begin{aligned} |u(x,y) - v(x,y)| &\leq \epsilon a(x,y) + \epsilon \int_0^{\alpha(x)} \int_{\beta(y)}^{\infty} d(s,t) W(|u(s,t) - v(x,y)|) \\ &\quad + b(x,y) \int_{\alpha(x_0)}^{\alpha(x)} c(s,y)(u(s,y) - v(x,y))\, ds \end{aligned} \tag{2.60}$$

Now, a suitable application of Theorem 2.1 with $f(x,y) = \epsilon$, $H(u) = u$, to (2.60) yields the required estimate, therefore

$$|u(x,y) - v(x,y)| \leq \epsilon p(x,y) \left\{ a(x,y) + G^{-1}\left[G(A) + \int_0^x \int_0^{\infty} d(s,t) W(\epsilon p(s,t))\, dtds\right]\right\}$$

for $x, y \in \mathbb{R}_+$, where p, G and G^{-1} are as defined in Theorem 2.1. Then the solution of problem (2.50) is stable.

Chapter 3

Some new non-linear Generalized Integral Inequalities in several variables And Applications

Pachpatte in [35] has presented the following integral inequalities

$$\begin{aligned} u^{p}(x,y) &= k+\int_{x_0}^{x}\int_{y_0}^{y}a(s,t)\,g_1(u(s,t))\,dtds \\ &\quad +\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}b(s,t)\,g_2(u(s,t))\,dtds. \end{aligned}$$

Khallaf and Smakdji [11] have studied the following type of integral inequality

$$\begin{aligned} u^{p}(x) &\leq c(x)+\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}(x^0)}^{\widetilde{\alpha}(x)}a_j(x,t)\,u(t)\,dt \\ &\quad +\sum_{k=1}^{n_2}\int_{\widetilde{\beta}(x^0)}^{\widetilde{\beta}(x)}b_k(x,t)\,u(t)\,dt. \end{aligned}$$

where $x=(x_1,x_2,...,x_n)$, $t=(t_1,t_2,...,t_n)$, $x^0=(x_1^0,x_2^0,...,x_n^0)$, and

$$\int_{\widetilde{\alpha}(x^0)}^{\widetilde{\alpha}(x)}dt=\int_{\alpha_{j1}(x_1^0)}^{\alpha_{j1}(x)}\int_{\alpha_{j2}(x_2^0)}^{\alpha_{j2}(x)}...\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x)}dt_n...dt_1,\ \ j=1,2,...,n_1.$$

$$\int_{\widetilde{\beta}(x^0)}^{\widetilde{\beta}(x)}dt=\int_{\beta_{k1}(x_1^0)}^{\beta_{k1}(x)}\int_{\beta_{k2}(x_2^0)}^{\beta_{k2}(x)}...\int_{\beta_{kn}(x_n^0)}^{\beta_{kn}(x)}dt_n...dt_1,\ \ k=1,2,...,n_2.$$

In this chapter we establish some non-linear retarded integral inequalities for functions of n independent variables, which can be used as handy tools in the theory of partial differential and integral equations. These new inequalities represent a generalization of the results obtained in [17]. Some applications of our results are also given.

3.1 Some non-linear Generalized Integral Inequalities With a term of Delay

Throughout, we define $I_i=[x_i,X_i)$ $i=1,2,...,n$ and $\Delta=I_1\times I_2\times...\times I_n$, $n\in\mathbb{N}$, and $n\geq 3$. The first-order partial derivative of a function $Z(x_1,x_2,...,x_n)$ for $x_i\in\mathbb{R}$ with respect to x_i is denoted as usual by $D_iZ(x_1,x_2,...,x_n)$. For $x=(x_1,x_2,...,x_n)$, $t=(t_1,t_2,...,t_n)$, $x^0=(x_1^0,x_2^0,...,x_n^0)$, we shall denote:

$$\begin{aligned}
\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} &= \int_{\alpha_{j_1 1}(x_1^0)}^{\alpha_{j_1 1}(x_1)} \int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)} ... \int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} dt_n...dt_1, \quad j_1 = 1, ..., m_1, \\
\int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} &= \int_{\alpha_{j_2 1}(x_1^0)}^{\alpha_{j_2 1}(x_1)} \int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)} ... \int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} dt_n...dt_1, \quad j_2 = 1, ..., m_2, \\
&\ . \\
&\ . \\
&\ . \\
\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} &= \int_{\alpha_{j_n 1}(x_1^0)}^{\alpha_{j_n 1}(x_1)} \int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)} ... \int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} dt_n...dt_1, \quad j_n = 1, ..., m_n.
\end{aligned}$$

with $m_1, m_2, ..., m_n \in \{1, 2, ...\}$. For $x, t \in \mathbb{R}^n$, we shall write $t \leq x$ whenever $t_i \leq x_i$, $i = 1, 2, ..., n$. We denote $D = D_1 D_2 ... D_n$, where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, ..., n$.

$$\begin{aligned}
\widetilde{\alpha}_{j_1}(t) &= (\alpha_{j_1 1}(t_1), \alpha_{j_1 2}(t_2), ..., \alpha_{j_1 n}(t_n)), \text{ for } j_1 = 1, 2, ..., m_1, \\
\widetilde{\alpha}_{j_2}(t) &= (\alpha_{j_2 1}(t_1), \alpha_{j_2 2}(t_2), ..., \alpha_{j_2 n}(t_n)), \text{ for } j_2 = 1, 2, ..., m_2, \\
&\ . \\
&\ . \\
&\ . \\
\widetilde{\alpha}_{j_n}(t) &= (\alpha_{j_n 1}(t_1), \alpha_{j_n 2}(t_2), ..., \alpha_{j_n n}(t_n)), \text{ for } j_n = 1, 2, ..., m_n.
\end{aligned}$$

We denote $\widetilde{\alpha}_{j_k}(t) \leq t$ for $k = 1, 2, ..., n$, $j_k = 1, 2, ..., m_k$ whenever $\widetilde{\alpha}_{j_k i}(t_i) \leq t_i$, for $i = 1, 2, ..., n$.

The following theorems deals some versions of non-linear integral inequalities, for functions of n independent variables with a term of delay

Theorem 3.1. *let $a_{j_1}, a_{j_2}, ..., a_{j_n} \in C(\Delta, \mathbb{R}_+)$, $\alpha_{j_k i} \in C(I_i, I_i)$ be non-decreasing functions for $j_k = 1, 2, ..., m_k$, $k = 1, 2, ..., n$, $i = 1, 2, ..., n$. with $\widetilde{\alpha}_{j_k}(x) \leq x$, and $w_j \in C(\mathbb{R}, \mathbb{R}_+)$ for $j = 1, 2, ..., n$, be a non-decreasing functions with $w_j(u) > 0$ for $u > 0$, $p > q \geq 0$ and $k \geq 0$ be constants. If $u \in C(\Delta, \mathbb{R}_+)$, and*

$$\begin{aligned}
u^p(x) \leq\ & k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s) u^q(s) w_1(u(s)) ds \\
& + \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s) u^q(s) w_2(u(s)) ds \\
& + ... + \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s) u^q(s) w_n(u(s)) ds, \qquad (3.1)
\end{aligned}$$

for any $x \in \Delta$. *Then*

$$\begin{aligned} u(x) \leq & \left\{ G^{-1} \left[G\left(k^{\frac{p-q}{p}}\right) + \frac{p-q}{p} \sum_{j_1=1}^{m_1} A_{1j_1}(x) + \right. \right. \\ & \left. \left. \frac{p-q}{p} \sum_{j_2=1}^{m_2} A_{2j_2}(x) + \ldots + \frac{p-q}{p} \sum_{j_n=1}^{m_n} A_{nj_n}(x) \right] \right\}^{\frac{1}{p-q}}. \end{aligned} \tag{3.2}$$

For $x^0 \leq x \leq x^1$ *where* $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)$ *and*

$$\begin{aligned} A_{1j_1}(x) &= \int_{\tilde{\alpha}_{j_1}(x^0)}^{\tilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, ds, \\ A_{2j_2}(x) &= \int_{\tilde{\alpha}_{j_2}(x^0)}^{\tilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, ds, \\ &\cdot \\ &\cdot \\ &\cdot \\ A_{nj_n}(x) &= \int_{\tilde{\alpha}_{j_n}(x^0)}^{\tilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, ds. \end{aligned} \tag{3.3}$$

$$G(r) = \int_{r_0}^{r} \frac{1}{w_1\left(s^{\frac{1}{p-q}}\right) + w_2\left(s^{\frac{1}{p-q}}\right) + \ldots. + w_n\left(s^{\frac{1}{p-q}}\right)} ds \quad r \geq r_0 > 0. \tag{3.4}$$

G^{-1} *denotes the inverse function of* G, *and real numbers* $x_i^1 \in I_i$ *for any* $i = 1, 2, \ldots, n$. *are chosen so that the quantity in the square brackets of (3.2) is in the range of* G.

Proof. Let $k > 0$, define $r(x)$ as the right side of (3.1), i.e

$$\begin{aligned} r(x) = & k + \sum_{j_1=1}^{m_1} \int_{\tilde{\alpha}_{j_1}(x^0)}^{\tilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, u^q(s)\, w_1(u(s))\, ds \\ & + \sum_{j_2=1}^{m_2} \int_{\tilde{\alpha}_{j_2}(x^0)}^{\tilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, u^q(s)\, w_2(u(s))\, ds \\ & + \ldots + \sum_{j_n=1}^{m_n} \int_{\tilde{\alpha}_{j_n}(x^0)}^{\tilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, u^q(s)\, w_n(u(s))\, ds. \end{aligned}$$

For $x \in \Delta$, clearly that, $r(x)$ is a positive non-decreasing function. From (3.1) we find

$$u(x) \leq r^{\frac{1}{p}}(x), \tag{3.5}$$

and

$$
\begin{aligned}
D_1 r(x) &= \sum_{j_1=1}^{m_1} \alpha'_{j_1 1}(x_1) \int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)} \dots \int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} a_{j_1}(\alpha_{j_1 1}(x_1), s_2, ..., s_n) \times \\
& u^q(\alpha_{j_1 1}(x_1), s_2, ..., s_n) w_1(u(\alpha_{j_1 1}(x_1), s_2, ..., s_n)) ds_n ... ds_2 \\
& + \sum_{j_2=1}^{m_2} \alpha'_{j_2 1}(x_1) \int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)} \dots \int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} a_{j_2}(\alpha_{j_2 1}(x_1), s_2, ..., s_n) \times \\
& u^q(\alpha_{j_2 1}(x_1), s_2, ..., s_n) w_2(u(\alpha_{j_2 1}(x_1), s_2, ..., s_n)) ds_n ... ds_2 \\
& + ... + \sum_{j_n=1}^{m_n} \alpha'_{j_n 1}(x_1) \int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)} \dots \int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} a_{j_n}(\alpha_{j_n 1}(x_1), s_2, ..., s_n) \times \\
& u^q(\alpha_{j_n 1}(x_1), s_2, ..., s_n) w_n(u(\alpha_{j_n 1}(x_1), s_2, ..., s_n)) ds_n ... ds_2 \\
&\leq \sum_{j_1=1}^{m_1} \alpha'_{j_1 1}(x_1) \int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)} \dots \int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} a_{j_1}(\alpha_{j_1 1}(x_1), s_2, ..., s_n) \times \\
& r^{\frac{q}{p}}(\alpha_{j_1 1}(x_1), s_2, ..., s_n) w_1\left(r^{\frac{1}{p}}(\alpha_{j_1 1}(x_1), s_2, ..., s_n)\right) ds_2 ... ds_n \\
& + \sum_{j_2=1}^{m_2} \alpha'_{j_2 1}(x_1) \int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)} \dots \int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} a_{j_2}(\alpha_{j_2 1}(x_1), s_2, ..., s_n) \times \\
& r^{\frac{q}{p}}(\alpha_{j_2 1}(x_1), s_2, ..., s_n) w_2\left(r^{\frac{1}{p}}(\alpha_{j_2 1}(x_1), s_2, ..., s_n)\right) ds_2 ... ds_n \\
& + ... + \sum_{j_n=1}^{m_n} \alpha'_{j_n 1}(x_1) \int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)} \dots \int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} a_{j_1}(\alpha_{j_n 1}(x_1), s_2, ..., s_n) \times \\
& r^{\frac{q}{p}}(\alpha_{j_n 1}(x_1), s_2, ..., s_n) w_n\left(r^{\frac{1}{p}}(\alpha_{j_n 1}(x_1), s_2, ..., s_n)\right) ds_2 ... ds_n
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{D_{x_1} r_1(x)}{r^{\frac{q}{p}}(x)} &\leq \sum_{j_1=1}^{m_1} \alpha'_{j_1 1}(x_1) \int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)} \dots \int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} a_{j_1}(\alpha_{j_1 1}(x_1), s_2, ..., s_n) \times \\
& w_1\left(r^{\frac{1}{p}}(\alpha_{j_1 1}(x_1), s_2, ..., s_n)\right) ds_n ... ds_2 \\
& + \sum_{j_2=1}^{m_2} \alpha'_{j_2 1}(x_1) \int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)} \dots \int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} a_{j_2}(\alpha_{j_2 1}(x_1), s_2, ..., s_n) \times \\
& w_2\left(r^{\frac{1}{p}}(\alpha_{j_2 1}(x_1), s_2, ..., s_n)\right) ds_n ... ds_2 \\
& + ... + \sum_{j_n=1}^{m_n} \alpha'_{j_n 1}(x_1) \int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)} \dots \int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} a_{j_n}(\alpha_{j_n 1}(x_1), s_2, ..., s_n) \times \\
& w_n\left(r^{\frac{1}{p}}(\alpha_{j_n 1}(x_1), s_2, ..., s_n)\right) ds_n ... ds_2.
\end{aligned}
\tag{3.6}
$$

Keeping $x_2, x_3, ..., x_n$ fixed in (3.6), and integrating from x_1^0 to x_1^1 with making the change of variable we get

$$\begin{aligned}\frac{p}{p-q}r^{\frac{p-q}{p}}(x) \leq\ & \frac{p}{p-q}k^{\frac{p-q}{p}} + \sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, w_1\left(r^{\frac{1}{p}}(s)\right) ds \\ & + \sum_{j_2=1}^{m_2}\int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, w_2\left(r^{\frac{1}{p}}(s)\right) ds \\ & + ... + \sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, w_n\left(r^{\frac{1}{p}}(s)\right) ds,\end{aligned}$$

it imply that

$$\begin{aligned}r^{\frac{p-q}{p}}(x) \leq\ & k^{\frac{p-q}{p}} + \frac{p-q}{p}\sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, w_1\left(r^{\frac{1}{p}}(s)\right) ds \\ & + \frac{p-q}{p}\sum_{j_2=1}^{m_2}\int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, w_2\left(r^{\frac{1}{p}}(s)\right) ds \\ & + ... + \frac{p-q}{p}\sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_1}(s)\, w_n\left(r^{\frac{1}{p}}(s)\right) ds.\end{aligned}$$

Let $v(x) = r^{\frac{p-q}{p}}(x)$ we find

$$\begin{aligned}v(x) \leq\ & k^{\frac{p-q}{p}} + \frac{p-q}{p}\sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, w_1\left(v^{\frac{1}{p-q}}(s)\right) ds \\ & + \frac{p-q}{p}\sum_{j_2=1}^{m_2}\int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, w_2\left(v^{\frac{1}{p-q}}(s)\right) ds \\ & + ... + \frac{p-q}{p}\sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_1}(s)\, w_n\left(v^{\frac{1}{p-q}}(s)\right) ds. \qquad (3.7)\end{aligned}$$

Setting $\overline{r}(x)$ as the right-hand side of (3.7), then we have $\overline{r}(x_1^0, x_2, ..., x_n) = k^{\frac{p-q}{p}}$

$$v(x) \leq \overline{r}(x), \qquad (3.8)$$

and

$$\frac{D_{x_1}\overline{r}(x)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}$$

$$\leq \frac{\frac{p-q}{p}\sum_{j_1=1}^{m_1}\alpha'_{j_1 1}(x_1)\int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)}...\int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)}a_{j_1}(\alpha_{j_1 1}(x_1),s_2,...,s_n)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$w_1\left(v^{\frac{1}{p-q}}(\alpha_{j_1 1}(x_1),s_2,...,s_n)\right)ds_n...ds_2$$

$$+\frac{\frac{p-q}{p}\sum_{j_2=1}^{m_2}\alpha'_{j_2 1}(x_1)\int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)}...\int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)}a_{j_2}(\alpha_{j_2 1}(x_1),s_2,...,s_n)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$w_2\left(v^{\frac{1}{p-q}}(\alpha_{1 j_2}(x_1),s_2,...,s_n)\right)ds_n...ds_2$$

$$+...+\frac{\frac{p-q}{p}\sum_{j_n=1}^{m_n}\alpha'_{j_n 1}(x_1)\int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)}...\int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)}a_{j_n}(\alpha_{j_n 1}(x_1),s_2,...,s_n)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$w_n\left(v^{\frac{1}{p-q}}(\alpha_{1 j_n}(x_1),s_2,...,s_n)\right)ds_n...ds_2$$

$$\leq \frac{\frac{p-q}{p}w_1\left(v^{\frac{1}{p-q}}(x)\right)\sum_{j_1=1}^{m_1}\alpha'_{j_1 1}(x_1)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$\int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)}...\int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)}a_{j_1}(\alpha_{j_1 1}(x_1),s_2,...,s_n)\,ds_n...ds_2$$

$$+\frac{\frac{p-q}{p}w_2\left(v^{\frac{1}{p-q}}(x)\right)\sum_{j_2=1}^{m_2}\alpha'_{j_2 1}(x_1)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$\int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)}...\int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)}a_{j_2}(\alpha_{j_2 1}(x_1),s_2,...,s_n)\,ds_n...ds_2$$

$$+...+\frac{\frac{p-q}{p}w_n\left(v^{\frac{1}{p-q}}(x)\right)\sum_{j_n=1}^{m_n}\alpha'_{j_n 1}(x_1)}{w_1\left(\overline{r}^{\frac{1}{p-q}}(x)\right)+...+w_n\left(\overline{r}^{\frac{1}{p-q}}(x)\right)}\times$$

$$\int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)}...\int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)}a_{j_n}(\alpha_{j_n 1}(x_1),s_2,...,s_n)\,ds_n...ds_2$$

$$
\begin{aligned}
&\leq \frac{p-q}{p}\sum_{j_1=1}^{m_1}\alpha'_{j_1 1}(x_1)\int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)}...\int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} a_{j_1}(\alpha_{j_1 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2 \\
&\quad \frac{p-q}{p}\sum_{j_2=1}^{m_2}\alpha'_{j_2 1}(x_1)\int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)}...\int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} a_{j_2}(\alpha_{j_2 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2 \\
&\quad +...+\frac{p-q}{p}\sum_{j_n=1}^{m_n}\alpha'_{j_n 1}(x_1)\int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)}...\int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} a_{j_n}(\alpha_{j_n 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2.
\end{aligned}
$$

By the definition of G we observe that from the last inequality

$$
\begin{aligned}
D_{x_1}G(\overline{r}(x)) &\leq \frac{p-q}{p}\sum_{j_1=1}^{m_1}\alpha'_{j_1 1}(x_1)\int_{\alpha_{j_1 2}(x_2^0)}^{\alpha_{j_1 2}(x_2)}...\int_{\alpha_{j_1 n}(x_n^0)}^{\alpha_{j_1 n}(x_n)} a_{j_1}(\alpha_{j_1 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2 \\
&\quad +\frac{p-q}{p}\sum_{j_2=1}^{m_2}\alpha'_{j_2 1}(x_1)\int_{\alpha_{j_2 2}(x_2^0)}^{\alpha_{j_2 2}(x_2)}...\int_{\alpha_{j_2 n}(x_n^0)}^{\alpha_{j_2 n}(x_n)} a_{j_2}(\alpha_{j_2 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2 \\
&\quad +...+\frac{p-q}{p}\sum_{j_n=1}^{m_n}\alpha'_{j_n 1}(x_1)\int_{\alpha_{j_n 2}(x_2^0)}^{\alpha_{j_n 2}(x_2)}...\int_{\alpha_{j_n n}(x_n^0)}^{\alpha_{j_n n}(x_n)} a_{j_n}(\alpha_{j_n 1}(x_1), s_2, ..., s_n)\, ds_n...ds_2.
\end{aligned}
$$

Keeping $x_2, x_3, ..., x_n$ fixed, and integrating from x_1^0 to x_1^1 with making the change of variable we get

$$
\begin{aligned}
G(\overline{r}(x)) &\leq G\left(\overline{r}\left(x_1^0, x_2, ..., x_n\right)\right) \\
&\quad +\frac{p-q}{p}\sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, ds+\frac{p-q}{p}\sum_{j_2=1}^{m_2}\int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, ds \\
&\quad +...+\frac{p-q}{p}\sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, ds
\end{aligned}
$$

then

$$
\begin{aligned}
G(\overline{r}(x)) &\leq G\left(k^{\frac{p-q}{p}}\right)+\frac{p-q}{p}\sum_{j_1=1}^{m_1}A_{1j_1}(x) \\
&\quad +\frac{p-q}{p}\sum_{j_2=1}^{m_2}A_{2j_2}(x)+...+\frac{p-q}{p}\sum_{j_n=1}^{m_n}A_{nj_n}(x),
\end{aligned}
\tag{3.9}
$$

from the last above inequality we show that

$$
\begin{aligned}
\overline{r}(x) &\leq G^{-1}\left[G\left(k^{\frac{p-q}{p}}\right)+\frac{p-q}{p}\sum_{j_1=1}^{m_1}A_{1j_1}(x)\right. \\
&\quad \left.+\frac{p-q}{p}\sum_{j_2=1}^{m_2}A_{2j_2}(x)+...+\frac{p-q}{p}\sum_{j_n=1}^{m_n}A_{nj_n}(x)\right],
\end{aligned}
\tag{3.10}
$$

for $x^0 \le x \le x^1$. In view of (3.5), (3.8) and (3.10) and by the fact $v(x) = r^{\frac{p-q}{p}}(x)$, we conclud the inequality (3.2). By continuity, (3.2) also holds for any $k \ge 0$. $\square$

Next, we give remarks and corollaries, from the above Theorem.

Remark 3.1. *If $n = 2$ and $w_1 = w_2$ in Theorem 3.1 we get theorem 2.2 in [17].*

Corollary 3.1. *Let the function $a_{j_1}, a_{j_2}, ..., a_{j_n}$, $\alpha_{j_k i}$ $(k = 1, 2, ..., n), (i = 1, 2, ..., n)$, $(j_k = 1, 2, ..., m_k)$, and the constants p, q and k be defined as in Theorem (3.1), and $w_j \in C(\mathbb{R}, \mathbb{R}_+)$ $(j = l+1, ..., n,\ 0 \le l \le n)$ be a non-decreasing functions with $w_j(u) > 0$ for $u > 0$ and*

$$\begin{aligned} u^p(x) \le\ & k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s) u^q(s) ds + \\ & \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s) u^q(s) ds + ... + \sum_{j_l=1}^{m_l} \int_{\widetilde{\alpha}_{j_l}(x^0)}^{\widetilde{\alpha}_{j_l}(x)} a_{j_l}(s) u^q(s) ds \\ & \sum_{j_{l+1}=1}^{m_{l+1}} \int_{\widetilde{\alpha}_{j_{l+1}}(x^0)}^{\widetilde{\alpha}_{j_{l+1}}(x)} a_{j_{l+1}}(s) u^q(s) w_{l+1}(u(s)) ds + ... + \\ & \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_n}(s) u^q(s) w_n(u(s)) ds, \end{aligned}$$

for any $x \in \Delta$. Then

$$\begin{aligned} u(x) \le\ & \left\{ G_1^{-1} \left[G_1 \left(k^{\frac{p-q}{p}} + \frac{p-q}{p} \sum_{j_1=1}^{m_1} A_{1j_1}(x) + \right.\right.\right. \\ & \left. \frac{p-q}{p} \sum_{j_2=1}^{m_2} A_{2j_2}(x) + ... + \frac{p-q}{p} \sum_{j_l=1}^{m_l} A_{lj_l}(x) \right) + \\ & \left.\left. \frac{p-q}{p} \sum_{j_{l+1}=1}^{m_{l+1}} A_{l+1j_{l+1}}(x) + ... + \frac{p-q}{p} \sum_{j_n=1}^{m_n} A_{nj_n}(x) \right] \right\}^{\frac{1}{p-q}}. \end{aligned}$$

For $x^0 \le x \le x^2$, where $x^2 = (x_1^2, x_2^2, ..., x_n^2)$ and $A_{kj_k}(x)$ where $(j_k = 1, 2, ..., m_k), (k = 1, 2, ..., n)$ are defined as in (3.3) and

$$G_1(r) = \int_{r_0}^{r} \frac{1}{w_{l+1}\left(s^{\frac{1}{p-q}}\right) + w_{l+2}\left(s^{\frac{1}{p-q}}\right) + ... + w_n\left(s^{\frac{1}{p-q}}\right)} ds \ \ r \ge r_0 > 0. \tag{3.11}$$

Remark 3.2. *If $n = 2$ and $l = 1$ in corollary 3.1 we get Theorem 2.1 in [17]*

Corollary 3.2. *Let the function $a_{j_1}, a_{j_2}, ..., a_{j_n}$, $\alpha_{j_k i}$, $(j_k = 1, 2, ..., m_k), (k = 1, 2, ..., n)$,and $(i =$*

$1, 2, ..., n)$, w_j $(j = 1, 2, ..., n)$ *and the constants* p, q *and* k *be defined as in Theorem (3.1) and*

$$\begin{aligned} u^p(x) &\leq k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, u^{p-1}(s)\, w_1(u(s))\, ds \\ &+ \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, u^{p-1}(s)\, w_2(u(s))\, ds \\ &+ ... + \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, u^{p-1}(s)\, w_n(u(s))\, ds, \end{aligned}$$

for any $x \in \Delta$. *Then*

$$\begin{aligned} u(x) &\leq G^{-1}\Bigg[G\left(k^{\frac{1}{p}}\right) + \frac{1}{p}\sum_{j_1=1}^{m_1} A_{1j_1}(x) + \\ &\frac{1}{p}\sum_{j_2=1}^{m_2} A_{2j_2}(x) + ... + \frac{1}{p}\sum_{j_n=1}^{m_n} A_{nj_n}(x)\Bigg]. \end{aligned}$$

For $x^0 \leq x \leq x^3$, *where* $x^3 = (x_1^3, x_2^3, ..., x_n^3)$, *and* $A_{kj_k}(x)$, *where* $(k = 1, 2, ..., n)$, $(j_k = 1, 2, ..., m_k)$ *are defined as in (3.3) and* G *is defined as in (3.4).*

Corollary 3.3. *Let the function* $a_{j_1}, a_{j_2}, ..., a_{j_n}$, $\alpha_{j_k i}$ $(j_k = 1, 2, ..., m_k)$, $(k = 1, 2, ..., n)$ *and* $(i = 1, 2, ..., n),$. *and the constants* p, q *and* k *be defined as in Theorem (3.1), and* $w_j \in C(\mathbb{R}, \mathbb{R}_+)$ $(j = l + 1, ..., n)$, *where* $0 \leq l \leq n$, *be a non-decreasing functions with* $w_j(u) > 0$, *for* $u > 0$, *and*

$$\begin{aligned} u^p(x) &\leq k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, u^{p-1}(s)\, ds \\ &+ \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, u^{p-1}(s)\, ds + ... + \sum_{j_l=1}^{m_l} \int_{\widetilde{\alpha}_{j_l}(x^0)}^{\widetilde{\alpha}_{j_l}(x)} a_{j_l}(s)\, u^{p-1}(s)\, ds \\ &+ \sum_{j_{l+1}=1}^{m_{l+1}} \int_{\widetilde{\alpha}_{j_{l+1}}(x^0)}^{\widetilde{\alpha}_{j_{l+1}}(x)} a_{j_{l+1}}(s)\, u^{p-1}(s)\, w_{l+1}(u(s))\, ds \\ &+ ... + \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, u^{p-1}(s)\, w_n(u(s))\, ds, \end{aligned}$$

for any $x \in \Delta$. Then

$$\begin{aligned} u(x) \leq\ & G_1^{-1}\Bigg[G_1\Bigg(k^{\frac{1}{p}} + \frac{1}{p}\sum_{j_1=1}^{m_1} A_{1j_1}(x) + \\ & \frac{1}{p}\sum_{j_2=1}^{m_2} A_{2j_2}(x) + ... + \frac{1}{p}\sum_{j_l=1}^{m_l} A_{lj_l}(x)\Bigg) + \\ & \frac{1}{p}\sum_{j_{l+1}=1}^{m_{l+1}} A_{l+1j_{l+1}}(x) + ... + \frac{1}{p}\sum_{j_n=1}^{m_n} A_{nj_n}(x)\Bigg]. \end{aligned}$$

For $x^0 \leq x \leq x^4$, where $x^4 = (x_1^4, x_2^4, ..., x_n^4)$, and $A_{kj_k}(x)$, where $(j_k = 1, 2, ..., m_k), (k = 1, 2, ..., n)$, are defined as in (3.3) and G_1 is defined as in (3.11).

Remark 3.3. *If $n = 2$ and $l = 1$ in corollary 3.3 we get corollary 2.1 in [17].*

Corollary 3.4. *Let the function $a_{j_1}, a_{j_2}, ..., a_{j_n}$, α_{ij_k} $(j_k = 1, 2, ..., m_k), (k = 1, 2, ..., n)$ and $(i = 1, 2, ..., n)$, and the constants p, q and k be defined an in theorem (3.1), and*

$$\begin{aligned} u^p(x) \leq\ & k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, u^{p-1}(s)\, ds \\ & + \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, u^{p-1}(s)\, ds + ... + \sum_{j_l=1}^{m_l} \int_{\widetilde{\alpha}_{j_l}(x^0)}^{\widetilde{\alpha}_{j_l}(x)} a_{j_l}(s)\, u^{p-1}(s)\, ds \\ & + \sum_{j_{l+1}=1}^{m_{l+1}} \int_{\widetilde{\alpha}_{j_{l+1}}(x^0)}^{\widetilde{\alpha}_{j_{l+1}}(x)} a_{j_{l+1}}(s)\, u^{p}(s)\, ds + ... + \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, u^{p}(s)\, ds, \end{aligned}$$

for any $x \in \Delta$. and $0 \leq l \leq n$. Then

$$\begin{aligned} u(x) \leq\ & \Bigg(k^{\frac{1}{p}} + \frac{1}{p}\sum_{j_1=1}^{m_1} A_{1j_1}(x) + \\ & \frac{1}{p}\sum_{j_1=1}^{m_2} A_{2j_2}(x) + ... + \frac{1}{p}\sum_{j_l=1}^{m_l} A_{lj_l}(x)\Bigg) \\ & \times \left[\exp\left(\frac{1}{p}\sum_{j_{l+1}=1}^{m_{l+1}} A_{l+1j_{l+1}}(x) + ... + \frac{1}{p}\sum_{j_n=1}^{m_n} A_{nj_n}(x)\right)\right]^{n-l-1}. \end{aligned}$$

For $x^0 \leq x \leq x^5$, where $x^5 = (x_1^5, x_2^5, ..., x_n^5)$, and $A_{j_k k}(x_1, x_2, ..., x_n)$ $(k = 1, 2, ..., n)$, are defined as in (3.3).

Corollary 3.5. *Let the function $a_{j_1}, a_{j_2}, ..., a_{j_n}$, $\alpha_{j_k i}$ $(j_k = 1, 2, ..., m_k), (k = 1, 2, ..., n)$ and $(i =$*

$1, 2, ..., n)$, *and the constants* p, q *and* k *be defined an in theorem (3.1), and*

$$\begin{aligned} u^p(x) &\leq k + \sum_{j_1=1}^{m_1} \int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} a_{j_1}(s)\, u^p(s)\, ds \\ &+ \sum_{j_2=1}^{m_2} \int_{\widetilde{\alpha}_{j_2}(x^0)}^{\widetilde{\alpha}_{j_2}(x)} a_{j_2}(s)\, u^p(s)\, ds + ... + \sum_{j_n=1}^{m_n} \int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} a_{j_n}(s)\, u^p(s)\, ds, \end{aligned}$$

for any $x \in \Delta$. *and* $0 \leq l \leq n$. *Then*

$$u(x) \leq k^{\frac{1}{p}} \left[\exp\left(\frac{1}{p} \sum_{j_1=1}^{m_1} A_{1j_1}(x) + ... + \frac{1}{p} \sum_{j_n=1}^{m_n} A_{nj_n}(x) \right) \right]^n .$$

For $x^0 \leq x \leq x^6$, *where* $x^6 = (x_1^6, x_2^6, ..., x_n^6)$, *and* $A_{j_k k}$ $(k = 1, 2, ..., n)$, *are defined as in (3.3).*

Remark 3.4. *for special cases to the functions of some inequalities of chapter 2 we find some inequalities of chapter 3 for* $n = 1$.

3.2 Some Apllications

we present three results of application to study respectively the boundless, uniqueness and stability of the solution of the following initial boundary value problem. We denote

$$\begin{aligned} u(x - h_i(x)) &= u\left(x_1 - h_{1i}^1(x_1), x_2 - h_{1i}^2(x_2), ..., x_n - h_{1i}^n(x_n)\right), \\ &\quad ..., u\left(x_1 - h_{m_i i}^1(x_1), x_2 - h_{m_i i}^2(x_2), ..., x_n - h_{m_i i}^n(x_n)\right), \end{aligned}$$

where $i = 1, 2, ..., n$. Consider the initial boundary value problem

$$D(x) = F(x, u(x - h_1(x)), u(x - h_2(x)), ..., u(x - h_n(x))). \tag{3.12}$$

With the given initial boundary conditions

$$\begin{aligned} u(x_1^0, x_2, ..., x_n) &= c_1(x_2, x_3, ..., x_n), \\ u(x_1, x_2^0, ..., x_n) &= c_2(x_1, x_3, ..., x_n), \\ &\vdots \\ u(x_1, x_2, ..., x_n^0) &= c_n(x_1, x_3, ..., x_{n-1}), \end{aligned} \tag{3.13}$$

$$u\left(x_1, ..., x_{i_1}^0, ..., x_{i_2}^0, ..., x_{i_k}^0, ..., x_n\right) = 0, \text{ for } 1 \leq i_1 < i_2 <, ..., < i_k \leq n. \tag{3.14}$$

where p is a constant, $F \in C\left(\Delta, \mathbb{R}^{m_1+m_2+...+m_n}, \mathbb{R}\right)$,
$c_j \in C^1\left(I_1 \times I_2 \times ... \times I_{j-1} \times I_{j+1} \times ... \times I_n, \mathbb{R}\right)$, $h_{j_k k}^i \in C^1(I_i, \mathbb{R})$ are non-increasing functions,

$x_i - h^i_{j_k k}(x_i) \geq 0$, $x_i - h^i_{j_k k}(x_i) \in C^1(I_i, I_i)$, $\left(h^i_{j_k k}\right)'(x_i) < 1$, $h^i_{j_k k}(x^0_i) = 0$, and

$$M^i_{j_k k} = \frac{1}{1 - \left(h^i_{j_k k}\right)'(x_i)},$$

for $1 \leq i, k \leq n$, $1 \leq j_k \leq m_k$, $x_i \in I_i$.

Our first result gives the bound on the solution of the problem (3.12)-(3.14).

Theorem 3.2. *Suppose that*

$$\begin{aligned} & \left|F\left(x, u_{11}, ..., u_{m_1 1}, u_{12}, ..., u_{m_2 2}, ..., u_{1n}, ..., u_{m_n n}\right)\right| \\ \leq & \sum_{j_1=1}^{m_1} a_{j_1}(x)\left|u_{j_1 1}\right|^{p-1} + \sum_{j_2=1}^{m_2} a_{j_2}(x)\left|u_{j_2 2}\right|^{p-1} + ... + \sum_{j_l=1}^{m_l} a_{j_l}(x)\left|u_{j_l l}\right|^{p-1} \\ & + \sum_{j_{l+1}=1}^{m_{l+1}} a_{j_{l+1}}(x)\left|u_{j_{l+1} l+1}\right|^{p} + ... + \sum_{j_n=1}^{m_n} a_{j_n}(x)\left|u_{j_n n}\right|^{p}, \end{aligned} \tag{3.15}$$

$$\left|c_1(x_2, x_3, ..., x_n) + c_2(x_1, x_3, ..., x_n) + ... + c_n(x_1, x_3, ..., x_{n-1})\right| \leq k, \tag{3.16}$$

where $a_{j_1}(x), a_{j_2}(x), ..., a_{j_n}(x)$, and k are defined as in theorem 3.1. If $u(x)$ is any solution of (3.12)-(3.14), then

$$\begin{aligned} |u(x)| \leq & \left(k^{\frac{1}{p}} + \frac{1}{p}\sum_{j_1=1}^{m_1}\overline{A_{1j_1}}(x) + \right. \\ & \left. \frac{1}{p}\sum_{j_2=1}^{m_2}\overline{A_{2j_2}}(x) + ... + \frac{1}{p}\sum_{j_l=1}^{m_l}\overline{A_{lj_l}}(x)\right) \\ & \times \left[\exp\left(\frac{1}{p}\sum_{j_{l+1}=1}^{m_{l+1}}\overline{A_{l+1j_{l+1}}}(x) + ... + \frac{1}{p}\sum_{j_1=1}^{m_n}\overline{A_{nj_n}}(x)\right)\right]^{n-l-1}. \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \overline{A_{1j_1}}(x) &= \int_{\widetilde{\alpha}_{j_1 1}(x^0)}^{\widetilde{\alpha}_{j_1 1}(x)} \overline{a_{j_1}}(s)\, ds, \\ \overline{A_{2j_2}}(x) &= \int_{\widetilde{\alpha}_{j_2 2}(x^0)}^{\widetilde{\alpha}_{j_2 2}(x)} \overline{a_{j_2}}(s)\, ds, \\ & . \\ & . \\ & . \\ \overline{A_{nj_n}}(x) &= \int_{\widetilde{\alpha}_{j_n n}(x^0)}^{\widetilde{\alpha}_{j_n n}(x)} \overline{a_{j_n}}(s)\, ds, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\overline{a_{j_1}}(x) = a_{j_1}\left(\left(\beta^1_{j_1 1}\right)^{-1}(x_1), \left(\beta^2_{j_1 1}\right)^{-1}(x_2), ..., \left(\beta^n_{j_1 1}\right)^{-1}(x_n)\right)\prod_{i=1}^n M^i_{j_1 1},\\
&\overline{a_{j_2}}(x) = a_{j_2}\left(\left(\beta^1_{j_2 2}\right)^{-1}(x_1), \left(\beta^2_{j_2 2}\right)^{-1}(x_2), ..., \left(\beta^n_{j_2 2}\right)^{-1}(x_n)\right)\prod_{i=1}^n M^i_{j_2 2},\\
&.\\
&.\\
&.\\
&\overline{a_{j_n}}(x) = a_{j_n}\left(\left(\beta^1_{j_n n}\right)^{-1}(x_1), \left(\beta^2_{j_n n}\right)^{-1}(x_2), ..., \left(\beta^n_{j_n n}\right)^{-1}(x_n)\right)\prod_{i=1}^n M^i_{j_n n}.
\end{aligned} \tag{3.19}$$

Where $\left(\beta^i_{j_k k}\right)(x_i) = x_i - h^i_{j_k k}(x_i)$ *and* $\widetilde{\beta}_{j_k k}(t) = \left(\beta^1_{j_k k}(t_1), \beta^2_{j_k k}(t_2), ..., \beta^n_{j_k k}(t_n)\right)$, *for* $k = 1, 2, ..., n,\ j_k = 1, 2, ..., m_k,$ *and* $i = 1, 2, ..., n.$

Proof. It is ease to observe that every solution $u(x)$ of (3.12)-(3.14) satisfies that equivalent integral equation

$$\begin{aligned}
u^p(x) \ &= \ c_1(x_2, x_3, ..., x_n) + c_2(x_1, x_3, ..., x_n) + ... + c_n(x_1, x_3, ..., x_{n-1}) + \\
&+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} ... \int_{x_n^0}^{x_n} F(s, u(s - h_1(s)), u(s - h_2(s)), ..., u(s - h_n(s)))\, ds.
\end{aligned} \tag{3.20}$$

Appling (3.15), (3.16) to (3.20) and changing the variables we obtain

$$\begin{aligned}
|u(x)|^p \ \leq \ & k + \sum_{j_1=1}^{m_1}\int_{\widetilde{\beta}_{j_1 1}(x^0)}^{\widetilde{\beta}_{j_1 1}(x)} \overline{a_{j_1}}(s)\,|u(s)|^{p-1}\, ds \\
&+ \sum_{j_2=1}^{m_2}\int_{\widetilde{\beta}_{j_2 2}(x^0)}^{\widetilde{\beta}_{j_2 2}(x)} \overline{a_{j_2}}(s)\,|u(s)|^{p-1}\, ds + ... + \sum_{j_l=1}^{m_l}\int_{\widetilde{\beta}_{j_l l}(x^0)}^{\widetilde{\beta}_{j_l l}(x)} \overline{a_{j_l}}(s)\,|u(s)|^{p-1}\, ds \\
&+ \sum_{j_{l+1}=1}^{m_{l+1}}\int_{\widetilde{\beta}_{j_{l+1} l+1}(x^0)}^{\widetilde{\beta}_{j_{l+1} l+1}(x)} \overline{a_{j_l}}(s)\,|u(s)|^{p}\, ds + ... + \sum_{j_n=1}^{m_n}\int_{\widetilde{\beta}_{j_n j_n}(x^0)}^{\widetilde{\beta}_{j_n j_n}(x)} \overline{a_{j_n}}(s)\,|u(s)|^{p}\, ds.
\end{aligned} \tag{3.21}$$

An application of corollary 3.4 to (3.21) yields (3.17). □

Our second result gives the uniqueness of the solution of the problem (3.12)-(3.14).

Theorem 3.3. *Let* $M^i_{j_k k},\ \beta^i_{j_k k}, a_{j_1}, a_{j_2}, ..., a_{j_n}, \overline{a_{j_1}}, \overline{a_{j_2}}, ..., \overline{a_{j_n}}\ (i = 1, 2, ...n), (j = 1, 2, ..., n),$ $(k = 1, 2, ..., n),$ *and* $(j_k = 1, 2, ..., m_k),$ *be as in Theorem (3.2). Suppose that the function* F *in (3.12) satisfies the condition*

$$\begin{aligned}
&|F(x, u_{11}, ..., u_{m_1 1}, u_{12}, ..., u_{m_2 2}, ..., u_{1n}, ..., u_{m_n n}) \\
&- F(x, v_{11}, ..., v_{m_1 1}, v_{12}, ..., v_{m_2 2}, ..., v_{1n}, ..., v_{m_n n})| \\
\leq\ & \sum_{j_1=1}^{m_1} a_{j_1}(x)\,|u_{j_1 1} - v_{j_1 1}|^p + \sum_{j_2=1}^{m_2} a_{j_2}(x)\,|u_{j_2 2} - v_{j_2 2}|^p \\
&+ ... + \sum_{j_n=1}^{m_n} a_{j_n}(x)\,|u_{j_n n} - v_{j_n n}|^p.
\end{aligned} \tag{3.22}$$

Then the problem (3.12)-(3.14) has at most one solution on Δ.

Proof. Let $u(x)$ and $v(x)$ be tow solutions of (3.12)-(3.14), then we have

$$\begin{aligned} & u^p(x) - v^p(x) \\ = & \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}...\int_{x_n^0}^{x_n} F\left(s, u\left(s-h_1\left(s_1\right)\right), u\left(s-h_2\left(s_2\right)\right), ..., u\left(s-h_n\left(s_n\right)\right)\right) \\ & -F\left(v\left(s-h_1\left(s_1\right)\right), v\left(s-h_2\left(s_2\right)\right), ..., v\left(s-h_n\left(s_n\right)\right)\right) ds. \end{aligned} \tag{3.23}$$

From (3.22),(3.23) making the changing of variables we get

$$\begin{aligned} \left|u^p(x) - v^p(x)\right| \leq & \sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)} \overline{a_{j_1}}(s)\left|u^p(s)-v^p(s)\right| ds \\ & +...+\sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)} \overline{a_{j_n}}(s)\left|u^p(s)-v^p(s)\right| ds. \end{aligned}$$

An application of Corollary 3.5 to the function $\left|u^p(x) - v^p(x)\right|^{\frac{1}{p}}$ show that

$$\left|u^p(x) - v^p(x)\right|^{\frac{1}{p}} \leq 0,$$

for any $x \in \Delta$. hance $u(x) = v(x)$. □

Our third result gives the stability of the solution of the problem (3.12)-(3.14).

Theorem 3.4. *Let* $M^i_{j_k k}$, $\beta^i_{j_k k}, a_{j_1}, a_{j_2}, ..., a_{j_n}, \overline{a_{j_1}}, \overline{a_{j_2}}, ..., \overline{a_{j_n}}$ $(i = 1, 2, ...n)$, $(j = 1, 2, ..., n)$, $(k = 1, 2, ..., n)$, *and* $(j_k = 1, 2, ..., m_k)$, *be as in Theorem (3.2) and let* $u(x)$ *and* $v(x)$ *be the solutions of (3.12) with the given initial boundary data*

$$\begin{aligned} u\left(x_1^0, x_2, ..., x_n\right) &= c_1\left(x_2, x_3, ..., x_n\right), \\ u\left(x_1, x_2^0, ..., x_n\right) &= c_2\left(x_1, x_3, ..., x_n\right), \\ &. \\ &. \\ &. \\ u\left(x_1, x_2, ..., x_n^0\right) &= c_n\left(x_1, x_3, ..., x_{n-1}\right), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} v\left(x_1^0, x_2, ..., x_n\right) &= d_1\left(x_2, x_3, ..., x_n\right), \\ v\left(x_1, x_2^0, ..., x_n\right) &= d_2\left(x_1, x_3, ..., x_n\right), \\ &. \\ &. \\ &. \\ v\left(x_1, x_2, ..., x_n^0\right) &= d_n\left(x_1, x_3, ..., x_{n-1}\right), \end{aligned} \tag{3.25}$$

where $c_j, d_j \in C^1\left(I_1 \times I_2 \times ... \times I_{j-1} \times I_{j+1} \times ... \times I_n, \mathbb{R}\right)$. *Suppose that the fonction* F *satisfies the condition (3.22), and*

$$\begin{aligned} &|c_1(x_2, x_3, ..., x_n) - d_1(x_2, x_3, ..., x_n) + \\ &c_2(x_1, x_3, ..., x_n) - d_2(x_1, x_3, ..., x_n) + ...+ \\ &c_n(x_2, x_3, ..., x_{n-1}) - d_n(x_2, x_3, ..., x_{n-1})| \quad < \epsilon^p. \end{aligned} \tag{3.26}$$

Where ϵ *is an arbitrary positive number. Then*

$$|u^p(x) - v^p(x)| \le \epsilon\left[\exp\left(\frac{1}{p}\sum_{j_1=1}^{m_1}\overline{A_{1j_1}}(x) + \frac{1}{p}\sum_{j_2=1}^{m_2}\overline{A_{2j_2}}(x) + ... + \frac{1}{p}\sum_{j_n=1}^{m_n}\overline{A_{nj_n}}(x)\right)\right]^n, \tag{3.27}$$

for $x \in \Delta$, *where* $\overline{A_{1j_1}}(x), \overline{A_{2j_2}}(x), ..., \overline{A_{nj_n}}(x)$ *are defined as in (3.18).*

Proof. we have $u(x)$ and $v(x)$ be solutions of (3.12),(3.24) and (3.12),(3.25) respectively. then we have

$$\begin{aligned} u^p(x) - v^p(x) &= c_1(x_2, x_3, ..., x_n) - d_1(x_2, x_3, ..., x_n) + \\ &\quad c_2(x_2, x_3, ..., x_n) - d_2(x_2, x_3, ..., x_n) + ... + \\ &\quad + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} ... \int_{x_n^0}^{x_n}\left[F(s, u(s - h_1(s_1)), u(s - h(s)), ..., u(s - h_n(s)))\right. \\ &\quad \left. - F(s, v(s - h_1(s_1)), v(s - h(s)), ..., v(s - h_n(s)))\right] ds, \end{aligned} \tag{3.28}$$

for $x \in \Delta$. From (3.22), (3.26) and (3.28), making change of variables we get

$$\begin{aligned} |u^p(x) - v^p(x)| &\le \epsilon^p + \sum_{j_1=1}^{m_1}\int_{\widetilde{\alpha}_{j_1}(x^0)}^{\widetilde{\alpha}_{j_1}(x)}\overline{a_{j_1}}(s)\,|u^p(s) - v^p(s)|\,ds \\ &\quad + ... + \sum_{j_n=1}^{m_n}\int_{\widetilde{\alpha}_{j_n}(x^0)}^{\widetilde{\alpha}_{j_n}(x)}\overline{a_{j_n}}(s)\,|u^p(s) - v^p(s)|\,ds. \end{aligned}$$

An application of corollary 3.5 to the function $|u^p(x) - v^p(x)|^{\frac{1}{p}}$ we obtain (3.27). Hence u^p dependents continuously on $c_1, c_2, ..., c_n$. □

Remark 3.5. *If* $n = 2$ *and* $l = 1$ *in Theorem 3.2, Theorem 3.3 and Theorem 3.4 we get Theorem 3.1, Theorem 3.2 and Theorem 3.3 respectively in [17].*

Chapter 4

Some New fractional integral inequalities

The literature on Gronwall type integral inequalities and their applications is vast; see [5, 24] and the references given therein. Usually, the integrals concerning this type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, D. Henry [16] used this type integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem.

In the first section of this chapter we give some necessary concepts of the generalized fractional and conformable fractional calculus. In the second section the main contribution using the method introduced by Zhu [50], novel weakly singular integral inequalities are established. In the third section, we study the following inequalities type

$$u(t) \leq a(t) + b(t) \int_a^t f(s)\, u(s)\, d_\alpha s + \int_a^t f(t)\, W\left(\int_a^s k(s,\tau)\, \Phi(u(\tau))\, d_\alpha \tau \right) d_\alpha s,$$

$$u(t) \leq a(t) + b(t) \int_a^t f(s)\, g(u(s))\, d_\alpha s + \int_a^t f(t)\, W\left(\int_a^s k(s,\tau)\, \Phi(u(\tau))\, d_\alpha \tau \right) d_\alpha s.$$

Where $a(.), b(.), f(.), W(.), \Phi(.)$ and $k(.,.)$ are given functions satisfied some conditions supposed later. This section is based on Rui A. C. Ferreira and Delfim F. M. Torres [14], we generalized the results in conformable fractional version integral inequalities with the help of the Katugampola conformable fractional calculus. In the fourth section, we give an application for the second and third section to illustrate the usefulness of our results, such that we present the existence, uniqueness and Ulam stability for the solution of the following problem

$$\begin{cases} {}^C D_{0^+}^{\beta;\chi} x(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where ${}^C D_{a^+}^{\beta;\chi}$ is the Caputo derivatives with respect to χ, $\beta \in (0,1)$ and the continuous function $f : J \times R \to R$, for the second section. And we gives a bound on the solution of the following integral equation

$$u(t) = k + \int_0^{\lambda(t)} F\left(s, u(s), \int_0^s K(\tau, u(\tau))\, d_\alpha \tau \right) d_\alpha s, \quad t \in [0,b],$$

for the third section.

4.1 Some necessary concepts of the generalized fractional and conformable fractional calculus

4.1.1 Some definitions

Let us introduce some preliminaries on fractional calculus (see [1, 2, 3]).

Definition 4.1. *Given $\beta > 0$ and $\chi \in C^1[a_1, a_2]$ such that $\chi'(t) > 0$ for every $t \in [a_1, a_2]$. The χ-Riemann– Liouville fractional integral of order β of a function $g \in L^1[a_1, a_2]$ is defined by*

$$I_{a_1}^{\beta,\chi} g(x) = \frac{1}{\Gamma(\beta)} \int_{a_1}^{x} \chi'(t) (\chi(x) - \chi(t))^{\beta-1} g(t) \, dt.$$

Definition 4.2. *Given $0 < \beta < 1$ and $\chi \in C^1[a_1, a_2]$ such that $\chi'(t) > 0$ for every $t \in [a_1, a_2]$. The χ-Riemann–Liouville fractional derivative of order β of a function g is defined by*

$$D_{a_1}^{\beta,\chi} g(x) = \frac{1}{\chi'(x)} \frac{d}{dx} I_{a_1}^{1-\beta,\chi} g(x).$$

Definition 4.3. *Given $0 < \beta < 1$ and $\chi \in C^1[a_1, a_2]$ such that $\chi'(t) > 0$ for every $t \in [a_1, a_2]$. The χ-Caputo fractional derivative of order β of a function g is defined by*

$$^{C}D_{a_1}^{\beta,\chi} g(x) = D_{a_1}^{\beta,\chi} (g(x) - g(a_1)).$$

Remark 4.1. *For certain special cases of $^{C}D_{a_1}^{\beta,\chi}$, we get the Caputo-Hadamard derivative [9], the Caputo derivative [20, 44] and the Caputo-Erdélyi-Kober derivative [47].*

Lemma 4.1. *For $\eta > 0$, we have*

$$\begin{aligned} I_{a_1}^{\beta,\chi} (\chi(x) - \chi(a_1))^{\eta} &= \frac{1}{\Gamma(\beta)} \int_{a_1}^{x} \chi'(t) (\chi(x) - \chi(t))^{\beta-1} (\chi(t) - \chi(a_1))^{\eta} \, dt \\ &= \frac{\Gamma(\eta+1)}{\Gamma(\eta+\beta+1)} (\chi(x) - \chi(a_1))^{\eta+\beta}. \end{aligned}$$

4.1.2 Katugampola conformable fractional integrals and derivatives

Katugampola conformable derivatives for $\alpha \in (0, 1]$ and $t \in [0, \infty)$ given by

$$D^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f\left(te^{\epsilon t^{-\alpha}}\right) - f(t)}{\epsilon}, \tag{4.2}$$

provided the limits exist (for detail see, [19]). If f is fully dierentiable at t; then

$$D^{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{4.3}$$

If the limit in (4.2) exists and is finit then A function f is $\alpha-$differentiable at a point $t \geq 0$.

Theorem 4.1. *[19]Let $\alpha \in (0, 1]$ and f, g be $\alpha-$ differentiable at a point $t > 0$. then*
1. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$,
2. $D^{\alpha}(\lambda)$, for all constant functions $f(t) = \lambda$,
3. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,
4. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{fD^{\alpha}(g) - gD^{\alpha}(f)}{g^2}$,

5. $D^{\alpha}(t^{n}) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$,

6. $D^{\alpha}(f \circ g) = f'(g(t))\, D^{\alpha}(g)(t)$ *for* f *is differentiable at* $g(t)$.

Definition 4.4. *[19](conformable fractional integral). Let* $\alpha \in (0,1]$ *and* $0 \leq a < b$. *A function* $f : [a,b] \to \mathbb{R}$ *is* α*-fractional integrale on* $[a,b]$ *if the integral*

$$\int_a^b f(s)\, d_\alpha s = \int_a^b f(s)\, s^{\alpha-1} ds,$$

exists and is finite. All α*-fractional integrable on* $[a,b]$ *is indicated by* $L^1_\alpha([a,b])$.

Remark 4.2. *[19]*

$$I_a^{\alpha}(f)(t) = I_a^1\left(t^{\alpha-1} f\right) = \int_a^t f(s)\, s^{\alpha-1} ds,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

4.2 Some new fractional integral inequalities with respect to another function

In this section, we present a new version of non-linear integral inequalities of fractional type with respect to another function

Theorem 4.2. *[8] Let* $\beta \in (0,1)$, $0 < T \leq \infty$, $\chi \in C^1[0,T)$ *such that* $\chi'(t) > 0$ *for every* $t \in [0,T)$, $a, v \in C([0,T), R_+)$, *and* $u \in C([0,T), R_+)$ *with*

$$u(t) \leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s)\,(\chi(t) - \chi(s))^{\beta-1} v(s)\, u(s)\, ds. \tag{4.4}$$

Then

$$u(t) \leq \left(\mathcal{A}(t) + \int_0^t \mathcal{G}(s)\, \mathcal{A}(s)\, e^{\int_s^t \mathcal{G}(\tau) d\tau} ds\right)^{\delta}. \tag{4.5}$$

If a *is non-decreasing on* $[0,T)$, *thus*

$$u(t) \leq \left(\mathcal{A}(t)\, e^{\int_0^t \mathcal{G}(s) ds}\right)^{\delta}.$$

In the case when $a(t) = 0$ *for every* $t \in [0,T)$, *we find*

$$u(t) \equiv 0,$$

where $\mathcal{A}(t) = 2^{\frac{1}{\delta}-1} a^{\frac{1}{\delta}}(t)$,
$\mathcal{G}(t) = \frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right)\Gamma\left(\frac{1-\beta}{1-\delta}\right)\right)^{\frac{1-\delta}{\delta}} (\chi(t) - \chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(t)\, v^{\frac{1}{\delta}}(t)$, *and* $0 < \delta < \beta < 1$.

Proof. Based on inequality (4.4), Lemma 4.1 and Holder inequality, we get

$$\begin{aligned}
u(t) &\leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) (\chi(t) - \chi(s))^{\beta-1} v(s) u(s) ds \\
&\leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (\chi'(s))^{1-\delta} (\chi(t) - \chi(s))^{\beta-1} (\chi(s) - \chi(0))^{\delta-\beta} \\
&\quad \times (\chi'(s))^{\delta} (\chi(s) - \chi(0))^{\beta-\delta} v(s) u(s) ds \\
&\leq a(t) + \frac{1}{\Gamma(\beta)} \left(\int_0^t \left[(\chi'(s))^{1-\delta} (\chi(t) - \chi(s))^{\beta-1} (\chi(s) - \chi(0))^{\delta-\beta} \right]^{\frac{1}{1-\delta}} ds \right)^{1-\delta} \\
&\quad \times \left(\int_0^t \left[(\chi(s) - \chi(0))^{\beta-\delta} (\chi'(s))^{\delta} v(s) u(s) \right]^{\frac{1}{\delta}} ds \right)^{\delta} \\
&\leq a(t) + \frac{1}{\Gamma(\beta)} \left(\int_0^t \left[\chi'(s) (\chi(t) - \chi(s))^{\frac{\beta-\delta}{1-\delta}-1} (\chi(s) - \chi(0))^{\frac{\delta-\beta}{1-\delta}} \right] ds \right)^{1-\delta} \\
&\quad \times \left(\int_0^t (\chi(s) - \chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(s) v^{\frac{1}{\delta}}(s) u^{\frac{1}{\delta}}(s) ds \right)^{\delta} \\
&\leq a(t) + \frac{1}{\Gamma(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{1-\delta} \\
&\quad \times \left(\int_0^t (\chi(s) - \chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(s) v^{\frac{1}{\delta}}(s) u^{\frac{1}{\delta}}(s) ds \right)^{\delta}.
\end{aligned}$$

In the fact of $(x_1 + x_2)^p \leq 2^{p-1} (x_1^p + x_2^p)$ for all $(x_1, x_2) \in \mathbb{R}_+^2$ and $p \geq 1$, we get

$$\begin{aligned}
u^{\frac{1}{\delta}}(t) &\leq 2^{\frac{1}{\delta}-1} \left(a^{\frac{1}{\delta}}(t) + \frac{1}{\Gamma^{\frac{1}{\delta}}(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{\frac{1-\delta}{\delta}} \right. \\
&\quad \left. \times \int_0^t (\chi(s) - \chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(s) v^{\frac{1}{\delta}}(s) u^{\frac{1}{\delta}}(s) ds \right).
\end{aligned}$$

By taking $w(t) = u^{\frac{1}{\delta}}(t)$, $\mathcal{A}(t) = 2^{\frac{1}{\delta}-1} a^{\frac{1}{\delta}}(t)$,
$\mathcal{G}(t) = \frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{\frac{1-\delta}{\delta}} (\chi(t) - \chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(t) v^{\frac{1}{\delta}}(t)$, the above inequality becomes

$$w(t) \leq \mathcal{A}(t) + \int_0^t \mathcal{G}(s) w(s) ds.$$

Using Lemma 2.2 in [50], we get inequality 4.5. The rest of the proof is obviously. □

Theorem 4.3. *Let $\beta > 0$, $0 < T \leq \infty$, $\chi \in C^1[0,T)$ such that $\chi'(t) > 0$ for every $t \in [0,T)$, $a, b, v \in C([0,T), R_+)$ and $u \in C([0,T), R_+)$ such that*

$$u(t) \leq a(t) + \frac{b(t)}{\Gamma(\delta)} \int_0^t \chi'(s) (\chi(t) - \chi(s))^{\beta-1} v(s) u(s) ds. \tag{4.6}$$

Then

$$u(t) \leq a(t) + \mathcal{B}(t) \frac{\left(\int_0^t \mathcal{G}(s)\,\mathcal{E}(s)\,\mathcal{A}^p(s)\,ds\right)^{\frac{1}{p}}}{1 - [1 - \mathcal{E}(t)]^{\frac{1}{p}}}, \tag{4.7}$$

with $\mathcal{E}(t) = \exp\left(-\int_0^t \mathcal{G}(s)\,\mathcal{B}^p(s)\,ds\right)$, $\mathcal{A}(t) = a(t)$,
$\mathcal{B}(t) = \frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} (\chi(t) - \chi(0))^{\beta-1+\frac{1}{q}}$, $\mathcal{G}(t) = \chi'(t)\,v^p(t)$, *and* $p, q \in (1, \infty)$ *with* $\frac{1}{q} + \beta > 1$ *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Choosing $q, p \in (1, \infty)$ with $\beta + \frac{1}{q} > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$. Using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} u(t) &\leq a(t) + \frac{b(t)}{\Gamma(\beta)} \int_0^t \chi'(s)\,(\chi(t) - \chi(s))^{\beta-1}\,v(s)\,u(s)\,ds \\ &\leq a(t) + \frac{b(t)}{\Gamma(\beta)} \left(\int_0^t \chi'(s)\,(\chi(t) - \chi(s))^{(\beta-1)q}\,ds\right)^{\frac{1}{q}} \\ &\quad \times \left(\int_0^t \chi'(s)\,(v(s)\,u(s))^p\,ds\right)^{\frac{1}{p}} \\ &\leq a(t) + \frac{b(t)}{\Gamma(\beta)\,(q(\beta-1)+1)^{\frac{1}{q}}} (\chi(t) - \chi(0))^{\beta-1+\frac{1}{q}} \\ &\quad \times \left(\int_0^t \chi'(s)\,v^p(s)\,u^p(s)\,ds\right)^{\frac{1}{p}}. \end{aligned}$$

By taking $\mathcal{A}(t) = a(t)$, $\mathcal{B}(t) = \frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} (\chi(t) - \chi(0))^{\beta-1+\frac{1}{q}}$, and $\mathcal{G}(t) = \chi'(t)\,v^p(t)$. We get

$$u(t) \leq \mathcal{A}(t) + \mathcal{B}(t) \left(\int_0^t \mathcal{G}(s)\,u^p(s)\,ds\right)^{\frac{1}{p}}.$$

Using Lemma 2.3 in [50] we get inequality (4.7). □

Theorem 4.4. *Let* $\beta > 0$, $0 < T \leq \infty$, $\chi \in C^1[0,T)$ *such that* $\chi'(t) > 0$ *for every* $t \in [0,T)$, $a, b, v \in C([0,T), R_+)$ *and* $u \in C([0,T), R_+)$ *such that*

$$u(t) \leq a(t) + \frac{b(t)}{\Gamma(\beta)} \int_0^t \chi'(s)\,(\chi(t) - \chi(s))^{\beta-1}\,v(s)\,u(s)\,ds. \tag{4.8}$$

Then

$$u(t) \leq \left(\mathcal{A}(t) + \mathcal{B}(t) \int_0^t \mathcal{G}(s)\,\mathcal{A}(s) \exp\left(\int_s^t \mathcal{G}(\tau)\,\mathcal{B}(\tau)\,d\tau\right) ds\right)^{\frac{1}{p}}, \tag{4.9}$$

where $\mathcal{A}(t) = 2^{p-1} a^p(t)$, $\mathcal{B}(t) = 2^{p-1} \left(\frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} (\chi(t) - \chi(0))^{\beta-1+\frac{1}{q}}\right)^p$,
$\mathcal{G}(t) = \chi'(t)\,v^p(t)$ *and* $p, q \in (0, \infty)$ *with* $\frac{1}{q} + \beta > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From the above Theorem 4.3 we have

$$\begin{aligned} u(t) &\leq a(t) + \frac{b(t)}{\Gamma(\beta)} \int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} v(s)\, u(s)\, ds \\ &\leq a(t) + \frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}} (\chi(t)-\chi(0))^{\beta-1+\frac{1}{q}} \\ &\quad \left(\int_0^t \chi'(s)\, v^p(s)\, u^p(s)\, ds\right)^{\frac{1}{p}}. \end{aligned}$$

Then

$$\begin{aligned} u^p(t) &\leq 2^{p-1}\left(a^p(t) + \left(\frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}}(\chi(t)-\chi(0))^{\beta-1+\frac{1}{q}}\right)^p \right. \\ &\quad \left. \times \int_0^t \chi'(s)\, v^p(s)\, u^p(s)\, ds\right). \end{aligned}$$

Let $w(t) = u^p(t)$, $\mathcal{A}(t) = 2^{p-1}a^p(t)$, $\mathcal{B}(t) = 2^{p-1}\left(\frac{b(t)}{\Gamma(\beta)(q(\beta-1)+1)^{\frac{1}{q}}}(\chi(t)-\chi(0))^{\beta-1+\frac{1}{q}}\right)^p$, and $\mathcal{G}(t) = \chi'(t)\, v^p(t)$. We have

$$w(t) \leq \mathcal{A}(t) + \mathcal{B}(t)\int_0^t \mathcal{G}(s)\, w(s)\, ds.$$

From Martyniuk and al. [31], we obtain inequality (4.9). □

Theorem 4.5. *Given $\beta \in (0,1)$, $0 < T \leq \infty$, $\chi \in C^1[0,T)$ such that $\chi'(t) > 0$ for every $t \in [0,T)$, $a(t)$ be a non-negative, non-decreasing C^1-function on $[0,T)$, $v, u \in C([0,T), R_+)$ and $w : [0,\infty) \to [0,\infty)$ be a non-decreasing, continuous function with*

$$u(t) \leq a(t) + \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} v(s)\, w(u(s))\, ds,\ t \in [0,T). \tag{4.10}$$

Then

$$u(t) \leq \left(\Omega^{-1}\left(\Omega(\mathcal{A}(t)) + \int_0^t \mathcal{G}(s)\, ds\right)\right)^{\delta},\ t \in [0,T_1), \tag{4.11}$$

where $0 < \delta < \beta < 1$, $\mathcal{A}(t) = 2^{\frac{1}{\delta}-1}a^{\frac{1}{\delta}}(t)$,
$\mathcal{G}(t) = \frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)}\left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right)\Gamma\left(\frac{1-\beta}{1-\delta}\right)\right)^{\frac{1-\delta}{\delta}}(\chi(t)-\chi(0))^{\frac{\beta-\delta}{\delta}}\chi'(t)\, v^{\frac{1}{\delta}}(t)$, $\Omega(x) = \int_{t_0}^x \frac{1}{\mu(t)}dt$, $\mu(t) = w^{\frac{1}{\delta}}(t^{\delta})$, $t_0 > 0$, Ω^{-1} is the inverse function of Ω, such that $\Omega(\mathcal{A}(t)) + \int_0^t \mathcal{G}(s)\, ds \in Dom(\Omega^{-1})$ for all $t \in [0,T_1]$, and for $T_1 \in (0,T)$.

Proof. From (4.10) and Holder inequality we get

$$\begin{aligned} u(t) &\leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} v(s) w(u(s))\, ds \\ &\leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (\chi'(s))^{1-\delta} (\chi(t)-\chi(s))^{\beta-1} (\chi(s)-\chi(0))^{\delta-\beta} \\ &\quad \times (\chi'(s))^{\delta} (\chi(s)-\chi(0))^{\beta-\delta} v(s) w(u(s))\, ds \\ &\leq a(t) + \frac{1}{\Gamma(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{1-\delta} \\ &\quad \times \left(\int_0^t \chi'(s) \left[(\chi(s)-\chi(0))^{\beta-\delta} v(s) w(u(s)) \right]^{\frac{1}{\delta}} ds \right)^{\delta}. \end{aligned}$$

Then

$$\begin{aligned} u^{\frac{1}{\delta}}(t) &\leq 2^{\frac{1}{\delta}-1} \left(a^{\frac{1}{\delta}}(t) + \frac{1}{\Gamma^{\frac{1}{\delta}}(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{\frac{1-\delta}{\delta}} \right. \\ &\quad \left. \times \int_0^t \chi'(s) (\chi(s)-\chi(0))^{\frac{\beta-\delta}{\delta}} v^{\frac{1}{\delta}}(s) w^{\frac{1}{\delta}}(u(s))\, ds \right). \end{aligned}$$

By taking $g(t) = u^{\frac{1}{\delta}}(t)$, $\mathcal{A}(t) = 2^{\frac{1}{\delta}-1} a^{\frac{1}{\delta}}(t)$,
$\mathcal{G}(t) = \frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)} \left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right) \Gamma\left(\frac{1-\beta}{1-\delta}\right) \right)^{\frac{1-\delta}{\delta}} (\chi(t)-\chi(0))^{\frac{\beta-\delta}{\delta}} \chi'(t) v^{\frac{1}{\delta}}(t)$, we find

$$g(t) \leq \mathcal{A}(t) + \int_0^t \mathcal{G}(s) \mu(g(s))\, ds.$$

Consider $V(t)$ be the right-hand side of above inequality. Therefore,

$$\mu(g(t)) [\mu(V(t))]^{-1} \leq 1,$$

and

$$\frac{V'(t)}{\mu(V(t))} = \frac{\mathcal{A}'(t) + \mathcal{G}(t)\mu(g(t))}{\mu(V(t))} \leq \frac{\mathcal{A}'(t)}{\mu(A(t))} + \mathcal{G}(t),$$

or

$$\frac{d}{dt}\Omega(V(t)) \leq \frac{d}{dt}\Omega(\mathcal{A}(t)) + \mathcal{G}(t).$$

By integrating both sides of last inequality from 0 to t, we get

$$\Omega(V(t)) \leq \Omega(\mathcal{A}(t)) + \int_0^t \mathcal{G}(s)\, ds,$$

and since Ω is an increasing function we get

$$g(t) \leq V(t) \leq \Omega^{-1}\left(\Omega(\mathcal{A}(t)) + \int_0^t \mathcal{G}(s)\, ds \right).$$

This achieves the proof. □

4.3 Some new conformable fractional integral inequalities

In this section, we present a new version of non-linear integral inequalities of conformable fractional type. We start by proving the following lemma, which we use in this section

Lemma 4.2. *suppose that $\lambda(.) \in C^1([a,b],\mathbb{R})$ is a non-decreasing function with $a \leq \lambda(t) \leq t$, for all $t \in [a,b]$. asumme that $u(.), a(.), b(.) \in C([a,b],\mathbb{R}_0^+)$ and let $(t,s) \to f(t,s) \in C([a,b] \times [a,\lambda(b)]\,\mathbb{R}_0^+)$ be non-decreasing in t for every s fixed. If*

$$u(t) \leq a(t) + b(t) \int_a^{\lambda(t)} f(t,s)\, u(s)\, d_\alpha s,$$

then

$$u(t) \leq a(t) + b(t) \int_a^{\lambda(t)} \exp\left(\int_s^{\lambda(t)} b(\tau)\, f(t,\tau)\, d_\alpha \tau\right) f(t,s)\, a(s)\, d_\alpha s.$$

Proof. The result is obvious for $t = a$. Let t_0 be an arbitrary number in $(a,b]$ and detine the function $z(.)$ as

$$z(t) = \int_a^{\lambda(t)} f(t_0,s)\, u(s)\, d_\alpha s, \quad t \in [a,t_0].$$

Then $u(t) \leq a(t) + b(t)\, z(t)$ for all $t \in [a,t_0]$, and $z(.)$ is non-decreasing. Hence

$$\begin{aligned} z'(t) &= f(t_0,\lambda(t))\, u(\lambda(t))\, \lambda^{\alpha-1}(t)\, \lambda'(t) \\ &\leq f(t_0,\lambda(t)) \left[a(\lambda(t)) + b(\lambda(t))\, z(\lambda(t))\right] \lambda^{\alpha-1}(t)\, \lambda'(t) \\ &\leq f(t_0,\lambda(t)) \left[a(\lambda(t)) + b(\lambda(t))\, z(t)\right] \lambda^{\alpha-1}(t)\, \lambda'(t). \end{aligned}$$

The last inequality can be rearranged as

$$z'(t) - f(t_0,\lambda(t))\, b((t))\, z(t)\, \lambda^{\alpha-1}(t)\, \lambda'(t) \leq f(t_0,\lambda(t))\, a(\lambda(t))\, \lambda^{\alpha-1}(t)\, \lambda'(t). \tag{4.12}$$

Multiplying both sides of inequality (4.12) by $\exp\left(-\int_a^{\lambda(t)} b(s)\, f(t_{0,}s)\, d_\alpha s\right)$, we get

$$\begin{aligned} &\left[z(t) \exp\left(-\int_a^{\lambda(t)} b(s)\, f(t_0,s)\, d_\alpha s\right)\right]' \\ \leq\ &\exp\left(-\int_a^{\lambda(t)} b(s)\, f(t_{0,}s)\, d_\alpha s\right) f(t_0,\lambda(t))\, a(\lambda(t))\, \lambda^{\alpha-1}(t)\, \lambda'(t). \end{aligned}$$

Integrating from a to t and noting that $z(a)=0$, we obtain

$$\begin{aligned} z(t) &\leq \exp\left(\int_a^{\lambda(t)} b(s) f(t_0,s)\, d_\alpha s\right) . \int_a^t \exp\left(-\int_a^{\lambda(s)} b(\tau) f(t_0,\tau)\, d_\alpha \tau\right) \\ &\quad \times f(t_{0,}\lambda(s))\, a(\lambda(s))\, \lambda^{\alpha-1}(s)\, \lambda'(s)\, ds \\ &= \int_a^t \exp\left(\int_{\lambda(s)}^{\lambda(t)} b(\tau) f(t_0,\tau)\, d_\alpha \tau\right) f(t_0,\lambda(s))\, a(\lambda(s))\, \lambda^{\alpha-1}(s)\, \lambda'(s)\, ds \\ &= \int_a^{\lambda(t)} \exp\left(\int_s^{\lambda(t)} b(\tau) f(t_0,\tau)\, d_\alpha \tau\right) f(t_0,s)\, a(s)\, d_\alpha s. \end{aligned}$$

Since $u(t) \leq a(t) + b(t)\, z(t)$, we have for $t = t_0$ that

$$u(t_0) \leq a(t_0) + b(t_0) \int_a^{\lambda(t_0)} \exp\left(\int_s^{\lambda(t_0)} b(\tau) f(t_0,\tau)\, d_\alpha \tau\right) f(t_0,s)\, a(s)\, d_\alpha s.$$

The intended conclusion follows from the arbitrariness of t_0. □

Remark 4.3. *If $f(t,s) = f(s)$ we get ([45] Theoreme 2.3).*

Remark 4.4. *If $\alpha = 1$, $b(t) = 1$, $\lambda(t) = t$ and $f(t,s) = f(s)$ we get([11] Lemma 1.1).*

Theorem 4.6. *Suppose that $\lambda(.), \beta(.) \in C^1([a,b],\mathbb{R})$ are non-decreasing functions with $\lambda(t), \beta(t) \in [a,t]$ for all $t \in [a,b]$. Assumme that $u(.), a(.), b(.) \in C([a,b],\mathbb{R}_0^+)$, $(t,s) \to f(t,s) \in C([a,b]\times[a,\lambda(b)],$ is non-decreasing in t for every s fixed, $g(.,.) \in C([a,b]\times[a,\beta(b)],\mathbb{R}_0^+)$, and $(s,\tau) \to k(s,\tau) \in C([a,\beta(b)]\times[a,\beta(b)],\mathbb{R}_0^+)$ is non-decreasing in s for every τ fixed. Let $W(.), \Phi(.) \in C(\mathbb{R}_0^+,\mathbb{R}_0^+)$ be non-decreasing functions,$\Phi(.)$ submultiplicative with $\Phi(x) > 0$ for $x \geq 1$. define*

$$G(x) = \int_0^x \frac{ds}{\Phi(1+W(s))}, \qquad x \geq 0,$$

$$\eta(\tau) = \max\left\{a(\tau), \int_a^{\beta(\tau)} g(\tau,\theta)\, d_\alpha \theta\right\}, \qquad \tau \in [a, \max\{\lambda(b), \beta(b)\}],$$

and

$$p(s) = \int_a^s k(s,\tau)\, \Phi\left(\eta(\tau) + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau,\theta)\, d_\alpha \theta\right) f(\tau,\xi)\, \eta(\xi)\, d_\alpha \xi\right) d_\alpha \tau.$$

If for $t \in [a,b]$

$$u(t) \leq a(t) + b(t) \int_a^{\lambda(t)} f(t,s)\, u(s)\, d_\alpha s + \int_a^{\beta(t)} g(t,s)\, W\left(\int_a^s k(s,\tau)\, \Phi(u(\tau))\, d_\alpha \tau\right) d_\alpha s, \quad (4.13)$$

then there exists $t_* \in (a, \beta(b)]$ such that $p(t) \in Dom(G^{-1})$ for all $t \in [a,t_*]$, $G^{-1}(.)$ the inverse *function of* $G(.)$ *, and*

$$u(t) \leq q(t) + b(t) \int_a^{\lambda(t)} \exp\left(\int_s^{\lambda(t)} b(\tau) f(t,\tau) d_\alpha \tau\right) f(t,s) q(s) d_\alpha s,$$

where

$$q(t) = a(t) + \int_a^{\beta(t)} g(t,s) W\left(G^{-1}(p(s))\right) d_\alpha s.$$

Proof. Let

$$z(t) = a(t) + \int_a^{\beta(t)} g(t,s) W\left(\int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau\right) d_\alpha s, \quad t \in [a,b].$$

From (4.13) we get

$$u(t) \leq z(t) + b(t) \int_a^{\lambda(t)} f(t,s) u(s) d_\alpha s. \tag{4.14}$$

Applying Lemma 4.2 to (4.14), we obtain

$$u(t) \leq z(t) + b(t) \int_a^{\lambda(t)} \exp\left(\int_s^{\lambda(t)} b(\tau) f(t,\tau) d_\alpha \tau\right) f(t,s) z(s) d_\alpha s. \tag{4.15}$$

In order to istimate $z(t)$, we define the function $v(.)$ by

$$v(s) = \int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau$$

therefore $z(t) = a(x) + \int_a^{\beta(t)} g(t,\theta) W(v(\theta)) d_\alpha \theta$, and

$$\begin{aligned} v(s) &\leq \int_a^s k(s,\tau) \Phi\left(z(\tau) + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau,\theta) d_\alpha \theta\right) f(\tau,\xi) z(\xi) d_\alpha \xi\right) d_\alpha \tau \\ &\leq \int_a^s k(s,\tau) \Phi\left[\eta(\tau)(1 + w(v(\tau)))\right. \\ &\quad \left. + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau,\theta) d_\alpha \theta\right) f(\tau,\xi) \eta(\xi) d_\alpha \xi (1 + w(v(\tau)))\right] d_\alpha \tau \\ &\leq \int_a^s k(s,\tau) \Phi\left[\eta(\tau)\right. \\ &\quad \left. + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau,\theta) d_\alpha \theta\right) f(\tau,\xi) \eta(\xi) d_\alpha \xi\right] \Phi(1 + w(v(\tau))) d_\alpha \tau \end{aligned}$$

Let $a < t_* \leq \beta(t)$ be a number such that $p(t) \in Dom(G^{-1})$ for all $t \in [a, t_*]$. Definre $r(.)$ on

$[a, s_0]$, where $a \leq s_0 \leq t_*$ is an arbitrary fixed number, by

$$\begin{aligned} r(s) &= \int_a^s k(s_0, \tau) \Phi \left[\eta(\tau) \right. \\ &\left. + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau, \theta) d_\alpha \theta \right) f(\tau, \xi) \eta(\xi) d_\alpha \xi \right] \Phi(1 + w(v(\tau))) d_\alpha \tau. \end{aligned}$$

Then

$$\begin{aligned} r'(s) &= k(s_0, s) \Phi \left[\eta(s) \right. \\ &\left. + b(s) \int_a^{\lambda(s)} \exp\left(\int_\xi^{\lambda(s)} b(\theta) f(\tau, \theta) d_\alpha \theta \right) f(s, \xi) \eta(\xi) d_\alpha \xi \right] \Phi(1 + w(v(s))) s^{\alpha - 1} \\ &\leq k(s_0, s) \Phi \left[\eta(s) \right. \\ &\left. + b(s) \int_a^{\lambda(s)} \exp\left(\int_\xi^{\lambda(s)} b(\theta) f(\tau, \theta) d_\alpha \theta \right) f(s, \xi) \eta(\xi) d_\alpha \xi \right] \Phi(1 + w(r(s))) s^{\alpha - 1}. \end{aligned}$$

That is

$$\begin{aligned} \frac{r'(s)}{\Phi(1 + w(r(s)))} &\leq k(s_0, s) \Phi \left[\eta(s) \right. \\ &\left. + b(s) \int_a^{\lambda(s)} \exp\left(\int_\xi^{\lambda(s)} b(\theta) f(s, \theta) d_\alpha \theta \right) f(s, \xi) \eta(\xi) d_\alpha \xi \right] s^{\alpha - 1}. \end{aligned}$$

by integrating the last inequality from a to s and using $G(r(a)) = 0$, we get

$$\begin{aligned} \int_a^s \frac{r'(s)}{\Phi(1 + w(r(s)))} ds &\leq \int_a^s k(s_0, \tau) \Phi \left[\eta(\tau) \right. \\ &\left. + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau, \theta) d_\alpha \theta \right) f(\tau, \xi) \eta(\xi) d_\alpha \xi \right] \tau^{\alpha - 1} d\tau, \end{aligned}$$

therefore

$$\begin{aligned} G(r(s)) &\leq \int_a^s k(s_0, \tau) \Phi \left[\eta(\tau) \right. \\ &\left. + b(\tau) \int_a^{\lambda(\tau)} \exp\left(\int_\xi^{\lambda(\tau)} b(\theta) f(\tau, \theta) d_\alpha \theta \right) f(\tau, \xi) \eta(\xi) d_\alpha \xi \right] d_\alpha \tau. \end{aligned}$$

The choice of t_* permits us to write $r(s_0) \leq G^{-1}(p(s_0))$. Since s_0 is arbitrary, we conclude that

$$r(s) \leq G^{-1}(p(s)), \ s \in [a, t_*]. \tag{4.16}$$

To complete the proof, we observe that for $a \leq s_0 \leq t_*$ the inequality $\beta(\lambda(s)) \leq t_*$ holds. Hence, we can insert inequality (4.16) into inequality (4.15). □

We present the definition of class H functions

Definition 4.5. *A functions* $g(.) \in C\left(\mathbb{R}_0^+, \mathbb{R}_0^+\right)$ *is said to belong to the class* H *if*
(1) $x \rightarrow g(x)$ *is non-decreasing for* $x \geq 0$ *and positive for* $x > 0$;
(2) *There exists a continuous function* $\Psi(.)$ *on* $\mathbb{R}_0^+$ *with* $g(\alpha x) \leq \Psi(\alpha) g(x)$ *for* $\alpha > 0$, $x \geq 0$.

Example 4.1. *If* $g(x) = x^m$, *for* $m > 0$ *then* $g(\alpha x) \leq \Psi(\alpha) g(x)$ *for any* $\alpha > 0$, $x > 0$. *then the function* g *is the class* H, *for* $\psi = g$

Lemma 4.3. *Suppose that* $\lambda(.) \in C^1([a,b], \mathbb{R}^+)$ *is a non-decreasing function with* $a \leq \lambda(t) \leq t$ *for all* $t \in [a,b]$. *Assume that* $u(.), a(.) \in C\left([a,b], \mathbb{R}_0^+\right)$ *with* $a(.)$ *apositive and non-decreasing function, and* $(t,s) \rightarrow f(t,s) \in C\left([a,b] \times [a; \lambda(b)], \mathbb{R}_0^+\right)$ *non-decreasing in* t *for every* s *fixed. If* $g(.) \in H$ *and*

$$u(t) \leq a(t) + \int_a^{\lambda(t)} f(t,s) g(u(s)) d_\alpha s, \tag{4.17}$$

Then there exists a function $\Psi(.)$ *and a number* $t_* \in (a,b]$ *that depends on* $\Psi(.)$ *such that*

$$G(1) + \int_a^{\lambda(t)} f(t,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s \in Dom\left(G^{-1}\right), \quad t \in [a, t_*], \tag{4.18}$$

and

$$u(t) \leq a(t) G^{-1}\left(G(1) + \int_a^{\lambda(t)} f(t,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s\right), \quad t \in [a, t_*],$$

where

$$G(x) = \int_{x_0}^{x} \frac{ds}{g(s)}, \quad x > 0, x_0 > 0.$$

and, as usual, $G^{-1}(.)$ *represents the inverse function of* $G(.)$.

Proof. Since $a(.)$ is positive and non-decreasing and $g(.) \in H$, we obtain from (4.17) that

$$\frac{u(t)}{a(t)} \leq 1 + \int_a^{\lambda(t)} \frac{f(t,s) g(u(s))}{a(s)} d_\alpha s \leq 1 + \int_a^{\lambda(t)} f(t,s) \frac{\Psi(a(s))}{a(s)} g\left(\frac{u(s)}{a(s)}\right) d_\alpha s$$

for some function $\Psi(.)$ as in the Definition 4.5. Let us now choose a number $a < t_* \leq b$ such that (4.18) holds, and define function $z(.)$ by

$$z(t) = 1 + \int_a^{\lambda(t)} f(t_0, s) \frac{\Psi(a(s))}{a(s)} g\left(\frac{u(s)}{a(s)}\right) d_\alpha s, \quad t \in [a; t_0]$$

Where $t_0 \in (a, t_*]$ is an arbitrary fixed number. Then, with $x(t) = \frac{u(t)}{a(t)}$, we have

$$\begin{aligned} z'(t) &= f(t_0, \lambda(t)) \frac{\Psi(a(\lambda(t)))}{a(\lambda(t))} g(x(\lambda(t))) \lambda^{1-\alpha}(t) \lambda'(t) \\ &\leq f(t_0, \lambda(t)) \frac{\Psi(a(\lambda(t)))}{a(\lambda(t))} g(z(t)) \lambda^{1-\alpha}(t) \lambda'(t), \end{aligned}$$

because $x(t) \leq z(t)$ and $z(t)$ is non-decreasing. Sinse $z(t)$ is positive, we can divide both sides of the last inequality by $g(z(t))$ and, after integrating both sides on $[a,t]$, we get

$$G(z(t)) \leq G(1) + \int_a^{\lambda(t)} f(t_0,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s.$$

Hence,

$$z(t_0) \leq G^{-1}\left(G(1) + \int_a^{\lambda(t_0)} f(t_0,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s\right).$$

Since $x(t_0) = \frac{u(t_0)}{a(t_0)} \leq z(t_0)$ and t_0 is arbitrary, the result follows for all $t \in (a, t_*]$. The case when $t = a$ is obvious. □

Theorem 4.7. *Let functions $u(.), f(.), g(.), W(.), \Phi(.), \lambda(.), \beta(.), p(.)$, and $G(.)$ be as in Theorem 4.6, and $a(.)$ be as in lemma 4.3. If $h(.) \in H$,*

$$\Omega(x) = \int_{x_0}^{x} \frac{ds}{h(s)}, \quad x > 0, x_0 > 0,$$

and

$$u(t) \leq a(t) + \int_a^{\lambda(t)} f(t,s) h(u(s)) d_\alpha s + \int_a^{\beta(t)} g(t,s) W\left(\int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau\right) d_\alpha s,$$

then there exists a function $\Psi(.)$ and a number $t'_* \in (a, \beta(t)]$ depending on $\Psi(.)$ such that, for all $t \in [a, t'_*]$,

$$\Omega(1) + \int_a^{\lambda(t)} f(t,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s \in Dom\left(\Omega^{-1}\right),$$

$$p(t) \in Dom\left(G^{-1}\right),$$

and

$$u(t) \leq \left[a(t) + \int_a^{\beta(t)} g(t,s) W\left(G^{-1}(p(s))\right) d_\alpha s\right] q(t),$$

where

$$q(t) = \Omega^{-1}\left(\Omega(1) + \int_a^{\lambda(t)} f(t,s) \frac{\Psi(a(s))}{a(s)} d_\alpha s\right).$$

Proof. Define function $z(.)$ by

$$z(t) = a(t) + \int_a^{\beta(t)} g(t,s) W\left(\int_a^s k(s,\tau) \Phi(u(\tau)) d_\alpha \tau\right) d_\alpha s, \quad t \in [a,b].$$

Clearly $z(.)$ is a positive and non-decreasing function. Hence, we can apply Lemma 2.2 to the inequality

$$u(t) \leq z(t) + \int_a^{\lambda(t)} f(t,s) h(u(s)) d_\alpha s,$$

to obtain

$$u(t) \leq z(t)\,\Omega^{-1}\left(\Omega(1) + \int_a^{\lambda(t)} f(t,s)\,\frac{\Psi(a(s))}{a(s)} d_\alpha s\right), \quad t \in [a, t_*],$$

for some function $\Psi(.)$ and some number $t_0 \in (a, b]$. An estimation of $z(t)$ can be obtained following the same procedure as in the proof of Theorem 4.6. After that, we obtain

$$z(t) \leq a(t) + \int_a^{\beta(t)} g(t,s)\,W\left(G^{-1}(p(s))\right) d_\alpha s, \quad t \in [a, t'_*],$$

where $G(.)$ and $p(.)$ are defined as in Theorem 4.6. □

4.4 Some Applications

We present two examples of application to study the existence and uniqueness and Ulam stability of the solution of Eq. (4.1).

Let $0 < T < \infty$. We consider the following assumptions:
(H_1) $f \in C([0,T] \times R, R)$ and there exist $l, k \in (C[0,T], R_+)$ such that

$$|f(t,x)| \leq l(t)\,|x| + k(t), \quad \forall t \in [0,T], \ \forall x \in \mathbb{R}. \tag{4.19}$$

(H_2) There exists $h \in C([0,T], \mathbb{R})$ with

$$|f(t,x) - f(t,y)| \leq h(t)\,|x-y|, \quad \forall t \in [0,T], \forall x, y \in \mathbb{R}.$$

The following theorem study the existence and uniqueness of the solution of Eq. (4.1).

Theorem 4.8. *Suppose that (H_1) is satisfied. Then there exist at least one solution for Eq. (4.1). Furthermore, if (H_2) is satisfied. Therefore Eq. (4.1) has a unique solution on $[0,T]$.*

Proof. Let consider the operator $H : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ given by

$$(Hx)(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s)\,(\chi(t) - \chi(s))^{\beta-1} f(s, x(s))\,ds.$$

According to condition (4.19) and the continuity f, it is clear that H is continuous and completely continuous. It remains to show that the set

$$\mathcal{F} = \{x \in C([0,T], \mathbb{R}),\ x = (\lambda H)(x),\ \text{for } \lambda \in (0,1)\}$$

is bounded, taking

$\mathcal{A}(t) = 2^{\frac{1}{\delta}-1}\left(|x_0| + \frac{\left(\frac{1-\delta}{\beta-\delta}\right)^{1-\delta}}{\Gamma(\beta)}(\chi(t)-\chi(0))^{\beta-\delta}\left(\int_0^t \chi'(s)\, k^{\frac{1}{\delta}}(s)\, ds\right)^{\delta}\right)^{\frac{1}{\delta}},$

$\mathcal{G}(t) = \frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)}\left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right)\Gamma\left(\frac{1-\beta}{1-\delta}\right)\right)^{\frac{1-\delta}{\delta}}(\chi(t)-\chi(0))^{\frac{\beta-\delta}{\delta}}\chi'(t)\, l^{\frac{1}{\delta}}(t)$

where $0<\delta<\beta<1$. Let $x \in \mathcal{F}$, then for $\lambda \in (0,1)$ and $t \in [0,T]$, we have

$$x(t) = \lambda\left(x_0 + \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} f(s,x(s))\, ds\right).$$

So

$$\begin{aligned} |x(t)| &\le |x_0| + \left|\frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} f(s,x(s))\, ds\right| \\ &\le |x_0| + \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} l(s)|x(s)|\, ds + \\ &\quad \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} k(s)\, ds \\ &\le |x_0| + \frac{\left(\frac{1-\delta}{\beta-\delta}\right)^{1-\delta}}{\Gamma(\beta)}(\chi(t)-\chi(0))^{\beta-\delta}\left(\int_0^t \chi'(s)\, k^{\frac{1}{\delta}}(s)\, ds\right)^{\delta} + \\ &\quad \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} l(s)|x(s)|\, ds. \end{aligned}$$

Using Theorem 4.2, we obtain

$$|x(t)| \le \left(\mathcal{A}(t) + \int_0^T \mathcal{G}(s)\mathcal{A}(s)\exp\left(\int_s^T \mathcal{G}(\tau)\, d\tau\right) ds\right)^{\delta},$$

for all $t \in [0,T]$. From Schaefer fixed point theorem we deduce that the operator H has at least one fixed point in $C([0,T],\mathbb{R})$ which is the solution Eq. (4.1).
If (H_2) is satisfied, we suppose that $x_1(t), x_2(t)$ are two solutions of Eq (4.1). Then

$$\begin{aligned} |x_1(t)-x_2(t)| &= \left|\frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1}(f(s,x_1(s))-f(s,x_2(s)))\, ds\right| \\ &\le \frac{1}{\Gamma(\beta)}\int_0^t \chi'(s)(\chi(t)-\chi(s))^{\beta-1} h(s)|x_1(s)-x_2(s)|\, ds. \end{aligned}$$

Using Theorem 4.2, we obtain $x_1 = x_2$. □

We consider the following inequality

$$\left|{}^{C}D_{a^+}^{\beta,\chi}y(t) - f(t,y(t))\right| \le \epsilon, \text{ for } t \in [0,T] \text{ and } \epsilon > 0. \tag{4.20}$$

Definition 4.6. *Eq. (4.1) is $\chi-$Ulam-Hyers stable if there exists $c>0$, such that for every $\epsilon>0$,* *and for every solution y of (4.20) there is a solution x of Eq. (4.1) with*

$$|x(t) - y(t)| \leq c\epsilon \left(\chi(t) - \chi(0)\right)^{\beta}.$$

Remark 4.5. *If y is a solution of (4.20) then y is a solution of*

$$\begin{aligned} &\left| y(t) - y(0) - \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, y(s))\, ds \right| \\ \leq\ & \frac{\epsilon}{\Gamma(\beta+1)} \left(\chi(t) - \chi(0)\right)^{\beta}. \end{aligned}$$

The following theorem study the $\chi-$Ulam stability of the solution of Eq. (4.1).

Theorem 4.9. *Suppose that (H_2) is satisfied. Then, Eq. (4.1) is $\chi-$Ulam-Hyers stable.*

Proof. Let y be a solution of (4.20) and x the unique solution of the following problem

$$\begin{cases} {}^{C}D_{0^+}^{\beta;\chi} x(t) = f(t, x(t)), \ \beta \in (0,1), t \in [0,T], \\ x(0) = y(0), \end{cases}$$

then

$$x(t) = y(0) + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, x(s))\, ds.$$

From Remark 4.5, we find

$$\begin{aligned} & |y(t) - x(t)| \\ \leq\ & \left| y(t) - y(0) - \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, x(s))\, ds \right| \\ \leq\ & \left| y(t) - y(0) - \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, y(s))\, ds \right. \\ & + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, y(s))\, ds \\ & \left. - \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} f(s, x(s))\, ds \right| \\ \leq\ & \frac{\epsilon}{\Gamma(\beta+1)} \left(\chi(t) - \chi(0)\right)^{\beta} \\ & + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} \left| f(s, y(s)) - f(s, x(s)) \right| ds \\ \leq\ & \frac{\epsilon}{\Gamma(\beta+1)} \left(\chi(t) - \chi(0)\right)^{\beta} \\ & + \frac{1}{\Gamma(\beta)} \int_0^t \chi'(s) \left(\chi(t) - \chi(s)\right)^{\beta-1} h(s) \left| y(s) - x(s) \right|. \end{aligned}$$

Using theorem 4.2, we get

$$|y(t)-x(t)| \leq 2^{1-\delta}\left(\frac{\epsilon}{\Gamma(\beta+1)}(\chi(t)-\chi(0))^{\beta}\right)\left(\exp\left(\int_0^T \mathcal{G}(s)\,ds\right)\right)^{\delta},$$

where $\mathcal{G}(t)=\frac{2^{\frac{1}{\delta}-1}}{\Gamma^{\frac{1}{\delta}}(\beta)}\left(\Gamma\left(\frac{\beta-\delta}{1-\delta}\right)\Gamma\left(\frac{1-\beta}{1-\delta}\right)\right)^{\frac{1-\delta}{\delta}}(\chi(t)-\chi(0))^{\frac{\beta-\delta}{\delta}}\chi'(t)\,h^{\frac{1}{\delta}}(t)$,
and $0<\delta<\beta<1$, the proof is complete. □

Example 4.2. *Let consider the following problem*

$$\begin{cases} D^{\frac{1}{2},\exp(t)}x(t)=t\arctan x(t)+\sin t,\ t\in\left[0,\frac{\pi}{4}\right], \\ x(0)=x_0. \end{cases} \tag{4.21}$$

Let

$$f(t,x(t))=t\arctan x(t)+\sin t.$$

For all $x,y\in\mathbb{R}$ and $t\in\left[0,\frac{\pi}{4}\right]$, we have

$$\begin{aligned} |f(t,x)-f(t,y)| &\leq t\,|\arctan x-\arctan y| \\ &\leq t\,|x-y|. \end{aligned}$$

Hence, the assumptions (H_2) is satisfied. It follows from Theorem 4.8 that the Eq. (4.21) has a unique solution on $\left[0,\frac{\pi}{4}\right]$.

In the second application we present an examples of application to gives a bound on the solution of the following retarded integral equation:

$$u(t)=k+\int_0^{\lambda(t)} F\left(s,u(s),\int_0^s K(\tau,u(\tau))\,d_\alpha\tau\right)d_\alpha s,\quad t\in[0,b], \tag{4.22}$$

where $k>0, b>0, \lambda(.)\in C^1([0,b],\mathbb{R})$ is a non-decreasing function with $0\leq\lambda(t)\leq t$, $u(.)\in C([0,b],\mathbb{R}), F\in C([0,b]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $K\in C([0,b]\times\mathbb{R},\mathbb{R})$. The following theorem gives a bound on the solution of integral equation (4.22).

Theorem 4.10. *Assume that functions $F(.,.,.)$ and $K(.,.)$ in (4.22) satisfy*

$$|K(t,u)|\leq k(t)\,\Phi(|u|), \tag{4.23}$$

$$|F(t,u,v)|\leq t\,|u|+|v|, \tag{4.24}$$

with $k(.)$ and $\Phi(.)$ defined as in Theorem 4.6. If $u(.)$ is a solution of (4.22), then

$$|u(t)|\leq q(t)+t\int_0^{\lambda(t)}\exp(t(\alpha(t)-s))\,q(s)\,d_\alpha s,\quad t\in[0,t_*],$$

for some $t_* \in (0, \lambda(b)]$ *such that*

$$p(t) \in Dom\left(G^{-1}\right), \quad t \in [0, t_*],$$

Here

$$q(t) = k + \int_0^{\lambda(t)} G^{-1}(p(s))\, d_\alpha s,$$

$$G(x) = \int_0^x \frac{ds}{\Phi(1+s)}, \quad x \geq 0,$$

$$p(s) = \int_0^s k(\tau)\, \Phi\left[\eta(\tau) + \tau \int_0^{\lambda(\tau)} \exp\left(\frac{\tau}{\alpha}\left(\lambda^\alpha(\tau) - \zeta^\alpha\right)\right) \eta(\zeta)\, d_\alpha \zeta\right] d_\alpha \tau,$$

$$\eta(\tau) = \max\left\{k, \frac{\lambda^\alpha(\tau)}{\alpha}\right\}, \quad \tau \in [0, \lambda(b)],$$

with $G^{-1}(.)$ *representing the inverse function of* $G(.)$.

Proof. Let $u(.)$ be a solution of equation (4.22). In view of (4.23) and (4.24), we get

$$|u(t)| \leq k + \int_0^{\lambda(t)} \left(t\,|u(s)| + \int_0^s k(\tau)\, \Phi(|u(\tau)|)\, d_\alpha \tau\right) d_\alpha s.$$

An application of Theorem 4.6 with $a(t) = k$, $\lambda(t) = \beta(t)$, $f(t,s) = t$, $b(t) = g(t,s) = 1$, and $W(u) = u$, gives the desired conclution:

$$|u(t)| \leq q(t) + t \int_0^{\lambda(t)} \exp\left(\frac{t}{\alpha}\left(\lambda^\alpha(t) - s^\alpha\right)\right) q(s)\, ds.$$

□

Conclusion

This thesis is devoted to some integral inequalities and applications for certain classes of partial differential equations. The important notion in this thesis is the study some non-linear integral inequalities for two-variable and n independent variables functions, finally some new fractional integral inequalities with singular kernels and using this type integral inequalities to prove the existence, uniqueness and Ulam stability for the solution of fractional Cauchy problem with respect to another function. This studies can be extend to more integral inequalities involving other types of Gronwall inequalities.

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Abstract

The aim of the present work is to give an exposition of the classical results about integral inequalities appeared in the mathematical researchs in recent years, and to establish some new integral inequalities, integrals inequalities for functions of several independent variables with a term of delay and also some new fractional integral inequalities.
Moreover we give some applications to certain classes of partial and fractional differential equations to illustrate the truth of our results.

Résumé

Le but de ce travail est de donner une exposition des résultats classiques de certaines inégalités intégrales apparues dans la recherché mathématique dans ces dernières années, et d' établir quelques nouvelles inégalités intégrales, inégalités intégrales pour des fonctions de plusieurs variables indépendantes avec un terme de retard et aussi des nouvelles inégalités intégrales fractionnaires.

De plus nous donnons quelque applications à certaines classes des equations aux dérivées partielles et fractionnaires pour illustrer fiabilité de nos résultats.

ملخص

الهدف من هذا العمل هو تقديم عرض حول بعض النتائج الكلاسيكية لبعض المتراجحات التكاملية التي دخلت مجال الرياضيات في السنوات الأخيرة و إيجاد بعض المتراجحات التكاملية الجديدة، المتراجحات التكاملية لدوال بعدة متغيرات مستقلة ذات حد متأخر وأيضا بعض المتراجحات التكاملية الكسرية.

علاوة على ذلك، نقدم بعض التطبيقات لفئات معينة من المعادلات التفاضلية الجزئية والكسرية لتوضيح صحة نتائجنا.