REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
MINISTERE DE L'ENSEIGNEMENT SUPERIEURE
ET DE LA RECHERCHE SCIENTIFIQUE
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DEPARTEMENT DE MATHEMATIQUES

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# THESE

**Présentée pour l'obtention du diplôme de :**

**DOCTORAT D'ETAT EN MATHEMATIQUES**

**Thème**

***ETUDE DES SYSTEMES LINEAIRES MULTIDIMENSIONNELS DECRITS PAR DES MODELES D'ETAT***

**Option**

**MATHEMATIQUES APPLIQUEES**

**Par :**

**DJEZZAR SALAH**

**Devant le jury :**

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| Président : | Mr. Merouani H. | Prof. | Univ. Ferhat Abbas Sétif |
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| | Mr. Ayadi A. | M. C. | Univ. Mentouri Constantine |
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**Soutenue le : .. / ....... / ....**

# *Dédicace*

*A ma très chère mère. A mon très cher père.*
*A ma chère femme.*
*A ma chère petite fille Safa.*
*A mes chers sœurs et frères.*
*A la famille Djezzar.*
*A la famille Bouzit.*
*A la famille Mahsni.*
*A tous mes collègues et amis.*
*Je dédie ce travail.*

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# Contents

## 0.1 Introduction

The research work presented here is divided into two (different) parts: one is concerned with system theory in the finite dimensional setting using matrix theory, i.e. the algebraic structure and the other is with the infinite dimensional case using the analytic stucture. The other related approach is the geometric one which is very much useful in the development of the subject see e.g. [8], [9], [34] and [45].

One of the important motivation behind the research work concerning canonical form system matrices (e.g. Smith form & companion form etc.) associated to different systems e.g. systems of P.D.E.and, R.D.D.E. etc. is that the crucial role that may be played by these canonical forms in unifying the research work in the theory of linear systems. They also play a fundamental role in the study of structural properties of systems as controllablity, observability and minimality.

In the first chapter, we present some preliminary results concerning the $1-D$ theory of linear systems, starting with the important notions of controllability, observability, companion form and Smith form over the ring of polynomials $\mathbb{R}[s]$ of one indeterminate. Then a system matrix representation, using the symbolic calculus (e.g. Laplace transform) following Rosenbrock, for ordinary, generalized ($1-D$ descriptor), and genral differential equations etc. is given and some matrix transformations between them are presented. We conclude this chapter by providing some canonical forms, under the introduced system matrix transformations, and controllability and observability properties via system matrix representations.

In chapter 2, we extend some notions and results from the previous chapter to the $2-D$ case, such as companion form and Smith form over the ring $\mathbb{R}[s,z]$. Following Frost and Boudellioua, a characterization result on $2-D$ companion forms is given. We also extend the notion of system matrices to the $2-D$ case, to modelise e.g. high order systems of partial (descriptor) and retarded delay differential equations and $2-D$ discrete systems etc.see [11], [12], [13]. Also the problem of relating these system matrices via matirx transformations as system equivalence, system similarity, strict system equivalence and restiricted system equivalence is discussed. Particularly, canonical forms of the matrices over $R[s,z]$, which arise from $2-D$

discrete models given by Roesser [39], are obtained under a similarity transformation. And the problem of obtaining canonical matrix forms of the above matrices is also considered, and particulary, an extension of a result concerninig matrices of the form $P(s, z) = sI - A(z)$ and which arise from retarded delay differential systems is established. Also the very close relation between $2 - D$ polynomial matrices and $2 - D$ state space notions such as controllability, observability, realizability etc. is pointed out. For more results on $2 - D$ polynomial matrices and $2 - D$ linear systems see [37], [38].

In chapter 3, we develop a $2-D$ discrete state space model for linear iterative circuits which can be regarded as a generalization of the well known state space model for single dimensional linear time discrete systems (for a comparison see [14]). This development will enclude the definition, formulation of a linear iterative circuit and the derivation of some basic concepts as the state transition matrix, modal controllability, modal observability etc.
we also note that by the end of this development a $2 - D$ unilateral linear iterative circuit representaion is given.

The study of the iterative circuits is limited here to the linear case (i.e. each cell perform a linear transformation), as this allows the use of linear transformation techniques wich considerably facilitate the analysis, design, and implimentation of such circuits. Linear iterative circuits may be used in applications such as encoding, decoding networks for linear codes, and image processing. For some other results see e.g. [20], [21], and [22].

We also present in this chapter a canonical form under strict system equivalence and the characterization result obtained in the previous chapter is extended here to the $k^{th}$ case. We also note that some of the results obtained in this chapter for $2 - D$ systems can be extended to the the $N - D$ case. We also introduced the multidimensional Laplace transform to get system matrices associated to high order partial differential equations, and the transformation of restricted system equivalence between them is considered.

Finally, we end this chapter by the study of a realization problem of a non- proper $2 - D$ transfer function.

In the last chapter (the second part of the thesis), we study an abstract ill-posed parabolic problem known as the final value problem (F.V.P.) of the following type

$$u'(t) + Au(t) = 0, \qquad 0 < t < T \tag{1}$$

$$u(T) = f \tag{2}$$

for some prescribed final value $f$ in a Hilbert space H. $A$ is a positive self-adjoint operator such that $0 \in \rho(A)$. Such problems are not well posed, that is, even if a unique solution exists on $[0, T]$ it need not depend continuously on the final value $f$. We note that this type of problems has been considered by many authors, using different approaches. Such authors as Lattes and Lions [31], Miller [32], and Showalter [42] have approximated (F.V.P.) by perturbing the operator $A$.

We also note that, in [1], [10], and [41] a similar problem is treated in a different way. By perturbing the final value condition, they approximate the problem (1), (2), with

$$u'(t) + Au(t) = 0, \; 0 < t < T, \tag{3}$$

$$u(T) + \alpha u(0) = f. \tag{4}$$

A similar approach known as the method of auxilliary boundary conditions was given in [33]. Also, we have to mention that the non standard conditions of the form (4) for parabolic equations have been considered in some recent papers [2], [3].

In this study, we perturb the final condition (2) to form an approximate non local problem depending on a small parameter, with boundary condition containing a derivative of the same order than the equation, as follows:

$$u'(t) + Au(t) = 0, \; 0 \le t \le T, \tag{5}$$

$$u(T) - \alpha u'(0) = f. \tag{6}$$

Following [10], this method is called quasi-boundary value method (Q.B.V.M.), and the related approximate problem is called quasi-boundary value problem (Q.B.V.P.). We show that the approximate problems are well posed and that their solutions $u_\alpha$ converge in $C^1([0, T], H)$ if and only if the original problem has a classical solution. We prove that this method gives a better approximation than many other quasi reversibility type methods e.g. [1], [10] and [31]. Finally, we obtain several other results, including some explicit convergence rates.

We end this thesis by giving a conclusion and references related to this research work.

# Chapter 1

# PRELIMINARY RESULTS (1-D SYSTEM THEORY)

## 1.1 Introduction

In 1-D case, Rosenbrock [40] has given system matrix representations for different types of 1-D systems e.g. 1-D ordinary differential systems, discrete and generalized differential systems, etc. Some important notions in system theory such as, state transition matrix, controllability, observability, state feedback, etc. were given in terms of these 1-D system matrices. In this chapter, we first, present these different notions in the one dimensional case (1-D case), then we try to give the analogous extensions of some of these notions to the multidimensional case, in particular to the bidimensional case (2-D case), which has many interesting physical applications(e.g. in image processing, iterative circuits coding and decoding theories etc.). We also present some results concerning the problem of bringing a class of system matrices to a canonical form under some matrix transformations (e.g. system equivalence, system simialrity, strict system equivalence and restricted system equivalence). For more results on 2-D polynomial matrices and 2-D systems see [37], [38].

## 1.2 Controllability and Observability

### 1.2.1 Controllability

Let the following controlled linear system given by

$$\begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^r$ and $A, B, C,$ and $D$ are matrices of dimensions respectively $n \times n$, $n \times m$, $r \times n$ and $r \times n$ , and they are respectively known as the dynamical matrix, controllability matrix, observability matrix and transmition matrix.

**Definition 1** *The system (1.1) is said to be completely controllable ( c.c.) if and only if for every $t_0 \in \mathbb{R}$ and every initial state $x(t_0) = x_0$ and every final state $x_f$, there exists a finite time $t_1 > t_0$ and a control $u(t)$, for $t \in [t_0, t_1]$ such that*

$$x(t_1) = x_f$$

**Remark 1** *We can define in a similar way the complete controllability of the folloiwng time varying system*

$$\begin{cases} x'(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}, \tag{1.2}$$

*where $A(t), B(t), C(t)$ and $D(t)$ are now time depending matrices of appropriate dimensions.*

We now give the dual notion to controllability, the observability of a system.

### 1.2.2 Observability

**Definition 2** *The system (1.1) is said to be completely observable ( c.o.) if and only if for an arbitrary $t_0 \in \mathbb{R}$ and an initial state $x(t_0) = x_0$, there exists a finite time $t_1 > t_0$ such that for given control variable $u(t)$, $t \in [t_0, t_1]$ and output vector $y(t)$, $t \in [t_0, t_1]$, we can determinate the state vector $x(t_0)$.*

In the following, we give two theorems which characterize the controllability and observability notions in algebraic terms.

**Theorem 1** *The system (1.1) is c.c. if and only if the block matrix (the controllability matrix)*

$$\mathbb{U} = \left[B, AB, A^2B, \cdots, A^{n-1}B\right]$$

*has rank equals n [5], i.e.*

$$rank \left[B, AB, A^2B, \cdots, A^{n-1}B\right] = n \tag{1.3}$$

**Theorem 2** *The system (1.1) is c.o. if and only if the block matrix (the observability matrix)*

$$\mathbb{V} = {}^t\left[C, CA, CA^2, \cdots, CA^{n-1}\right]$$

*has rank equals n [5], i.e.*

$$rank\ {}^t\left[C, CA, CA^2, \cdots, CA^{n-1}\right] = n \tag{1.4}$$

*The following theorem shows that the notion of observability is a dual notion to controllability.*

**Theorem 3** *The system (1.1) is c.c. if and only if the dual system*

$$\begin{cases} x'(t) &= -{}^tAx(t) + {}^tCu(t) \\ y(t) &= {}^tBx(t) \end{cases}, \tag{1.5}$$

*is c.o.[5].*

## 1.3 Canonical Matrix Form Problem over R[s]

### 1.3.1 Companion Form over the Ring R[s]

We show here that a companion form matrix associated to a differential equation or differential system can be obtained by two different approaches, one is algebraic and the other is geometric.

First, let us be given the following ordinary differential equation (o.d.e.) of order $n$ with constant coefficients

$$z^{(n)}(t) + k_1 z^{(n-1)}(t) + ... + k_{n-1}z'(t) + k_n z(t) = u\,(t)\,. \tag{1.6}$$

Then (1.6) can be transformed [4] into the canonical form (companion form) system given by

$$w^{'} = Cw + du \tag{1.7}$$

where $d = [0, 0, \cdots, 1]^t$ and $C$ is the $n \times n$ constant companion form matrix given by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_n & -k_{n-1} & -k_{n-2} & \cdots & -k_1 \end{bmatrix}. \tag{1.8}$$

We note that a system of the form

$$x^{'} = Ax + Bu \tag{1.9}$$

can be transformed by a non singular transformation $w = Tx$ into the companion form system

(??) if and only if it is completely controllable. For a proof of this result see [5].

Now, we give the following result which can be considered as a generalization of the above result given in Barnett [4] to the case of an $n^{th}$ order o.d.e. of variable coefficients.

**Theorem 4** *Let the following differential equation of order n with variable coefficients be given*

$$z^{(n)}(t) + k_1\,(t)\,z^{(n-1)}(t) + ... + k_{n-1}\,(t)\,z'(t) + k_n\,(t)\,z(t) = \beta\,(t)\,u\,(t)\,. \tag{1.10}$$

*Then (1.10) can be transformed into the canonical form*

$$w' = C\,(t)\,w + d\,(t)\,u\,(t) \tag{1.11}$$

*where $C(t)$, is the companion form matrix associated with* (1.10), *and it is given by*

$$C(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & 1 \\ -k_n(t) & -k_{n-1}(t) & -k_{n-2}(t) & \cdots & -k_1(t) \end{bmatrix},$$

*and*

$$d(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta(t) \end{bmatrix}^t$$

We note that $w = \left[z,\ z', z'', \cdots, z^{(n-1)}\right]^t$, and the solution of (1.10) is by definition the first component of $w$ i.e. $z$, and not the whole vector $w$.

**Proof.** By using the following change of variables

$$\begin{cases} w_1(t) & = & z(t) \\ w_2(t) & = & z^{(1)}(t) \\ \vdots & \vdots & \vdots \\ w_n(t) & = & z^{(n-1)}(t) \end{cases},$$

we obtain

$$\begin{bmatrix} w_1'(t) & = & z^{(1)}(t) & = & w_2(t) \\ w_2'(t) & = & z^{(2)}(t) & = & w_3(t) \\ \vdots & = & \vdots & = & \vdots \\ w_{n-1}'(t) & = & z^{(n-1)}(t) & = & w_n(t) \\ w_n'(t) & = & z^{(n)}(t) & = & -k_1(t)\, z^{(n-1)}(t) - \ldots - k_n(t)\, z(t) + \beta(t) \\ w_n'(t) & & & = & -k_1(t)\, w_n(t) - \ldots - k_n(t)\, w_1(t) + \beta(t) \end{bmatrix}$$

Now, letting

$$w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^t.$$

Then, we get the required canonical system in companion form

$$w'(t) = C(t)w(t) + d(t)u(t),$$

where $C(t)$ is the $n \times n$ companion form matrix given by

$$C(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_n(t) & -k_{n-1}(t) & -k_{n-2}(t) & \cdots & -k_1(t) \end{bmatrix} \tag{1.12}$$

and

$$d(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \beta(t) \end{bmatrix}$$

■

Now, before we present the geometric approach, we first give the following definition concerning the polynomial matrices in one indeterminate $s$ i.e. matrices with entries in the ring of polynomials $\mathbb{R}[s]$.

**Definition 3** *A matrix $M(s)$ with entries in the polynomial ring $\mathbb{R}[s]$ of one- indeterminate $s$ and with coefficients in $\mathbb{R}$ is called a polynomial matrix.The ring of such polynomial matrices is denoted by $\mathbb{M}_n(\mathbb{R}[s])$.*

Let $A : X \to X$ be an arbitrary endomorphism of a vector space of dimension $n$ on a field $\mathbb{F}$, $k(\lambda)$ and $m(\lambda)$ are respectively its characteristic and minimal polynomials.

We know from [30] that if $A$ is cyclic (i.e. $k(\lambda) = m(\lambda)$), then there exists a cyclic generator $y$ of $X$ relatively to $A$ (i.e. such that $y, Ay, A^2y, \cdots, A^{n-1}y$ are linearly independent).

Now, assume that $A$ is a cyclic endomorphism of $X$ with generator $y$ and define the auxiliary polynomials as follows:

$$\begin{array}{rcl} m^0(\lambda) & = & m(\lambda) = \lambda^n - (a_1 + a_2\lambda + \cdots + a_n\lambda^{n-1}) \\ m^{(1)}(\lambda) & = & \lambda^{n-1} - (a_2 + a_3\lambda + \cdots + a_n\lambda^{n-2}) \\ \vdots & \vdots & \vdots \\ m^{(n-1)}(\lambda) & = & \lambda - a_n \\ m^{(n)}(\lambda) & = & 1 \end{array},$$

from this we get

$$\lambda m^{(i)}(\lambda) = m^{(i-1)}(\lambda) + a_i m^{(n)}(\lambda), \quad i \in \{1, 2, \cdots, n\}, \tag{1.13}$$

using the generator vector $y$ and replacing $\lambda$ by $A$ in (1.13) we obtain

$$Am^{(i)}(A)y = m^{(i-1)}(A)y + a_i m^{(n)}(A)y, \quad i \in \{1, 2, \cdots, n\},$$

Now, defining the vectors $e_i$ by

$$e_i = \begin{cases} m^{(i)}(A)y, & i \in \{1, 2, \cdots, n\} \\ 0 & \text{if } \; i = 0 \end{cases},$$

then $\{e_i\}_{i \in \{1,2,\cdots,n\}}$ is a basis for $X$, and we have

$$Ae_i = e_{i-1} + a_i e_n, \quad i \in \{1, 2, \cdots, n\}$$

hence

$$\begin{cases} Ae_1 & = & 0e_1 + 0e_2 \cdots + 0e_{n-1} + a_1 e_n \\ Ae_2 & = & 1e_1 + 0e_2 + \cdots + 0e_{n-1} + a_2 e_n \\ \vdots & \vdots & \vdots \\ Ae_{n-1} & = & 0e_1 + \cdots + 1e_{n-2} + 0e_{n-1} + a_{n-1} e_n \\ Ae_n & = & 0e_1 + \cdots + 0e_{n-2} + 1e_{n-1} + a_n e_n \end{cases}.$$

Therefore, the companion form matrix associated with the endomorphism $A$ in the basis

$\{e_i\}_{i\in\{1,2,\cdots,n\}}$ is given by

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & \cdots & a_{n-1} & a_n \end{bmatrix}.$$

Now, we give a generalization of the previous result to the case of the ring of polynomials $\mathbb{R}[s]$.

Let $X = (\mathbb{R}[s])^n$ be the free modulus of finite type over the ring $\mathbb{R}[s]$, then we give the following theorem

**Theorem 5** *Let $A(s) : (\mathbb{R}[s])^n \longrightarrow (\mathbb{R}[s])^n$ be an endomorphism over the $\mathbb{R}[s]$-modulus$(\mathbb{R}[s])^n$, and $k(\lambda(s))$ be the characteristic polynomial of $A(s)$, (i.e. $k(\lambda(s)) = \det(\lambda(s)I - A(s))$, $\lambda(s) \in \mathbb{R}[s]$. Then*

$$k(A(s)) = 0 \tag{1.14}$$

***Proof.*** *Since $\mathbb{R}[s]$ is a commutative ring, then we can apply the Caley-Hamilton thoerem to get (1.14).* ■

An endomorphism $A(s) : (\mathbb{R}[s])^n \longrightarrow (\mathbb{R}[s])^n$ is said to be cyclic if its minimal polynomial $m(A(s)$ coincide with its characteristic polynomial $k(A(s))$ ( i.e. $m(A(s)) = k(A(s))$. Recall that the minimal polynomial of $A(s)$ is defined here to be the polynomial of least degree such that

$$m(A(s)) = 0.$$

Now, using [30], there exists a cyclic generator $y(s)$ of $(\mathbb{R}[s])^n$ relatively to $A(s)$ such that

$$y(s), A(s)y(s), \cdots, A^{n-1}(s)y(s).$$

are linearly independent.

Now, define the vectors $e_i(s)$ by

$$\begin{cases} e_i(s) &= m^{(i)}(A(s))y(s), \quad i \in \{1, 2, \cdots n\} \\ e_0(s) &= 0 \end{cases},$$

where $m^{(i)}(\lambda(s))$ are the auxiliary polynomials given by

$$\begin{array}{rcl} m^{0}(\lambda(s)) & = & m(\lambda(s)) = \lambda(s)^{n} - (a_1(s) + a_2(s)\lambda(s) + \cdots + a_n(s)\lambda^{n-1}(s)) \\ m^{(1)}(\lambda(s)) & = & \lambda(s)^{n-1} - (a_2(s) + a_3\lambda(s) + \cdots + a_n\lambda(s)^{n-2}) \\ \vdots & \vdots & \vdots \\ m^{(n-1)}(\lambda(s)) & = & \lambda(s) - a_n(s) \\ m^{(n)}(\lambda(s)) & = & 1 \end{array} .$$

Since for $p_i(s) \in \mathbb{R}[s]$, $i = 1, 2, \cdots, n$

$$\sum_{i=1}^{n} p_i(s)e_i(s) = 0 \Rightarrow \sum_{i=1}^{n} p_i(s)m^{(i)}(A(s))y(s) = 0,$$

and this implies that

$$\sum_{i=1}^{n} p_i(s)[A^{n-i}(s) - a_{i+1}(s) - a_{i+2}(s)A(s) - \cdots - a_n(s)A^{n-i-1}(s)]y(s) = 0$$

which in turn implies that

$$\begin{gathered} p_1(s)[A^{n-1}(s)y(s) + [-p_1(s)a_n(s) + p_2(s)]A^{n-2}(s)y(s) + [-p_2(s)a_n(s) + p_3(s)]\times \\ A^{n-3}(s)y(s) + \cdots + [-p_1(s)a_3(s) - p_2(s)a_4(s) - \cdots - p_{n-1}(s)]A(s)y(s) + \\ [-p_1(s)a_2(s) - p_2(s)a_3(s) - \cdots - p_{n-1}(s)a_n(s) + p_n(s)]y(s) = 0. \end{gathered}$$

Since $y(s), A(s)y(s), A^2(s)y(s), \cdots, A^{n-1}(s)y(s)$ are linearly independent, then we have

$$\left\{ \begin{array}{c} p_1(s) = 0 \\ -p_1(s)a_n(s) + p_2(s) = 0 \\ -p_2(s)a_n(s) + p_3(s) = 0 \\ -p_1(s)a_3(s) - p_2(s)a_4(s) - \cdots - p_{n-1}(s) = 0 \\ -p_1(s)a_2(s) - p_2(s)a_3(s) - \cdots - p_{n-1}(s)a_n(s) + p_n(s) = 0 \end{array} \right. \tag{1.15}$$

Hence from (1.15) we get

$$p_1(s) = p_2(s) = p_3(s) = \cdots = p_n(s) = 0.$$

So$\{e_i(s)\}_{i=1,2,\cdots,n}$ are linearly independent, which shows that they form a basis for the modulus $X = (\mathbb{R}[s])^n$. Hence,

$$A(s)e_i(s) = e_{i-1}(s) + a_i(s)e_n(s), \quad i = 1, 2, \cdots, n.$$

And so the companion form associated with the endomorphism $A(s)$ is given by

$$C(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots\cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1(s) & a_2(s) & a_3(s) & \cdots & a_{n-1}(s) \end{bmatrix}. \qquad (1.16)$$

we note that the transformation used here to get the companion form (1.16) is given by

$$T(s) = [e_1(s), e_2(s), \cdots, e_n(s)] \qquad (1.17)$$

### 1.3.2 Smith Form over the Ring R[s]

**Definition 4** *Let $P(s)$ be a polynomial matrix, over the polynomial ring $\mathbb{R}[s]$, of order $n \times m$ and of rank $r$. Then, the Smith form matrix $S(s)$ of $P(s)$ is defined by*

$$S(s) = \begin{bmatrix} i_1(s) & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \cdots & \cdots & 0 \\ \vdots & \ddots & i_r(s) & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \qquad (1.18)$$

*where $i_1(s), i_2(s), \cdots, i_r(s)$ are the invariant polynomials of the matrix $P(s)$ and which satisfies the following properties:*

- $i_k(s)$ divides $i_{k-1}(s)$, $\forall k = \overline{2, r}$

- $i_k(s) = \frac{d_k(s)}{d_{k-1}(s)}$, $\quad d_0(s) = 1$.

**Definition 5** *Two polynomial matrices $P(s)$ and $Q(s)$ of orders $m \times q$ and $n \times p$ respectively, over the ring of polynomials $\mathbb{R}[s]$, are said to be equivalent if there exist two unimodular (i.e.with non- zero constant determinants) polynomial matrices $M(s)$ and $N(s)$ of orders $m \times n$ and $p \times q$ respectively, such that*

$$P(s) = M(s)Q(s)N(s).$$

*We note that two matrices $P(s)$ and $Q(s)$ are equivalent if one is obtained from the other by a sequence of elementary operations undertaken on the rows and columns of the other (i.e.$P(s)$ and $Q(s)$ are products of elementary matrices).*

**Definition 6** *Two polynomial matrices $P(s)$ and $Q(s)$ of the same order, over the ring of polynomials $\mathbb{R}[s]$, are said to be similar if there exist a square unimodular (i.e.with non- zero constant determinant) polynomial matrix $M(s)$ such that*

$$P(s) = M^{-1}(s)Q(s)M(s).$$

# 1.4 System Matrix Representation (1-D Case)

## 1.4.1 System Matrix Representation for O.D.E.

Let the following differential system

$$\begin{cases} x' &= Ax + Bu \\ y &= Cx + D(u + u' + \cdots) \end{cases}, \tag{1.19}$$

where $x$ is the state vector, $u$, $u'$ and $y$ is the input vector (the control variable) and its derivative and the output vector respectively. $A, B, C$, and $D$ are matrices matrices of appropriate dimensions

If we assume zeros initial conditions and take Laplace transform of (1.19) we get

$$\begin{cases} s\overline{x} &= A\overline{x} + B\overline{u} \\ \overline{y} &= C\overline{x} + D(s)\overline{u} \end{cases}. \tag{1.20}$$

**Definition 7** *The system matrix*

$$M(s) = \begin{bmatrix} sI_n - A & \vdots & B \\ \cdots\cdots\cdots & \vdots & \cdots \\ -C & \vdots & D(s) \end{bmatrix}, \tag{1.21}$$

*introduced by Rosenbrock [40] in 1-D case, is known as the state space system matrix representation of the differential system (1.19).*

. In a similar way, we can obtain different types of system matrices over the ring of polynomials $\mathbb{R}[s]$ in one variable $s$ i.e. in the $1-D$ case, for the following different types of systems:

$$\begin{cases} Ex' &= Ax + Bu \\ y &= Cx + Du \end{cases} \tag{1.22}$$

where $A, B, C$, $D$ and $E$ are matrices of appropriate dimensions with $E$ is may be a singular matrix.

Where, again by taking Laplace transform and assuming zeros initial conditions we obtain the system matrix representation of (1.22)

$$N(s) = \begin{bmatrix} sE - A & \vdots & B \\ \cdots\cdots\cdots & \vdots & \cdots \\ -C & \vdots & D(s) \end{bmatrix} \tag{1.23}$$

and for the more general system e.g.

$$\begin{cases} T(s)\overline{\xi} &= U(s)\overline{u} \\ \overline{y} &= V(s)\overline{\xi} + W(s)\overline{u} \end{cases}$$

we have the following system matrix

$$P(s) = \begin{bmatrix} T(s) & \vdots & U(s) \\ \cdots\cdots\cdots & \vdots & \cdots \\ -V(s) & \vdots & W(s) \end{bmatrix} \tag{1.24}$$

### 1.4.2 System Matrix Transformations

Rosenbrock [40] has introduced the notions of system matrix representation and system matrix transformation between such system matrices. Verghese and al [44] have studied and developed the so called generalized (or descriptor) state space systems. For the 1-D case, these systems (1.22) give rise to matrices over $R[s]$ of the form

$$\begin{bmatrix} sE - A & \vdots & B \\ \cdots & \vdots & \cdots \\ -C & \vdots & 0 \end{bmatrix} \tag{1.25}$$

where $A, B$ and $C$ are matrices of appropriate dimensions and $E$ is an $n \times n$ matrix which may be singular.

Matrices in (1.25) are extensions of matrices of the form

$$\begin{bmatrix} sI_n - A & \vdots & B \\ \cdots & \vdots & \cdots \\ -C & \vdots & 0 \end{bmatrix} \tag{1.26}$$

**Definition 8** *Let*

$$P_i(s) = \begin{bmatrix} T_i(s) & \vdots & U_i(s) \\ \cdots & \vdots & \cdots \\ -V_i(s) & \vdots & W_i(s) \end{bmatrix}, \; i = 1, 2,$$

*be two* $(r+m) \times (r+l)$ *polynomial system matrices over* $\mathbb{R}[s]$. *We say that* $P_1(s)$ *is strictly system equivalent (s.s.e.) to* $P_2(s)$ *if there exist* $(r+m) \times (r+l)$ *system matrices*

$$L_1(s) = \begin{bmatrix} M(s) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ X(s) & \vdots & I_m \end{bmatrix},$$

*and*

$$L_2(s) = \begin{bmatrix} N(s) & \vdots & Y(s) \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I_l \end{bmatrix},$$

such that

$$P_1(s) = L_1(s) P_2(s) L_2(s),$$

where $M(s)$ and $N(s)$ are $r \times r$ square unimodular matrices and $X$, $Y$ are matrices of orders $m \times r$ and $r \times l$ respectively.

**Remark 2** *(1.25) is strictly system equivalent to (1.26) if $E$ is regular.*

**Definition 9** *Let*

$$P_i(s) = \begin{bmatrix} T_i(s) & \vdots & U_i(s) \\ \cdots & \vdots & \cdots \\ -V_i(s) & \vdots & W_i(s) \end{bmatrix}, \ i = 1, 2,$$

*be two $(r+m) \times (r+l)$ polynomial system matrices over $\mathbb{R}[s]$. We say that $P_1(s)$ is restricted system equivalent (r.s..e.) to $P_2(s)$ if there exists $(r+m) \times (r+l)$ system matrices*

$$L_1(s) = \begin{bmatrix} M(s) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I_m \end{bmatrix},$$

*and*

$$L_2(s) = \begin{bmatrix} N(s) & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & I_l \end{bmatrix},$$

such that

$$P_1(s) = L_1(s) P_2(s) L_2(s),$$

where $M(s)$ and $N(s)$ are $r \times r$ square unimodular matrices over the ring of polynomials $\mathbb{R}[s]$.

**Remark 3** *We note that if in the definition of the transformation of strict system equivalence (s.s.e.) we replace $X(s)$ and $Y(s)$ by 0 (the null matrix) we get the transformation of restricted system equivalence (r.s.e.). For some interesting results on strict (respectively, restricted ) system equivalence see [15].*

### 1.4.3 A Canonical Form Under System Matrix Transformation

We give in the following a canonical form of a polynomial matrix over the ring of polynomials $\mathbb{R}[s]$ of one indeterminate, known as Smith form, under the equivalence system matrix transformation

**Theorem 6** *Every polynomial matrix $P(s)$, over the ring $\mathbb{R}[s]$, of order $n \times m$ and rank $r$ is equivalent to its Smith form given in (1.18).*

**Proof.** Since $\mathbb{R}[s]$ is a principal ideal domain, then a proof of the above theorem can be obtained in a similar way as in [19] using the elementary operations on the polynomial matrix $P(s)$. And so for more details see [19]. ■

### 1.4.4 Controllability and Observability Properties via System Matrix Representations

Now, we give some purely algebraic criterias on the characterization of the controllability and observability of the systems represented by some of the matrices given above.

**Theorem 7** *The system described by the system matrix $M(s)$, given in (1.21) is completely controllable if and only if the folowing matrix ( the controllability matrix):*

$$\mathbb{C}(A, B) = \left[B,\ AB, \ldots,\ A^{n-1}B\right]$$

*has rank $n$.*

Similarly, by using the theorem of duality, we can get the following analogous characterization result for complete observability.

**Theorem 8** *The system described by the system matrix $M(s)$, given in (1.21) is completely observable if and only if the following matrix ( the observability matrix):*

$$\mathbb{O}(C, A) = \left[C,\ CA, \ldots,\ CA^{n-1}\right]^T$$

*has rank $n$.*

For a proof of the above two theorems see [4].

**Remark 4** *The above characterazition theorems can be generalized to systems described by system matrices over the ring of polynomials* $\mathbb{R}[s, z]$.

**Remark 5** *The above matrices (1.25), (1.26) will be extended in chapter* 2 *to the* $2-D$ *case i.e. to matrices over* $\mathbb{R}[s, z]$ *in two indeterminates* $s$ *and* $z$ *of an appropriate form.*

# Chapter 2

# EXTENSION OF SOME RESULTS TO 2-D CASE

## 2.1 Introduction

We try in this chapter to present a generalization of some of the notions and results that are given in the $1-D$ system theory case of the previous chapter. We start by extending the two important notions, in the theory of linear systems over the ring of polynomials in two indeterminates $\mathbb{R}[s,z]$, concerning Smith form and Companion form matrices to the $2-D$ case. First, we give a $2-D$ Companion form matrix associated with a $2-D$ characteristic polynomial.

## 2.2 Canonical Matrix Form Problem over R[s,z]

### 2.2.1 A Companion Form over the Ring R[s,z]

Let the following discrete system used in Gregor[23] and which was introduced by Roesser [39] in modeling the bidimensional image processing:

$$x' = Ax + Bu , \tag{2.1}$$

where

$$x = \begin{bmatrix} x^h(i,j) \\ \\ x^v(i,j) \end{bmatrix}, \quad x' = \begin{bmatrix} x^h(i+1,j) \\ \\ x^v(i,j+1) \end{bmatrix}$$

where $x^h \in \mathbb{R}^{n_1}$ , $x^v \in \mathbb{R}^{n_2}$ are the horizontal and vertical states of the state vector $x$, $u \in \mathbb{R}^p$ is the input vector, and $A$ and $B$ are constant matrices in $\mathbb{M}_{n,n}(\mathbb{R})$ and $\mathbb{M}_{n,p}(\mathbb{R})$ respectively.

**Definition 10** *Let $P$ be a square matrix of order $n \times n$. We define the bidimensional characteristic matrix of $P$ and the characteristic polynomial of $P$ as follows:*

$$\begin{aligned} A(s,z) &= \Lambda I_n - P \\ a(s,z) &= det(\Lambda I_n - P), \end{aligned}$$

*where*

$$\Lambda = sI_{n_1} \oplus zI_{n_2},$$

*with $\oplus$ denotes the direct sum.*

**Definition 11** *The matrix $P$ which can be written in the following form*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

*where the matrices $P_{i,j} \in \mathbb{M}_{n_i,n_j}(\mathbb{R}), i,j = 1,2$, is known as the Companion matrix of the $2-D$ characteristic polynomial $a(s,z)$, and is characterized here by the fact that the $rankP_{12}$ or $rankP_{21}$ is equal to 1. We note that the matrix $P$ above can be choosen in arbitrary way since the use of a non- singular transformation does not change the $rankP_{12}$ or $rankP_{21}$. We also note that the in the $2-D$ case, the use of the previously given direct methods to construct the Companion form matrix is usually complicated if*

$$\min(rankP_{12}, rankP_{21}) > 1.$$

*And so in this case, the construction of the companion form matrix for an arbitrary polynomial $a(s,z)$ and so an arbitrary matrix $A(s,z)$ can be obtained by using the elementary operations* *on the ring of polynomials* $\mathbb{R}[s, z]$ *and the augmented operator* $f$ *defined by*

$$f : \mathbb{M}_n(\mathbb{R}[s, z]) \to \mathbb{M}_{n+1}(\mathbb{R}[s, z])$$

*such that*

$$f(T(s,z)) = \begin{bmatrix} 1 & 0 \\ 0 & T(s,z) \end{bmatrix}, \quad T(s,z) \in \mathbb{M}_n(\mathbb{R}[s,z])$$

*For more details concerning this case see Galkowski [18].*

**Remark 6** *For polynomial matrices* $P(s, z)$ *over the ring* $\mathbb{R}[s, z]$ *which has the following form*

$$P(s,z) = sI_n - A(z), \tag{2.2}$$

*the companion form associated to this matrix (2.2) can be obtained in a similar way as in the* $1-D$ *case. And it has the form*

$$\overline{P}(s,z) = sI_n - C(z),$$

*where* $C(z)$ *is the companion matrix given by*

$$C(z) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ -a_n(z) & -a_{n-1}(z) & \cdots & -a_1(z) \end{bmatrix}. \tag{2.3}$$

### 2.2.2 A Companion Form for $2-D$ Polynomials

Now, our aim is to give a $2-D$ canonical form matrix which is analogous to the $1-D$ companion form matrix given above.

We consider the same $2-D$ discrete system given above (by Roesser [39] for $2-D$ image processing). If inputs and outputs are neglected in the model equations, then the equations of the system have the form

$$\begin{cases} x^h(i+1,j) &= A_1 x^h(i,j) + A_2 x^u(i,j) \\ x^u(i,j+1) &= A_3 x^h(i,j) + A_4 x^u(i,j) \end{cases}$$

Now, let $S$ ( resp.$Z$ ) be an operator that has the effect of advacing the horizontal coordinate ( resp. the vertical coordinate) upon which it is operating. The effect of these operators on the state vectors is

$$\left\{\begin{array}{rcl} x^h(i+1,j) & = & sx^h(i,j) \\ x^u(i,j+1) & = & zx^u(i,j) \end{array}\right. .$$

Then, we have

$$\begin{aligned} (sI_{n_1} - A_1)\, x^h(i,j) - A_2 x^u(i,j) &= 0 \\ -A_3 x^h(i,j) + (zI_{n_2} - A_4)\, x^u(i,j) &= 0, \end{aligned}$$

and so we have:

$$\begin{bmatrix} sI_{n_1} - A_1 & -A_2 \\ -A_3 & zI_{n_2} - A_4 \end{bmatrix} T(i,j) = 0,$$

where

$$T(i,j) = \begin{bmatrix} x^h(i,j) \\ x^u(i,j) \end{bmatrix}.$$

The above equation represents a system of homogeneous equations in the elements of $T(i,j)$. For the system to have a non trivial solution for $T(i,j)$, the transformation represented by the matrix must be singular.

**Definition 12** *The matrix*

$$wI - A = (s,z)\, I - A$$

$$= \begin{bmatrix} sI_{n_1} - A_1 & -A_2 \\ -A_3 & zI_{n_2} - A_4 \end{bmatrix}$$

*obtained above is said to be the two-dimensional characteristic matrix of the partitioned matrix* $A$*, where*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Our aim now, is to find a matrix $A$ given as above where $A_1$ is $n_1 \times n_1$ ,and $A_4$ is $n_2 \times n_2$, $A_2$ is $n_1 \times n_2$ and $A_3$ is $n_2 \times n_1$. Such that the determinant of the characteristic matrix

$$wI - A = (s, z)\, I - A$$

$$= \begin{bmatrix} sI_{n_1} - A_1 & -A_2 \\ -A_3 & zI_{n_2} - A_4 \end{bmatrix}$$

is given by the following polynomial

$$\begin{aligned} d(s,z) &= \sum_{j=0}^{n_2} P_j(s)\; z^{n_2 - j} \\ &= \sum_{i=0}^{n_1} Q_i(z)\; s^{n_1 - i} \end{aligned}$$

where $P_0(s)$ and $Q_0(z)$ are monic polynomials and have degrees $n_1$ and $n_2$ respectively. also $P_j(s)$ , $j = \overline{1, n_2}$ ( respectively, $Q_i(z)$ , $i = \overline{1, n_1}$) have degrees less or equal to $n_1$ (respectively, $n_2$ ) and such that the matrix $A$ is in a form which is similar to the $1-D$ companion form . Here we mean by $A$ is in a $2-D$ companion form the following :

$A_1$ and $A_4$ are in companion forms and moreover $A_2$ is such that all the elements above the diagonal of the over all matrix $A$ are zero except for the elements on the superdiagonal which are all equal to one.

In the following we present a companion form for $2-D$ polynomials

**Proposition 1** *Let $d(s,z)$ be a $2-D$ polynomial given as above, then the $2-D$ companion matrix of $d(s,z)$ is given by* =

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

$$
= \begin{bmatrix}
0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\
P(0,n_1) & P(0,n_1-1) & \dots & P(0,1) & 1 & 0 & \dots & 0 \\
R_3(1,1) & R_3(1,2) & \dots & R_3(1,n_1) & 0 & 1 & \dots & 0 \\
R_3(2,1) & R_3(2,2) & \dots & R_3(2,n_1) & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & 0 \\
R_3(n_2-1,1) & R_3(n_2-1,2) & \dots & R_3(n_2-1,n_1) & 0 & 0 & \dots & 1 \\
R_3(n_2,1) & R_3(n_2,2) & \dots & R_3(n_2,n_1) & Q(0,n_2) & Q(0,n_2-1) & \dots & Q(0,1)
\end{bmatrix} \tag{2.4}
$$

Where $C_1, C_4$ are the $n_1 \times n_1, n_2 \times n_2$ companion matrices of $P_0(s)$ and $Q_0(z)$ respectively.i.e..

$\det(sI_{n1} - C_1) = P_0(s) = s^{n1} + P(0,1)\,s^{n1-1} + P(0,2)\,s^{n1-2} + \cdots + P(0,n_1)$,

and,

$$det\,(zI_{n2} - C_4) = Q_0(z) = z^{n2} + Q(0,1)\,z^{n2-1} + q(0,2)\,z^{n2-2} + \cdots + Q(0,n_2),$$

where, $C_2$ is $n_1 \times n_2$ matrix and has all its columns zero except for the first one which is given by $E_{n1}$(the first column of $I_{n1}$), $C_3$ is $n_2 \times n_1$ matrix and its elements $R_3(i,j)$ are determinated uniquely and recursively from the following formula:

$$R_3(i,j) = Q(0,i)P(0,n_1-j+1) - P(i,n_1-j+1) - \sum_{k=1}^{i-1} Q(0,i-k)R_3(k,j) \tag{2.5}$$

where $P(i,j)$ and $Q(i,j)$ are defined by the following:

$$P_i(s) = \sum_{j=0}^{n_1} P(i,j)\,s^{n_1-j}, i = \overline{0,n_2},$$

and

$$Q_i(z) = \sum_{j=0}^{n_1} Q(i,j)\,s^{n_1-j}, i = \overline{0,n_2}$$

Furthermore, if $d(s,z)$ is separable ie. can be written as a product of two $1-D$ polynomials, then $C_3$ is taken to be the null matrix. For the proof of this result see [6].

The matrices considered here are associated with partial differential equations or $2-D$ discrete systems. For some results concerning these matrices see [11, 12, 13].
Canonical forms of these matrices over $R[s,z]$,and wich arise from $2-D$ discrete models given by [39] are obtained under a similarity transformation. And since canonical forms play a fondamental role in the modern theory of linear systems, we present here a particular type of canonical forms, for a comparision see [24], wich is known as companion matrix form. Finally, using a result of [6, 16], we give a necessary and sufficient condition for a matrix $A$ of the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

to be equivalent to the $2-D$ companion form.

Now, we give the following characterization theorem:

**Theorem 9** *A necessary and sufficient condition for a matrix*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

*as the one given above, to be equivalent to the companion form (2.4) is that its characteristic matrix* $wI - A \equiv (s,z)\,I - A$ *is equivalent to the Smith form:*

$$S(s,z) = \begin{bmatrix} I_{n_1+n_2-1} & 0 \\ 0 & \det(wI - A) \end{bmatrix}$$

**Proof.** Suppose that the matrix $A$ is equivalent to the companion form, then it is clear from the form of the matrix $wI - C$ that this latter matrix is equivalent to the Smith form $S(s,z)$ given above. Hence $wI - A$ is equivalent to the Smith form $S(s,z)$, and so the necessity condition is established.
To prove the sufficiency, we suppose that the matrix $A$ is equivalent to the Smith form $S(s,z)$, then by the previous proposition there exists a companion form for the polynomial $d(s,z)$ ie. there exists a matrix $C$ in the form (2.4) such that $|wI - C| = d(s,z)$. And so we get $wI - A$ and $wI - C$ are equivalent since they are both equivalent to the Smith form. ∎

**Example 1** *Let the* $2-D$ *polynomial* $d(s,z)$ *given by*

$$\begin{aligned} d(s,z) &= (s^2+1)z^2 + (s^2+2s+1)z + 3s^2 + s + 2 \\ &= (z^2+z+3)s^2 + (2z+1)s + z^2 + z + 1 \end{aligned}$$

*where the polynomials* $P_0(s)$ *and* $Q_0(z)$ *are given by*

$$P_0(s) = s^2+1, Q_0(z) = z^2+z+3$$

*and the polynomial coefficients* $P_i(s)$ *and* $Q_i(z)$ *are given by*

$$P_i(s) = \sum_{j=0}^{2} P(i,j)\, s^{2-j}, i = \overline{1,2},$$

*and*

$$Q_i(z) = \sum_{j=0}^{2} Q(i,j)\, z^{2-j}, i = \overline{1,2}.$$

*And the elements of the matrix* $C_3$ *are calculated from the formula (2.5). And so we obtain the elements* $C_i, i = \overline{1,4}$ *of the bloc matrix* $C$ *which are given by*

$$C_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}.$$

*And so the over all matrix* $C$ *has the following companion form*

$$C = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ \hdashline 0 & -2 & 0 & 1 \\ 2 & 1 & -3 & -1 \end{array}\right].$$

**Conclusion 1** *A particular matrix form , known as* $2-D$ *companion form is obtained for* $2-D$ *discrete systems. Furtur research work can be carried out by using this canonical form to link certain notions of* $2-D$ *system theory (e.g. controllability, observability, realizability etc.), as for the case of* $1-D$ *linear systems.*

### 2.2.3 The Smith Form over the Ring R[s,z]

Some notions and results that have been given in the first chapter concerning Smith form on the ring of polynomials $\mathbb{R}[s]$ of one indeterminate are generalized here to the bidimensional case, i.e. over the ring of polynomials $\mathbb{R}[s,z]$ in two indeterminates $s$ and $z$. So we begin by giving the following definitions concerning the equivalence and similarity transformations between polynomial matrices on the ring $\mathbb{R}[s,z]$.

**Definition 13** *A polynomial matrix of order $n$, over the ring $\mathbb{R}[s,z]$ of polynomials in two indeterminates $s$ and $z$ with coefficients in $\mathbb{R}$, is defined as a matrix $P(s,z)$ with entries in $\mathbb{R}[s,z]$ i.e.*

$$P(s,z) = [P_{i,j}(s,z)]_{i,j=1}^{n}, \dots, P_{i,j}(s,z) \in \mathbb{R}[s,z].$$

**Definition 14** *Two polynomial matrices $P_1(s,z)$ and $P_2(s,z)$ of orders $n \times m$ and $p \times q$ respectively, are said to be equivalent if there exists two polynomial matrices $M(s,z)$ and $N(s,z)$ of orders $n \times p$ and $q\times$ $m$ respectively such that*

$$P_1(s,z) = M(s,z)P_2(s,z)N(s,z).$$

**Definition 15** *Two square polynomial matrices $P_1(s,z)$ and $P_2(s,z)$ of the same order $n$, are said to be similar if there exists a square unimodular polynomial matrix $M(s,z)$ of order $n$ such that*

$$P_1(s,z) = M(s,z)P_2(s,z)M^{-1}(s,z).$$

**Remark 7** *In contrast to the one dimensional case, the ring of polynomials of two variables $s$ and $z$, $\mathbb{R}[s,z]$ is not a principal ideal domain [7].*

In 1-D case a Smith form of a matrix $M(s)$ over $\mathbb{R}[s]$ can always be defined and it is equivalent to its Smith form. For the 2-D case a Smith form of a polynomial matrix $M(s,z)$ over the ring $\mathbb{R}[s,z]$ of polynomials of two indeterminates can also always be defined. However, this Smith form is not, in general, equivalent to $M(s,z)$, as it can be shown later on in this work. Now, we assume, in all of the forthcoming definitions, that $\Re$ is an integral domain (that is, $\Re$ is a unitary commutative ring which has no zeros divisors).

**Definition 16** *We say that $\Re$ is a greatest common divisor domain (g.c.d.d) if any two elements in $\Re$ possess a greatest common divisor (g.c.d).*

**Definition 17** *$\Re$ is called a Bezout domain (B.d) if any finitely generated ideal of $\Re$ is principal.*

**Definition 18** *$\Re$ is called a Smith domain (S.d) if the following hold:*

1. $\Re$ is a B.d.

2. For any non-zeros coprime $a$, $b$, $c \in \Re$ there exists $s, t \in \Re$ such that $sa$ and $sb + tc$ are coprime.

Now, we recall the following result.

**Theorem 10** *Let $R$ be a ring such that : every two elements of $R$ have a greatest commun divisor. Then, a necessary condition for a matrix $M$ in the ring of rectangular matrices $M_{n\times m}(R)$ over the ring $R$ to be equivalent to its Smith form is that $R$ must be a principal ideal domain [29].*

**Remark 8** *We note that, in contrast to the $1-D$ case, the ring of polynomials in two indeterminates $\mathbb{R}[s,z]$ is not a principal ideal domain, and so as a result of the above theorem we do not have an equivalence transformation between a polynomial matrix over $\mathbb{R}[s,z]$ and its Smith form.*

Now, we extend the definition (1.18) of the Smith form matrix to the ring of polynomials in two variables $s$ and $z$, $\mathbb{R}[s,z]$ as follows:

Let $T(s,z)$ be a polynomial matrix of order $p \times q$, over $\mathbb{R}[s,z]$, we define the Smith form $S(s,z)$ of $T(s,z)$ by:

$$S(s,z) = \begin{cases} \begin{bmatrix} F(s,z) & 0 \end{bmatrix}; & if \quad p < q, \\ \text{or,} \quad F(s,z); & if \quad p = q, \\ \begin{bmatrix} F(s,z) \\ 0 \end{bmatrix}; & if \quad p > q \end{cases} \tag{2.6}$$

where,

$$F(s,z) = diag\{i_1(s,z), i_2(s,z), \ldots, i_m(s,z)\}, \qquad \text{with } m = \min(p,q),$$

and the elements $i_k(s,z), k = 1, 2, \cdots, m.$ are known as the invariant polynomials over $\mathbb{R}[s,z]$ of the polynomial matrix $T(s,z)$, and they are given by

$$i_k(s,z) = \begin{cases} \frac{d_k(s,z)}{d_{k-1}(s,z)}, & k = 1, 2, \ldots, r \\ \cdots\cdots\cdots 0 \cdots\cdots\cdots, & k = r+1, r+2, \ldots, m \end{cases}, \tag{2.7}$$

where r is the rank of $T(s,z)$, and $d_0(s,z) = 1$, and the determinantal divisor $d_k(s,z)$ is the greatest common divisor of all the $k^{th}$ order minors of $T(s,z)$. We note that all $i_k(s,z)$ in (2.7) which are not identically zeros, are monic over $\mathbb{R}[s,z]$, and they satisfy the following divisibility property:

$$i_1(s,z)/i_2(s,z)/\cdots\cdots\cdots/i_r(s,z) \tag{2.8}$$

**Theorem 11** *If $\Re$ is a g.c.d.d then a necessary condition for a matrix $M \in \Re^{l\times m}$ $(l+m>2)$ to be equivalent to its Smith form $S$ is that $\Re$ is a B.d.*

For a proof of this theorem see [26]

**Remark 9** *We can define a Smith form $S(s,z)$ (2.6) for any matrix $M(s,z) \in \mathbb{R}[s,z]$, since, $\mathbb{R}[s,z]$ is a g.c.d.d., but not a B.d since the finitely generated ideals in $\mathbb{R}[s,z]$ are not necessarily principals ( e.g. the ideal generated by s and z). And so as $\mathbb{R}[s,z]$ is not a B.d, then a matrix $M(s,z)$ over $\mathbb{R}[s,z]$ is, in general, not always equivalent to its Smith form. To overcome this difficulty, we use instead of $\mathbb{R}[s,z]$ the ring $\mathbb{R}(z)[s]$ or $\mathbb{R}(s)[z]$, and for this vision of $\mathbb{R}[s,z]$,* *we can establish an equivalance of* $1-D$ *matrix with a Smith form over* $\mathbb{R}(z)[s]$ *or* $\mathbb{R}(s)[z]$. *However, this approach can give a Smith form of a matrix with entries which contain rational elements. But [29] suggested the ideas of renormalising the resulting transformations to get Smith forms by transformation over* $\mathbb{R}(z)[s]$ *or* $\mathbb{R}(s)[z]$. *And one problem which arises from this approach is that the resulting Smith form is not necessarily unique.*

Because of the important role that is played by canonical forms in many research areas especially in control theory (e.g. Smith form), so our aim is to establish necessary and sufficient conditions for a matrix $M(s,z)$ to be equivalent to its Smith form. We try here to generalize a result given in [6]. For some other results on these canonical forms see, [13] and [16].

Now, in the following, we give a theorem which can be considered as an extension of a characterization result [6] for a polynomial matrix over $\mathbb{R}[s,z]$ to be equivalent to its Smith form.

**Theorem 12** *Let* $P(s,z)$ *be a square matrix of order* $n$ *over the ring of polynomials* $\mathbb{R}[s,z]$ *such that*

$$P(s,z) = sI_n - A(z).$$

*Then,* $P(s,z)$ *is equivalent to the Smith form*

$$S(s,z) = \begin{bmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & i_{n-2}(s,z) & 0 & 0 \\ 0 & 0 & i_{n-1}(s,z) & 0 \\ 0 & 0 & 0 & i_n(s,z) \end{bmatrix}, \tag{2.9}$$

*if and only if the polynomial matrix* $A(z)$ *over the ring* $\mathbb{R}[z]$ *is similar over the ring* $\mathbb{R}[z]$ *to the companion matrix* $C(z)$ *of the following form*

$$C(z) = \begin{bmatrix} C_1(z) & 0 & 0 \\ 0 & C_2(z) & 0 \\ 0 & 0 & C_3(z) \end{bmatrix}, \tag{2.10}$$

*where the matrices $C_i(z)$, $i = 1, 2, 3$ are polynomial square matrices of order $n_i$ in companion form over the ring $\mathbb{R}[z]$ and such that their characteristic polynomials are given by*

$$\begin{aligned}\det(sI_{n_1} - C_1(z)) &= i_{n-2}(s, z) \\ \det(sI_{n_2} - C_2(z)) &= i_{n-1}(s, z)\ , \\ \det(sI_{n_3} - C_3(z)) &= i_n(s, z)\end{aligned}$$

**Proof.** To prove the necessity, we assume that $T(s, z) = sI_n - A(z)$ is equivalent over $\mathbb{R}[s, z]$ to $S(s, z)$ in (2.9). By elementary raws and columns operations on $S(s, z)$ ,we get

$$\widetilde{S}(s, z) = \begin{bmatrix} I_{n_1-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{n-2}(s, z) & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{n-1}(s, z) & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_3-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_n(s, z) \end{bmatrix} \qquad (2.11)$$

and so $T(s, z)$ is equivalent to the Smith form $\widetilde{S}(s, z)$ in (2.11) (since elementary raws and columns operations on a matrix preseve equivalence). And since the matrix $\widetilde{S}(s, z)$ given in (2.11) is clearly equivalent to the block matrix

$$sI_n - C(z) = \begin{bmatrix} sI_{n_1} - C_1(z) & 0 & 0 \\ 0 & sI_{n_2} - C_2(z) & 0 \\ 0 & 0 & sI_{n_3} - C_3(z) \end{bmatrix}$$

where $C_i(z)$, are $n_i \times n_i$, $i = \overline{1,3}$ square matrices in companion form respectively, such that

$\det(sI_{n_1} - C_1(z)) = q_{n-2}(s, z)$, $det(sI_{n_2} - C_2(z)) = q_{n-1}(s, z)$, and $det(sI_{n_3} - C_3(z)) = q_n(s, z)$. Hence, the matrix $T(s, z) = sI_n - A(z)$ is equivalent to $sI_n - C(z)$, and since a transformation of equivalence between such matrices (resulting from r.d.d. systems) can be replaced by a similarity transformation, then $A(z)$ is also similar to

$$C(z) = \begin{bmatrix} C_1(z) & 0 & 0 \\ 0 & C_2(z) & 0 \\ 0 & 0 & C_3(z) \end{bmatrix},$$

which ends the proof of necessity.

Now, we have to prove the sufficiency. Since $A(z)$ is similar to the matrix $C(z)$ given in (2.10), then $sI_n - A(z)$ is also similar to $sI_n - C(z)$ and because this last matrix is in companion form and is equivalent to its Smith form

$$S(s,z) = \begin{bmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & q_{n-2}(s,z) & 0 & 0 \\ 0 & 0 & q_{n-1}(s,z) & 0 \\ 0 & 0 & 0 & q_n(s,z) \end{bmatrix}.$$

Then so it is $sI_n - A(z)$. This ends the proof of the theorem. ■

**Remark 10** *Since the block matrix* $[sI_n - C(z) \quad E_n]$ *has no zeros, where* $C(z)$ *is the companion matrix given in (2.3), so the matrix* $\overline{P}(s,z) = sI_n - C(z)$ *is equivalent to the same Smith form as the matrix* $P(s,z) = sI_n - A(z)$.

**Conclusion 2** *In this previous work, we tried to present some types of canonical forms for matrices over the ring of polynomials* $\mathbb{R}[s,z]$ *(e.g. Smith form), and we generalized a result concerning necessary and sufficient conditions for a matrix over* $\mathbb{R}[s,z]$ *to be equivalent to a given Smith form. However, the work on canonical forms needs more investigation, especially, in the case of matrices of multivariate polynomials.*

In the following example we will just show how the rows and columns operations are applied on $S(s,z)$.

**Example 2** *For* $n = 6$, *let* $S(s,z)$ *be the matrix of the form*

$$S(s,z) = \begin{bmatrix} I_3 & 0 & 0 & 0 \\ 0 & q_4(s,z) & 0 & 0 \\ 0 & 0 & q_5(s,z) & 0 \\ 0 & 0 & 0 & q_6(s,z) \end{bmatrix},$$

*which is the same as the matrix*

$$S(s,z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_4(s,z) & 0 & 0 \\ 0 & 0 & 0 & 0 & q_5(s,z) & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6(s,z) \end{bmatrix}.$$

*Note that the degrees of* $q_4(s,z)$, $q_5(s,z)$ *and* $q_6(s,z)$ *in* $s$ *is* 1. *Now if we change line* 2 *with* 4 *and then column* 4 *with* 2 *in the above matrix we get*

$$S_1(s,z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_4(s,z) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_5(s,z) & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6(s,z) \end{bmatrix},$$

*and if we then change line* 4 *with* 5 *and column* 4 *with* 5 *we obtain*

$$S_2(s,z) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_4(s,z) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_5(s,z) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_6(s,z) \end{bmatrix}.$$

*We note here that,* $I_{n_1-1} = [1]$ ; $I_{n_2-1} = [1]$ ; $I_{n_3-1} = [1]$ .

## 2.3 System Matrix Representation (2-D Case)

### 2.3.1 System Matrix Representation for P.D.E.

We know from 1-D case, that the systems of the form (1.19), (1.22), and for more general systems [40], give rise to matrices (1.21), (1.23) and (1.24) respectively over $R[s]$ of the following form:

$$M(s) = \begin{bmatrix} sI_n - A & \vdots & B \\ \cdots\cdots\cdots & \vdots & \cdots \\ -C & \vdots & D(s) \end{bmatrix},$$

$$N(s) = \begin{bmatrix} sE - A & \vdots & B \\ \cdots\cdots\cdots & \vdots & \cdots \\ -C & \vdots & D(s) \end{bmatrix},$$

and

$$P(s) = \begin{bmatrix} T(s) & \vdots & U(s) \\ \cdots\cdots & \vdots & \cdots \\ -V(s) & \vdots & W(s) \end{bmatrix},$$

where $A$, $B$, $C$, $D(s)$, $T(s)$, $U(s)$, $V(s)$, and $W(s)$ are $n \times n$, $n \times l$, $m \times n$, $m \times l$, $r \times r$, $r \times l$, $m \times r$, $m \times l$ matrices respectively, and $E$ is an $n \times n$ square matrix may by singular.

The above system matrices given in (1.21), (1.23) and (1.24) are obtained using the symbolic calculus(e.g. Laplace transform). And the great interest in these system matrices is due to the fact that they are very much useful in control theory, since the very important notions such as controllability, observability, stability and feed back etc. can be described in terms of these matrices [21, 22], and [25], [27]. Moreover, these system matrices contain all the mathematical information about the system which is needed to describe its properties and behaviour. The extension of the above system matrices to the 2-D case to represent e.g. systems of partial and retarded delay differential equations etc. is given. Also the problem of relating these system matrices via matirx transformations as system equivalence, system similarity, strict system equivalence and restiricted system equivalence is discussed. And the problem of obtaining canonical forms of the above matrices is considered.

The above system matrix representaions are extended to the 2-D case as follows:

$$M(s,z) = \begin{bmatrix} sI_n - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zI_m - A_4 & \vdots & B_2 \\ \cdots & \cdots & \vdots & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix}, \tag{2.12}$$

and

$$N(s,z) = \begin{bmatrix} sE_1 - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zE_2 - A_4 & \vdots & B_2 \\ \cdots & \cdots & \vdots & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix} \tag{2.13}$$

where $\mathrm{A}_i$, $B_j$, $C_k$, $i = \overline{1,4}$; $j$, $k = \overline{1,2}$ are matrices of appropriate dimensions, and $E_l$, $l = \overline{1,2}$ are square matrices may be singular. We note that the above system matrices in (2.12), (2.13) are obtained in connection with 2-D discrete equations [21, 22], and 2-D generalized (descriptor) systems which can be regarded as a generalization of the $1-D$ descriptor systems given in [40], respectively. And the following system matrices are obtained in connection with retaded delay differential systems, partial differential systems, and genral differatial systems:

$$R(s,z) = \begin{bmatrix} sI_n - A(z) & \vdots & B(s,z) \\ \cdots & \vdots & \cdots \\ -C(s,z) & \vdots & 0 \end{bmatrix}, \tag{2.14}$$

and

$$G(s,z) = \begin{bmatrix} T(s,z) & \vdots & U(s,z) \\ \cdots & \vdots & \cdots \\ -V(s,z) & \vdots & 0 \end{bmatrix}, \tag{2.15}$$

where the above matrices in (2.14), (2.15) $A(z)$, $B(s,z)$, $C(s,z)$, $T(s,z)$, $U(s,z)$, $-V(s,z)$ are of appropriate dimensions. For more results on these system matices see, [13]. In the following, we give some of the systems that can give rise to the above system matrices in the $2-D$ case:

Let the system of retarded delay differential equations given by

$$\begin{cases} x'(t) - \sum_{i=1}^{r} A_i x(t-ih) = \sum_{j=1}^{s} B_j u(t-jh) \\ y(t) = \sum_{k=1}^{q} C_k x(t-kh) \end{cases} \tag{2.16}$$

where $x(t)$ is an $n-$column state vector, $u(t)$ is an $m-$column control vector, and $y(t)$ is a $p-$column out put vector, $h$ is a positif constant and $A_i,\ B_{j,}\ C_k,\ \ 1 \le i \le r, 1 \le j \le s, 1 \le k \le q$, are $n \times n$, $n \times m$, and $p \times n$ constant matrices respectively. By taking Laplace transform of (2.16) and assuming zeros initial conditons we get the following system matrix

$$M(s,z) = \begin{bmatrix} A(s,z) & B(z) \\ -C(z) & 0 \end{bmatrix} \tag{2.17}$$

with $A(s,z) = sI_n - A(z)$, where $s$ and $z$ stand for differential and delay operators respectively.

Also a matrix of the form (2.17) may arise in connection with a partial differential system of the form

$$\begin{cases} \frac{\partial X}{\partial t} & = & \sum\limits_{i=0}^{g} A_i \frac{\partial^i X}{\partial \tau^i} + \sum\limits_{j=0}^{q} B_j \frac{\partial^j X}{\partial \tau^j} u(t,\tau) \\ Y & = & \sum\limits_{k=0}^{r} C_k \frac{\partial^k X}{\partial \tau^k}, \end{cases} \tag{2.18}$$

In this case $X(t,\tau), u(t,\tau)$ and $Y(t,\tau)$ will be vector functions of $t$ which will usually be time, and $\tau$ which will usually be spacial variable.

As in the case of system matrices over $\mathbb{R}[s]$, the matrix

$$M_c = [sI_n - A(z) \quad B(z)]$$

describes the controllability properties of the system (2.17) whereas the matrix

$$M_o = \begin{bmatrix} sI_n - A(z) \\ -C(z) \end{bmatrix},$$

describes the observability properties of the system (2.17). A state system matrix of the form (2.12) may arise from a $2-D$ discrete system of the form

$$\begin{cases} X^h(i+1,j) &= A_1 X^h(i,j) + A_2 X^v(i,j) + B_1 u(i,j) \\ X^v(i,j+1) &= A_3 X^h(i,j) + A_4 X^v(i,j) + B_2 u(i,j) \\ Y(i,j) &= C_1 X^h(i,j) + C_2 X^v(i,j), \end{cases} \tag{2.19}$$

This model is due to [21, 22], in which the local state $X$ is divided into a horizontal state vector $X^h(i,j)$ and a vertical state vector $X^v(i,j)$ which are propagated respectively horizontally and vertically by first order difference equations, $u(i,j)$ is the input vector, $Y(i,j)$ is the output vector,and $A_1, A_2, A_3, A_4, B_1, B_2$, $C_1$and $C_2$ are real constant matrices of appropriate dimensions.

. If we take the $(s,z)$ transform (the two dimensional Laplace transform) of (2.19) and taking zeros boundary conditions on $X^h(0,j)$, and $X^v(i,0)$, we get the system matrix (2.16). The controllability ( resp. observability.) properties of (2.19) are described by the system matrix

$$\begin{bmatrix} sI_n - A_1 & -A_2 & \vdots & B_1 \\ \cdots\cdots & \cdots\cdots & \vdots & \cdots \\ -A_3 & zI_m - A_4 & \vdots & B_2 \end{bmatrix},$$

respectively, by the system matrix

$$\begin{bmatrix} sI_n - A_1 & \vdots & -A_2 \\ -A_3 & \vdots & zI_m - A_4 \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ -C_1 & \vdots & -C_2 \end{bmatrix},$$

### 2.3.2 System Matrix Transformations

**Definition 19** *Two system matrices of the form (2.13) are said to be restricted system equivalent (r.s.e.) if they are related by the transformation of the type*

$$\begin{bmatrix} M_1 & 0 & \vdots & 0 \\ 0 & M_2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & I_p \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zE_2 - A_4 & \vdots & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} N_1 & 0 & \vdots & 0 \\ 0 & N_2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & I_l \end{bmatrix} = \begin{bmatrix} sE_1' - A_1' & -A_2' & \vdots & B_1' \\ -A_3' & zE_2' - A_4' & \vdots & B_2' \\ \cdots & \cdots & \cdots & \cdots \\ -C_1' & -C_2' & \vdots & 0 \end{bmatrix} \tag{2.20}$$

*where $M_1, M_2, N_1,$ and $N_2$ are matrices of appropriate dimensions.*

1. The transformation in (2.20) is a special case of strict -system equivalence (s.s.e.).

2. If $E_1$ and $E_2$ in (2.20) are singular, then $p(s,z)$ is restricted system equivalent (r.s.e.) to a system matrix $p'(s,z)$ of the form (2.12). We note that this type of matrix arises in the state space model used by Givon-Roesser [21] in describing 2-D discrete systems.

This restricted system matrix equivalence transformation preserves the $p \times l$ rational transfer function matrix given by

$$G(s,z) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -A_2 \\ -A_3 & zE_2 - A_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Now, we present a special type of equivalence transformation. ie. a similarity transformation between system matrices of the form (2.12) given above

Let $P(s,z)$ and $\overline{P}(s,z)$ be two block polynomial matrices of the form (2.12), then there exists a transformation of similarity between these two matrices of the following form :

$$\begin{bmatrix} H_1 & 0 & \vdots & 0 \\ 0 & H_2 & \vdots & 0 \\ \cdots & \cdots & \cdot & \cdots \\ 0 & 0 & \vdots & I_p \end{bmatrix} \begin{bmatrix} sI_n - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zI_m - A_4 & \vdots & B_2 \\ \cdots & \cdots & \cdot & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} H_1^{-1} & 0 & \vdots & 0 \\ 0 & H_2^{-1} & \vdots & 0 \\ \cdots & \cdots & \cdot & \cdots \\ 0 & 0 & \vdots & I_l \end{bmatrix} = \begin{bmatrix} sI_n - \overline{A}_1 & -\overline{A}_2 & \vdots & \overline{B}_1 \\ -\overline{A}_3 & zI_m - \overline{A}_4 & \vdots & \overline{B}_2 \\ \cdots & \cdots & \cdot & \cdots \\ -\overline{C}_1 & -\overline{C}_2 & \vdots & 0 \end{bmatrix}$$

**Remark 11** *The above transformation preserves the T. F. M., and the order $n+m$ of the system.*

### 2.3.3 A Canonical Form Under a System Matrix Transformation

Now, we give the conditions under which a canonical form is obtained using a transformation of restricted system equivalence (r.s.e.).

**Theorem 13** *Let $p(s,z)$ a $p \times l$ matrix in state space form (2.13) such that*

$$|sE_1 - A_1| \neq 0 \text{ and } |zE_2 - A_4| \neq 0.$$

*Then $p(s,z)$ is r.s.e. to a canonical system matrix of the form*

$$\begin{bmatrix} sI_r - \overline{A}_1 & 0 & \vdots & -\overline{A}_{21} & -\overline{A}_{22} & \vdots & B_{1s} \\ 0 & I_{n-r} - sJ_1 & \vdots & -\overline{A}_{23} & -\overline{A}_{24} & \vdots & B_{1f} \\ -\overline{A}_{31} & -\overline{A}_{32} & \vdots & zI_t - \overline{A}_4 & 0 & \vdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\overline{A}_{33} & -\overline{A}_{34} & \vdots & 0 & I_{m-t} - zJ_2 & \vdots & B_{2f} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -C_{1s} & -C_{1f} & \vdots & -C_{2s} & -C_{2f} & \vdots & 0 \end{bmatrix} \tag{2.21}$$

*where $\overline{A}_1, \overline{A}_2$ are $r \times r$ and $t \times t$ matrices respectively ($r =$deg $|sE_1 - A_1|$ , $t =$deg $|zE_2 - A_4|$, $J_1$ and $J_2$ are in jordan cnonical form).*

**Remark 12** *If the matrices*

$$M_1 = \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix},$$

*and*

$$M_2 = \begin{bmatrix} zE_2 - A_4 & B_2 \end{bmatrix}$$

*have full rank, $\forall s, z \in \mathbb{C}^2$, then the matrices $\overline{A}_1, \overline{A}_4, B_{1s}$ and $B_{2s}$ in (2.21) can be choosen to be in canonical forms.*

*to obtain these canonical forms and the proof of the previous theorem see Gantmacher [19]. For further results on canonical forms see also [15] and [16].*

### 2.3.4 Controllability and Observability Properties via System Matrix Representations

There exists a very close relation between $1-D$ ( resp.$2-D$ ) polynomial matrices and $1-D$ ( resp.$2-D$ ) state space notions such as controllability, observability, and realizability, etc., see [40].

We note that, as in the case of system matrices over $\mathbb{R}[s]$, the matrix

$$M_c = [sI_n - A(z) \quad B(z)]$$

describes the controllability properties of the system (2.17), whereas the matrix

$$\begin{bmatrix} sI_n - A(z) \\ -C(z) \end{bmatrix},$$

describes the observability properties of the system (2.17). We also can describe the controllability and observability properties of the system (2.19) using the associated system matrices

$$\begin{bmatrix} sI_n - A_1 & -A_2 & \vdots & B_1 \\ \cdots\cdots & \cdots\cdots & \vdots & \cdots \\ -A_3 & zI_m - A_4 & \vdots & B_2 \end{bmatrix}$$

and

$$\begin{bmatrix} sI_n - A_1 & \vdots & -A_2 \\ -A_3 & \vdots & zI_m - A_4 \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ -C_1 & \vdots & -C_2 \end{bmatrix}$$

respectively.

Now, we try to get a companion form matrix representation for the matrix $A(s,z) = sI_n - A(z)$ using a controllability property.

**Theorem 14** *Suppose that the system is $\mathbb{R}^n[z]$ controllable i.e. the following rank condition is satisfied*

$$rank\begin{bmatrix} B(z) & A(z)B(z) & \cdots & A^{n-1}(z)B(z) \end{bmatrix} = n, \forall z \in C,$$

then the matrix

$A(s,z) = sI_n - A(z)$ over $\mathbb{R}[s,z]$ can be transformed by an equivalence transformation into the canonical form (companion form)

$$\widetilde{A}(s,z) = sI_n - C(z)$$

where $C(z)$ is the companion matrix ovre $\mathbb{R}[z]$ given by

$$C(z) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ a_n(z) & a_{n-1}(z) & \cdots & \cdots & a_1(z) \end{bmatrix}$$

whre $a_k(z)$ are the cefficients in the characteristic polynomial of $A(z)$ i.e. $|sI_n - A(z)| = \sum_{k=1}^{n} -a_k(z)\, s^{n-k},\ a_0(z) = 1$

**Proof.** using a characterization result given in [6], and that the fact

$$\left[\ |sI_n - A(z)| \ \vdots \ E_n \ \right]$$

has no zeros, we get that the matrix $\widetilde{A}(s,z) = sI_n - C(z)$, is equivalent to the Smith form

$$S(s,z) = \begin{bmatrix} I_{n-1} & 0 \\ 0 & |sI_n - A(z)| \end{bmatrix}$$

and since, $|sI_n - C(z)| = |sI_n - A(z)|$, then $A(s,z) = sI_n - A(z)$ is equivalent over $R[s,z]$ to $\widetilde{A}(s,z) = sI_n - C(z)$. ■

**Conclusion 3** *In this chapter, we tried to present some types of canonical forms for matrices over the ring of polynomials $\mathbb{R}[s,z]$ (e.g. Smith form), and to generalize a result concerning necessary and sufficient conditions for a matrix over $\mathbb{R}[s,z]$ to be equivalent to a given Smith form. However, the work on canonical forms needs more investigation, especially, in the case of matrices of multivariate polynomials.*

# Chapter 3

# APPLICATION TO A PHYSICAL PROBLEM (ITERATIVE CIRCUITS)

## 3.1 Introduction

In this chapter we apply some of the results obtained previously to the study and development of a 2-D discrete state space model for linear iterative circuits which can be regarded as a generalization of the well known state space model for single dimensional linear time discrete systems (for a comparison see [14]). This development will enclude the definition, formulation of a linear iterative circuit and the derivation of some basic concepts such as the state transition matrix, modal controllability, modal observability etc.
You find in figure (1) bellow a 2-D unilateral linear iterative circuit representaion.

The study of the iterative circuits is limited here to the linear case (i.e. each cell perform a linear transformation), as this allows the use of linear transformation techniques wich considerably facilitate the analysis, design, and implimentation of such circuits. Linear iterative circuits may be used in applications such as encoding, decoding networks for linear codes, and image processing. For some results related to this see e.g. [20], [21], and [22].

## 3.2 Application to Iterative Circuits

### 3.2.1 Definition and Formulation of a Linear Iterative Circuit

1. Let $U(Y, X^h, X^v)$ denote the linear vector space of all primary inputs ( respectively, primary outputs, horizantal states, and vertical states) over a finite field $F$. And $f$ ( respectively, $g$ ) denote the linear transformation for the output total secondary state

$$f : X^h \times X^v \times U \to X^h \times X^v$$

(respectively, for the primary output $g$),

$$g : X^h \times X^v \times U \to Y$$

The six tuple $T = \langle U, Y, X^h, X, f, g \rangle$ is called a 2-D linear iterative circcuit.

2. The state space equations, representing a 2-D linear iterative state space model, are formulated as follows :

$$\begin{cases} X^h(i+1,j) = A_1 X^h(i,j) + A_2 X^v(i,j) + B_1 U(i,j) \\ X^v(i,j+1) = A_3 X^h(i,j) + A_4 X^v(i,j) + B_2 U(i,j) \\ Y(i,j) = C \begin{bmatrix} X^h(i,j) \\ X^v(i,j) \end{bmatrix} \end{cases} \tag{3.1}$$

And a system matrix representation is given by

$$\begin{bmatrix} sI_n & -A_1 & -A_2 & \vdots & B_1 \\ & -A_3 & zI_m - A_4 & \vdots & B_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots \\ & -C_1 & -C_2 & \vdots & 0 \end{bmatrix} \tag{3.2}$$

**Remark 13** *Physically, combinational circuits composed of identical cells that are interconnected in the form of a regular pattern are called iterative circuits. See figure ( 1 ) given bellow.*

### 3.2.2 The General Response Formula and the Transition Matrix

Let $E$ and $F$ be the advance operators defined by

$$X^h(i+1,j) = EX^h(i,j), \text{ and } X^v(i,j+1) = FX^v(i,j)$$

we want to find a closed-form expression for the secondary output $Y(i,j)$ in terms of the inputs to the circuit.

**Proposition 2** *for all $i, j \geq 0$, we have*

$$Y(i,j) = [C_1 \quad C_2] \; X(i,j) + D \; U(i,j)$$

*where*

$$X(i,j) = \begin{bmatrix} X^h(i,j) \\ X^v(i,j) \end{bmatrix}$$

*is the total secondary state into the $(i,j)^{th}$ cell.*

**Proof.** is obvious from the equations of the model. ■

**Definition 20** *Let*

$$A = \begin{bmatrix} A1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

*where $A_i$, $i = \overline{1,4}$, are matrices of appropriate dimensions, and $A$ is then the transformation matrix for the secondary state. We define the transition matrix of a $2-D$ unilateral iterative circuit as the $(i,j)^{th}$ power of $A$ as follows:*

$$\begin{aligned} A^{i,,j} &= 0, \text{ for } \; i < 0 \; \text{ or } j < 0, \quad A^{0,0} = I, \text{ and} \\ A^{i,,j} &= A^{1,0}A^{i-1,,j} + A^{0,1}A^{i,,j-1} \; \text{ for } (i,j) > (0,0)\,. \end{aligned}$$

*where*

$$A^{1,0} = \begin{bmatrix} A1 & A_2 \\ 0 & 0 \end{bmatrix}, \text{ and } \; A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

**Theorem 15** *For all $i, j \geq 0$, we have*

$$X(i,j) = \sum_{k=0}^{j} A^{i,,j-k} \begin{bmatrix} X^h(0,k) \\ 0 \end{bmatrix} + \sum_{r=0}^{i} A^{i-r,,j} \begin{bmatrix} 0 \\ X^v(r,0) \end{bmatrix}$$

$$+\sum_{(0,0)\leq(r,k)\leq(i,j)} \left\{ A^{i-r-1,,j-k} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A^{i-r,,j-k-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} U(r,k) \right\}$$

For a proof see [21].

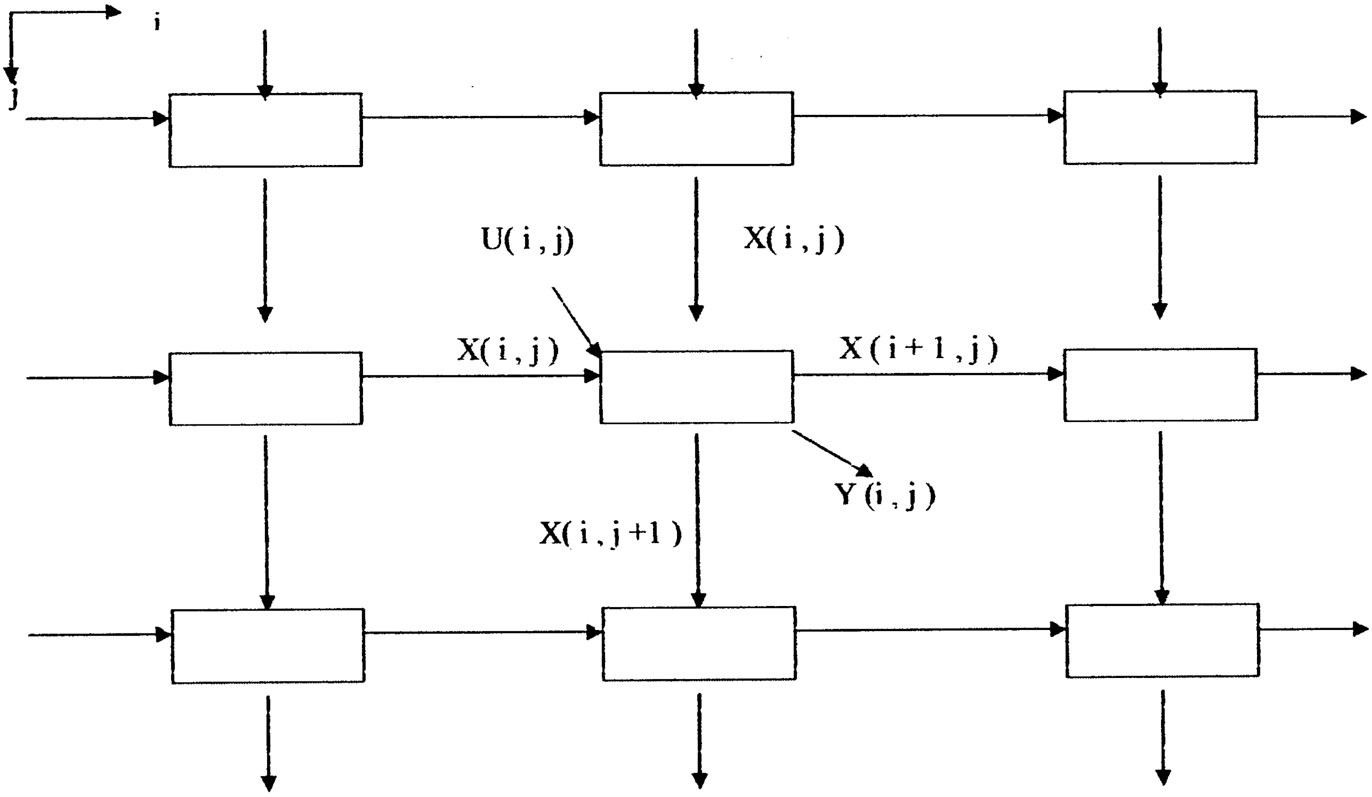


Fig. (1) a 2-D unilateral linear iterative circuit.

### 3.2.3 Modal Controllabilty and Modal Observability

The notions of controllability and observability introduced by Roesser in [39] and which look like as natural generalizations of the 1-D case are in fact just local, they are not closely related to the notions of minimality, they do not yield a canonical decomposition of the state space, and they are not compatible with the introduced class of similarity transformations to reduce the model. So we need to reformulate these notions to get compatible results.

**Remark 14** *In the 1-D case we note that a system which is represented by a state space system matrix of the form*

$$\begin{bmatrix} sI - A & \vdots & B \\ \cdots\cdots & \vdots & \cdots \\ -C & \vdots & 0 \end{bmatrix} \tag{3.3}$$

*is controllable if and only if* $sI - A,\ B$ *are left coprime, and is observable if and only if* $C,\ sI - A$ *are right coprime.*

**Remark 15** *A generalization of this approach to the 2-D case for systems described by system matrices of the form given in (3.2) can be obtained if we adopt the following definition.*

**Definition 21** *A system described by (3.2) is said to be modally controllable (respectively modally observable) if*

$$\begin{bmatrix} sI_n - A_1 & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & zI_m - A_4 \end{bmatrix},\ B \text{ are left coprime.}$$

$$\text{( respectively, } C, \begin{bmatrix} sI_n - A_1 & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & zI_m - A_4 \end{bmatrix}, \text{ are right coprime).}$$

For more results on modal controllability and modal observability, see Kung et all [35] and [29].

### 3.2.4 A Canonical Form Under Strict System Equivalence

In the following theorem we give a more general transformation of strict system equivalence (s.s.e.) which yields a canonical form of the system matrix (3.2). For other results on canonical forms see Boudellioua [6], Frost and Boudellioua [15], [16] and [17].

**Theorem 16** *Let $p(s,z)$ be a $(n+m+1)\times(n+m+1)$ system matrix in the state space form (3.2), having no input decoupling zeros and a transfer function with numerator depending on s only. Then, $p(s,z)$ is s.s.e. to a canonical form system matrix of the following form:*

$$\overline{p}(s,z) = \begin{bmatrix} sI_n & -F_1 & -\overline{A}_2 & \vdots & 0 \\ & -\overline{A}_3 & zI_m - F_4 & \vdots & E_m \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots \\ & -\overline{C}_1 & -\overline{C}_2 & \vdots & 0 \end{bmatrix} \tag{3.4}$$

*where $F_1, F_2$ are respectively, $n\times n$ and $m\times m$ companion matrices, and $\overline{A}_2 = [E_n \;\; 0]$. The elements of $F_1$, $F_4$, and $\overline{A}_3$ are uniquely determined by the characteristic polynomial of the block matrix*

$$A = \begin{bmatrix} A1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

**Proof.** since both $p(s,z)$ and $\overline{p}(s,z)$ have no input decoupling zeros, it follows that they are s.s.e. to the polynomial system matrices

$$\begin{bmatrix} I_{n+m-1} & 0 & \vdots & 0 \\ 0 & d & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -n(s) & \vdots & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} I_{n+m-1} & 0 & \vdots & 0 \\ 0 & \overline{d} & \vdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -\overline{n}(s,z) & \vdots & 0 \end{bmatrix}$$

respectively. And by a suitable choice of $F_1$, $F_4$, and $\overline{A}_3$ we can make $d = \overline{d}$ [6]. Now, we have to show that $n(s) = \overline{n}(s,z)$

Let $\overline{g}(s,z)$ be the transfer function corresponding to $\overline{p}(s,z)$ i.e.

$$\overline{g}(s,z) = [\overline{C}_1 \;\; \overline{C}_2] \begin{bmatrix} sI_n - F_1 & -\overline{A}_2 \\ -\overline{A}_3 & zI_m - F_4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ E_m \end{bmatrix}$$

and using shur's formula and some operations on matrices with

$$\overline{A}_2 = [E_n \quad 0], \text{ and } \overline{A}_3 = [t_1\ t_2\ \cdots\ t_m]^t$$

we get the following

$$\overline{g}(s,z) = [\overline{C}_1 \quad \overline{C}_2] \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \\ p(s) \\ p(s)z - t_1 v_1 \\ \vdots \\ p(s)z^{m-1} - t_1 v_1 z^{m-2} - \cdots t_{m-1} v_1 \end{bmatrix} \times \frac{1}{p(s)\overline{q}(s,z)},$$

and so

$$\overline{g}(s,z) = [\overline{C}_1 \quad \overline{C}_2] \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \\ p(s) \\ p(s)z - t_1 v_1 \\ \vdots \\ p(s)z^{m-1} - t_1 v_1 z^{m-2} - \cdots t_{m-1} v_1 \end{bmatrix}$$

where, $p(s) = |sI_n - F_1\ |$ and $\overline{q}(s,z) = zI - F - Q$ with the matrix $Q$ equals

$$\begin{bmatrix} t_1 v_1 \\ t_2 v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ t_{m-1} v_1 \\ t_m v_1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 \\ s \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ s^{n-2} \\ s^{n-1} \end{bmatrix}$$

Let $\overline{C}_1 = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \cdots & \overline{C}_{1n} \end{bmatrix}$ and $\overline{C}_2 = \begin{bmatrix} \overline{C}_{21} & \overline{C}_{22} & \cdots & \overline{C}_{2m} \end{bmatrix}$, $n(s) = e_n s^n + e_{n-1}s^{n-1} + \cdots e_0$, and $p(s) = s^n + a_1 s^{n-1} + \cdots a_n$. Then $\overline{n}(s,z)$ can be made equal to $n(s)$ by letting

$$\overline{C}_{1i} = e_{i-1} - e_n a_{n-i+1} \ , \overline{C}_{21} = e_n \ \text{ and } \ \overline{C}_{2j} = 0, \ \ i = \overline{1,n} \ \text{ and j} = \overline{2,m} \ .$$

It follows that $p(s,z)$ and $\overline{p}(s,z)$ are s.s.e. ■

### 3.2.5 Extension to the k$^{th}$ Case of the Previous Characterization Result

Now, we give a result concerning matrices in the state space form $A(s,z) = sI_n - A(z)$ and which can be considered as an extension of a result given in [12].

**Theorem 17** *The matrix $A(s,z) = sI_n - A(z)$ is equivalent to the Smith form*

$$S(s,z) = \begin{bmatrix} I_{n-k} & 0 & 0 & \cdots & 0 \\ 0 & d_{n-(k-1)}(s,z) & 0 & \cdots & 0 \\ 0 & 0 & d_{n-(k-2)}(s,z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & d_n(s,z) \end{bmatrix}, \ k \in N^* \qquad (3.5)$$

*iff the matrix $A(z)$ is simialr over $R[z]$ to the block companion matrix*

$$C(z) = \begin{bmatrix} C_1(z) & 0 & 0 & \cdots & 0 \\ 0 & C_2(z) & 0 & \cdots & 0 \\ 0 & 0 & C_3(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C_k(z) \end{bmatrix} \tag{3.6}$$

*where $C_i(z)$ are $n_i \times n_i$ $\left(i = \overline{1,k}\right)$ companion matrices having characteristic polynomials $d_{n-(k-i)}(s,z)$, $\left(i = \overline{1,k}\right)$ i.e.$|sI_n - C(z)| = d_{n-(k-i)}(s,z)$, where $n_i$ $\left(i = \overline{1,k}\right)$ are given by the degrees in $s$ of $d_{n-(k-i)}(s,z)$, $\left(i = \overline{1,k}\right)$*

**Proof.** To show the necessity, we suppose that $A(s,z) = sI_n - A(z)$ is equivalent over $R[s,z]$ to $S(s,z)$ in (3.5). Then by elementary rows and columns operations on $S(s,z)$, we get $S(s,z)$ equivalent to

$$S_k(s,z) = \begin{bmatrix} I_{n_1-1} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & d_{n-(k-1)} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I_{n_2-1} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & d_{n-(k-2)} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & I_{n_3-1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{n_k-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & d_n \end{bmatrix} \tag{3.7}$$

where $d_{n-(k-i)}$ denotes $d_{n-(k-i)}(s,z)$, $1 \leq i \leq k$. Since these operations on a matrix preserve equivalence, then $A(s,z)$ is equivalent to $S_k(s,z)$ in (3.7). And since the matrix $S_k(s,z)$ in (3.7) is clearly equivalent to the block companion matrix

$$sI_n - C(z) = \begin{bmatrix} sI_{n_1} - C_1(z) & 0 & \cdots & 0 \\ 0 & sI_{n_2} - C_2(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & sI_{n_k} - C_k(z) \end{bmatrix} \tag{3.8}$$

where $C_i(z)$, are $n_i \times n_i$ $\left(i = \overline{1,k}\right)$ are square matrices in companion form respectively, such that

$$det\left(sI_{ni} - C_i(z)\right) = d_{n-(k-i)}(s,z), \quad i = \overline{1,k} \quad (k \geq 2).$$

Hence, the matix $A(s,z) = sI_n - A(z)$ is equivalent to $sI_n - C(z)$, and since this equivalence transformation (between these system matrices) can be replaced by a similarity transformation, then $A(z)$ is similar to

$$C(z) = \begin{bmatrix} C_1(z) & 0 & \cdots & 0 \\ 0 & C_2(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_k(z) \end{bmatrix} \quad (3.9)$$

this ends the proof of necessity.

The proof of sufficiency: if we assume that $A(z)$ is similar to $C(z)$ in (3.9), then $sI_n - A(z)$ is also similar to $sI_n - C(z)$ in (3.8), and since this last matrix is in companion form and is equivalent to its Smith form

$$S(s,z) = \begin{bmatrix} I_{n-k} & 0 & \cdots & 0 \\ 0 & d_{n-(k-1)}(s,z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n(s,z) \end{bmatrix}$$

then, so it is $sI_n - A(z)$, which ends the proof of the theorem.

**Example 3** *Now, as an illustrative example, for* $n = 2k$, $k \in N^*$, *let* $S(s,z)$ *be the matrix of the form*

$$S(s,z) = \begin{bmatrix} I_{2k-k} & 0 & \cdots & 0 \\ 0 & d_{2k-(k-1)}(s,z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{2k}(s,z) \end{bmatrix}. \quad (3.10)$$

*By* $(k-1)$ *operations on the rows (respectively columns) of the matrix* $S(s,z)$ *in* (3.10), *we* *obtain*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d_{2k-(k-1)} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{2k-(k-2)} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d_{2k-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & d_{2k} \end{bmatrix}$$

*where $d_{2k-i}$ denote $d_{2k-i}(s,z)$, $i = \overline{0, k-1}$, and in the above case we have*

$I_{n_1-1} = [1]$ ; $I_{n_2-1} = [1]$ ; ...; $I_{n_k-1} = [1]$. i.e. $I_{n_i}$ $\left(i = \overline{1,k}\right)$ are all matrices of one element equals 1.

We note that the above resut is a generalization of a previous result given in [12] for the case of $k = 3$. ■

**Remark 16** *In the above work, a companion form for a matrix of the form $A(s,z) = sI_n - A(z)$, which arises in the study of e.g. retarded delay differential equations was presented, and a characterization result concerning these matrices was extended to a more general case (the $k^{th}$ case). A similar study can be investigated for more general matrices $A(s,z)$ arising from singular retarded delay differential equations.*

**Conclusion 4** *In this chapter, the state space equations representing a 2-D discrete state space model describing a linear iterative circuit have been given. Some basic concepts such as, state space transition matrix, modal controllability, and modal observablity are derived. Also a 2-D system matrix represenrting this model is given and a canonical form under a more general transformation of strict system equivalence (s.s.e.) is obtained. For further research work on this model, we can consider the extension of the above results to polynomial rings of more then two varibles s and z e.g. to the $N-D$ case. We also note that the problem related to the notions of modal controllability and modal observabilty needs to be more investigated.*

## 3.3 A Canonical Form Under the Multidimensional Laplace Transform

**Definition 22** *We define the multidimensional Laplace transform by*

$$\begin{aligned} \pounds_m [Y(t)] &= \int_0^\infty \cdots \int_0^\infty Y(t_1, t_2, \cdots, t_m) e^{-s_1 t_1 - s_2 t_2 \cdots - s_m t_m} dt_1 dt_2 \cdots dt_m \\ &= Y(s_1, s_2, \cdots, s_m), \end{aligned} \tag{3.11}$$

*where* $s_k = \alpha_k + i\beta_k \in C$, $(k = \overline{1,m})$, $Y(t) \in S$, *where* $S$ *denotes a certain class of functions for which the integral in (3.11) exists.*

### 3.3.1 System Matrices Assosciated to High Order Partial Differential Equations

In the following we try to show over an example how the use of the multidimensional Laplace transform can give rise to system matrices of the form in (3) above.

The following example shows how the use of Laplace transform given in (3.11) ($m = 2$) on the following partial differential equation (p.d.e) yields a matrix of the form given in (2.12). Consider the wave equation

$$\frac{\partial^2}{\partial^2 t_1} - \frac{\partial^2}{\partial^2 t_2} = U(t_1, t_2), \quad 0 < t_1 < \infty, \quad 0 < t_2 < \infty \tag{3.12}$$

with the boundary conditions

$$\begin{cases} Y(0, t_2) = a(t_2) & ; \quad \pounds_2 [a(t_2)] = \hat{a}(s_2) \\ Y(t_1, 0) = b(t_1) & ; \quad \pounds_2 [b(t_1)] = \hat{b}(s_1) \\ \frac{\partial Y}{\partial t_1} \mid_{t_1=0} = c(t_2) & ; \quad \pounds_2 [c(t_2)] = \hat{c}(s_2) \\ \frac{\partial Y}{\partial t_2} \mid_{t_2=0} = d(t_1) & ; \quad \pounds_2 [d(t_1)] = \hat{d}(s_1) \end{cases} \tag{3.13}$$

since

$$\pounds_2 \left[ \frac{\partial^2 Y}{\partial t_1^2} \right] = s_1^2 Y(s_1, s_2) - s_1 Y(0, s_2) - Y_{t_1}(0, s_2)$$

and

$$\pounds_2 \left[ \frac{\partial^2 Y}{\partial t_2^2} \right] = s_2^2 Y(s_1, s_2) - s_2 Y(s_1, 0) - Y_{t_2}(s_1, 0).$$

Then by taking Laplace transform of both sides in (3.12), replacing the above two equalities and using (3.13) we get

$$Y(s_1,s_2)=\frac{1}{s_1^2-s_2^2}U(s_1,s_2)+\frac{\hat{c}(s_2)-\hat{d}(s_1)+s_1\hat{a}(s_2)-s_2\hat{b}(s_1)}{s_1^2-s_2^2} \tag{3.14}$$

Now we have to give a realization of the transfer function $\frac{1}{s_1^2-s_2^2}$ in order to obtain a state space representation in the form (2.12). To this end, we follow the 2-step realization procedure given by Zak [46]:

The first level realization of $g(s,z)=\frac{1}{s_1^2-s_2^2}$ has the form

$$A(s_2)=\begin{bmatrix}0 & 1\\ s_2^2 & 0\end{bmatrix},\quad B(s_2)=\begin{bmatrix}1\\ 0\end{bmatrix},\quad C(s_2)=\begin{bmatrix}1 & 0\end{bmatrix},\quad D(s_2)=0$$

which yields

$$g(s,z)=C(s_2)\left[s_1I-A(s_2)\right]^{-1}B(s_2)+D(s_2)$$

where $A(s_2)$, $B(s_2)$, $C(s_2)$ and $D(s_2)$ can be regarded as 1-D non proper, in general, transfer matrices in themselves. So by realizing each of them following the 2-level realization procedure we get

$$\begin{aligned}
A(s_2) &= C_1^A\left[s_2I_{n_A}-A_1^A\right]^{-1}B_1^A+C_2^A\left[s_2J^A-I\right]^{-1}B_2^A+D^A,\\
B(s_2) &= C_1^B\left[s_2I_{n_B}-A_1^B\right]^{-1}B_1^B+C_2^B\left[s_2J^B-I\right]^{-1}B_2^B+D^B,\\
C(s_2) &= C_1^C\left[s_2I_{n_C}-A_1^C\right]^{-1}B_1^C+C_2^C\left[s_2J^C-I\right]^{-1}B_2^C+D^C,\\
D(s_2) &= C_1^D\left[s_2I_{n_D}-A_1^D\right]^{-1}B_1^D+C_2^D\left[s_2J^D-I\right]^{-1}B_2^D+D^D,
\end{aligned}$$

and since

$$A(s_2)=\begin{bmatrix}0 & 1\\ s_2^2 & 0\end{bmatrix}\in R^{2\times2}\left[s_2\right],$$

then

$$A(s_2)=C_2^A\left[s_2J^A-I\right]^{-1}B_2^A+D^A.$$

We can choose

$$C_2^A=\begin{bmatrix}0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix},\quad B_2^A=\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix},$$

and

$$D^A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This yields

$$A(s_2) = \begin{bmatrix} 0 & 1 \\ s_2^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & s_2 & 0 \\ 0 & -1 & s_2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

And so a state representation is given by

$$\begin{cases} \begin{bmatrix} I_2 & 0 \\ 0 & J^A \end{bmatrix} \begin{bmatrix} s_1 & X^1 \\ s_2 & X^2 \end{bmatrix} = \begin{bmatrix} D^A & C_2^A \\ B_2^A & I_3 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} + \begin{bmatrix} D^B \\ 0 \end{bmatrix} U \\ Y = \begin{bmatrix} D^C & \vdots & 0 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} \end{cases} \quad (3.15)$$

with $E_1 = I_2$ and $E_2 = J^A$ (Jordan form).

Note that the system matrix associated to the system in (3.15) has the same form as the matrix given in (2.12). If we now assume that the initial conditions have the form

$$\begin{bmatrix} \hat{a}(s_2) \\ \hat{b}(s_2) \\ -\hat{c}(s_1) \\ -\hat{d}(s_1) \\ 0 \end{bmatrix}. \quad (3.16)$$

Then the 2-D Laplace transform relating $Y(s_1, s_2)$ and $U(s_1, s_2)$ with the initial conditions (3.16), caculated from (3.15), has the form given in (3.14). (note that the order of this realization is 5).

### 3.3.2 Restricted System Equivalence Transformation (For System Matrices Assosciated to High Order Partial Differential Equations)

**Definition 23** *Two system matrices of the form (2.13) are said to be restricted system equivalent if they are related by the transformation of the type*

$$\begin{bmatrix} M_1 & 0 & \vdots & 0 \\ 0 & M_2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & I_p \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zE_2 - A_4 & \vdots & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} N_1 & 0 & \vdots & 0 \\ 0 & N_2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & I_l \end{bmatrix} = \begin{bmatrix} sE_1' - A_1' & -A_2' & \vdots & B_1' \\ -A_3' & zE_2' - A_4' & \vdots & B_2' \\ \cdots & \cdots & \cdots & \cdots \\ -C_1' & -C_2' & \vdots & 0 \end{bmatrix} \tag{3.17}$$

*where $M_1, M_2, N_1,$ and $N_2$ are matrices of appropriate dimensions.*

1. The transformation in (2.20) is a special case of strict -system equivalence (s.s.e.).

2. If $E_1$ and $E_2$ in (2.20) are singular, then $p(s,z)$ is restricted system equivalent (r.s.e.) to a system matrix $p'(s,z)$ of the form

$$p'(s,z) = \begin{bmatrix} sI_n - A_1 & -A_2 & \vdots & B_1 \\ -A_3 & zI_m - A_4 & \vdots & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ -C_1 & -C_2 & \vdots & 0 \end{bmatrix}. \tag{3.18}$$

   We note that this type of matrix arises in the state space model used by Givon-Roesser [21] in describing 2-D discrete systems.

3. This restricted system equivalence transformation preserves the $p \times l$ rational transfer

function matrix given by

$$G(s,z) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -A_2 \\ -A_3 & zE_2 - A_4 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

Now, we give the conditions under which a canonical form is obtained using a transformation of restricted system equivalence (r.s.e.).

**Theorem 18** *Let $p(s,z)$ a $p \times l$ matrix in state space form (2.13) such that*

$$|sE_1 - A_1| \neq 0 \text{ and } |zE_2 - A_4| \neq 0.$$

*Then, $p(s,z)$ is r.s.e. to a canonical system matrix of the form*

$$\begin{bmatrix} sI_r - \overline{A}_1 & 0 & \vdots & -\overline{A}_{21} & -\overline{A}_{22} & \vdots & B_{1s} \\ 0 & I_{n-r} - sJ_1 & \vdots & -\overline{A}_{23} & -\overline{A}_{24} & \vdots & B_{1f} \\ -\overline{A}_{31} & -\overline{A}_{32} & \vdots & zI_t - \overline{A}_4 & 0 & \vdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\overline{A}_{33} & -\overline{A}_{34} & \vdots & 0 & I_{m-t} - zJ_2 & \vdots & B_{2f} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -C_{1s} & -C_{1f} & \vdots & -C_{2s} & -C_{2f} & \vdots & 0 \end{bmatrix} \tag{3.19}$$

*where $\overline{A}_1, \overline{A}_2$ are $r \times r$ and $t \times t$ matrices respectively ($r =$deg $|sE_1 - A_1|$ , $t =$deg $|zE_2 - A_4|$, $J_1$ and $J_2$ are in jordan cnonical form).*

**Remark 17** *If the matrices $\begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix}$ and $\begin{bmatrix} zE_2 - A_4 & B_2 \end{bmatrix}$ have full rank, $\forall (s,z) \in \mathbb{C}^2$, then the matrices $\overline{A}_1, \overline{A}_4, B_{1s}$ and $B_{2s}$ in (3.19) can be choosen to be in canonical forms. To obtain these canonical forms and the proof of the previous theorem see Gantmacher [19]. For further results on canonical forms see also [15] and [16].*

### 3.3.3 The Realization Problem of a Non-Proper 2-D Transfer Function

For a proper rational transfer function in two indeterminates there are available algorithms which provide low orders minimal realizations. For the 2-D non proper transfer functions, several authors such as, Kung and al [29] and [35], Sontag [43] have suggested canonical realizations for a special class of 2-D transfer functions (e.g. with separable denominator, etc.).

Recently, Boudellioua [6] solved this problem for a non-proper transfer function which has a denominator depends only on one variable. Here we gave the solution of the realization problem of a non-proper $2-D$ transfer function in the form

$$g(s,z)=\frac{\overline{n}(s,z)}{d(s,z)} \tag{3.20}$$

for which

$$\overline{n}(s,z)=r(s,z)d(s,z)+n(s),$$

where

$$d(s,z)=k_0(s)z^m+k_1(s)z^{m-1}+\cdots+k_m(s)$$

($k_0(s)$ is monic and deg $k_0(s)=n$, deg $k_j(s)\leq n,\ \ j=1,2,\cdots,m$), and

$$n(s)=e_ns^n+e_{n-1}s^{n-1}+\cdots+e_0$$

are factor coprime, and

$$r(s,z)=r_{q+1}(s)z^q+r_q(s)z^{q-1}+\cdots+r_1(s)$$

where

$$r_i(s)=\sum_{j=1}^{l+1}w_{ij}s^{l-j+1}, \qquad i=1,2,\cdots,q+1$$

and $l=\deg_s r(s,z)$.

Now, since $g(s,z)$ can be written in the form

$$g(s,z)=g_1(s,z)+r(s,z)$$

where

$$g_1(s,z)=\frac{n(s)}{d(s,z)}.$$

The realization of $g_1(s,z)$ was given by Boudellioua [6] and it has the form

$$\begin{bmatrix} sI_n-F_1 & -\overline{A}_2 & \vdots & 0 \\ -\overline{A}_3 & zI_m-F_4 & \vdots & E_m \\ \cdots & \cdots & \vdots & \cdots \\ -\overline{C}_1 & -\overline{C}_2 & \vdots & 0 \end{bmatrix} \tag{3.21}$$

where $F_1$ and $F_2$ are respectively $n \times n$ and $m \times m$ companion matrices and $\overline{A}_2 = \begin{bmatrix} E_n & 0 \end{bmatrix}$, the elements of $F_1$ , $F_2$ and $\overline{A}_3$ are uniquely determined by the 2-D characteristic polynomial of $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$.

For the realization of $r(s,z)$ (which is a polynomial of the ring $R[s][z]$ ) we can use Jordan block matrices, and so it can be verified that a realization of $r(s,z)$ is given by

$$\begin{bmatrix} I_{l+1} - sJ_1 & 0 & \vdots & E_{l+1} \\ -\overline{w} & I_{q+1} - zJ_2 & \vdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ 0 & -E_1^t & \vdots & 0 \end{bmatrix} \tag{3.22}$$

where, $\overline{w} = (w_{ij})$ , $1 \leq i \leq q+1, \ 1 \leq j \leq l+1$.

Combining the realizations in (3.21) and (3.22) we obtain a realization of $g(s,z)$ which is given by the system matrix

$$\begin{bmatrix} sI_n - F_1 & -\overline{A}_2 & 0 & 0 & \vdots & 0 \\ -\overline{A}_3 & zI_m - F_4 & 0 & 0 & \vdots & E_m \\ 0 & 0 & I_{l+1} - sJ_1 & 0 & \vdots & E_{l+1} \\ 0 & 0 & -\overline{w} & I_{q+1} - zJ_2 & \vdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ -\overline{C}_1 & -\overline{C}_2 & 0 & -E_1^t & \vdots & 0 \end{bmatrix} \tag{3.23}$$

where $F_1, F_4, \overline{A}_2, \overline{A}_3, \overline{C}_1$, and $\overline{C}_2$ are the matrices obtained in the realization of the 2-D proper transfer function $g_0(s,z)$.

Now, by elementary row and column operations on the system matrix (3.23) we get the system matrix

$$\begin{bmatrix} sI_n - F_1 & 0 & -\overline{A}_2 & 0 & \vdots & 0 \\ 0 & I_{l+1} - sJ_1 & 0 & 0 & \vdots & E_{l+1} \\ -\overline{A}_3 & 0 & zI_m - F_4 & 0 & \vdots & E_m \\ 0 & -\overline{w} & 0 & I_{q+1} - zJ_2 & \vdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots \\ -\overline{C}_1 & 0 & -\overline{C}_2 & -E_1^t & \vdots & 0 \end{bmatrix} \tag{3.24}$$

which is in the required canonical form (3.19).

# Chapter 4

# A CONTROL WITH THE INITIAL VALUE OF AN ILL-POSED PROBLEM ( INFINITE DIMENSIONAL CASE)

## 4.1 Introduction

We consider the following final value problem (F.V.P.)

$$u'(t) + Au(t) = 0, \qquad 0 < t < T \tag{4.1}$$

$$u(T) = f \tag{4.2}$$

for some prescribed final value $f$ in a Hilbert space H. $A$ is a positive self-adjoint operator such that $0 \in \rho(A)$. Such problems are not well posed, that is, even if a unique solution exists on $[0, T]$ it need not depend continuously on the final value $f$. We note that this type of problems has been considered by many authors, using different approaches. Such authors as Lattes and Lions [31], Miller [32], and Showalter [42] have approximated (F.V.P.) by perturbing the operator $A$.

In [1], [10], and [41] a similar problem is treated in a different way. By perturbing the final value condition, they approximate the problem (4.1), (4.2), with

$$\begin{aligned} u'(t) + Au(t) &= 0, \ 0 < t < T, \\ u(T) + \alpha u(0) &= f. \end{aligned} \tag{4.3}$$

A similar approach known as the method of auxiliary boundary conditions was given in [33]. Also, we have to mention that the non standard conditions of the form (4.3) for parabolic equations have been considered in some recent papers [2], [3].

In this work, we perturb the final condition (4.2) to form an approximate non local problem depending on a small parameter, with boundary condition containing a derivative of the same order than the equation, as follows:

$$\begin{gathered} u'(t) + Au(t) = 0, \ 0 \le t \le T, \\ u(T) - \alpha u'(0) = f. \end{gathered} \tag{4.4}$$

Following [10], this method is called quasi-boundary value method, and the related approximate problem is called quasi-boundary value problem (Q.B.V.P.). We show that the approximate problems are well posed and that their solutions $u_\alpha$ converge in $C^1([0,T],H)$ if and only if the original problem has a classical solution. We prove that this method gives a better approximation than many other quasi reversibility type methods e.g. [1], [10] and [31]. Finally, we obtain several other results, including some explicit convergence rates.

## 4.2 Preliminary Notions

This section is devoted to definitions of some of the basic concepts related to linear operators in Hilbert spaces, mainly spectral resolution and spectral representation for a self-adjoint unbounded operator on a Hilbert space $H$ and includes a brief account of some fundamental properties and results concerning them.

### 4.2.1 Spectral Resolution

**Definition 24** *A spectral resolution* $\{E_\lambda\}$ *on a Hilbert space* $H$ *is a fuction*

$$E : \mathbb{R} \to B(H)$$

*having the following properties:*

1. $E(\lambda)$ is an orthogonal projection for every $\lambda \in \mathbb{R}$,
2. $E(\lambda) \leq E(\mu)$ for $\lambda \leq \mu$ (monotonicity),
3. $E(\lambda + \varepsilon) \to E(\lambda)$ for all $\lambda \in \mathbb{R}$, as $\varepsilon \to 0^+$ (right continuity),
4. $E(\lambda) \to 0$ as $\lambda \to -\infty$, and $E(\lambda) \to I$ as $\lambda \to +\infty$.

In the following, we give some fundamental properties concerning the spectral resolution $\{E_\lambda\}$:

- $E_\mu E_\lambda = E_\lambda$, if $\mu \geq \lambda$,
- If $TA = AT$, for $T \in B(H)$, then, $TE_\lambda = E_\lambda T$, and $E_\lambda A = AE_\lambda$,
- $E_\lambda^2 = E_\lambda$.

### 4.2.2 Spectral Decomposition for an Unbounded Operator

**Definition 25** *Let* $A$ *be a self-adjoint unbounded operator on a Hilbert space* $H$*. The following representation of* $A$

$$A = \int_{-\infty}^{+\infty} \lambda dE_\lambda,$$

*is known as the spectral decomposition of* $A$ *relatively to the spectral resolution* $\{E_\lambda\}$.

We note that there exists a unique spectral resolution $\{E_\lambda\}$ for which the operator $A$ is represented in the above form. We also have the following spectral decompositions:

$$Af = \int_{-\infty}^{+\infty} \lambda dE_\lambda f, \ \forall f \in D(A),$$

where the domain of the operator $A$ is given by

$$D(A) = \{f \in H : \|Af\| < \infty\},$$

and

$$\|Af\|^2 = \int_{-\infty}^{+\infty} \lambda^2 d\|E_\lambda f\|^2.$$

We also have

$$\langle Af, g\rangle = \int_{-\infty}^{+\infty} \lambda d \langle E_\lambda f, g\rangle, \ \forall f, g \in H.$$

## 4.3 An Abstract Ill-Posed Parabolic Problem

### 4.3.1 The Resolution and Bound Estimates of the Approximate Problem

**Definition 26** *A function $u : [0, T] \to H$ is called a classical solution of the (F.V.P.) problem (4.1),(4.2) (respectively Q.B.V.P. (4.1),(4.4)) if $u \in C^1([0, T], H)$, $u(t) \in D(A)$ for every $t \in [0, T]$ and satisfies the equation (4.1) and the final condition(4.2) ( respectively the boundary condition(4.4)).*

Now, let $\{E_\lambda\}_{\lambda>0}$ be a spectral resolution (spectral measure)associated to the operator $A$ in the Hilbert space $H$, then for all $f \in H$, we can write

$$f = \int_0^\infty dE_\lambda f \tag{4.5}$$

If the problem (F.V.P.) (4.1), (4.2) (respectively (Q.B.V.P.) (4.1), (4.4)) admits a solution $u$ (respectively $u_\alpha$), then this solution can be represented by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f, \tag{4.6}$$

respectively,

$$u_\alpha(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \tag{4.7}$$

**Theorem 19** *For all $f \in H$, the functions $u_\alpha$ given by (4.7) are classical solutions to the (Q.B.V.P.) (4.1), (4.4) and we have the following estimate*

$$\|u_\alpha(t)\| \le \frac{T}{\alpha\left(1+\ln\frac{T}{\alpha}\right)}\|f\|, \ \forall t \in [0,T], \tag{4.8}$$

*where $\alpha < eT$.*

**Proof.** If we assume that the functions $u_\alpha$ given in (4.7) are defined for all $t \in [0,T]$, then, it is easy to show that $u_\alpha \in C^1([0,T],H)$ and

$$u'_\alpha(t) = \int_0^\infty \frac{-\lambda e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f. \tag{4.9}$$

Since

$$Au_\alpha(t) = \int_0^\infty \frac{\lambda e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f, \tag{4.10}$$

then,

$$\begin{aligned} \|Au_\alpha(t)\|^2 &= \int_0^\infty \frac{\lambda^2 e^{-2\lambda t}}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \\ &\le \frac{1}{\alpha^2}\int_0^\infty d\|E_\lambda f\|^2 = \frac{1}{\alpha^2}\|f\|^2, \end{aligned}$$

and this shows that $u_\alpha(t) \in D(A)$ and so $u_\alpha \in C([0,T], D(A))$. From (4.9) and (4.10) we see that function $u_\alpha$ given in (4.7) is a classical solution to the (Q.B.V.P.) problem (4.1), (4.4). Now, using (4.7) we have

$$\|u_\alpha(t)\|^2 \le \int_0^\infty \frac{1}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2, \tag{4.11}$$

if we put

$$h(\lambda) = \left(\alpha\lambda + e^{-\lambda T}\right)^{-1}, \text{ for } \lambda > 0,$$

then,

$$\sup_{\lambda>0} h(\lambda) = h\left(\frac{\ln(\frac{T}{\alpha})}{T}\right), \tag{4.12}$$

and this yields

$$\begin{aligned} \|u_\alpha(t)\|^2 &\le \left[\frac{T}{\alpha\left(1+\ln(\frac{T}{\alpha})\right)}\right]^2 \int_0^\infty d\|E_\lambda f\|^2 \\ &= \left[\frac{T}{\alpha\left(1+\ln(\frac{T}{\alpha})\right)}\right]^2 \|f\|^2. \end{aligned}$$

This shows that the integral defining $u_\alpha(t)$ exists for all $t \in [0,T]$ and we have the desired estimate. ■

**Remark 18** *One advantage of this method of regularization is that the order of the error, introduced by small changes in the final value $f$, is less than the order given in [10].*

### 4.3.2 Some Convergence Results

Now, we give the following convergence result

**Theorem 20** *For every $f \in H, \|u_\alpha(T) - f\|$ tends to zero as $\alpha$ tends to zero. That is $u_\alpha(T)$ converges to $f$ in $H$.*

**Proof.** Let $\varepsilon > 0$, choose $\eta > 0$ for which

$$\int_\eta^\infty d\|E_\lambda f\|^2 < \frac{\varepsilon}{2}.$$

From (4.7), we have

$$\|u_\alpha(T) - f\|^2 = \int_0^\infty \frac{\alpha^2\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2,$$

then,

$$\|u_\alpha(T) - f\|^2 \leq \alpha^2 \int_0^\eta \frac{\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2},$$

so by choosing $\alpha$ such that

$$\alpha^2 < \varepsilon \left( 2 \int_0^\eta \lambda^2 e^{2\lambda T} \|E_\lambda f\|^2 \right)^{-1},$$

we obtain the proof of the theorem. ■

**Theorem 21** *For every $f \in H$, the (FVP) problem (4.1), (4.2) has a classical solution $u$ given by (4.6), if and only if the sequence $(u'_\alpha(0))_{\alpha>0}$ converge in $H$. Furthermore, we then have that $u_\alpha(t)$ converges to $u(t)$ as $\alpha$ tends to zero in $C^1([0,T], H)$.*

**Proof.** If we assume that the (F.V.P.) problem (4.1), (4.2) has a classical solution $u$, then we have

$$\begin{aligned} \|u'_\alpha(0) - u'(0)\|^2 &= \int_0^\infty \frac{\alpha^2\lambda^4 e^{2\lambda T}}{(\alpha\lambda + e^{-\lambda T})^2}\|dE_\lambda f\|^2 \\ &\leq \alpha^2 \int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2 + \int_\eta^\infty \frac{\alpha^2\lambda^4 e^{2\lambda T}}{\alpha^2\lambda^2} d\|E_\lambda f\|^2 \\ &< \alpha^2 \int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2}, \end{aligned}$$

so by choosing $\alpha$ such that $\alpha^2 < \varepsilon \left(2\int_0^\eta \lambda^4 e^{4\lambda T} d\|E_\lambda f\|^2\right)^{-1}$, we obtain

$$\|u'_\alpha(0) - u'(0)\|^2 < \varepsilon,$$

this shows that $\|u'_\alpha(0) - u'(0)\|$ tends to zero as $\alpha$ tends to zero. Since

$$\begin{aligned} \|u'_\alpha(t) - u'(t)\|^2 &\leq \int_0^\infty \lambda^2 \left(\frac{1}{\alpha\lambda + e^{-\lambda T}} - e^{\lambda T}\right)^2 d\|E_\lambda f\|^2 \\ &= \|u'_\alpha(0) - u'(0)\|^2, \end{aligned}$$

then $\|u'_\alpha(t) - u'(t)\|$ tends to zero as $\alpha$ tends to zero uniformly in $t$, for every $t \in [0, T]$.

Now, we show that $\|u_\alpha(t) - u(t)\|$ tends to zero as $\alpha$ tends to zero uniformly in t, for every $t \in [0, T]$. To this end, let $t \in [0, T]$, and since

$$\begin{aligned} \|u_\alpha(t) - u(t)\|^2 &\leq \int_0^\infty \frac{\alpha^2\lambda^2 e^{2\lambda T}}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \\ &= \|u_\alpha(0) - u(0)\|^2. \end{aligned}$$

So, it is sufficient to show that $u_\alpha(0)$ converges to $u(0)$ as $\alpha$ tends to zero. To this end, we compute

$$\begin{aligned} \|u_\alpha(0) - u(0)\|^2 &\leq \alpha^2 \int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2 + \int_\eta^\infty e^{2\lambda T} d\|E_\lambda f\|^2 \\ &\leq \alpha^2 \int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2 + \int_\eta^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2, \end{aligned}$$

for $\eta$ quite large, and since $u(0) \in D(A)$, then we have

$$\|u_\alpha(0) - u(0)\|^2 \leq \alpha^2 \int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2 + \frac{\varepsilon}{2},$$

and so by choosing $\alpha$ such that $\alpha^2 < \left(2\int_0^\eta \lambda^2 e^{4\lambda T} d\|E_\lambda f\|^2\right)^{-1}$ we get

$$\|u_\alpha(0) - u(0)\|^2 < \varepsilon.$$

Thus $u_\alpha(0)$ converges to $u(0)$ as $\alpha$ tends to zėro, which in turn gives that $u_\alpha(t)$ converges to $u(t)$ as $\alpha$ tends to zero, uniformly in t, for every $t \in [0, T]$. Combining all these convergence results, we conclude that $u_\alpha(t)$ converges to $u(t)$ in $C^1([0, T], H)$.

Now, assume that $(u'_\alpha(0))_{\alpha>0}$ converges in $H$. Since

$$u_\alpha(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f,$$

is a classical solution to the (Q.B.V.P.) problem (4.1), (4.4), then we have

$$u'_\alpha(t) = \int_0^\infty \frac{-\lambda e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f,$$

hence

$$\|u'_\alpha(0)\|^2 = \int_0^\infty \frac{\lambda^2}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2.$$

Now, using the dominated convergence theorem we get

$$\|\lim_{\alpha\downarrow 0} u'_\alpha(0)\|^2 = \int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2,$$

and so it is easy to see that the function $u(t)$ defined by

$$u(t) = \int_0^\infty e^{\lambda(T-t)} dE_\lambda f,$$

is a classical solution to the (F.V.P.) problem (4.1), (4.2). This ends the proof of the theorem. ■

### 4.3.3 A Comparison of the Error Estimates

**Theorem 22** *If the function u given by (4.6) is a classical solution of the (F.V.P.) problem (4.1), (4.1), and $u_\alpha^\delta$ is a solution of the (Q.B.V.P.) problem (4.1), (4.4) for $f \approx f_\delta$, such that $\|f - f_\delta\| < \delta$, then we have*

$$\|u(0) - u_\alpha^\delta(0)\| \le c\left(1 + \ln\frac{T}{\delta}\right)^{-1}, \tag{4.13}$$

*where $c = T(1 + \|Au(0)\|)$.*

**Proof.** Suppose that the function $u$ given by (4.6) is a classical solution to the (F.V.P.) problem (4.1), (4.2), and let's denote by $u_\alpha^\delta$ a solution of the (Q.B.V.P.) problem (4.1), (4.4) for $f \approx f_\delta$, such that

$$\|f - f_\delta\| < \delta.$$

Then, $u_\alpha^\delta(t)$ is given by

$$u_\alpha^\delta(t) = \int_0^\infty \frac{e^{-\lambda t}}{\alpha\lambda + e^{-\lambda T}} dE_\lambda f_\delta, \ \forall t \in [0, T], \tag{4.14}$$

where $f_\delta = \int_0^\infty dE_\lambda f_\delta$. From (4.6) and (4.14), we have

$$\|u(0) - u_\alpha^\delta(0)\| \leq \Delta_1 + \Delta_2,$$

where $\Delta_1 = \|u(0) - u_\alpha(0)\|$, and $\Delta_2 = \|u_\alpha(0) - u_\alpha^\delta(0)\|$. Using (4.12), we get

$$\Delta_1 \leq \frac{T}{(1 + \ln(\frac{T}{\alpha}))} \left( \int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2 \right)^{\frac{1}{2}},$$

and

$$\Delta_2 \leq \frac{T\delta}{\alpha\left(1 + \ln\frac{T}{\alpha}\right)},$$

then,

$$\Delta_1 \leq \frac{T\|Au(0)\|}{1 + \ln\frac{T}{\alpha}}, \tag{4.15}$$

and

$$\Delta_2 \leq \frac{T\delta}{\alpha\left(1 + \ln\frac{T}{\alpha}\right)}. \tag{4.16}$$

From (4.15) and (4.16), we obtain

$$\|u_\alpha(0) - u_\alpha^\delta(0)\|^2 \leq \frac{T\|Au(0)\|}{\left(1 + \ln\frac{T}{\alpha}\right)} + \frac{T\delta}{\alpha\left(1 + \ln\frac{T}{\alpha}\right)},$$

then, for the choice $\alpha = \delta$, we get

$$\|u_\alpha(0) - u_\alpha^\delta(0)\|^2 \leq \frac{T\left(1 + \|Au(0)\|\right)}{\left(1 + \ln\frac{T}{\alpha}\right)}.$$

This ends the proof of the theorem. ■

**Remark 19** *From (4.13), for $T > e^{-1}$ we get*

$$\|u(0) - u_\alpha^\delta(0)\| \leq c \left( \ln \frac{1}{\delta} \right)^{-1},$$

**Proposition 3** *Under the hypothesis of the above theorem, if we denote by $U_\alpha^\delta$ the solution of the approximate (F.V.P.) problem (4.1), (4.2) for $f \approx f_\delta$, using the quasireversibility method [31], we obtain the following estimate*

$$\|u(0) - U_\alpha^\delta(0)\| \leq c_1 \left( \ln \frac{1}{\delta} \right)^{-\frac{2}{3}}.$$

**Proof.** A proof can be given in a similar way as in [33]. ■

### 4.3.4 Some Explicit Convergence Rates

**Theorem 23** *If there exists an $\varepsilon \in ]0, 2[$ so that*

$$\int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} \|dE_\lambda f\|^2,$$

*converges, then $\|u_\alpha(T) - f\|$ converges to zero as $\alpha$ tends to zero with order $\alpha^\varepsilon \varepsilon^{-2}$.*

**Proof.** Let $\varepsilon \in ]0, 2[$ such that $\int_0^\infty \lambda^\varepsilon e^{\varepsilon \lambda T} \|dE_\lambda f\|^2$converges, and let $\beta \in ]0, 2[$. For a fix $\lambda > 0$, and if we define a function $g_\lambda(\alpha) = \frac{\alpha^\beta}{\left(\alpha\lambda + e^{-\lambda T}\right)^2}$. Then we can show that

$$g_\lambda(\alpha) \leq g_\lambda(\alpha_0), \ \forall \alpha > 0, \tag{4.17}$$

where $\alpha_0 = \frac{\beta e^{-\lambda T}}{(2-\beta)\lambda}$. Furthermore, from (4.7), we have

$$\|u_\alpha(T) - f\|^2 = \alpha^{2-\beta} \int_0^\infty \lambda^2 g_\lambda(\alpha)\, dE_\lambda f. \tag{4.18}$$

Hence from (4.17) and (4.18) we obtain

$$\|u_\alpha(T) - f\|^2 \leq \alpha^{2-\beta} \left( \frac{\beta}{2-\beta} \right)^\beta \int_0^\infty \lambda^{2-\beta} e^{(2-\beta)\lambda T} d\|E_\lambda f\|^2.$$

If we choose $\beta = (2 - \varepsilon)$, we have

$$\begin{aligned} \|u_\alpha(T) - f\|^2 &\leq \alpha^\varepsilon \left(\frac{2^2}{\varepsilon^2}\right) \int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T} d\|E_\lambda f\|^2 \\ &= \left(4 \int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T} d\|E_\lambda f\|^2\right), \end{aligned}$$

hence

$$\|u_\alpha(T) - f\|^2 \leq c_\varepsilon \alpha^\varepsilon \varepsilon^{-2},$$

with $c_\varepsilon = 4\int_0^\infty \lambda^\varepsilon e^{\varepsilon\lambda T} d\|E_\lambda f\|^2$. ■

Now, we give the following corollary.

**Corollary 1** *If there exists an* $\varepsilon > 0$ *so that*

$$\int_0^\infty \lambda^{(\varepsilon+2)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2, \tag{4.19}$$

*and*

$$\int_0^\infty \lambda^\varepsilon e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2, \tag{4.20}$$

*converge, then* $u_\alpha$ *converges to* $u$ *as* $\alpha$ *tends to zero in* $C^1([0,T], H)$ *with order of convergence* $\alpha^\varepsilon \varepsilon^{-2}$.

**Proof.** If we assume that (4.19) is satisfied, then

$$\int_0^\infty \lambda^2 e^{2\lambda T} d\|E_\lambda f\|^2,$$

converges, and so the function $u(t)$ given by (4.6) is a classical solution of the (F.V.P.) problem (4.1), (4.2). Now, using the following inequalities

$$\begin{aligned} \|u'_\alpha(0) - u'(0)\|^2 &= \int_0^\infty \frac{\alpha^2\lambda^4 e^{2\lambda T}}{(\alpha\lambda + e^{-\lambda T})^2} d\|E_\lambda f\|^2 \\ &\leq \alpha^{2-\beta} \int_0^\infty \lambda^4 \frac{\alpha^\beta}{(\alpha\lambda + e^{-\lambda T})^2} e^{2\lambda T} d\|E_\lambda f\|^2 \\ &\leq \alpha^{2-\beta} \int_0^\infty \lambda^4 g_\lambda(\alpha)\, e^{2\lambda T} d\|E_\lambda f\|^2 \\ &\leq \alpha^{2-\beta} \left(\frac{\beta}{2-\beta}\right)^\beta \int_0^\infty \lambda^{4-\beta} e^{(4-\beta)\lambda T} d\|E_\lambda f\|^2, \end{aligned} \tag{4.21}$$

where $c_\varepsilon = 4\int_0^\infty \lambda^{(\varepsilon+\varepsilon)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2$, and setting $\beta = 2-\varepsilon$, in (4.21), we obtain

$$\|u'_\alpha(0) - u'(0)\|^2 \leq c_\varepsilon \alpha^\varepsilon \varepsilon^{-2},$$

where $c_\varepsilon = 4\int_0^\infty \lambda^{(\varepsilon+2)} e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2$. And since

$$\|u'_\alpha(t) - u'(t)\|^2 \leq \|u'_\alpha(0) - u'(0)\|^2,$$

then $u'_\alpha(t)$ converges to $u'(t)$ uniformly in $t$, for all $t \in [0,T]$ ,with order of convergnece $\alpha^\varepsilon \varepsilon^{-2}$.

Now, if we assume that (4.20) is satisfied, then

$$\|u_\alpha(0) - u(0)\|^2 = \alpha^{2-\beta} \int_0^\infty \lambda^2 g_\lambda(\alpha) e^{2\lambda T} d\|E_\lambda f\|^2,$$

and proceding in a similar way as in the proof of the previous theorem, we get

$$\|u_\alpha(0) - u(0)\|^2 \leq \alpha^{2-\beta} \left(\frac{\beta}{2-\beta}\right)^\beta \int_0^\infty \lambda^{2-\beta} e^{(4-\beta)\lambda T} d\|E_\lambda f\|^2, \tag{4.22}$$

again, by setting $\beta = 2-\varepsilon$ in (4.22), we obtain

$$\|u_\alpha(0) - u(0)\|^2 \leq c'_\varepsilon \alpha^\varepsilon \varepsilon^{-2},$$

where, $c'_\varepsilon = 4\int_0^\infty \lambda^\varepsilon e^{(\varepsilon+2)\lambda T} d\|E_\lambda f\|^2$. Now, using the inequality

$$\|u_\alpha(t) - u(t)\|^2 \leq \|u_\alpha(0) - u(0)\|^2,$$

we see that $u_\alpha(t)$ converges to $u(t)$ uniformly in $t$, for all $t \in [0,T]$ ,with order of convergnece $\alpha^\varepsilon \varepsilon^{-2}$. Combining all these convergence results, we see that $u_\alpha$ converges to $u$ as $\alpha$ tends to zero in $C^1([0,T], H)$, with order of convergence $\alpha^\varepsilon \varepsilon^{-2}$. ■

### 4.3.5 Conclusion

We note that one advantage of this method of regularization is that the order of the error, introduced by small changes in the final value $f$, is less than the order given in [10]. And we also conclude that the regularization method used here gives a better approximation than many other quasi reversibility type methods e.g. [1], [10], and [31]. We also recommend for future research work the use of this approach to treat other problems described by more general types of partial differential equations.

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# Résumé

Le présent travail est composé de deux parties :

La première partie est consacrée à l'étude de certaines classes de matrices, sur l'anneau des polynômes à deux variables $\Re[s,z]$, associées en particulier aux différents systèmes linéaires différentiels. En suite les résultas obtenus sont appliqués à l'étude d'un problème modélisé par un système d'équations bidimensionnel (2-D) .

Dans la deuxième partie on étudie un problème de contrôle par la condition initiale d'un problème parabolique abstrait mal posé à coefficient opératoriel auto-adjoint non borné.

# Abstract

The present work is composed of two parts:

The first part is devoted to the study of some classes of matrices, over the ring of polynomials in two variables $\Re[s,z]$, associated in particular with different differential linear systems. Then, the obtained results are applied to the study of a problem described by a bidimensional (2-D) system of equations.

In the second part we study a control probem by the initial condition of an ill-posed abstract parabolic problem with an unbounded self-adjoint operatorial coefficient.

# ملخص

هذا العمل ينقسم إلى قسمين :

القسم الأول يتطرق إلى دراسة صنف من المصفوفات، على حلقة كثيرات الحدود لمتغيرين، $\Re[s,z]$ مرفقة خاصة بجمل معدلات خطية مختلفة. ثم نطبق النتائج المحصل عليها في دراسة مسألة موصوفة بجملة معادلات ثنائية البعد 2-D.

في القسم الثاني ندرس مسألة تحكمية بواسطة الشرط الابتدائي لمسألة تكافئية مجردة مطروحة بشكل سيئ وذات معامل مؤثري ذاتي القرينة وغير محدود.

# Résumé

Le présent travail est composé de deux parties :

La première partie est consacrée à l'étude de certaines classes de matrices, sur l'anneau des polynômes à deux variables $\Re[s,z]$, associées en particulier aux différents systèmes linéaires différentiels. En suite les résultas obtenus sont appliqués à l'étude d'un problème modélisé par un système d'équations bidimensionnel (2-D) .

Dans la deuxième partie on étudie un problème de contrôle par la condition initiale d'un problème parabolique abstrait mal posé à coefficient opératoriel auto-adjoint non borné.