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Theme

Asymptotic analysis of a Signorini problem with Coulomb friction for shallow shells. Dynamical case

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Dedication

To the spirit of my father "may ALLAH have mercy on him",

To my mother,

To my wife,

To my daughters Youssera, Fâtima az-Zahra and Zaineb,

To my son Mohammed Tahar,

To my family.

Acknowledgement

First of all, In the name of ALLAH, the beneficent, the merciful. Praise be to ALLAH, the lord of the worlds.

Foremost, I would like to thank my supervisor, Prof. CHACHA Djamel Ahmed, for having accepted to supervise my thesis. Thank you for guiding me to this fascinating research topic. I am grateful for all those opportunities, to make interesting research and to publish.

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I would like to thank my colleagues Mohammed El hadi and Ismail. Many thanks go in particular to Abdellah, for helping me.

I will never forget my teachers, from the primary to the university, to every one help me in my studies. I thank them.

Of course, my gratitude goes to my family. First, to my mother, for her unconditional support with respect to all my aspirations. I sincerely thank my wife, for the continuous support, encouragement and help.

Contents

Notations and definitions

Notations

ω: connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary, being locally on a single side of its boundary.

γ: boundary of ω, $0 \in \gamma$.

$\gamma(y)$: arc joining 0 to the point $y \in \gamma$.

γ_1: relatively open subset of γ such that $length \gamma_1 > 0$.

$\gamma_2 = \gamma \backslash \gamma_1$.

$d\gamma$: length element along γ.

(ν_α): unit outer normal vector along the boundary γ.

(τ_α): unit tangent vector along the boundary γ, related by $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$.

$(x_\alpha) = (x_1, x_2)$: generic point in $\bar{\omega}$.

$\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, $\partial_{\alpha\beta} = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}$.

$\partial_\nu = \nu_\alpha \partial_\alpha$: outer normal derivative operator.

$\partial_\tau = \tau_\alpha \partial_\alpha$: tangential derivative operator.

$\Delta = \partial_{\alpha\alpha}$: Laplacian operator.

$\Delta^2 = \Delta\Delta = \partial_{\alpha\alpha}\partial_{\beta\beta}$: biharmonic operator.

$\Omega = \omega \times]-1, 1[$.

$\Gamma_\pm = \omega \times \{\pm 1\}$.

$d\Gamma$: area element along $\partial\Omega$.

$x = (x_i) = (x_1, x_2, x_3)$: generic point in $\bar{\Omega}$.

$\partial_i = \frac{\partial}{\partial x_i}$.

$\gamma_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$.

$\partial_\alpha^\theta = \partial_\alpha - \partial_\alpha\theta\partial_3$, $\partial_3^\theta = \partial_3$.

$\gamma_{ij}^\theta(\mathbf{v}) = \frac{1}{2}\left(\partial_i^\theta v_j + \partial_j^\theta v_i\right)$.

δ_{ij}: Kronecker symbols.

$\Lambda \geq 0$: friction coefficient.

λ^ε, μ^ε: Lamé constants of the material.

$\Omega^\varepsilon = \omega \times] - \varepsilon, \varepsilon[$.

$\Gamma_\pm^\varepsilon = \omega \times \{\pm\varepsilon\}$.

$d\Gamma^\varepsilon$: area element along $\partial\Omega^\varepsilon$.

$x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3)$: generic point in $\bar{\Omega}^\varepsilon$.

$\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$.

$\theta^\varepsilon : \bar{\omega} \to \mathbb{R}$: mapping for defining the middle surface $\bar{\hat{\omega}}^\varepsilon$ of the shallow shell.

$\bar{\hat{\omega}}^\varepsilon = \{(x_1, x_2, \theta^\varepsilon(x_1, x_2)) \in \mathbb{R}^3, (x_1, x_2) \in \omega\}$: middle surface of the shallow shell.

$\mathbf{a}_3^\varepsilon = (|\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1)$: continuously varying unit vector normal to the middle surface $\bar{\hat{\omega}}^\varepsilon$.

$\boldsymbol{\Theta}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$: mapping for defining the reference configuration of the shallow shell.

$\boldsymbol{\Theta}^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \mathbf{a}_3^\varepsilon(x_1, x_2)$.

$\hat{\Omega}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\Omega^\varepsilon)$.

$\hat{\gamma}_1^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\gamma_1)$.

$\hat{\Gamma}_+^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\Gamma_+^\varepsilon)$: upper face of $\bar{\hat{\Omega}}^\varepsilon$.

$\hat{\Gamma}_-^\varepsilon = \boldsymbol{\Theta}^\varepsilon(\Gamma_-^\varepsilon)$: lower face of $\bar{\hat{\Omega}}^\varepsilon$.

$\boldsymbol{\Theta}^\varepsilon(\gamma \times [-\varepsilon, \varepsilon])$: lateral face of $\bar{\hat{\Omega}}^\varepsilon$.

$\bar{\hat{\Omega}}^\varepsilon$: reference configuration of the shallow shell.

$d\hat{\Gamma}^\varepsilon$: area element along $\partial\hat{\Omega}^\varepsilon$.

$\hat{\mathbf{n}}^\varepsilon = (\hat{n}_i^\varepsilon)$: unit outer normal vector along the boundary of $\hat{\Omega}^\varepsilon$.

$(\hat{f}_i^\varepsilon)$: applied body forces of density in interior of shallow shell.

$(\hat{g}_i^\varepsilon)$: applied surface forces of density on upper and lower (or lower) faces of shallow shell.

$(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$: applied surface forces of von Kármán type on lateral face of shallow shell.

$\hat{x}^\varepsilon = \boldsymbol{\Theta}^\varepsilon(x^\varepsilon)$: generic point in $\bar{\hat{\Omega}}^\varepsilon$.

$\hat{\partial}_i^\varepsilon = \frac{\partial}{\partial \hat{x}_i^\varepsilon}$.

$\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)$: unknown vector field.

$\hat{E}_{ij}^\varepsilon(\hat{\mathbf{v}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{v}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{v}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{v}_m^\varepsilon)$: Green-Saint Venant strain tensor field.

$\hat{\gamma}_{ij}^\varepsilon(\hat{\mathbf{v}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{v}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon)$: linearized strain tensor.

$\hat{\sigma}_{ij}^\varepsilon$: second Piola-Kirchhoff stresses.

$\hat{v}_N^\varepsilon$: normal components of $\hat{\mathbf{v}}^\varepsilon$.

$\hat{\mathbf{v}}_T^\varepsilon$: tangential components of $\hat{\mathbf{v}}^\varepsilon$.

$\hat{G}_N^\varepsilon$: contact force.

$\hat{\mathbf{G}}_T^\varepsilon$: friction force.

$\frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}$: velocity.

$\frac{\partial \hat{u}_N^\varepsilon}{\partial t}$: normal velocity.

$\frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t}$: tangential velocity.

$\hat{\rho}^\varepsilon$: mass density.

$\hat{\mathbf{p}}^\varepsilon$, $\hat{\mathbf{q}}^\varepsilon$: given initial data.

$\hat{C}^\varepsilon = (\hat{c}_{ijkl}^\varepsilon)$: compliance tensor.

$\hat{A}^\varepsilon = (\hat{a}_{ijkl}^\varepsilon)$: rigidity tensor.

$\hat{d}^\varepsilon$: gap function.

$\rightharpoonup$: weak convergence.

$\rightarrow$: strong convergence.

Definitions

$W^{s,p}(.)$, $(s \in \mathbb{R},\ p \geq 1)$: usual Sobolev space.

$\|\ \|_{s,p,.}$: norm in $W^{s,p}(.)$.

$|\ |_{s,p,.}$: semi-norm in $W^{s,p}(.)$, $(s \in \mathbb{N})$.

$H^s(.) = W^{s,2}(.)$, $\|\ \|_{s,.} = \|\ \|_{s,2,.}$ and $|\ |_{s,.} = |\ |_{s,2,.}$.

$$\mathbf{V}(\hat{\Omega}^\varepsilon) = \left\{ \begin{array}{c} \hat{\mathbf{v}}^\varepsilon = (\hat{v}_i^\varepsilon) \in W^{1,4}(\hat{\Omega}^\varepsilon; \mathbb{R}^3); \hat{v}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{v}_3^\varepsilon = 0 \\ \text{on } \boldsymbol{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon]) \end{array} \right\}.$$

$$\mathbf{K}(\hat{\Omega}^\varepsilon) = \left\{ \hat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\hat{\Omega}^\varepsilon); \hat{v}_N^\varepsilon \leq \hat{d}^\varepsilon \ on \ \hat{\Gamma}_+^\varepsilon \right\}.$$

$$\mathbf{V}(\Omega^\varepsilon) = \left\{ \begin{array}{c} \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in W^{1,4}(\Omega^\varepsilon; \mathbb{R}^3); v_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } v_3^\varepsilon = 0 \\ \text{on } \gamma_1 \times [-\varepsilon, \varepsilon] \end{array} \right\}.$$

$\mathbf{K}(\Omega^\varepsilon) = \{\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon); v_N^\varepsilon \le d^\varepsilon \ on\ \Gamma_+^\varepsilon\}$.

$\mathbf{V}(\Omega) = \left\{ \begin{array}{c} \mathbf{v} = (v_i) \in W^{1,4}(\Omega;\mathbb{R}^3); v_\alpha \text{ independent of } x_3 \text{ and } v_3 = 0 \\ \text{on } \gamma_1 \times [-1,1] \end{array} \right\}$.

$V_1(\Omega) = V_2(\Omega) = \left\{ \ v \in W^{1,4}(\Omega); v \text{ independent of } x_3 \text{ on } \gamma_1 \times [-1,1] \ \right\}$.

$V_3(\Omega) = \left\{ \ v \in W^{1,4}(\Omega); v = 0 \text{ on } \gamma_1 \times [-1,1] \ \right\}$.

$\mathbf{K}(\varepsilon)(\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega); v_N(\varepsilon) \le d(\varepsilon) \ on\ \Gamma_+\}$.

$\mathbf{K}(\Omega) = \{\mathbf{v} \in \mathbf{V}(\Omega); v_3 \le d \ on\ \Gamma_+\}$ with $d(\varepsilon) = d + O(\varepsilon)$.

$\mathbf{V}_{KL}(\Omega) = \left\{ \begin{array}{c} \mathbf{v} = (v_i) \in H^1(\Omega;\mathbb{R}^3); v_\alpha \text{ independent of } x_3 \text{ and } v_3 = 0 \\ \text{on } \gamma_1 \times [-1,1], \partial_i v_3 + \partial_3 v_i = 0 \text{ in } \Omega \end{array} \right\}$.

$\mathbf{V}(\omega) = \{\eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_1\}$.

$V(\omega) = \{\eta \in H^2(\omega); \eta = \partial_\nu \eta = 0 \text{ on } \gamma_1\}$.

$\mathcal{V}(\hat{\Omega}^\varepsilon) = \left\{ \begin{array}{c} \hat{\mathbf{v}}^\varepsilon \in W^{1,4}(\hat{\Omega}^\varepsilon;\mathbb{R}^3); \frac{\partial \hat{\mathbf{v}}^\varepsilon}{\partial t} \in W^{1,4}(\hat{\Omega}^\varepsilon;\mathbb{R}^3),\ \hat{v}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \\ \text{and } \hat{v}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon,\varepsilon]) \end{array} \right\}$.

$\mathcal{K}(\hat{\Omega}^\varepsilon) = \left\{ \ \hat{\mathbf{v}}^\varepsilon \in \mathcal{V}(\hat{\Omega}^\varepsilon); \hat{v}_N^\varepsilon \le \hat{d}^\varepsilon \text{ on } \hat{\Gamma}_+^\varepsilon \ \right\}$.

$\mathcal{V}(\Omega^\varepsilon) = \left\{ \begin{array}{c} \mathbf{v}^\varepsilon = (v_i^\varepsilon) \in W^{1,4}(\Omega^\varepsilon;\mathbb{R}^3); \frac{\partial \mathbf{v}^\varepsilon}{\partial t} \in W^{1,4}(\Omega^\varepsilon;\mathbb{R}^3),\ v_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \\ \text{and } v_3^\varepsilon = 0 \text{ on } \gamma_1 \times [-\varepsilon,\varepsilon] \end{array} \right\}$.

$\mathcal{K}(\Omega^\varepsilon) = \left\{ \ \mathbf{v}^\varepsilon \in \mathcal{V}(\Omega^\varepsilon); v_N^\varepsilon \le d^\varepsilon \text{ on } \Gamma_+^\varepsilon \ \right\}$.

$\mathcal{V}(\Omega) = \left\{ \begin{array}{c} \mathbf{v} = (v_i) \in W^{1,4}(\Omega;\mathbb{R}^3); \frac{\partial \mathbf{v}}{\partial t} \in W^{1,4}(\Omega;\mathbb{R}^3),\ v_\alpha \text{ independent of } x_3 \\ \text{and } v_3 = 0 \text{ on } \gamma_1 \times [-1,1] \end{array} \right\}$.

$\mathcal{V}_1(\Omega) = \mathcal{V}_2(\Omega) = \left\{ \begin{array}{c} v \in W^{1,4}(\Omega); \frac{\partial v}{\partial t} \in W^{1,4}(\Omega),\ v \text{ independent of } x_3 \\ \text{on } \gamma_1 \times [-1,1] \end{array} \right\}$.

$\mathcal{V}_3(\Omega) = \left\{ \ v \in W^{1,4}(\Omega); \frac{\partial v}{\partial t} \in W^{1,4}(\Omega), v = 0 \text{ on } \gamma_1 \times [-1,1] \ \right\}$.

$\mathcal{K}(\varepsilon)(\Omega) = \left\{ \ \mathbf{v} \in \mathcal{V}(\Omega); v_N(\varepsilon) \le d(\varepsilon) \text{ on } \Gamma_+ \ \right\}$.

$\mathcal{K}(\Omega) = \left\{ \ \mathbf{v} \in \mathcal{V}(\Omega); v_3 \le d \text{ on } \Gamma_+ \ \right\}$.

$\mathcal{V}_{KL}(\Omega) = \left\{ \ \mathbf{v} \in \mathcal{V}(\Omega); \partial_i v_3 + \partial_3 v_i = 0 \text{ in } \Omega \ \right\}$.

$\mathcal{K}_{KL}(\Omega) = \left\{ \ \mathbf{v} \in \mathcal{V}_{KL}(\Omega); v_3 \le d \text{ on } \Gamma_+ \ \right\}$.

$\mathcal{K}(\omega) = \{\eta \in \mathbf{V}(\omega); \eta_3 \le d \text{ in } \omega\}$.

$\mathcal{K} = \{\eta \in V(\omega); \eta \le d \text{ in } \omega\}$.

$\widetilde{\mathcal{K}} = \{\eta \in V(\omega); \eta \le \widetilde{d} \text{ in } \omega\}$.

$\Sigma(\hat{\Omega}^\varepsilon) = L^2(\hat{\Omega}^\varepsilon;\mathbb{S}^3)$.

$\Sigma(\Omega^\varepsilon) = L^2(\Omega^\varepsilon;\mathbb{S}^3)$.

$\Sigma(\Omega) = L^2(\Omega;\mathbb{S}^3)$.

$\mathbb{M}^3$: space of matrix of order 3.

$\mathbb{S}^n$: space of symmetric tensors of order n.

Conventions

Latin indices: belong to the set {1,2,3}.

Greek indices: belong to the set {1,2}.

The summation convention with respect to repeated indices is systematically used.

ε: designates a parameter that is > 0 and approaches zero.

2ε: thickness of the shallow shell.

Exponents iso and anis: corresponds to a problem respectively designate isotropic and anisotropic material.

Indices sta, dyn and c: corresponds to a problem respectively designate static, dynamical and contact cases.

Introduction

Historical notes

From one century ago it has been appear the justification of the classical von Kármán's theory of plates. The von Kármán equations, originally proposed by Theodore von Kármán [vK10] in 1910, which play an important role in applied mathematics.

The Marguere-von Kármán equations are two-dimensional equations for a nonlinearly elastic shallow shell subjected to boundary conditions analogous to those of von Kármán equations for plate. They were initially proposed by Marguerre [Mar38] in 1938 and von Kármán and Tsien [vKT39] in 1939.

Since these equations attracted the attention of several researchers. However, the majority of the results obtained were on the static or semi-static models. The questions of existence, unicity, regularity and stability were the subject of several research tasks. We quote among them the works carried out by Kesavan and Srikanth [KS83], Kavian and Rao [KR93], Rao [Rao95], Léger and Miara [LM05], Devdariani, Janjgava and Gulua [DJG06].

Since the remarkable work of Ciarlet and Paumier [CP86] in 1986 on the justification of the Marguerre-von Kármán equations in Cartesian coordinates by means of the formal asymptotic expansions method applied in the form of the displacement-stress approach, the mathematical analysis of these equations knew much progress and developments. Thus Andreoiu-Banica [AB99] in 1999 justified these equations in curvilinear coordinates. Then and within the same preceding framework Gratie [Gra02] in 2002 has generalized these equations, where only a portion of the lateral face is subjected to boundary conditions of von Kármán type, the remaining portion being free. She shows that the leading term of the asymptotic expansion is characterized by a two-dimensional boundary value problem called the generalized Marguerre-von Kármán equations. Then Ciarlet and Gratie [CG06a, CG06b] in 2006 have described and analyzed two models of generalized von Kármán plates and generalized Marguerre-von Kármán shallow shells, and have established an existence theorem for the generalized Marguerre-von Kármán equations.

In the dynamical case, some studies were done for the linearized isotropic homogeneous elastic thin shells, Xiao [Xia98, Xia01a, Xia01b, Xia99] studied the two-dimensional linear dynamic equations of membrane shells, flexural shells and Koiter shells, and proved the existence and uniqueness of solutions to the dynamic equations for Koiter shells. Ye [Ye03] improved Xiao's results on membrane shells and extended them to the generalized dynamic membrane shells. Yan [Yan06] justified the two-dimensional equations of linear dynamic shallow shells with variable thickness. The existence and uniqueness of a strong, global in time, solution of the time-dependent von Kármán equations have been established by Puel and Tucsnak [PT96]. In this direction, see also Lions [Lio69, Theorem 4.1], Koch and Stahel [KS93], Böhm [Böh96], Tataru and Tucsnak [TT97], Chueshov and Lasiecka [CL04, CL07]. Li and Bai [LB09], Li [Li09, Li10] in 2009-2010 extended the study of the Marguerre-von Kármán equations to the viscoelastic case. In the same way, we quote the works [CGB10, CGB13], where we identified the dynamical equations of generalized Marguerre-von Kármán shallow shells and we established the existence of solutions to these equations using compactness method.

It is well-known that the analysis of convergence of the nonlinear three-dimensional models towards the two-dimensional models in the elastostatic or elastodynamic cases is very difficult problem. To our knowledge the method of gamma convergence is the only method employed effectively until now for such problems. We quote the recent work of Abels et al [AMM11] where the asymptotic behaviour of solutions of three-dimensional nonlinear elastodynamics thin plate is studied. They showed that three-dimensional solutions of the nonlinear elastodynamic equation converge to solutions of the time-dependent von Kármán plate equation. The same question for the nonlinear elastodynamic Marguerre-von Kármán shallow shells equations remains open.

The problem of a unilateral contact with Coulomb friction attracted attention of many research workers both in engineering and mathematics. In the case of linearly thin elastic structures, Paumier [Pau02] studied the asymptotic modeling of Signorini with Coulomb friction in the Kirchhoff-Love theory of plates by using a convergence method. In the same way but for the frictionless case, Léger and Miara [LM08, LM11] extended the study to the elastic shallow shell. More recently, Ben Belgacem et al. [BBBT02] modeled the obstacle problem without friction for Naghdi shell. In this direction, see also Kikuchi and Oden [KO88] and the references therein for the contact problems in elasticity. For the nonlinear case, Chacha and Bensayah [CB08] studied the asymptotic modeling of a Coulomb frictional Signorini problem for the von Kármán plates using the formal asymptotic expansion method. In this direction, we quote our work [BCG13] for justification of the generalized Marguerre-von Kármán equations with Signorini conditions. In the dynamical case, we refer to Eck et al. [EJK05] for further references for linear elasticity and we quote the important result of Bock and Jarušek [BJ09] for von Kármán equations.

For anisotropic materials, the justification by asymptotic analysis has been done for: linearly elastic plates by Destuynder [Des80], linearly elastic shells by Caillerie and Sanchez-Palencia [CSP95] and Giroud [Gir98], see also Sanchez-Hubert and Sanchez-Palencia [SHSP92]. In this way but for nonlinear case, we refer to Begehr, Gilbert and Lo [BGL91], Gilbert and Vashakmadze [GV00] for plates and Chacha and Miloudi [CM12] for shells.

For numerical approximations, some studies have been done for the von Kármán equations. Miyoshi [Miy76] studied the mixed finite element method for these equations. Kesavan [Kes79, Kes80] proposed an iterative finite element method of the bifurcation branches near simple eigenvalues of the linearized problem of von Kármán equations and mixed finite element method for the same problem. Brezzi [Bre78] and Brezzi et al. [BRR80, BRR81] analyzed a finite element approximations of von Kármán plate bending equations and studied a Hellan-Herrmann-Johnson mixed finite element scheme for the von Kármán equations. Reinhart [Rei82] proposed an approximation of the von Kármán equations using a Hermann-Miyoshi finite element scheme. Ciarlet et al. [CGK07] studied the finite element method for the generalized von Kármán equations. Recently, we extend the results of Ciarlet et al. [CGK07] to the generalized Marguerre-von Kármán equations in [GC14].

Organization of the thesis

The objective of this thesis is to study the asymptotic modeling of three-dimensional problems of nonlinearly elastic shallow shells, in dynamical case, with and without unilateral contact. More precisely, the aim of this study is to derive, mathematical justification of a two-dimensional models for nonlinearly elastodynamic shallow shell problems, with and without unilateral contact. This contact is modeled by the Signorini conditions with Coulomb's law friction. The derivation of the two-dimensional models is done using an asymptotic analysis. Also, to study the numerical approximation of the generalized Marguerre-von Kármán equations.

This thesis is organized as follows:

The first Chapter, concerns the mathematical models of three-dimensional problems of nonlinear elasticity shallow shell with and without unilateral contact, in static and dynamical case.

The second Chapter, concerns the formal derivation of the two-dimensional dynamical model for thin elastic shallow shell of generalized Marguerre-von Kármán type with homogeneous and isotropic material, starting from the three-dimensional nonlinear elastodynamics problem. It extended the model obtained by Gratie [Gra02] to the dynamical case. This work was published in [CGB10].

In addition, concerns the study of existence solutions to dynamical equations of generalized Marguerre-von Kármán shallow shells, which identified in this Chapter. Uusing compactness method of Lions [Lio69]. It generalizes the study carried out by Ciarlet and Gratie [CG06b] to the dynamical case. This work was published in [CGB13].

The third Chapter, concerns the formal derivation of the two-dimensional dynamical model of generalized Marguerre-von Kármán shallow shell with nonhomogeneous and anisotropic material.

The fourth Chapter, concerns the asymptotic modeling of the three-dimensional model of generalized Marguerre-von Kármán shallow shell with homogeneous and isotropic material under Signorini conditions in elastostatic. This work was published in [BCG13].

The fifth Chapter, concerns the generalization of the results obtained in the fourth Chapter to the dynamical case.

In addition, concerns the study of existence solutions to dynamical contact equations of generalized Marguerre-von Kármán shallow shells, which identified in this Chapter. Using penalization method.

The sixth Chapter, concerns the generalization of the results obtained in the fifth Chapter to the nonhomogeneous and anisotropic material.

The seventh Chapter, concerns the analysis a finite element approximations of generalized Marguerre-von Kármán equations. This work was published in [GC14].

Realized works

Publications:

- D.A. Chacha, A. Ghezal and A. Bensayah. Modélisation asymptotique d'une coque peu-profonde de Marguerre-von Kármán généralisée dans le cas dynamique. Revue ARIMA, 13: 63–76, 2010.

- D.A. Chacha, A. Ghezal and A. Bensayah. Existence result for a dynamical equations of generalized Marguerre-von Kármán shallow shells. Journal Elasticity, 111: 265–283, 2013.

- A. Bensayah, D.A. Chacha and A. Ghezal. Asymptotic modelling of a Signorini problem of generalized Marguerre-von Kármán shallow shells. Applicable Analysis, 92(9): 1848–1862, 2013.

- A. Ghezal and D.A. Chacha. Convergence of finite element approximations for generalized Marguerre-von Kármán equations. Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics 87, 97–106, 2014.

International conferences:

- D.A. Chacha, A. Ghezal and A. Bensayah. Modélisation asymptotique d'une coque peu-profonde de Marguerre-von Kármán généralisée dans le cas dynamique. TAM-TAM'09, Kenitra, Maroc, 2009.

- A. Ghezal, A. Bensayah and D.A. Chacha. Asymptotic modeling of dynamic generalized Marguerre-von Kármán shallow shells in curvilinear coordinates. TAM-TAM'11, Sousse, Tunisie, 2011.

- A. Ghezal and D.A. Chacha. Finite element approximations of the generalized Marguerre-von Kármán equations. GICAM'13, Gulf International Conference on Applied Mathematics, Kuwait, 2013.

- A. Ghezal and D.A. Chacha. Asymptotic analysis of nonhomogeneous generalized Marguerre-von Kármán shallow shells in anisotropic elastodynamic. The 2^{nd} Abu Dhabi University Annual International Conference: Mathematical Science and Its Applications, United Arab Emirates, 2013.

Chapter 1

Mathematical models of nonlinearly elastic shallow shell with and without unilateral contact

In this Chapter, we give the definitions of the kinds of nonlinearly elastic shallow shell and the associated three-dimensional mathematical models with and without unilateral contact.

1.1 Definitions

Definition 1.1 *(Shallow shell) A shell is shallow if, in its reference configuration, the deviation of the middle surface from a plane is (up to an additive constant) of the order of the thickness of the shell (see Figure 1.1).*

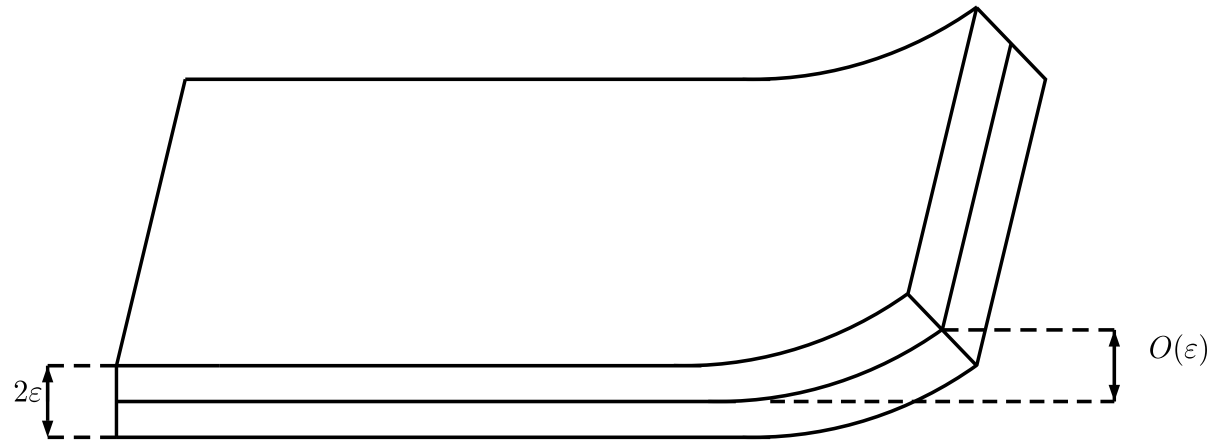

Figure 1.1

Let ω be a connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, ω being locally on a single side of γ, we assume $0 \in \gamma$ and we denote by $\gamma(y)$ the arc joining 0 to the point $y \in \gamma$. Let γ_1 be a relatively open subset of γ such that $length \gamma_1 > 0$ and $length \gamma_2 > 0$, where $\gamma_2 = \gamma \backslash \gamma_1$. The unit outer normal vector (ν_α) and the unit tangent vector (τ_α) along the boundary γ are related by $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$. The outer normal and tangential derivative operators $\nu_\alpha \partial_\alpha$ and $\tau_\alpha \partial_\alpha$ along γ are denoted respectively by ∂_ν and ∂_τ.

For any $\varepsilon > 0$, let $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, $\Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}$ and $\theta^\varepsilon : \bar{\omega} \to \mathbb{R}$ is a function of class C^3 that satisfies $\theta^\varepsilon = \partial_\nu \theta^\varepsilon = 0$ on γ_1. We define the mapping

$$\mathbf{\Theta}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3 : \mathbf{\Theta}^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \mathbf{a}_3^\varepsilon(x_1, x_2),$$

for all $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon$, where $\mathbf{a}_3^\varepsilon$ is a continuously varying unit vector normal to the middle surface $\mathbf{\Theta}^\varepsilon(\bar{\omega})$. For small enough ε, the mapping $\mathbf{\Theta}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbf{\Theta}^\varepsilon(\bar{\Omega}^\varepsilon)$ is a C^1 – diffeomorphism (see [CP86]). We let $\hat{\Omega}^\varepsilon = \mathbf{\Theta}^\varepsilon(\Omega^\varepsilon)$, $\hat{\gamma}_1^\varepsilon = \mathbf{\Theta}^\varepsilon(\gamma_1)$, $\hat{\Gamma}^\varepsilon_\pm = \mathbf{\Theta}^\varepsilon(\Gamma^\varepsilon_\pm)$ and we denote by $\hat{x}^\varepsilon = \mathbf{\Theta}^\varepsilon(x^\varepsilon)$ a generic point in $\bar{\hat{\Omega}}^\varepsilon$, $(\hat{n}_i^\varepsilon)$ is the unit outer normal vector along the boundary of the set $\hat{\Omega}^\varepsilon$. The set $\bar{\hat{\Omega}}^\varepsilon$ in the absence of applied forces is called the reference configuration of the shell.

Following the definition proposed by Ciarlet and Paumier [CP86], we say that a shell $\bar{\hat{\Omega}}^\varepsilon$ is shallow if there exists a function $\theta \in C^3(\bar{\omega})$ independent of ε such that $\theta^\varepsilon(x_1, x_2) = \varepsilon\theta(x_1, x_2)$, for all $(x_1, x_2) \in \bar{\omega}$ (see Figure 1.2).

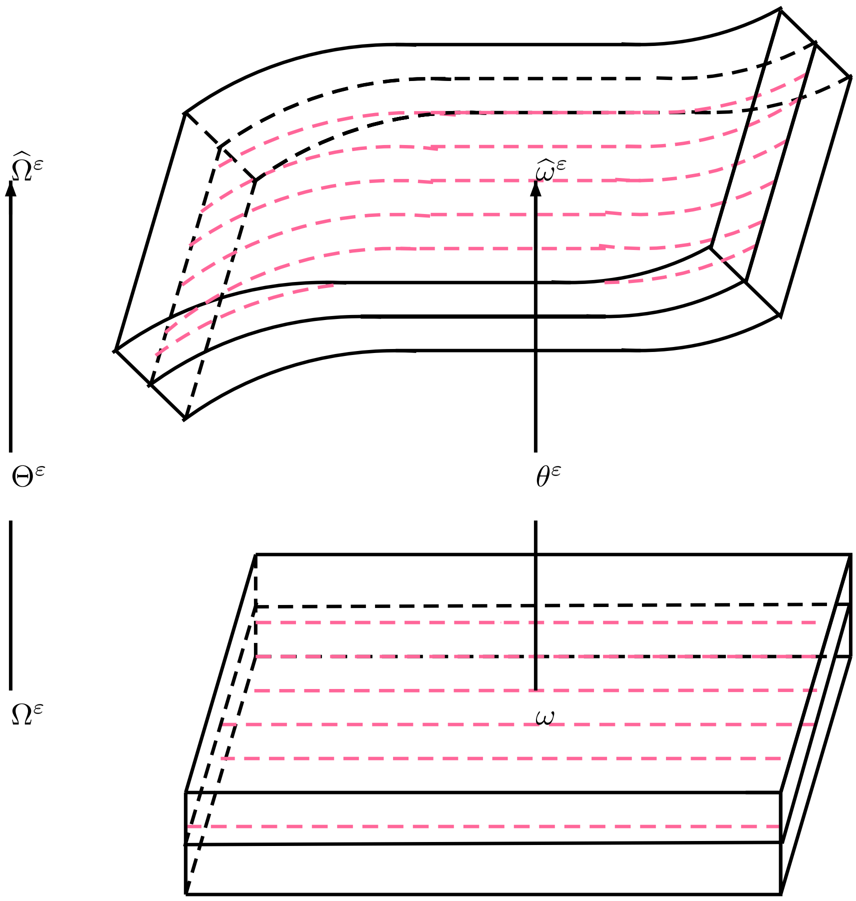

Figure 1.2

Definition 1.2 *(Clamped shallow shell) If the shallow shell is subjected to a boundary condition of place* $\hat{u}_i^\varepsilon = 0$ *on the lateral face* $\mathbf{\Theta}^\varepsilon(\gamma \times [-\varepsilon, \varepsilon])$.

Definition 1.3 *(Forces of von Kármán's type) These specific surfaces forces acting on the portion* $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ *of its lateral face, are horizontal and their resultant after integration across the thickness of the shallow shell is given along* $\mathbf{\Theta}^\varepsilon(\gamma_1)$ *and that, accordingly,* *the admissible displacements along* $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ *are those whose horizontal components are independent of the vertical variable and whose vertical component vanishes.*

These boundary conditions read as:

$$\begin{cases} \hat{u}^\varepsilon_\alpha \text{ independent of } \hat{x}^\varepsilon_3 \text{ and } \hat{u}^\varepsilon_3 = 0 \text{ on } \mathbf{\Theta}^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right), \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}^\varepsilon_{\alpha\beta} + \hat{\sigma}^\varepsilon_{k\beta}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_\alpha) \circ \mathbf{\Theta}^\varepsilon\}\nu_\beta d x^\varepsilon_3 = \hat{h}^\varepsilon_\alpha \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1. \end{cases} \tag{1.1}$$

Definition 1.4 *(Classical Marguerre-von Kármán shallow shell) If the shallow shell is subjected to applied surface forces of von Kármán's type on the all lateral face* $\mathbf{\Theta}^\varepsilon(\gamma \times [-\varepsilon, \varepsilon])$.

Definition 1.5 *(Generalized Marguerre-von Kármán shallow shell) If the shallow shell is subjected to applied surface forces of von Kármán's type on the portion* $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ *of its lateral face, the remaining portion* $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ *being free.*

1.2 Three-dimensional problem of nonlinear elasticity in Cartesian coordinates

Consider a nonlinearly shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε.

The shell is subjected to body forces of density $(\hat{f}^\varepsilon_i)$ in its interior $\hat{\Omega}^\varepsilon$ and to surface forces of density $(\hat{g}^\varepsilon_i)$ on its upper and lower faces $\hat{\Gamma}^\varepsilon_+$ and $\hat{\Gamma}^\varepsilon_-$. We assume that these densities do not depend on the unknown.

The unknown in the three-dimensional formulation of clamped shallow shell is the displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}^\varepsilon_i)(\hat{x}^\varepsilon, t)$, where the functions $\hat{u}^\varepsilon_i : \bar{\hat{\Omega}}^\varepsilon \to \mathbb{R}$ are their Cartesian components that the body undergoes when it is subjected to applied forces. The unknown $\hat{\mathbf{u}}^\varepsilon$ satisfies the following equations of equilibrium

$$\begin{cases} -\hat{\partial}^\varepsilon_j(\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i) = \ \hat{f}^\varepsilon_i \text{ in } \hat{\Omega}^\varepsilon, \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j = 0 \text{ on } \mathbf{\Theta}^\varepsilon(\gamma \times [-\varepsilon, \varepsilon]), \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j = \hat{g}^\varepsilon_i \text{ on } \mathbf{\Theta}^\varepsilon\left(\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-\right), \end{cases}$$

where the matrix $\hat{\sigma}^\varepsilon = (\hat{\sigma}^\varepsilon_{ij}) : \bar{\hat{\Omega}}^\varepsilon \to \mathbb{M}^3$ called the second Piola- Kirchhoff stress tensor field.

Let the Green-Saint Venant strain tensor field associated with an arbitrary displacement field $\hat{\mathbf{v}}^\varepsilon$ of the reference configuration $\bar{\hat{\Omega}}^\varepsilon$ be defined by

$$\hat{E}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) = \frac{1}{2}(\hat{\partial}^\varepsilon_i \hat{v}^\varepsilon_j + \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i + \hat{\partial}^\varepsilon_i \hat{v}^\varepsilon_m \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_m).$$

We assume that the elastic material constituting the shell is nonhomogeneous and anisotropic, and that the reference configuration is a natural state. The consideration of the fundamental principle of material frame-indifference implies that the stress tensor $\hat{\sigma}^\varepsilon$ satisfies the following Hooke's law

$$\hat{\sigma}^\varepsilon_{ij} = \hat{A}^\varepsilon_{ijkl}(\hat{x}^\varepsilon)\hat{E}^\varepsilon_{kl}(\hat{\mathbf{u}}^\varepsilon),$$

which is called the constitutive equation in Cartesian coordinates of the material, where $\hat{A}^\varepsilon = (\hat{a}^\varepsilon_{ijkl})$ is the rigidity tensor.

In the important special case where the elastic material is homogeneous and isotropic and the reference configuration is a natural state, the Hooke's law is of the following specific form

$$\hat{\sigma}^\varepsilon_{ij} = \lambda^\varepsilon \hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon),$$

where λ^ε and μ^ε are the two Lamé constants of the material. In this case, the material is called a Saint Venant-Kirchhoff material.

1.3 Signorini contact conditions and Coulomb friction law

The most models for the description of contact and friction are the Signorini contact condition and the Coulomb friction law. The Signorini condition models was proposed in 1933 by Signorini [Sig33] and the Coulomb law of friction was proposed in 1781 by [Cou81].

We consider that the shallow shell is in contact on $\hat{\Gamma}^\varepsilon_+$ with a rigid foundation. Let $\hat{d}^\varepsilon$ be the normal gap, or separation, between the contact boundary and the rigid surface, measured along the outward normal direction to $\hat{\Gamma}^\varepsilon_+$. The fact that the elastic shallow shell cannot penetrate the foundation means that the normal displacement $\hat{u}^\varepsilon_N$ satisfies

$\hat{u}_N^\varepsilon \leq \hat{d}^\varepsilon$ on $\hat{\Gamma}_+^\varepsilon$. Moreover the contact force $\hat{G}_N^\varepsilon$ are compressive hence $\hat{G}_N^\varepsilon \leq 0$ and either there is contact $\hat{u}_N^\varepsilon = \hat{d}^\varepsilon$ or the surface is free $\hat{G}_N^\varepsilon = 0$, which means $\hat{G}_N^\varepsilon(\hat{u}_N^\varepsilon - \hat{d}^\varepsilon) = 0$.

Thus the Signorini contact model is then given by the complementarity condition

$$\hat{u}_N^\varepsilon \leq \hat{d}^\varepsilon,\ \hat{G}_N^\varepsilon \leq 0,\ \hat{G}_N^\varepsilon(\hat{u}_N^\varepsilon - \hat{d}^\varepsilon) = 0.$$

Let $\Lambda \geq 0$ be the friction coefficient, then the static Coulomb friction law reads as follows

$$\begin{cases} |\hat{\mathbf{G}}_T^\varepsilon| < \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \hat{\mathbf{u}}_T^\varepsilon = 0 \text{ on } \hat{\Gamma}_+^\varepsilon, \\ |\hat{\mathbf{G}}_T^\varepsilon| = \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \exists\delta > 0,\ \hat{\mathbf{u}}_T^\varepsilon = -\delta\hat{\mathbf{G}}_T^\varepsilon \text{ on } \hat{\Gamma}_+^\varepsilon. \end{cases}$$

It states that, if $|\hat{\mathbf{G}}_T^\varepsilon| < \Lambda|\hat{G}_N^\varepsilon|$, then the shallow shell sticks to the foundation, that is $\hat{\mathbf{u}}_T^\varepsilon = 0$. If $|\hat{\mathbf{G}}_T^\varepsilon| = \Lambda|\hat{G}_N^\varepsilon|$, then the shallow shell slip to the foundation, that is $\hat{\mathbf{u}}_T^\varepsilon = -\delta\hat{\mathbf{G}}_T^\varepsilon$.

The dynamical Coulomb friction law reads as follows

$$\begin{cases} |\hat{\mathbf{G}}_T^\varepsilon| < \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = 0 \text{ on } \hat{\Gamma}_+^\varepsilon \times]0, +\infty[\,, \\ |\hat{\mathbf{G}}_T^\varepsilon| = \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \exists\delta > 0,\ \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = -\delta\hat{\mathbf{G}}_T^\varepsilon \text{ on } \hat{\Gamma}_+^\varepsilon \times]0, +\infty[\,. \end{cases}$$

It states that, if $|\hat{\mathbf{G}}_T^\varepsilon| < \Lambda|\hat{G}_N^\varepsilon|$, then the shallow shell sticks to the foundation, that is $\frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = 0$. If $|\hat{\mathbf{G}}_T^\varepsilon| = \Lambda|\hat{G}_N^\varepsilon|$, then the shallow shell slip to the foundation, that is $\frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = -\delta\hat{\mathbf{G}}_T^\varepsilon$.

1.4 Formulation of problems without unilateral contact

In this Section, we consider a nonlinearly shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε. The shell is subjected to body forces of density $(\hat{f}_i^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$ and to surface forces of density $(\hat{g}_i^\varepsilon)$ on its upper and lower faces $\hat{\Gamma}_+^\varepsilon$ and $\hat{\Gamma}_-^\varepsilon$. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

1.4.1 Generalized Marguerre-von Kármán shallow shell in elastostatic

Consider a nonlinearly elastostatic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The unknown in the three-dimensional formulation is the displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon)$, where the functions $\hat{u}_i^\varepsilon$ are their Cartesian components, satisfies the following three-dimensional boundary value problem

$$\left\{\begin{array}{l} -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ \left\{\begin{array}{l} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right), \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\}\nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1, \end{array}\right. \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } \gamma_2 \times [-\varepsilon, \varepsilon]\,, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon, \end{array}\right. \tag{1.2}$$

where

$$\left\{\begin{array}{l} \hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon), \\ \hat{E}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon). \end{array}\right. \tag{1.3}$$

This problem studied by Gratie [Gra02] in detail. In addition, when $\gamma_1 = \gamma$, we obtain the classical Marguerre-von Kármán equations, which have been studied by Ciarlet and Paumier [CP86].

1.4.2 Generalized Marguerre-von Kármán shallow shell in elastodynamic

Consider a nonlinearly elastodynamic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The unknown in the three-dimensional formulation is the displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$, where the functions $\hat{u}_i^\varepsilon$ are their Cartesian components satisfies the following three-dimensional boundary value problem

$$\left\{\begin{array}{l}\hat{\rho}^{\varepsilon}\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial t^2}-\hat{\partial}_j^{\varepsilon}(\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})=\ \hat{f}_i^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}\times]0,+\infty[\,,\\ \left\{\begin{array}{l}\hat{u}_{\alpha}^{\varepsilon}\text{ independent of }\hat{x}_3^{\varepsilon}\text{ and }\hat{u}_3^{\varepsilon}=0\text{ on }\boldsymbol{\Theta}^{\varepsilon}\left(\gamma_1\times[-\varepsilon,\varepsilon]\right)\times]0,+\infty[\,,\\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^{\varepsilon}+\hat{\sigma}_{k\beta}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_{\alpha}^{\varepsilon})\circ\boldsymbol{\Theta}^{\varepsilon}\}\nu_{\beta}dx_3^{\varepsilon}=\hat{h}_{\alpha}^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\gamma_1\times]0,+\infty[\,,\end{array}\right.\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=0\text{ on }(\gamma_2\times[-\varepsilon,\varepsilon])\times]0,+\infty[\,,\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=\hat{g}_i^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\left(\Gamma_+^{\varepsilon}\cup\Gamma_-^{\varepsilon}\right)\times]0,+\infty[\,,\\ \hat{\mathbf{u}}^{\varepsilon}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{p}}^{\varepsilon}\text{ and }\frac{\partial\hat{\mathbf{u}}^{\varepsilon}}{\partial t}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{q}}^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon},\end{array}\right.\tag{1.4}$$

where

$$\left\{\begin{array}{l}\hat{\rho}^{\varepsilon}:\ \text{the mass density},\\ \hat{\mathbf{p}}^{\varepsilon},\hat{\mathbf{q}}^{\varepsilon}:\ \text{the given initial data.}\end{array}\right.\tag{1.5}$$

This problem is studied in Chapter 2.

1.4.3 Generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shell in elastodynamic

Consider a nonlinearly elastodynamics shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^{\varepsilon}$, with thickness 2ε. We assume that the elastic material constituting the shell is nonhomogeneous and anisotropic, and that the reference configuration is a natural state.

The unknowns displacement field $\hat{\mathbf{u}}^{\varepsilon}=(\hat{u}_i^{\varepsilon})(\hat{x}^{\varepsilon},t)$ and stress field $\hat{\sigma}^{\varepsilon}=(\hat{\sigma}_{ij}^{\varepsilon})(\hat{x}^{\varepsilon},t)$ satisfies the following three-dimensional boundary value problem in Cartesian coordinates:

$$\left\{\begin{array}{l}\hat{\rho}^{\varepsilon}\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial t^2}-\hat{\partial}_j^{\varepsilon}(\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})=\ \hat{f}_i^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}\times]0,+\infty[\,,\\ \left\{\begin{array}{l}\hat{u}_{\alpha}^{\varepsilon}\text{ independent of }\hat{x}_3^{\varepsilon}\text{ and }\hat{u}_3^{\varepsilon}=0\text{ on }\boldsymbol{\Theta}^{\varepsilon}\left(\gamma_1\times[-\varepsilon,\varepsilon]\right)\times]0,+\infty[\,,\\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^{\varepsilon}+\hat{\sigma}_{k\beta}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_{\alpha}^{\varepsilon})\circ\boldsymbol{\Theta}^{\varepsilon}\}\nu_{\beta}dx_3^{\varepsilon}=\hat{h}_{\alpha}^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\gamma_1\times]0,+\infty[\,,\end{array}\right.\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=0\text{ on }(\gamma_2\times[-\varepsilon,\varepsilon])\times]0,+\infty[\,,\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=\hat{g}_i^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\left(\Gamma_+^{\varepsilon}\cup\Gamma_-^{\varepsilon}\right)\times]0,+\infty[\,,\\ (A\hat{\sigma}^{\varepsilon})_{ij}=\hat{\gamma}_{ij}^{\varepsilon}(\hat{\mathbf{u}}^{\varepsilon})+\frac{1}{2}\hat{\partial}_i^{\varepsilon}\hat{u}_l^{\varepsilon}\hat{\partial}_j^{\varepsilon}\hat{u}_l^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}\times]0,+\infty[\,,\\ \hat{\mathbf{u}}^{\varepsilon}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{p}}^{\varepsilon}\text{ and }\frac{\partial\hat{\mathbf{u}}^{\varepsilon}}{\partial t}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{q}}^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon},\end{array}\right.\tag{1.6}$$

where

$$\left\{\begin{array}{l}\hat{\gamma}_{ij}^{\varepsilon}(\hat{\mathbf{u}}^{\varepsilon})=\frac{1}{2}(\hat{\partial}_i^{\varepsilon}\hat{u}_j^{\varepsilon}+\hat{\partial}_j^{\varepsilon}\hat{u}_i^{\varepsilon}),\\ \hat{\rho}^{\varepsilon}:\ \text{the mass density},\\ \hat{\mathbf{p}}^{\varepsilon},\hat{\mathbf{q}}^{\varepsilon}:\ \text{the given initial data.}\end{array}\right.\tag{1.7}$$

The mapping A is defined by

$$(A\hat{\sigma}^{\varepsilon})_{ij}=\hat{c}_{ijkl}^{\varepsilon}\hat{\sigma}_{kl}^{\varepsilon},$$

where $\hat{C}^{\varepsilon}=(\hat{c}_{ijkl}^{\varepsilon})$ is the compliance tensor and $\hat{A}^{\varepsilon}=(\hat{a}_{ijkl}^{\varepsilon})$ the associated rigidity tensor.

This problem is studied in Chapter 3.

1.5 Formulation of problems with unilateral contact

In this Section, we consider a nonlinearly shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε. The shell is subjected to vertical body forces of density $(\hat{f}_i^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$, its lower face $\hat{\Gamma}_-^\varepsilon$ subjected to a surface forces of density $(\hat{g}_i^\varepsilon)$. We suppose also that this shell is in unilateral contact with Coulomb friction at the upper face $\hat{\Gamma}_+^\varepsilon$ and Λ its frictional coefficient, such that $\hat{d}^\varepsilon$ is the gap function which describes the distance between the upper face and the rigid foundation measured in the normal direction. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

1.5.1 Signorini problem with Coulomb friction of generalized Marguerre-von Kármán shallow shell in elastostatic

Consider a nonlinearly elastostatic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The problem consists of finding the displacement field $\hat{\mathbf{u}}^\varepsilon$ and the contact force $\hat{\mathbf{G}}^\varepsilon$ which satisfy the following problem in Cartesian coordinates:

$$\begin{cases} -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \ \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ \begin{cases} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right), \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\}\nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1, \end{cases} \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } \gamma_2 \times [-\varepsilon, \varepsilon], \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \Gamma_-^\varepsilon, \\ \hat{u}_N^\varepsilon \leq \hat{d}^\varepsilon, \hat{G}_N^\varepsilon \leq 0, \hat{G}_N^\varepsilon(\hat{u}_N^\varepsilon - \hat{d}^\varepsilon) = 0 \; on \; \hat{\Gamma}_+^\varepsilon, \\ |\hat{\mathbf{G}}_T^\varepsilon| < \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = 0 \; on \; \hat{\Gamma}_+^\varepsilon, \\ |\hat{\mathbf{G}}_T^\varepsilon| = \Lambda|\hat{G}_N^\varepsilon| \Rightarrow \exists \delta > 0, \; \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = -\delta \hat{\mathbf{G}}_T^\varepsilon \; on \; \hat{\Gamma}_+^\varepsilon, \end{cases} \tag{1.8}$$

where

$$\begin{cases} \hat{u}_N^\varepsilon = \hat{\mathbf{u}}^\varepsilon \hat{\mathbf{n}}^\varepsilon, \hat{\mathbf{u}}_T^\varepsilon = \hat{\mathbf{u}}^\varepsilon - \hat{u}_N^\varepsilon \hat{\mathbf{n}}^\varepsilon, \\ \hat{G}_N^\varepsilon = \hat{\mathbf{G}}^\varepsilon \hat{\mathbf{n}}^\varepsilon, \hat{\mathbf{G}}_T^\varepsilon = \hat{\mathbf{G}}^\varepsilon - \hat{G}_N^\varepsilon \hat{\mathbf{n}}^\varepsilon. \end{cases} \tag{1.9}$$

This problem is studied in Chapter 4.

1.5.2 Signorini problem with Coulomb friction of generalized Marguerre-von Kármán shallow shell in elastodynamic

Consider a nonlinearly elastodynamic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The unknowns displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$, stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)(\hat{x}^\varepsilon, t)$ and the contact force $\hat{\mathbf{G}}^\varepsilon$ satisfies the following three-dimensional boundary value problem in Cartesian coordinates:

$$\begin{cases} \hat{\rho}^\varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} - \hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[\,, \\ \begin{cases} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\} \nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1 \times]0, +\infty[\,, \end{cases} \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } (\gamma_2 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \Gamma_-^\varepsilon \times]0, +\infty[\,, \\ \hat{u}_N^\varepsilon \le \hat{d}^\varepsilon, \hat{G}_N^\varepsilon \le 0, \hat{G}_N^\varepsilon(\hat{u}_N^\varepsilon - \hat{d}^\varepsilon) = 0 \; on \; \hat{\Gamma}_+^\varepsilon \times]0, +\infty[\,, \\ |\hat{\mathbf{G}}_T^\varepsilon| < \Lambda |\hat{G}_N^\varepsilon| \Rightarrow \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = 0 \; on \; \hat{\Gamma}_+^\varepsilon \times]0, +\infty[\,, \\ |\hat{\mathbf{G}}_T^\varepsilon| = \Lambda |\hat{G}_N^\varepsilon| \Rightarrow \exists \delta > 0, \; \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = -\delta \hat{\mathbf{G}}_T^\varepsilon \; on \; \hat{\Gamma}_+^\varepsilon \times]0, +\infty[\,, \\ \hat{\mathbf{u}}^\varepsilon(\hat{x}^\varepsilon, 0) = \hat{\mathbf{p}}^\varepsilon, \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}(\hat{x}^\varepsilon, 0) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \end{cases} \tag{1.10}$$

where

$$\begin{cases} \frac{\partial \hat{u}_N^\varepsilon}{\partial t} = \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t} \hat{\mathbf{n}}^\varepsilon, \\ \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t} - \frac{\partial \hat{u}_N^\varepsilon}{\partial t} \hat{\mathbf{n}}^\varepsilon. \end{cases} \tag{1.11}$$

This problem is studied in Chapter 5.

1.5.3 Signorini problem with Coulomb friction of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shell in elastodynamic

Consider a nonlinearly elastodynamics shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε. We assume that the elastic material constituting the shell is nonhomogeneous and anisotropic, and that the reference configuration is a natural state.

The unknowns displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$, stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)(\hat{x}^\varepsilon, t)$ and the contact force $\hat{\mathbf{G}}^\varepsilon$ satisfies the following three-dimensional boundary value problem in

Cartesian coordinates:

$$\left\{\begin{array}{l}\hat{\rho}^{\varepsilon}\frac{\partial^2\hat{u}_i^{\varepsilon}}{\partial t^2}-\hat{\partial}_j^{\varepsilon}(\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})=\ \hat{f}_i^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}\times]0,+\infty[\,,\\ \left\{\begin{array}{l}\hat{u}_{\alpha}^{\varepsilon}\text{ independent of }\hat{x}_3^{\varepsilon}\text{ and }\hat{u}_3^{\varepsilon}=0\text{ on }\boldsymbol{\Theta}^{\varepsilon}\left(\gamma_1\times[-\varepsilon,\varepsilon]\right)\times]0,+\infty[\,,\\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^{\varepsilon}+\hat{\sigma}_{k\beta}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_{\alpha}^{\varepsilon})\circ\boldsymbol{\Theta}^{\varepsilon}\}\nu_{\beta}dx_3^{\varepsilon}=\hat{h}_{\alpha}^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\gamma_1\times]0,+\infty[\,,\end{array}\right.\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=0\text{ on }(\gamma_2\times[-\varepsilon,\varepsilon])\times]0,+\infty[\,,\\ (\hat{\sigma}_{ij}^{\varepsilon}+\hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{n}_j^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}=\hat{g}_i^{\varepsilon}\circ\boldsymbol{\Theta}^{\varepsilon}\text{ on }\Gamma_-^{\varepsilon}\times]0,+\infty[\,,\\ (A\hat{\sigma}^{\varepsilon})_{ij}=\hat{\gamma}_{ij}^{\varepsilon}(\hat{\mathbf{u}}^{\varepsilon})+\frac{1}{2}\hat{\partial}_i^{\varepsilon}\hat{u}_l^{\varepsilon}\hat{\partial}_j^{\varepsilon}\hat{u}_l^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}\times]0,+\infty[\,,\\ \hat{u}_N^{\varepsilon}\leq\hat{d}^{\varepsilon},\hat{G}_N^{\varepsilon}\leq 0,\hat{G}_N^{\varepsilon}(\hat{u}_N^{\varepsilon}-\hat{d}^{\varepsilon})=0\ on\ \hat{\Gamma}_+^{\varepsilon}\times]0,+\infty[\,,\\ |\hat{\mathbf{G}}_T^{\varepsilon}|<\Lambda|\hat{G}_N^{\varepsilon}|\Rightarrow\frac{\partial\hat{\mathbf{u}}_T^{\varepsilon}}{\partial t}=0\ on\ \hat{\Gamma}_+^{\varepsilon}\times]0,+\infty[\,,\\ |\hat{\mathbf{G}}_T^{\varepsilon}|=\Lambda|\hat{G}_N^{\varepsilon}|\Rightarrow\exists\delta>0,\ \frac{\partial\hat{\mathbf{u}}_T^{\varepsilon}}{\partial t}=-\delta\hat{\mathbf{G}}_T^{\varepsilon}\ on\ \hat{\Gamma}_+^{\varepsilon}\times]0,+\infty[\,,\\ \hat{\mathbf{u}}^{\varepsilon}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{p}}^{\varepsilon},\frac{\partial\hat{\mathbf{u}}^{\varepsilon}}{\partial t}\left(\hat{x}^{\varepsilon},0\right)=\hat{\mathbf{q}}^{\varepsilon}\text{ in }\hat{\Omega}^{\varepsilon}.\end{array}\right. \tag{1.12}$$

This problem is studied in Chapter 6.

1.6 Conclusion

In this Chapter, we present a new three-dimensional models for generalized Marguerre-von Kármán shallow shell, two models without contact and three models with contact. These models are reduced (lower-dimensional) to two-dimensional models using asymptotic analysis in Chapters 2-6.

Part I

Problems without unilateral contact

Chapter 2

Dynamical equations of generalized Marguerre-von Kármán shallow shells

In a recent work Gratie [Gra02] has generalized the classical Marguerre-von Kármán equations studied by Ciarlet and Paumier in [CP86], where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type and the remaining portion being free. In this Chapter, we extend formally this study to dynamical case. To this end, we have identified the dynamical equations of generalized Marguerre-von Kármán shallow shells. This work was published in [CGB10]. Then, we establish the existence of solutions to these equations. This results was published in [CGB13].

2.1 Asymptotic Modeling of generalized Marguerre-von Kármán shallow shell in dynamical case

2.1.1 Three-dimensional problem

Consider a nonlinearly elastodynamic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The shell is subjected to vertical body forces of density $(\hat{f}_i^\varepsilon) = (0, 0, \hat{f}_3^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$ and to vertical surface forces of density $(\hat{g}_i^\varepsilon) = (0, 0, \hat{g}_3^\varepsilon)$ on its upper and lower faces $\hat{\Gamma}_+^\varepsilon$ and $\hat{\Gamma}_-^\varepsilon$. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

The unknown in the three-dimensional formulation is the displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$, where the functions $\hat{u}_i^\varepsilon$ are their Cartesian components, which satisfies the following three-dimensional boundary value problem:

$$(C.\hat{P}^\varepsilon)_{dyn}^{iso} \begin{cases} \hat{\rho}^\varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} - \hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \ \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[\,, \\ \begin{cases} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon\,(\gamma_1 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\}\nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1 \times]0, +\infty[\,, \end{cases} \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } (\gamma_2 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \left(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon\right) \times]0, +\infty[\,, \\ \hat{\mathbf{u}}^\varepsilon\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{p}}^\varepsilon \text{ and } \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \end{cases}$$

where

$$\begin{cases} \hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon), \\ \hat{E}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon), \\ \hat{\rho}^\varepsilon : \text{ the mass density}, \\ \hat{\mathbf{p}}^\varepsilon, \hat{\mathbf{q}}^\varepsilon : \text{ the given initial data.} \end{cases} \tag{2.1}$$

First, we rewrite the above boundary value problem $(C.\hat{P}^\varepsilon)_{dyn}^{iso}$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem:

$$(V.\hat{P}^\varepsilon)_{dyn}^{iso} \begin{cases} \text{Find } \hat{\mathbf{u}}^\varepsilon(\hat{x}^\varepsilon, t) \in \mathbf{V}(\hat{\Omega}^\varepsilon)\ \forall t \geq 0, \text{ such that,} \\ \frac{d^2}{dt^2}\left\{\hat{\rho}^\varepsilon \int_{\hat{\Omega}^\varepsilon} \hat{u}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon\right\} + \int_{\hat{\Omega}^\varepsilon}(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon = \int_{\hat{\Omega}^\varepsilon} \hat{f}_3^\varepsilon \hat{v}_3^\varepsilon d\hat{x}^\varepsilon \\ + \int_{\hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon} \hat{g}_3^\varepsilon \hat{v}_3^\varepsilon d\hat{\Gamma}^\varepsilon + \int_{\hat{\gamma}_1^\varepsilon}\{\int_{-\varepsilon}^{\varepsilon}(\hat{v}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon)\,dx_3^\varepsilon\}\hat{h}_\alpha^\varepsilon d\hat{\gamma}^\varepsilon, \\ \forall \hat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\hat{\Omega}^\varepsilon), \forall t > 0, \\ \hat{\mathbf{u}}^\varepsilon\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{p}}^\varepsilon \text{ and } \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon. \end{cases}$$

Next, we formulate the variational problem $(V.\hat{P}^\varepsilon)^{iso}_{dyn}$ on a domain Ω^ε.

Proposition 2.1 *Let there be a given C^1-diffeomorphism $\mathbf{\Theta}^\varepsilon$ that satisfies the orientation-preserving condition. Then the variational problem $(V.\hat{P}^\varepsilon)^{iso}_{dyn}$ is equivalent to the following variational problem*

$$(P^\varepsilon)^{iso}_{dyn}\begin{cases} \textit{Find } \mathbf{u}^\varepsilon(x^\varepsilon,t)\in\mathbf{V}(\Omega^\varepsilon)\ \forall t\geq 0,\ \textit{such that,} \\ \frac{d^2}{dt^2}\left\{\rho^\varepsilon\int_{\Omega^\varepsilon}u_i^\varepsilon v_i^\varepsilon\delta^\varepsilon dx^\varepsilon\right\}+\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon b_{kj}^\varepsilon\partial_k^\varepsilon v_i^\varepsilon\delta^\varepsilon dx^\varepsilon \\ +\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon b_{ki}^\varepsilon\partial_k^\varepsilon u_l^\varepsilon b_{mj}^\varepsilon\partial_m^\varepsilon v_l^\varepsilon\delta^\varepsilon dx^\varepsilon=\int_{\Omega^\varepsilon}f_3^\varepsilon v_3^\varepsilon\delta^\varepsilon dx^\varepsilon+\int_{\Gamma_+^\varepsilon\cup\Gamma_-^\varepsilon}g_3^\varepsilon v_3^\varepsilon\delta^\varepsilon\beta^\varepsilon d\Gamma^\varepsilon \\ +\int_{\gamma_1}h_\alpha^\varepsilon\{\int_{-\varepsilon}^{\varepsilon}v_\alpha^\varepsilon dx_3^\varepsilon\}d\gamma,\forall\mathbf{v}^\varepsilon\in\mathbf{V}(\Omega^\varepsilon),\forall t>0, \\ \mathbf{u}^\varepsilon(x^\varepsilon,0)=\mathbf{p}^\varepsilon\ \textit{and}\ \frac{\partial\mathbf{u}^\varepsilon}{\partial t}(x^\varepsilon,0)=\mathbf{q}^\varepsilon\ \textit{in}\ \Omega^\varepsilon, \end{cases}$$

where

$$\begin{cases} u_i^\varepsilon=\hat{u}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon, f_i^\varepsilon=\hat{f}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon, \\ g_i^\varepsilon=\hat{g}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon, h_\alpha^\varepsilon=\hat{h}_\alpha^\varepsilon\circ\mathbf{\Theta}^\varepsilon, \\ p_i^\varepsilon=\hat{p}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon, q_i^\varepsilon=\hat{q}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon, \end{cases} \tag{2.2}$$

$$\begin{cases} \hat{\partial}_j^\varepsilon\hat{v}_i^\varepsilon=b_{kj}^\varepsilon(x^\varepsilon)\partial_k^\varepsilon v_i^\varepsilon(x^\varepsilon), \\ d\hat{x}^\varepsilon=\delta^\varepsilon dx^\varepsilon, \\ d\hat{\Gamma}^\varepsilon=\delta^\varepsilon\beta^\varepsilon d\Gamma^\varepsilon, \end{cases} \tag{2.3}$$

such that

$$\begin{cases} \nabla^\varepsilon\mathbf{\Theta}^\varepsilon(x^\varepsilon)=(\partial_j^\varepsilon\Theta_i^\varepsilon(x^\varepsilon))\ \forall x^\varepsilon\in\bar{\Omega}^\varepsilon, \\ \delta^\varepsilon(x^\varepsilon)=\det\nabla^\varepsilon\mathbf{\Theta}^\varepsilon(x^\varepsilon)\ \forall x^\varepsilon\in\bar{\Omega}^\varepsilon, \\ b_{ij}^\varepsilon(x^\varepsilon)=(\{\nabla^\varepsilon\mathbf{\Theta}^\varepsilon(x^\varepsilon)\}^{-1})_{ij}\ \forall x^\varepsilon\in\bar{\Omega}^\varepsilon, \\ \beta^\varepsilon(x^\varepsilon)=\{b_{3i}(x^\varepsilon)b_{3i}(x^\varepsilon)\}^{\frac{1}{2}}\ \forall x^\varepsilon\in(\Gamma_+^\varepsilon\cup\Gamma_-^\varepsilon). \end{cases}$$

Proof.

In the problem $(V.\hat{P}^\varepsilon)^{iso}_{dyn}$, using the relations (2.2)-(2.3), we obtain the problem $(P^\varepsilon)^{iso}_{dyn}$. ■

2.1.2 Asymptotic analysis

We first transform $(P^\varepsilon)^{iso}_{dyn}$ into a problem posed over an open set independent of ε. Accordingly, let

$\Omega = \omega \times]-1,1[$, $\Gamma_\pm = \omega \times \{\pm 1\}$ and to any point $x \in \bar{\Omega}$, we associate the point $x^\varepsilon \in \bar{\Omega}^\varepsilon$ by the bijection $\pi^\varepsilon : x = (x_1, x_2, x_3) \in \bar{\Omega} \to x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$.

To the functions $\mathbf{u}^\varepsilon$, $\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$ and $\sigma^\varepsilon \in \Sigma(\Omega^\varepsilon)$, we associate the scaled functions $\mathbf{u}(\varepsilon)$, $\mathbf{v}$ and $\sigma(\varepsilon)$ defined by

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon, t) = \varepsilon^2 u_\alpha(\varepsilon)(x,t), u_3^\varepsilon(x^\varepsilon, t) = \varepsilon u_3(\varepsilon)(x,t), \\ v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x), v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x), \\ \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon, t) = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)(x,t), \sigma_{\alpha 3}^\varepsilon(x^\varepsilon, t) = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x,t), \\ \sigma_{33}^\varepsilon(x^\varepsilon, t) = \varepsilon^4 \sigma_{33}(\varepsilon)(x,t), \end{cases} \tag{2.4}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon$.

Next, we make the following assumptions: there exists constants $\lambda > 0$, $\mu > 0$, $\rho > 0$ and for some $T > 0$, the functions $f_3 \in L^2(0,T; L^2(\Omega))$, $g_3 \in L^2(0,T; L^2(\Gamma_+ \cup \Gamma_-))$, $h_\alpha \in L^2(0,T; L^2(\gamma_1))$, $\theta \in C^3(\bar{\omega})$ independent of ε and $\mathbf{p}(\varepsilon) \in \mathbf{V}(\Omega)$, $\mathbf{q}(\varepsilon) \in L^2(\Omega; \mathbb{R}^3)$, such that

$$\begin{cases} \lambda^\varepsilon = \lambda, \mu^\varepsilon = \mu, \rho^\varepsilon = \varepsilon^2 \rho, \\ f_3^\varepsilon(x^\varepsilon, t) = \varepsilon^3 f_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ g_3^\varepsilon(x^\varepsilon, t) = \varepsilon^4 g_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in (\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon), \\ h_\alpha^\varepsilon(y_1, y_2, t) = \varepsilon^2 h_\alpha(y_1, y_2, t) \ \forall (y_1, y_2) \in \gamma_1, \\ \theta^\varepsilon(x_1, x_2) = \varepsilon \theta(x_1, x_2) \ \forall (x_1, x_2) \in \bar{\omega}, \\ p_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 p_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ p_3^\varepsilon(x^\varepsilon) = \varepsilon p_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 q_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q_3^\varepsilon(x^\varepsilon) = \varepsilon q_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon. \end{cases} \tag{2.5}$$

Using the scalings (2.4) and the assumptions (2.5), we obtain

Theorem 2.1 *The scaled displacement field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ satisfies the following variational problem*

$$(P(\varepsilon))_{dyn}^{iso} \begin{cases} \textit{Find } \mathbf{u}(\varepsilon)(x,t) \in \mathbf{V}(\Omega) \ \forall t \in [0,T], \ \textit{such that,} \\ A^t(\mathbf{u}(\varepsilon), \mathbf{v}) + B^\theta(\sigma(\varepsilon), \mathbf{v}) + 2C^\theta(\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) = F(\mathbf{v}) \\ +\varepsilon^2 R(\varepsilon; \sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}), \forall \mathbf{v} \in \mathbf{V}(\Omega), \forall t \in]0,T[, \\ \mathbf{u}(\varepsilon)(x,0) = \mathbf{p}(\varepsilon) \ \textit{and} \ \frac{\partial \mathbf{u}(\varepsilon)}{\partial t}(x,0) = \mathbf{q}(\varepsilon) \ \textit{in } \Omega, \end{cases}$$

where

$$\begin{cases} A^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=-\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3\left(\varepsilon\right)v_3dx\right\}, \\ B^\theta\left(\sigma\left(\varepsilon\right),\mathbf{v}\right)=-\int_\Omega\sigma_{ij}\left(\varepsilon\right)\gamma_{ij}^\theta(\mathbf{v})dx, \\ C^\theta\left(\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=-\frac{1}{2}\int_\Omega\sigma_{ij}\left(\varepsilon\right)\partial_i^\theta u_3\left(\varepsilon\right)\partial_j^\theta v_3dx, \\ F\left(\mathbf{v}\right)=-\int_\Omega f_3v_3dx-\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma-\int_{\gamma_1}h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma, \\ \partial_\alpha^\theta v=\partial_\alpha v-\partial_\alpha\theta\partial_3 v,\partial_3^\theta v=\partial_3 v,\gamma_{ij}^\theta(\mathbf{v})=\frac{1}{2}\left(\partial_i^\theta v_j+\partial_j^\theta v_i\right). \end{cases}$$

Proof.

The proof is similar to that of Theorem 3.1 in [CP86],
we have

$$\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon b_{kj}^\varepsilon\partial_k^\varepsilon v_i^\varepsilon\delta^\varepsilon dx^\varepsilon=\varepsilon^5\int_\Omega\sigma_{ij}\left(\varepsilon\right)\gamma_{ij}^\theta\left(\mathbf{v}\right)dx+\varepsilon^7\varrho_B\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{v}\right),$$

$$\begin{aligned}\int_{\Omega^\varepsilon}\sigma_{ij}^\varepsilon b_{ki}^\varepsilon\partial_k^\varepsilon u_l^\varepsilon b_{mj}^\varepsilon\partial_m^\varepsilon v_l^\varepsilon\delta^\varepsilon dx^\varepsilon &= \varepsilon^5\int_\Omega\sigma_{ij}\left(\varepsilon\right)\partial_i^\theta u_3\left(\varepsilon\right)\partial_j^\theta v_3dx \\ &+ \varepsilon^7\varrho_C\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right),\end{aligned}$$

$$\int_{\Omega^\varepsilon}f_3^\varepsilon v_3^\varepsilon\delta^\varepsilon dx^\varepsilon+\int_{\Gamma_+^\varepsilon\cup\Gamma_-^\varepsilon}g_3^\varepsilon v_3^\varepsilon\delta^\varepsilon\beta^\varepsilon d\Gamma^\varepsilon+\int_{\gamma_1}h_\alpha^\varepsilon\left\{\int_{-\varepsilon}^\varepsilon v_\alpha^\varepsilon dx_3^\varepsilon\right\}d\gamma =$$
$$\varepsilon^5\left(\int_\Omega f_3v_3dx+\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma+\int_{\gamma_1}h_\alpha\left\{\int_{-1}^1 v_\alpha dx_3\right\}d\gamma\right)+\varepsilon^7\varrho_F\left(\varepsilon;\mathbf{v}\right).$$

In addition

$$\frac{d^2}{dt^2}\left\{\rho^\varepsilon\int_{\Omega^\varepsilon}u_i^\varepsilon v_i^\varepsilon\delta^\varepsilon dx^\varepsilon\right\}=\varepsilon^5\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3\left(\varepsilon\right)v_3dx\right\}+\varepsilon^7\varrho_D(\varepsilon;\mathbf{u}(\varepsilon),\mathbf{v}).$$

Then

$$-\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3(\varepsilon)v_3dx\right\}-\int_\Omega\sigma_{ij}(\varepsilon)\gamma_{ij}^\theta(\mathbf{v})dx-\int_\Omega\sigma_{ij}(\varepsilon)\partial_i^\theta u_3(\varepsilon)\partial_j^\theta v_3dx=$$
$$-\int_\Omega f_3v_3dx-\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma-\int_{\gamma_1}h_\alpha\left\{\int_{-1}^1 v_\alpha dx_3\right\}d\gamma+\varepsilon^2R\left(\varepsilon;\sigma(\varepsilon),\mathbf{u}(\varepsilon),\mathbf{v}\right).$$

The remainders ϱ_B, ϱ_C, ϱ_F and ϱ_D are bounded for all $0\le\varepsilon\le\varepsilon_0$, (see [CP86]). ■

We use technics of asymptotic analysis employed by Ciarlet [Cia90] in $(P(\varepsilon))_{dyn}^{iso}$ (ε designates a parameter approaches zero), we obtain

Theorem 2.2 *We suppose that*

$$(\mathbf{u}(\varepsilon), \sigma(\varepsilon)) = (\mathbf{u}^0, \sigma^0) + \varepsilon(\mathbf{u}^1, \sigma^1) + \varepsilon^2(\mathbf{u}^2, \sigma^2) + \cdots,$$

with

$$\mathbf{u}^0 = \left(u_i^0\right) \in \mathbf{V}(\Omega),\ \partial_3 u_3^0 \in C^0\left(\bar{\Omega}\right),$$

$$\mathbf{u}^p = (u_i^p) \in W^{1,4}(\Omega; \mathbb{R}^3),\ \forall p \geq 1,\ \sigma_{ij}^0 = \sigma_{ji}^0 \in L^2(\Omega).$$

Then the leading term $\mathbf{u}^0 \in \mathbf{V}_{KL}(\Omega)$ $\forall t \in [0,T]$ *(Kirchhoff-Love displacement field) is solution of the problem*

$$(P_{KL})_{dyn}^{iso} \begin{cases} \textit{Find } \mathbf{u}^0 \in \mathbf{V}_{KL}(\Omega)\ \forall t \in [0,T],\ \textit{such that,} \\ \frac{d^2}{dt^2}\left\{\rho \int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\beta v_\alpha dx + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha (u_3^0 + \theta) \partial_\beta v_3 dx = \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_+ \cup \Gamma_-} g_3 v_3 d\Gamma + 2\int_{\gamma_1} h_\alpha v_\alpha d\gamma, \forall \mathbf{v} \in \mathbf{V}_{KL}(\Omega), \forall t \in]0,T[\,, \\ \mathbf{u}^0(x,0) = \mathbf{p}^0 \textit{ and } \frac{\partial \mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0 \textit{ in } \Omega, \end{cases}$$

where

$$\begin{cases} \sigma_{\alpha\beta}^0 = \frac{2\lambda\mu}{\lambda+2\mu} \bar{E}_{\sigma\sigma}^0(\mathbf{u}^0)\delta_{\alpha\beta} + 2\mu \bar{E}_{\alpha\beta}^0(\mathbf{u}^0), \\ \bar{E}_{\alpha\beta}^0(\mathbf{u}^0) = \frac{1}{2}(\partial_\alpha u_\beta^0 + \partial_\beta u_\alpha^0 + \partial_\alpha u_3^0 \partial_\beta u_3^0 + \partial_\alpha \theta \partial_\beta u_3^0 + \partial_\beta \theta \partial_\alpha u_3^0). \end{cases} \tag{2.6}$$

Proof.

We introduce the formal series expansions of the scaled displacement and the scaled stresses $(\mathbf{u}(\varepsilon), \sigma(\varepsilon))$ into the variational problem $(P(\varepsilon))_{dyn}^{iso}$ and cancel the successive powers of ε, until we can fully identify the leading term, for more details see [Cia90, Cia97].
■

2.1.3 Equivalence with a two-dimensional problem

We show that the leading term $\mathbf{u}^0$ of the asymptotic expansions $\mathbf{u}(\varepsilon)$ is characterized by the following two-dimensional problem

Theorem 2.3 *The components of the leading term* $\mathbf{u}^0 = (u_i^0)$ *are of the form* $u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ *and* $u_3^0 = \zeta_3$ *with* $\zeta = (\zeta_i) \in \mathbf{V}(\omega)$ $\forall t \in [0,T]$*, where the field* ζ *satisfies the following limit scaled two-dimensional displacement problem*

$$(P(\omega))_{dyn}^{iso}\begin{cases} 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m_{\alpha\beta}(\nabla^2\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega \\ + \int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \forall\eta\in\mathbf{V}(\omega), \forall t \in]0,T[, \\ \zeta_3(.,0) = p_3^0 \text{ and } \frac{\partial\zeta_3}{\partial t}(.,0) = q_3^0 \text{ in } \omega, \end{cases}$$

where

$$\begin{cases} m_{\alpha\beta}(\nabla^2\zeta_3) = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\}, \\ \bar{N}_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu}\bar{E}_{\sigma\sigma}^0(\zeta)\,\delta_{\alpha\beta} + 4\mu\bar{E}_{\alpha\beta}^0(\zeta), \\ \bar{E}_{\alpha\beta}^0(\zeta) = \frac{1}{2}\left(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3 + \partial_\alpha\zeta_3\partial_\beta\zeta_3\right), \\ \mathrm{p}_3 = \int_{-1}^1 f_3 dx_3 + g_3(.,+1) + g_3(.,-1). \end{cases}$$

Proof. Since $\mathbf{v} = (v_i) \in \mathbf{V}_{KL}(\Omega)$, we conclude that there exists $\eta = (\eta_i) \in \mathbf{V}(\omega)$, such that $v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3$ and $v_3 = \eta_3$ (see [Cia97, Theorem 1.4-4]).

(i) We take $\mathbf{v} = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$, with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu\eta_3 = 0$ on γ_1, we get

$$\frac{d^2}{dt^2}\{\rho\int_\Omega \zeta_3\eta_3 dx\} + \int_\Omega -x_3\sigma_{\alpha\beta}^0\partial_{\alpha\beta}\eta_3 dx + \int_\Omega \sigma_{\alpha\beta}^0\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\Omega f_3\eta_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3\eta_3 d\Gamma.$$

(ii) We take $\mathbf{v} = (\eta_1, \eta_2, 0)$, with $\eta_\alpha \in H^1(\omega)$, we get

$$\int_\Omega \sigma_{\alpha\beta}^0\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

(iii) Using Fubini's Formula: $\int_\Omega F dx = \int_\omega\left\{\int_{-1}^1 F dx_3\right\} d\omega$, we obtain

$$\frac{d^2}{dt^2}\{\rho\int_\Omega \zeta_3\eta_3 dx\} = 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega,$$

$$\int_\Omega -x_3\sigma_{\alpha\beta}^0\partial_{\alpha\beta}\eta_3 dx = -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega,$$

$$\int_\Omega f_3\eta_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3\eta_3 d\Gamma = \int_\omega \{\int_{-1}^1 f_3 dx_3 + g_3(.,+1) + g_3(.,-1)\}\eta_3 d\omega$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = \int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

Then

$$2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega$$
$$+\int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \{\int_{-1}^1 f_3 dx_3 + g_3(.,+1) + g_3(.,-1)\}\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

■

Next, we write the two-dimensional boundary value problem as an equivalent variational problem $(\bar{P}(\omega))^{iso}_{dyn}$, using Green's formulas.

Theorem 2.4 *If $\zeta = (\zeta_i)$ is a solution of the problem $(P(\omega))^{iso}_{dyn}$ sufficiently regular, then is also a solution of the following two-dimensional problem*

$$(\bar{P}(\omega))^{iso}_{dyn}\begin{cases} \textit{Find } \zeta \in \mathbf{V}(\omega)\ \forall t\in[0,T], \textit{ such that,} \\ 2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\zeta_3) - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = \mathrm{p}_3 \textit{ in } \omega\times]0,T[, \\ \partial_\beta\bar{N}_{\alpha\beta} = 0 \textit{ in } \omega\times]0,T[, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \textit{ on } \gamma_1\times]0,T[, \\ \bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha \textit{ on } \gamma_1\times]0,T[, \\ m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\tau_\beta) = 0 \textit{ on } \gamma_2\times]0,T[, \\ \bar{N}_{\alpha\beta}\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[, \\ \zeta_3(.,0) = p^0_3 \textit{ and } \frac{\partial\zeta_3}{\partial t}(.,0) = q^0_3 \textit{ in } \omega. \end{cases}$$

Proof.

Applying the Green formulas, we obtain

$$
\begin{aligned}
-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega &= \int_\gamma \{(\partial_\alpha m_{\alpha\beta})\,\nu_\beta + \partial_\tau\,(m_{\alpha\beta}\nu_\alpha\tau_\beta)\}\,\eta_3 d\gamma \\
&- \int_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma - \int_\omega (\partial_{\alpha\beta}m_{\alpha\beta})\,\eta_3 d\omega,
\end{aligned}
$$

$$
\begin{aligned}
\int_\omega \bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\,\partial_\beta\eta_3 d\omega &= -\int_\omega \left\{\partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right)\right\}\eta_3 d\omega \\
&+ \int_\gamma \left(\bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right)\nu_\beta\eta_3 d\gamma,
\end{aligned}
$$

$$
\int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = -\int_\omega \left(\partial_\beta\bar{N}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \bar{N}_{\alpha\beta}\nu_\beta\eta_\alpha d\gamma.
$$

Thus

$$
\begin{aligned}
\int_\omega \left[2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right) - p_3\right]\eta_3 d\omega &\quad - \\
\int_\omega \left(\partial_\beta\bar{N}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \left(\bar{N}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma &\quad + \\
\int_{\gamma_2} \left\{\left[\partial_\alpha m_{\alpha\beta} + \bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right]\nu_\beta + \partial_\tau\,(m_{\alpha\beta}\nu_\alpha\tau_\beta)\right\}\eta_3 d\gamma &= 0,
\end{aligned}
$$

for all $\eta = (\eta_\alpha, \eta_3) \in \mathbf{V}(\omega)$. The functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by

$$
\widetilde{h}_\alpha = h_\alpha \ \text{ on } \ \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \ \text{ on } \ \gamma_2 \times [0,T].
$$

These equations imply that all the factors of η_α, η_3, and $\partial_\nu\eta_3$ vanish in their respective domains of integration. Then we get

$$
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \ \partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right) = \mathrm{p}_3 \text{ in } \omega \times \left]0,T\right[,
$$

and

$$
\partial_\beta\bar{N}_{\alpha\beta} = 0 \ \text{ in } \omega \times \left]0,T\right[,
$$

so that

$$
\partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha\,(\zeta_3+\theta)\right) = \bar{N}_{\alpha\beta}\partial_{\alpha\beta}\,(\zeta_3+\theta) \ \text{ in } \omega \times \left]0,T\right[,
$$

consequently

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3 + \theta\right) = \mathrm{p}_3 \text{ in } \omega \times]0, T[\,.$$

For boundary conditions, we get

$$\bar{N}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha = 0 \;\text{ on } \gamma \times]0, T[\,,$$

thus

$$\bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha \;\text{ on } \gamma_1 \times]0, T[\,,$$

and

$$\bar{N}_{\alpha\beta}\nu_\beta = 0 \;\text{ on } \gamma_2 \times]0, T[\,.$$

We also get

$$m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \;\text{ on } \gamma_2 \times]0, T[\,,$$

and

$$\left[\partial_\alpha m_{\alpha\beta} + \bar{N}_{\alpha\beta}\partial_\alpha\left(\zeta_3 + \theta\right)\right]\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \;\text{ on } \gamma_2 \times]0, T[\,,$$

since $\bar{N}_{\alpha\beta}\nu_\beta = 0$ on $\gamma_2 \times]0, T[$, we conclude that

$$\partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2 \times]0, T[\,.$$

■

2.1.4 Equivalence with the dynamical equations of generalized Marguerre-von Kármán shallow shells

We rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))^{iso}_{dyn}$ as dynamical equations of generalized Marguerre-von Kármán shallow shells as follows

Theorem 2.5 *Assume that the set ω is simply-connected and that its boundary γ is smooth enough. Let $\zeta = (\zeta_i)$ be a solution of $(\bar{P}(\omega))^{iso}_{dyn}$ with the regularity $\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^4(\omega)$ $\forall t \in [0, T]$. Then*

a) *The functions* $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ *defined by :*

$$\widetilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0,T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ *and satisfy the compatibility conditions :*

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma = 0.$$

b) *Furthermore, there exists a function* $\Phi \in H^4(\omega)$*, uniquely defined by the relations* $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$*, such that*

$$\bar{N}_{11} = 2\partial_{22}\Phi,\ \bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi,\ \bar{N}_{22} = 2\partial_{11}\Phi.$$

c) *Finally, the pair* $(\zeta_3, \Phi) \in H^4(\omega) \times H^4(\omega)\ \forall t \in [0,T]$*, satisfies the following scaled dynamic equations of generalized Marguerre-von Kármán shallow shells*

$$(P)^{iso}_{dyn} \begin{cases} 2\rho\frac{\partial^2\zeta_3}{\partial t^2} + \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 = 2\,[\Phi, \zeta_3 + \theta] + \mathrm{p}_3 \text{ in } \omega \times]0,T[\,, \\ \Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}\,[\zeta_3, \zeta_3 + 2\theta] \text{ in } \omega \times]0,T[\,, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma_1 \times]0,T[\,, \\ m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \Phi = \Phi_0 \text{ and } \partial_\nu\Phi = \Phi_1 \text{ on } \gamma \times]0,T[\,, \\ \zeta_3\,(.,0) = p_3^0 \text{ and } \frac{\partial\zeta_3}{\partial t}\,(.,0) = q_3^0 \text{ in } \omega, \end{cases}$$

where

$$\begin{cases} \Phi_0(y) = -y_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \\ \Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma,\ y = (y_1, y_2) \in \gamma, \\ [\Phi, \zeta] = \partial_{11}\Phi\partial_{22}\zeta + \partial_{22}\Phi\partial_{11}\zeta - 2\partial_{12}\Phi\partial_{12}\zeta. \end{cases}$$

Proof.

a) From the definition of $\bar{N}_{\alpha\beta}$, and since $\bar{N}_{\alpha\beta}\nu_\beta = 2\widetilde{h}_\alpha$ on γ, we conclude that $\widetilde{h}_\alpha \in H^{\frac{3}{2}}(\gamma)$.

Hence $\widetilde{h}_\alpha$ belong to the space $\in H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions (see [CG01, Theorem 4]).

b) **(i)** Since the set ω is simply-connected and by using the generalized Poincaré theorem (see [Sch66, Theorem VI,p.59],[CG01, Theorem 7]), the equation $\partial_\beta \bar{N}_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in D'(\omega)$, unique up to the addition of constants, such that $\bar{N}_{1\alpha} = 2\partial_2\psi_\alpha,\ \bar{N}_{2\alpha} = -2\partial_1\psi_\alpha$.

(ii) Since the equation $\bar{N}_{12} = \bar{N}_{21}$ implies that $\partial_\alpha\psi_\alpha = 0$. Another application of the same result shows that there exist a distribution $\Phi \in D'(\omega)$, unique up to the addition of polynomials of degree ≤ 1, such that $\psi_1 = \partial_2\Phi,\ \psi_2 = -\partial_1\Phi$, so that $\bar{N}_{11} = 2\partial_{22}\Phi,\ \bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi,\ \bar{N}_{22} = 2\partial_{11}\Phi\ in\ \omega$.

c) **(i)** From $\bar{N}_{\alpha\beta}\nu_\beta = 2\widetilde{h}_\alpha$ on $\gamma\times]0,T[$, we obtain

$$\widetilde{h}_1 = \frac{1}{2}\bar{N}_{1\beta}\nu_\beta = \partial_\tau(\partial_2\Phi),$$

$$\widetilde{h}_2 = \frac{1}{2}\bar{N}_{2\beta}\nu_\beta = -\partial_\tau(\partial_1\Phi),$$

then for all $y \in \gamma$, we get

$$\partial_1\Phi(y) = -\int_{\gamma(y)} \widetilde{h}_2 d\gamma \ \text{ et } \ \partial_2\Phi(y) = \int_{\gamma(y)} \widetilde{h}_1 d\gamma,$$

$$\partial_\nu\Phi(y) = -\nu_1(y)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \nu_2(y)\int_{\gamma(y)} \widetilde{h}_1 d\gamma,$$

$$\partial_\tau\Phi(y) = -\tau_1(y)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \tau_2(y)\int_{\gamma(y)} \widetilde{h}_1 d\gamma.$$

Thus

$$\Phi = \Phi_0 \ \text{ and } \ \partial_\nu\Phi = \Phi_1 \text{ on } \gamma\times]0,T[.$$

(ii) Since $-\partial_{\alpha\beta}m_{\alpha\beta} = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3,\ \bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = 2[\Phi,\zeta_3+\theta]$, so that

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} + \frac{8\mu\,(\lambda+\mu)}{3\,(\lambda+2\mu)}\Delta^2\zeta_3 = 2\,[\Phi,\zeta_3+\theta] + p_3 \ \text{ in } \omega\times\,]0,T[\,.$$

(iii) From $\Delta^2\Phi = \frac{1}{2}\Delta\bar{N}_{\alpha\alpha}$ and $\partial_{\alpha\beta}\bar{N}_{\alpha\beta} = 0$, we get

$$\Delta^2\Phi = -\frac{\mu\,(3\lambda+2\mu)}{2\,(\lambda+\mu)}\,[\zeta_3,\zeta_3+2\theta] \ \text{ in } \omega\times\,]0,T[\,.$$

■

2.2 Existence result for a dynamical equations of generalized Marguerre-von Kármán shallow shells

2.2.1 Dynamical equations of generalized Marguerre-von Kármán shallow shells

Theorem 2.6 *Assume that the set ω is simply-connected and that the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ $\forall t \in [0,T]$ satisfy the compatibility conditions. Let $\chi \in H^2(\omega)$ be the unique solution in the sense of distributions of*

$$\begin{cases} \Delta^2\chi = 0 \ \textit{in } \omega, \\ \chi = \Phi_0 \ \textit{and } \partial_\nu\chi = \Phi_1 \ \textit{on } \gamma, \\ \Phi_0 \in H^{\frac{3}{2}}(\gamma), \Phi_1 \in H^{\frac{1}{2}}(\gamma) \end{cases} \tag{2.7}$$

and let

$$E = \frac{\mu\,(3\lambda+2\mu)}{\lambda+\mu},\ \xi = \sqrt{E}\zeta_3,\ \widetilde{\theta} = \sqrt{E}\theta,\ f = \sqrt{E}\mathrm{p}_3,\ \widetilde{\Phi} = \Phi - \chi. \tag{2.8}$$

The pair $(\zeta_3,\Phi) \in H^4(\omega)\times H^4(\omega)$ $\forall t \in [0,T]$, satisfies the scaled dynamical equations of generalized Marguerre-von Kármán shallow shells in the sense of distributions, if and only if, the pair $(\xi,\widetilde{\Phi}) \in V(\omega)\times H_0^2(\omega)$ $\forall t \in [0,T]$, satisfies

$$(\mathcal{P})^{iso}_{dyn} \begin{cases} 2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\xi) = 2[\widetilde{\Phi}+\chi,\xi+\widetilde{\theta}] + f \ \textit{in } \omega\times]0,T[, \\ \Delta^2\widetilde{\Phi} = -\frac{1}{2}[\xi,\xi+2\widetilde{\theta}] \ \textit{in } \omega\times]0,T[, \\ \xi = \partial_\nu\xi = 0 \ \textit{on } \gamma_1\times]0,T[, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \ \textit{on } \gamma_2\times]0,T[, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \ \textit{on } \gamma_2\times]0,T[, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \ \textit{on } \gamma\times]0,T[, \\ \xi(.,0) = \xi_0(.) \ \textit{and } \frac{\partial\xi}{\partial t}(.,0) = \xi_1(.) \ \textit{in } \omega. \end{cases}$$

Proof.

By classical elliptic theory, there exists a unique function $\chi \in H^2(\omega)$ such that $\Delta^2\chi = 0$ in ω, $\chi = \Phi_0$ and $\partial_\nu \chi = \Phi_1$ on γ (see [Cia97, Theorem 5.6-1]). Letting $\widetilde{\Phi} = \Phi - \chi$, we clearly have

$$\begin{cases} \Delta^2\widetilde{\Phi} = \Delta^2\Phi \text{ in } \omega \times]0,T[\,, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \text{ on } \gamma \times]0,T[\,. \end{cases}$$

Using the functions ξ, $\widetilde{\theta}$, f and $\widetilde{\Phi}$ defined in (2.8), the scaled dynamical equations of generalized Marguerre-von Kármán shallow shells presented in Theorem 2.5 is equivalent to the scaled problem $(\mathcal{P})^{iso}_{dyn}$. ■

2.2.2 Existence theory

The asymptotic analysis carried out in the first part in this Chapter is purely formal. In what follows, we establish the existence of solutions to the dynamical equations of generalized Marguerre-von Kármán shallow shells, by adapting a compactness method.

First, we use the following Lemma

Lemma 2.1 *If* $(\xi, \eta, \chi) \in [H^2(\omega)]^3$ *such that*

$$\xi = \partial_\nu \xi = 0 \text{ on } \gamma_1 \text{ and } \chi = \partial_\nu \chi = 0 \text{ on } \gamma_2,$$

then

$$\int_\omega [\xi, \eta]\chi d\omega = \int_\omega [\chi, \eta]\xi d\omega. \tag{2.9}$$

Proof.

Since $\overline{C^\infty(\bar{\omega})} = H^2(\omega)$, let the functions ξ, η, and χ in $C^\infty(\bar{\omega})$, we write

$$[\xi, \eta] = \partial_{11}(\partial_{22}\eta.\xi) + \partial_{22}(\partial_{11}\eta.\xi) - 2\partial_{12}(\partial_{12}\eta.\xi).$$

Integrating by parts, we obtain

$$\int_\omega [\xi, \eta]\chi d\omega - \int_\omega [\chi, \eta]\xi d\omega =$$
$$\int_\gamma \chi\{\partial_{22}\eta\partial_1\xi\nu_1 + \partial_{11}\eta\partial_2\xi\nu_2 - \partial_{12}\eta\partial_2\xi\nu_1 - \partial_{12}\eta\partial_1\xi\nu_2\}d\gamma$$
$$- \int_\gamma \xi\{\partial_{22}\eta\partial_1\chi\nu_1 + \partial_{11}\eta\partial_2\chi\nu_2 - \partial_{12}\eta\partial_2\chi\nu_1 - \partial_{12}\eta\partial_1\chi\nu_2\}d\gamma.$$

If $\xi = \partial_\nu \xi = 0$ on γ_1 and $\chi = \partial_\nu \chi = 0$ on γ_2, consequently

$$\int_\omega [\xi, \eta]\chi d\omega - \int_\omega [\chi, \eta]\xi d\omega = 0.$$

■

Theorem 2.7 *Assume $f \in L^2(0,T; L^2(\omega))$, $\xi_0 \in V(\omega)$ and $\xi_1 \in L^2(\omega)$, then there exists a solution $(\xi, \widetilde{\Phi})$ to the problem $(\mathcal{P})^{iso}_{dyn}$, such that*

$$\begin{cases} \xi \in L^\infty(0,T; V(\omega)), \\ \frac{\partial \xi}{\partial t} \in L^\infty(0,T; L^2(\omega)), \\ \widetilde{\Phi} \in L^\infty(0,T; H^2_0(\omega)). \end{cases} \quad (2.10)$$

Proof.

Denote by G_2 the inverse of Δ^2 with homogenous Dirichlet boundary condition in ω (the Green operator), we write

$$\widetilde{\Phi} = -\frac{1}{2} G_2 \left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[\,.$$

Then

$$2\rho \frac{\partial^2 \xi}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \xi\right) = 2\left[-\frac{1}{2} G_2 \left[\xi, \xi + 2\widetilde{\theta}\right] + \chi, \xi + \widetilde{\theta}\right] + f \text{ in } \omega \times]0,T[\,.$$

From (2.10), we get

$$\left[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}\right] \in L^\infty(0,T; L^1(\omega)),$$

and for the first equation in $(\mathcal{P})^{iso}_{dyn}$, we have

$$\frac{\partial^2 \xi}{\partial t^2} \in L^\infty(0,T; H^{-1}(\omega)),$$

so that the initial conditions make sense.

Step 1: (Faedo-Galerkin approximation)

Let w_i, $i \geq 1$ denote an orthonormal basis of the Hilbert space $V(\omega)$ and let V_m denote, for each integer $m \geq 1$, the subspace of $V(\omega)$ spanned by the functions w_i, $1 \leq i \leq m$.

We construct the Faedo-Galerkin approximation $\xi_m(t)$ of a solution in the form

$$\xi_m(t) = \sum_{i=1}^m \alpha_{im}(t) w_i.$$

So the function $\xi_m(t)$ is the solution of the following approximate problem

$$(\mathcal{P}_m)^{iso}_{dyn} \begin{cases} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega \ - \int_\omega \partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right) w_j d\omega = \\ 2\int_\omega \left[-\frac{1}{2}G_2\left[\xi_m(t), \xi_m(t)+2\widetilde{\theta}\right] + \chi, \xi_m(t)+\widetilde{\theta}\right] w_j d\omega + \int_\omega f w_j d\omega, \\ 1 \leq j \leq m \text{ in } \omega \times]0,T[\,, \\ \xi_m(t) = \partial_\nu \xi_m(t) = 0 \text{ on } \gamma_1 \times]0,T[\,, \\ m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right) \nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \partial_\alpha m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\left(\nabla^2\xi_m(t)\right)\nu_\alpha \tau_\beta\right) = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \xi_m\left(.,0\right) = \xi_{0m}(.) \text{ and } \frac{\partial \xi_m}{\partial t}\left(.,0\right) = \xi_{1m}(.) \text{ in } \omega, \end{cases}$$

and we have

$$\xi_{0m} \in V_m \text{ and } \xi_{0m} \to \xi_0 \text{ in } V(\omega),\ \ \xi_{1m} \in V_m \text{ and } \xi_{1m} \to \xi_1 \text{ in } L^2(\omega).$$

Now, define

$$\widetilde{\Phi}_m(t) = -\frac{1}{2}G_2\left[\xi_m(t), \xi_m(t)+2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[\,, \tag{2.11}$$

and note that

$$\Delta^2 \widetilde{\Phi}_m(t) = -\frac{1}{2}\left[\xi_m(t), \xi_m(t)+2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[\,, \tag{2.12}$$

$$\widetilde{\Phi}_m(t) \in H^2_0(\omega), \tag{2.13}$$

so that we may rewrite the first equation of $(\mathcal{P}_m)^{iso}_{dyn}$ as

$$\begin{aligned} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega + a(\xi_m(t), w_j) - 2\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t)+\widetilde{\theta}\right] w_j d\omega \ &= \\ 2\int_\omega \left[\chi, \xi_m(t)+\widetilde{\theta}\right] w_j d\omega + \int_\omega f w_j d\omega\ ,\ 1 \leq j \leq m \ \text{ in } \omega \times]0,T[\,, \end{aligned} \tag{2.14}$$

where

$$a(\xi,\eta) = \frac{2E}{3(1-\sigma^2)} \int_\omega [\Delta\xi\Delta\eta - (1-\sigma)\{\partial_{11}\xi\partial_{22}\eta + \partial_{22}\xi\partial_{11}\eta - 2\partial_{12}\xi\partial_{12}\eta\}]d\omega. \tag{2.15}$$

The constants $E > 0$ and $\sigma \in]0,\frac{1}{2}[$ are respectively the Young's modulus and the Poisson's coefficient of the constitutive elastic material of the shallow shells.

In general $\widetilde{\Phi}_m(t)$ is not in V_m, one assures the existence of $\xi_m(t)$, and therefore of $\widetilde{\Phi}_m(t)$, in an interval $[0, t_m]$, $t_m > 0$ (see [Lio69, Theorem 4.1]).

Step 2: (A priori estimates)

Multiplying $\frac{d\alpha_{jm}(t)}{dt}$ on both sides of (2.14) and summing on the index j, we obtain

$$2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2}\frac{\partial \xi_m(t)}{\partial t} d\omega + a(\xi_m(t), \frac{\partial \xi_m(t)}{\partial t})$$
$$-2\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t)+\widetilde{\theta}\right]\frac{\partial \xi_m(t)}{\partial t} d\omega = 2\int_\omega \left[\chi, \xi_m(t)+\widetilde{\theta}\right]\frac{\partial \xi_m(t)}{\partial t} d\omega$$
$$+\int_\omega f\frac{\partial \xi_m(t)}{\partial t} d\omega \text{ in } \omega\times]0,T[. \tag{2.16}$$

Since we have

$$2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2}\frac{\partial \xi_m(t)}{\partial t} d\omega = \rho\frac{d}{dt}\int_\omega |\frac{\partial \xi_m(t)}{\partial t}|^2 d\omega = \rho\frac{d}{dt}||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega},$$

and since a is elliptic, we conclude that there exists a constant $\alpha > 0$ such that

$$a(\xi_m(t), \xi_m(t)) \;\; \geq \;\; \alpha||\xi_m(t)||^2_{V(\omega)},$$

thus

$$a(\xi_m(t), \frac{\partial \xi_m(t)}{\partial t}) \;\; = \;\; \frac{d}{2dt}a(\xi_m(t), \xi_m(t)) \geq \frac{\alpha}{2}\frac{d}{dt}||\xi_m(t)||^2_{V(\omega)}.$$

Since $\widetilde{\Phi}_m(t) \in H^2_0(\omega)$, we infer by use of [Cia97, Theorem 5.8-2] that

$$\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t)+\widetilde{\theta}\right]\frac{\partial \xi_m(t)}{\partial t} d\omega = \int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+\widetilde{\theta}\right]\widetilde{\Phi}_m(t) d\omega.$$

Using (2.12), we get

$$\begin{aligned}
\frac{\partial}{\partial t}\Delta^2\widetilde{\Phi}_m(t) &= \Delta^2\frac{\partial \widetilde{\Phi}_m(t)}{\partial t} \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\left[\xi_m(t), \xi_m(t)+2\widetilde{\theta}\right] \\
&= -\frac{1}{2}\left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+2\widetilde{\theta}\right] - \frac{1}{2}\left[\xi_m(t), \frac{\partial \xi_m(t)}{\partial t}\right] \\
&= -\left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+\widetilde{\theta}\right],
\end{aligned}$$

which yields

$$\begin{aligned}
-2\int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+\widetilde{\theta}\right]\widetilde{\Phi}_m(t) d\omega &= 2\int_\omega [\Delta^2\frac{\partial \widetilde{\Phi}_m(t)}{\partial t}]\widetilde{\Phi}_m(t) d\omega \\
&= 2\int_\omega [\Delta\frac{\partial \widetilde{\Phi}_m(t)}{\partial t}][\Delta\widetilde{\Phi}_m(t)] d\omega \\
&= \frac{d}{dt}\int_\omega |\Delta\widetilde{\Phi}_m(t)|^2 d\omega \\
&= \frac{d}{dt}||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega}.
\end{aligned}$$

Since $\frac{\partial \xi_m(t)}{\partial t} \in V(\omega)$, i.e., $\frac{\partial \xi_m(t)}{\partial t} = \partial_\nu[\frac{\partial \xi_m(t)}{\partial t}] = 0$ on γ_1 and $\chi = \partial_\nu \chi = 0$ on γ_2, then applying Lemma 2.1, gives

$$\begin{aligned} 2\int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta}\right] \frac{\partial \xi_m(t)}{\partial t} d\omega &= 2\int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right] \chi d\omega \\ &= -2\int_\omega \Delta^2 \frac{\partial \widetilde{\Phi}_m(t)}{\partial t} \chi d\omega, \end{aligned}$$

and we have

$$\int_\omega \Delta^2 \frac{\partial \widetilde{\Phi}_m(t)}{\partial t} . \chi d\omega = \frac{d}{dt}\int_\omega \Delta^2 \widetilde{\Phi}_m(t) . \chi d\omega - \int_\omega \Delta^2 \widetilde{\Phi}_m(t) . \frac{\partial \chi}{\partial t} d\omega.$$

From (2.7) and (2.13), it follows that

$$\frac{d}{dt}\int_\omega \Delta^2 \widetilde{\Phi}_m(t) . \chi d\omega = \frac{d}{dt}\int_\omega \widetilde{\Phi}_m(t) . \Delta^2 \chi d\omega = 0,$$

and since the function χ is independent of t, so that

$$\int_\omega \Delta^2 \widetilde{\Phi}_m(t) . \frac{\partial \chi}{\partial t} d\omega = 0,$$

thus

$$\int_\omega \Delta^2 \frac{\partial \widetilde{\Phi}_m(t)}{\partial t} . \chi d\omega = 0.$$

Then (2.16) can be written as

$$\frac{d}{dt}\{\rho || \frac{\partial \xi_m(t)}{\partial t} ||^2_{0,\omega} + \frac{1}{2} a(\xi_m(t), \xi_m(t)) + ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega}\} = \int_\omega f \frac{\partial \xi_m(t)}{\partial t} d\omega,$$

which, by integration from 0 to t, yields

$$\begin{aligned} \int_0^t \frac{d}{d\tau}\{\rho || \frac{\partial \xi_m(\tau)}{\partial \tau} ||^2_{0,\omega} + \frac{1}{2} a(\xi_m(\tau), \xi_m(\tau)) + ||\Delta \widetilde{\Phi}_m(\tau)||^2_{0,\omega}\} d\tau = \\ \int_0^t \{\int_\omega f \frac{\partial \xi_m(\tau)}{\partial \tau} d\omega\} d\tau. \end{aligned}$$

Hence, there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} \rho || \frac{\partial \xi_m(t)}{\partial t} ||^2_{0,\omega} + \frac{\alpha}{2} ||\xi_m(t)||^2_{V(\omega)} + ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega} \leq C_1 \int_0^t ||f||^2_{0,\omega} d\tau \\ + C_2 \int_0^t || \frac{\partial \xi_m(\tau)}{\partial \tau} ||^2_{0,\omega} d\tau + \rho || \frac{\partial \xi_m(0)}{\partial t} ||^2_{0,\omega} + \frac{\alpha}{2} ||\xi_m(0)||^2_{V(\omega)} + ||\Delta \widetilde{\Phi}_m(0)||^2_{0,\omega}. \end{aligned}$$

Since

$$\Delta^2 \widetilde{\Phi}_m(0) = -\frac{1}{2}\left[\xi_m(0), \xi_m(0) + 2\widetilde{\theta}\right],$$

then, there exists a constant $C_3 > 0$ such that

$$||\Delta \widetilde{\Phi}_m(0)||_{0,\omega} \leq C_3.$$

Thus, there exists a constant $C_4 > 0$ such that

$$\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(t)||^2_{V(\omega)} + ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega} \leq C_4 + C_2 \int_0^t ||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega} d\tau,$$

for all $t \in [0, T]$, which implies that $t_m = T$.

Then, via Gronwall's inequality, we conclude that

$$\xi_m(t) \in L^\infty(0, T; V(\omega)), \tag{2.17}$$

$$\frac{\partial \xi_m(t)}{\partial t} \in L^\infty(0, T; L^2(\omega)), \tag{2.18}$$

$$\widetilde{\Phi}_m(t) \in L^\infty(0, T; H_0^2(\omega)). \tag{2.19}$$

Step 3: (Passing to the limit)
From (2.17)-(2.19), we observe that there exists $\xi_n(t)$ and $\widetilde{\Phi}_n(t)$ such that (weak convergence is denoted $\rightharpoonup$)

$$\xi_n(t) \rightharpoonup \xi(t) \text{ in } L^\infty(0, T; V(\omega)) \text{ weak}*,$$

$$\frac{\partial \xi_n(t)}{\partial t} \rightharpoonup \frac{\partial \xi(t)}{\partial t} \text{ in } L^\infty(0, T; L^2(\omega)) \text{ weak}*,$$

$$\widetilde{\Phi}_n(t) \rightharpoonup \widetilde{\Phi}(t) \text{ in } L^\infty(0, T; H_0^2(\omega)) \text{ weak} * .$$

According to the Rellich-Kondrachoff theorem [LM68, Chap. 1, Theorem 16.1], the compact imbedding of $H^2(\omega \times]0, T[)$ into $L^2(\omega \times]0, T[)$ implies that

$$\xi_n(t) \rightarrow \xi(t) \text{ in } L^2(\omega \times]0, T[). \tag{2.20}$$

Let ϕ_j, $1 \leq j \leq j_0$ be functions of $C^1([0, T])$ such that

$$\phi_j(T) = 0 \text{ and } \psi = \sum_{j=1}^{j_0} \phi_j \otimes w_j. \tag{2.21}$$

For $m=n>j_0$, we obtain

$$2\rho\int_\omega \frac{\partial^2\xi_n(t)}{\partial t^2}\psi(t)d\omega + a(\xi_n(t),\psi(t)) - 2\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega \;=\; 2\int_\omega \left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega + \int_\omega f\psi(t)d\omega \text{ in } \omega\times\,]0,T[\,.$$

Thus

$$2\rho\int_0^T\{\int_\omega \frac{\partial^2\xi_n(t)}{\partial t^2}\psi(t)d\omega\}dt + \int_0^T a(\xi_n(t),\psi(t))dt$$
$$-2\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt = 2\int_0^T\{\int_\omega\left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt$$
$$+\int_0^T\{\int_\omega f\psi(t)d\omega\}dt \text{ in } \omega\times\,]0,T[\,,$$

and we have

$$\begin{aligned}\int_0^T\{\int_\omega \frac{\partial^2\xi_n(t)}{\partial t^2}\psi(t)d\omega\}dt &= -\int_0^T\{\int_\omega \frac{\partial\xi_n(t)}{\partial t}\frac{\partial\psi(t)}{\partial t}d\omega\}dt\\ +\int_\omega \frac{\partial\xi_n(T)}{\partial t}\psi(T)d\omega - \int_\omega\frac{\partial\xi_n(0)}{\partial t}\psi(0)d\omega &= -\int_0^T\{\int_\omega \frac{\partial\xi_n(t)}{\partial t}\frac{\partial\psi(t)}{\partial t}d\omega\}dt\\ &- \int_\omega \xi_{1n}\psi(0)d\omega.\end{aligned}$$

Since $\psi(T)=0$, we also obtain

$$\begin{aligned}-2\rho\int_0^T\{\int_\omega \frac{\partial\xi_n(t)}{\partial t}\frac{\partial\psi(t)}{\partial t}d\omega\}dt + \int_0^T a(\xi_n(t),\psi(t))dt &-\\ 2\int_0^T\{\int_\omega\left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt &=\\ 2\int_0^T\{\int_\omega\left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt + \int_0^T\{\int_\omega f\psi(t)d\omega\}dt &+\\ 2\rho\int_\omega \xi_{1n}\psi(0)d\omega \text{ in } \omega\times\,]0,T[\,. \end{aligned}\tag{2.22}$$

From (2.13), we get

$$\int_0^T\{\int_\omega\left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt = \int_0^T\{\int_\omega\left[\widetilde{\Phi}_n(t),\psi(t)\right](\xi_n(t)+\widetilde{\theta})d\omega\}dt,$$

and we have

$$[\widetilde{\Phi}_n(t),\psi(t)] \rightharpoonup [\widetilde{\Phi}(t),\psi(t)] \text{ in } L^2(\omega\times]0,T[).$$

Then, because $\xi_n(t) \to \xi(t)$ in $L^2(\omega\times]0,T[)$, we obtain

$$\begin{aligned}\int_0^T \{\int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt &\to \int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \psi(t)\right](\xi(t)+\widetilde{\theta})d\omega\}dt \\ &= \int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt.\end{aligned}$$

We have

$$[\chi, \xi_n(t)+\widetilde{\theta}] \rightharpoonup [\chi, \xi(t)+\widetilde{\theta}] \text{ in } L^2(\omega\times]0,T[),$$

thus

$$\int_0^T \{\int_\omega \left[\chi, \xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt \to \int_0^T \{\int_\omega \left[\chi, \xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt.$$

Then passing to the limit in (2.22), we obtain

$$\begin{aligned}-2\rho\int_0^T \{\int_\omega \frac{\partial \xi(t)}{\partial t}\frac{\partial \psi(t)}{\partial t}d\omega\}dt + \int_0^T a(\xi(t),\psi(t))dt &- \\ 2\int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt &= \\ 2\int_0^T \{\int_\omega \left[\chi, \xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt + \int_0^T \{\int_\omega f\psi(t)d\omega\}dt &+ \\ 2\rho\int_\omega \xi_1\psi(0)d\omega \text{ in } \omega\times\left]0,T\right[, &\end{aligned} \tag{2.23}$$

for all ψ of the form (2.21).

Passing to the limit, we deduce that (2.23) still true for all $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ and $\psi(T)=0$ (this comes from the density of functions of the form (2.21) in the space of functions $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ with $\psi(T)=0$ see [DL72, LM68]).

Then $(\xi, \widetilde{\Phi})$ satisfies

$$2\rho\frac{\partial^2 \xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}\left(\nabla^2\xi\right) = 2\left[\widetilde{\Phi}+\chi, \xi+\widetilde{\theta}\right] + f \text{ in } \omega\times\left]0,T\right[,$$

and

$$\frac{\partial \xi}{\partial t}(0) = \xi_1.$$

Taking into account (2.17) and (2.18), and applying [Lio69, Lemma 1.2], we deduce that

$$\xi_n(0) \rightharpoonup \xi(0) \text{ in } L^2(\omega),$$

and we obtain

$$\xi_n(0) = \xi_{0n} \quad \to \quad \xi_0 \text{ in } V(\omega),$$

with the consequence that

$$\xi(0) = \xi_0.$$

It remains to be shown that

$$\Delta^2 \widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times \left]0, T\right[.$$

Noting that

$$[\xi_n(t), \xi_n(t) + 2\widetilde{\theta}] \to [\xi(t), \xi(t) + 2\widetilde{\theta}] \text{ in } D'(\omega \times]0, T[),$$

if $\phi \in D(\omega \times]0, T[)$ we obtain

$$[\phi, \xi_n(t) + 2\widetilde{\theta}] \rightharpoonup [\phi, \xi(t) + 2\widetilde{\theta}] \text{ in } L^2(\omega \times]0, T[),$$

and, from (2.20), we deduce that

$$\begin{aligned}
\int_0^T \{\int_\omega \left[\xi_n(t), \xi_n(t) + 2\widetilde{\theta}\right] \phi d\omega\} dt &= \int_0^T \{\int_\omega \left[\phi, \xi_n(t) + 2\widetilde{\theta}\right] \xi_n(t) d\omega\} dt \\
&\to \int_0^T \{\int_\omega \left[\phi, \xi(t) + 2\widetilde{\theta}\right] \xi(t) d\omega\} dt \\
&= \int_0^T \{\int_\omega \left[\xi(t), \xi(t) + 2\widetilde{\theta}\right] \phi d\omega\} dt.
\end{aligned}$$

Finally, passing to the limit in (2.11) for $m = n$, we have

$$\widetilde{\Phi}(t) = -\frac{1}{2} G_2 \left[\xi(t), \xi(t) + 2\widetilde{\theta}\right] \text{ in } \omega \times \left]0, T\right[,$$

which yields

$$\Delta^2 \widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times \left]0, T\right[$$

■

2.3 Conclusion

The application of the asymptotic expansions method to the three-dimensional nonlinear elastodynamic shallow shells, with a specific class of boundary conditions of generalized Marguerre-von Kármán type, shows that the leading term of the asymptotic expansions is characterized by two-dimensional dynamical boundary value problem called the dynamical equations of generalized Marguerre-von Kármán shallow shells, which depends on the Airy function Φ and the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell.

The application of the compactness method to the dynamical equations of generalized Marguerre-von Kármán shallow shells, shows that there exists a solution to these equations.

Chapter 3

Dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells

In this Chapter, we extend formally the study of the second Chapter to nonhomogeneous anisotropic material. More precisely, we considered a three-dimensional dynamical model for a nonlinearly elastic shallow shell with a specific class of boundary conditions of generalized Marguerre-von Kármán type, made of a general nonhomogeneous anisotropic material.

3.1 Setting of the problem

Consider a nonlinearly elastodynamics shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε. We assume that the elastic material constituting the shell is nonhomogeneous and anisotropic, and that the reference configuration is a natural state.

The shell is subjected to vertical body forces of density $(\hat{f}_i^\varepsilon) = (0, 0, \hat{f}_3^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$ and to vertical surface forces of density $(\hat{g}_i^\varepsilon) = (0, 0, \hat{g}_3^\varepsilon)$ on its upper and lower faces $\hat{\Gamma}_+^\varepsilon$ and $\hat{\Gamma}_-^\varepsilon$. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

The unknowns displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$ and stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)(\hat{x}^\varepsilon, t)$ satisfy the following three-dimensional boundary value problem in Cartesian coordinates:

$$(C.\hat{P}^\varepsilon)_{dyn}^{anis} \begin{cases} \hat{\rho}^\varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} - \hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[\,, \\ \begin{cases} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon (\gamma_1 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\} \nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1 \times]0, +\infty[\,, \end{cases} \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) \hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } (\gamma_2 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) \hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \left(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon\right) \times]0, +\infty[\,, \\ (A\hat{\sigma}^\varepsilon)_{ij} = \hat{\gamma}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) + \frac{1}{2} \hat{\partial}_i^\varepsilon \hat{u}_l^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_l^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[\,, \\ \hat{\mathbf{u}}^\varepsilon (\hat{x}^\varepsilon, 0) = \hat{\mathbf{p}}^\varepsilon \text{ and } \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t} (\hat{x}^\varepsilon, 0) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \end{cases}$$

where

$$\begin{cases} \hat{\gamma}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon), \\ \hat{\rho}^\varepsilon : \text{ the mass density}, \\ \hat{\mathbf{p}}^\varepsilon, \hat{\mathbf{q}}^\varepsilon : \text{ the given initial data}. \end{cases} \tag{3.1}$$

The mapping A is defined by

$$(A\hat{\sigma}^\varepsilon)_{ij} = \hat{c}_{ijkl}^\varepsilon \hat{\sigma}_{kl}^\varepsilon,$$

where $\hat{C}^\varepsilon = (\hat{c}_{ijkl}^\varepsilon)$ is the compliance tensor. We suppose that the associated rigidity tensor $\hat{A}^\varepsilon = (\hat{a}_{ijkl}^\varepsilon)$ satisfy the following conditions

$$\begin{cases} \hat{a}_{ijkl}^\varepsilon(\hat{x}^\varepsilon) \in L^\infty(\hat{\Omega}^\varepsilon), \\ \hat{a}_{ijkl}^\varepsilon = \hat{a}_{jikl}^\varepsilon = \hat{a}_{klij}^\varepsilon = \hat{a}_{klji}^\varepsilon \\ \exists c > 0,\ \hat{a}_{ijkl}^\varepsilon \hat{\tau}_{kl}^\varepsilon \hat{\tau}_{ij}^\varepsilon \geq c \hat{\tau}_{ij}^\varepsilon \hat{\tau}_{ij}^\varepsilon,\ \hat{\tau}_{ij}^\varepsilon = \hat{\tau}_{ji}^\varepsilon. \end{cases}$$

First, we rewrite the above boundary value problem $(C.\hat{P}^\varepsilon)_{dyn}^{anis}$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem:

$$(V.\hat{P}^\varepsilon)^{anis}_{dyn}\left\{\begin{array}{l}\text{Find } (\hat{\mathbf{u}}^\varepsilon,\hat{\sigma}^\varepsilon)\in\mathbf{V}(\hat{\Omega}^\varepsilon)\times\Sigma(\hat{\Omega}^\varepsilon)\ \forall t\geq 0,\ \text{such that,}\\ \frac{d^2}{dt^2}\left\{\hat{\rho}^\varepsilon\int_{\hat{\Omega}^\varepsilon}\hat{u}^\varepsilon_i\hat{v}^\varepsilon_i d\hat{x}^\varepsilon\right\}+\int_{\hat{\Omega}^\varepsilon}\hat{\sigma}^\varepsilon_{ij}\hat{\gamma}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)d\hat{x}^\varepsilon+\int_{\hat{\Omega}^\varepsilon}\hat{\sigma}^\varepsilon_{ij}\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_l d\hat{x}^\varepsilon=\\ \int_{\hat{\Omega}^\varepsilon}\hat{f}^\varepsilon_3\hat{v}^\varepsilon_3 d\hat{x}^\varepsilon+\int_{\hat{\Gamma}^\varepsilon_+\cup\hat{\Gamma}^\varepsilon_-}\hat{g}^\varepsilon_3\hat{v}^\varepsilon_3 d\hat{\Gamma}^\varepsilon+\int_{\hat{\gamma}^\varepsilon_1}\{\int_{-\varepsilon}^{\varepsilon}(\hat{v}^\varepsilon_\alpha\circ\boldsymbol{\Theta}^\varepsilon)\,dx^\varepsilon_3\}\hat{h}^\varepsilon_\alpha d\hat{\gamma}^\varepsilon,\\ \forall\hat{\mathbf{v}}^\varepsilon\in\mathbf{V}(\hat{\Omega}^\varepsilon),\forall t>0,\\ \int_{\hat{\Omega}^\varepsilon}(A\hat{\sigma}^\varepsilon)_{ij}\hat{\tau}^\varepsilon_{ij}d\hat{x}^\varepsilon-\int_{\hat{\Omega}^\varepsilon}\hat{\tau}^\varepsilon_{ij}\hat{\gamma}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)d\hat{x}^\varepsilon-\frac{1}{2}\int_{\hat{\Omega}^\varepsilon}\hat{\tau}^\varepsilon_{ij}\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l d\hat{x}^\varepsilon=0\\ \forall\hat{\tau}^\varepsilon\in\Sigma(\hat{\Omega}^\varepsilon),\forall t>0,\\ \hat{\mathbf{u}}^\varepsilon(\hat{x}^\varepsilon,0)=\hat{\mathbf{p}}^\varepsilon\text{ and }\frac{\partial\hat{\mathbf{u}}^\varepsilon}{\partial t}(\hat{x}^\varepsilon,0)=\hat{\mathbf{q}}^\varepsilon\text{ in }\hat{\Omega}^\varepsilon.\end{array}\right.$$

In order to transform the problem $(V.\hat{P}^\varepsilon)^{anis}_{dyn}$ into problem posed over the cylindrical domain Ω^ε, we use the one to one mapping $(\boldsymbol{\Theta}^\varepsilon)^{-1}$ and the relations (2.3).

Let there be a given C^1-diffeomorphism that satisfies the orientation-preserving condition. Then the variational problem $(V.\hat{P}^\varepsilon)^{anis}_{dyn}$ is equivalent to the following variational problem:

$$(P^\varepsilon)^{anis}_{dyn}\left\{\begin{array}{l}\text{Find } (\mathbf{u}^\varepsilon,\sigma^\varepsilon)\in\mathbf{V}(\Omega^\varepsilon)\times\Sigma(\Omega^\varepsilon)\ \forall t\geq 0,\ \text{such that,}\\ \frac{d^2}{dt^2}\left\{\rho^\varepsilon\int_{\Omega^\varepsilon}u^\varepsilon_i v^\varepsilon_i\delta^\varepsilon dx^\varepsilon\right\}+\int_{\Omega^\varepsilon}\sigma^\varepsilon_{ij}b^\varepsilon_{kj}\partial^\varepsilon_k v^\varepsilon_i\delta^\varepsilon dx^\varepsilon\\ +\int_{\Omega^\varepsilon}\sigma^\varepsilon_{ij}b^\varepsilon_{ki}\partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj}\partial^\varepsilon_m v^\varepsilon_l\delta^\varepsilon dx^\varepsilon=\int_{\Omega^\varepsilon}f^\varepsilon_3 v^\varepsilon_3\delta^\varepsilon dx^\varepsilon+\int_{\Gamma^\varepsilon_+\cup\Gamma^\varepsilon_-}g^\varepsilon_3 v^\varepsilon_3\delta^\varepsilon\beta^\varepsilon d\Gamma^\varepsilon\\ +\int_{\gamma_1}h^\varepsilon_\alpha\{\int_{-\varepsilon}^{\varepsilon}v^\varepsilon_\alpha dx^\varepsilon_3\}d\gamma,\forall\mathbf{v}^\varepsilon\in\mathbf{V}(\Omega^\varepsilon),\forall t>0,\\ \int_{\Omega^\varepsilon}(A\sigma^\varepsilon)_{ij}\tau^\varepsilon_{ij}\delta^\varepsilon dx^\varepsilon-\int_{\Omega^\varepsilon}\tau^\varepsilon_{ij}b^\varepsilon_{kj}\partial^\varepsilon_k u^\varepsilon_i\delta^\varepsilon dx^\varepsilon\\ -\frac{1}{2}\int_{\Omega^\varepsilon}\tau^\varepsilon_{ij}b^\varepsilon_{ki}\partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj}\partial^\varepsilon_m u^\varepsilon_l\delta^\varepsilon dx^\varepsilon=0\\ \forall\tau^\varepsilon\in\Sigma(\Omega^\varepsilon),\forall t>0,\\ \mathbf{u}^\varepsilon(x^\varepsilon,0)=\mathbf{p}^\varepsilon\text{ and }\frac{\partial\mathbf{u}^\varepsilon}{\partial t}(x^\varepsilon,0)=\mathbf{q}^\varepsilon\text{ in }\Omega^\varepsilon,\end{array}\right.$$

where

$$\left\{\begin{array}{l}u^\varepsilon_i=\hat{u}^\varepsilon_i\circ\boldsymbol{\Theta}^\varepsilon, v^\varepsilon_i=\hat{v}^\varepsilon_i\circ\boldsymbol{\Theta}^\varepsilon,\ \sigma^\varepsilon_{ij}=\hat{\sigma}^\varepsilon_{ij}\circ\boldsymbol{\Theta}^\varepsilon,\ \tau^\varepsilon_{ij}=\hat{\tau}^\varepsilon_{ij}\circ\boldsymbol{\Theta}^\varepsilon,\\ (A\sigma^\varepsilon)_{ij}=(A\hat{\sigma}^\varepsilon)_{ij}\circ\boldsymbol{\Theta}^\varepsilon,\ c^\varepsilon_{ijkl}=\hat{c}^\varepsilon_{ijkl}\circ\boldsymbol{\Theta}^\varepsilon,\\ f^\varepsilon_3=\hat{f}^\varepsilon_3\circ\boldsymbol{\Theta}^\varepsilon,\ g^\varepsilon_3=\hat{g}^\varepsilon_3\circ\boldsymbol{\Theta}^\varepsilon,\ h^\varepsilon_\alpha=\hat{h}^\varepsilon_\alpha\circ\boldsymbol{\Theta}^\varepsilon,\\ p^\varepsilon_i=\hat{p}^\varepsilon_i\circ\boldsymbol{\Theta}^\varepsilon,\ q^\varepsilon_i=\hat{q}^\varepsilon_i\circ\boldsymbol{\Theta}^\varepsilon.\end{array}\right.$$

3.2 Asymptotic analysis

3.2.1 The scaled three-dimensional problem

We transform $(P^\varepsilon)^{anis}_{dyn}$ into a problem posed over an open set independent of ε. Accordingly, let the bijection $\pi^\varepsilon: x=(x_1,x_2,x_3)\in\bar{\Omega}\to x^\varepsilon=(x_1,x_2,\varepsilon x_3)\in\bar{\Omega}^\varepsilon$.

First, we note that, if θ^ε satisfies $\theta^\varepsilon=\varepsilon\theta$ with $\theta\in C^3(\bar{\omega})$. Then there exists $\varepsilon_0=\varepsilon_0(\theta)>0$ such that the Jacobian matrix $\nabla^\varepsilon\Theta^\varepsilon(x^\varepsilon)$ is invertible for all $x^\varepsilon\in\bar{\Omega}^\varepsilon$ and all ε with $0\leq\varepsilon\leq\varepsilon_0$. Let the functions $b_{ij}(\varepsilon)$, $\delta(\varepsilon)$ and $\beta(\varepsilon)$ be defined by

$$\begin{cases} b_{ij}(\varepsilon)(x) = b_{ij}^{\varepsilon}(x^{\varepsilon}), \\ \delta(\varepsilon)(x) = \delta^{\varepsilon}(x^{\varepsilon}), \\ \beta(\varepsilon)(x) = \beta^{\varepsilon}(x^{\varepsilon}), \end{cases} \tag{3.2}$$

where

$$\begin{cases} b_{\alpha\beta}(\varepsilon)(x) = \delta_{\alpha\beta} + \varepsilon^2 b_{\alpha\beta}^{\sharp}(\varepsilon; x_1, x_2), \\ b_{\alpha 3}(\varepsilon)(x) = \varepsilon(\partial_{\alpha}\theta(x_1, x_2) + \varepsilon^2 b_{\alpha 3}^{\sharp}(\varepsilon; x_1, x_2)), \\ b_{3\beta}(\varepsilon)(x) = -\varepsilon(\partial_{\beta}\theta(x_1, x_2) + \varepsilon^2 b_{3\beta}^{\sharp}(\varepsilon; x_1, x_2)), \\ b_{33}(\varepsilon)(x) = 1 + \varepsilon^2 b_{33}^{\sharp}(\varepsilon; x_1, x_2), \\ \delta(\varepsilon)(x) = 1 + \varepsilon^2 \delta^{\sharp}(\varepsilon; x_1, x_2), \\ \beta(\varepsilon)(x) = 1 + \varepsilon^2 \beta^{\sharp}(\varepsilon; x_1, x_2), \end{cases}$$

and there exists a positive constant c such that

$$\sup_{\substack{0 \le \varepsilon \le \varepsilon_0 \\ (x_1, x_2) \in \bar{\omega}}} |b_{ij}^{\sharp}(\varepsilon; x_1, x_2)| \le c,$$

$$\sup_{\substack{0 \le \varepsilon \le \varepsilon_0 \\ (x_1, x_2) \in \bar{\omega}}} |\delta^{\sharp}(\varepsilon; x_1, x_2)| \le c.$$

$$\sup_{\substack{0 \le \varepsilon \le \varepsilon_0 \\ (x_1, x_2) \in \bar{\omega}}} |\beta^{\sharp}(\varepsilon; x_1, x_2)| \le c.$$

To the functions $\mathbf{u}^{\varepsilon}$, $\mathbf{v}^{\varepsilon} \in \mathbf{V}(\Omega^{\varepsilon})$ and σ^{ε}, $\tau^{\varepsilon} \in \Sigma(\Omega^{\varepsilon})$, we associate the scaled functions $\mathbf{u}(\varepsilon)$, $\mathbf{v} \in \mathbf{V}(\Omega)$ and $\sigma(\varepsilon)$, $\tau \in \Sigma(\Omega)$ defined by

$$\begin{cases} u_{\alpha}^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon^2 u_{\alpha}(\varepsilon)(x, t), u_3^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon u_3(\varepsilon)(x, t), \\ v_{\alpha}^{\varepsilon}(x^{\varepsilon}) = \varepsilon^2 v_{\alpha}(x), v_3^{\varepsilon}(x^{\varepsilon}) = \varepsilon v_3(x), \\ \sigma_{\alpha\beta}^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)(x, t), \sigma_{\alpha 3}^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x, t), \\ \sigma_{33}^{\varepsilon}(x^{\varepsilon}, t) = \varepsilon^4 \sigma_{33}(\varepsilon)(x, t), \\ \tau_{\alpha\beta}^{\varepsilon}(x^{\varepsilon}) = \varepsilon^2 \tau_{\alpha\beta}(x), \tau_{\alpha 3}^{\varepsilon}(x^{\varepsilon}) = \varepsilon^3 \tau_{\alpha 3}(x), \\ \tau_{33}^{\varepsilon}(x^{\varepsilon}) = \varepsilon^4 \tau_{33}(x), \end{cases} \tag{3.3}$$

for all $x^{\varepsilon} = \pi^{\varepsilon} x \in \bar{\Omega}^{\varepsilon}$.

Next, we make the following assumptions : there exists constant $\rho > 0$ and for some $T > 0$, the functions $f_3 \in L^2(0, T; L^2(\Omega))$, $g_3 \in L^2(0, T; L^2(\Gamma_+ \cup \Gamma_-))$, $h_{\alpha} \in L^2(0, T; L^2(\gamma_1))$, $\theta \in C^3(\bar{\omega})$ independent of ε and $\mathbf{p}(\varepsilon) \in \mathbf{V}(\Omega)$, $\mathbf{q}(\varepsilon) \in L^2(\Omega; \mathbb{R}^3)$, such that

$$\left\{\begin{array}{l}\rho^\varepsilon=\varepsilon^2\rho,\\ f_3^\varepsilon(x^\varepsilon,t)=\varepsilon^3 f_3(x,t)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon,\\ g_3^\varepsilon(x^\varepsilon,t)=\varepsilon^4 g_3(x,t)\ \forall x^\varepsilon=\pi^\varepsilon x\in(\Gamma_+^\varepsilon\cup\Gamma_-^\varepsilon),\\ h_\alpha^\varepsilon(y_1,y_2,t)=\varepsilon^2 h_\alpha(y_1,y_2,t)\ \forall(y_1,y_2)\in\gamma_1,\\ \theta^\varepsilon(x_1,x_2)=\varepsilon\theta(x_1,x_2)\ \forall(x_1,x_2)\in\bar{\omega},\\ p_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^2 p_\alpha(\varepsilon)(x)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon,\\ p_3^\varepsilon(x^\varepsilon)=\varepsilon p_3(\varepsilon)(x)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon,\\ q_\alpha^\varepsilon(x^\varepsilon)=\varepsilon^2 q_\alpha(\varepsilon)(x)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon,\\ q_3^\varepsilon(x^\varepsilon)=\varepsilon q_3(\varepsilon)(x)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon,\\ c_{ijkl}^\varepsilon(x^\varepsilon)=c_{ijkl}(\varepsilon)(x)\ \forall x^\varepsilon=\pi^\varepsilon x\in\Omega^\varepsilon.\end{array}\right. \tag{3.4}$$

Using the relations (3.2), the scalings (3.3) and the assumptions (3.4), we obtain

Theorem 3.1 *The scaled displacement-stress fields* $(\mathbf{u}(\varepsilon),\sigma(\varepsilon))$ *satisfies the following variational problem:*

$$(P(\varepsilon))_{dyn}^{anis}\left\{\begin{array}{l}\textit{Find }(\mathbf{u}(\varepsilon),\sigma(\varepsilon))\in\mathbf{V}(\Omega)\times\Sigma(\Omega)\ \forall t\in[0,T],\ \textit{such that,}\\ D^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)+B^\theta\left(\sigma\left(\varepsilon\right),\mathbf{v}\right)+2C^\theta\left(\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=F\left(\mathbf{v}\right)\\ +\varepsilon^2 R\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right),\forall\mathbf{v}\in\mathbf{V}(\Omega),\forall t\in\,]0,T[\,,\\ A(\sigma\left(\varepsilon\right),\tau)-B^\theta\left(\tau,\mathbf{u}\left(\varepsilon\right)\right)-C^\theta\left(\tau,\mathbf{u}\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right)\right)=\\ \varepsilon^2 S\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\tau\right),\forall\tau\in\Sigma(\Omega),\forall t\in\,]0,T[\,,\\ \mathbf{u}\left(\varepsilon\right)\left(x,0\right)=\mathbf{p}\left(\varepsilon\right)\ \textit{ and }\ \frac{\partial\mathbf{u}(\varepsilon)}{\partial t}\left(x,0\right)=\mathbf{q}\left(\varepsilon\right)\ \textit{ in }\Omega,\end{array}\right.$$

where

$$\left\{\begin{array}{l}A(\sigma\left(\varepsilon\right),\tau)=\int_\Omega c_{\alpha\beta\gamma\delta}(\varepsilon)\sigma_{\gamma\delta}(\varepsilon)\tau_{\alpha\beta}dx,\\ B^\theta\left(\tau\left(\varepsilon\right),\mathbf{v}\right)=\int_\Omega\tau_{ij}\left(\varepsilon\right)\gamma_{ij}^\theta(\mathbf{v})dx,\\ C^\theta\left(\tau\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=\frac{1}{2}\int_\Omega\tau_{ij}\left(\varepsilon\right)\partial_i^\theta u_3\left(\varepsilon\right)\partial_j^\theta v_3dx,\\ D^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3\left(\varepsilon\right)v_3dx\right\},\\ F\left(\mathbf{v}\right)=\int_\Omega f_3v_3dx+\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma+\int_{\gamma_1}h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma,\\ \partial_\alpha^\theta v=\partial_\alpha v-\partial_\alpha\theta\partial_3 v,\ \partial_3^\theta v=\partial_3 v,\ \gamma_{ij}^\theta(\mathbf{v})=\frac{1}{2}\left(\partial_i^\theta v_j+\partial_j^\theta v_i\right),\end{array}\right.$$

and the remainders R *and* S *are bounded.*

Proof.

We have

$$\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{kj} \partial^\varepsilon_k v^\varepsilon_i \delta^\varepsilon dx^\varepsilon = \varepsilon^5 \int_\Omega \sigma_{ij}(\varepsilon)\, \gamma^\theta_{ij}(\mathbf{v})\, dx + \varepsilon^7 \varrho_B(\varepsilon; \sigma(\varepsilon), \mathbf{v}),$$

$$\begin{aligned}\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{ki} \partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj} \partial^\varepsilon_m v^\varepsilon_l \delta^\varepsilon dx^\varepsilon &= \varepsilon^5 \int_\Omega \sigma_{ij}(\varepsilon)\, \partial^\theta_i u_3(\varepsilon)\, \partial^\theta_j v_3 dx \\ &+ \varepsilon^7 \varrho_C(\varepsilon; \sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}),\end{aligned}$$

$$\int_{\Omega^\varepsilon} f^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} g^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon + \int_{\gamma_1} h^\varepsilon_\alpha \left\{ \int_{-\varepsilon}^{\varepsilon} v^\varepsilon_\alpha dx^\varepsilon_3 \right\} d\gamma =$$
$$\varepsilon^5 \left(\int_\Omega f_3 v_3 dx + \int_{\Gamma_+ \cup \Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha \left\{ \int_{-1}^{1} v_\alpha dx_3 \right\} d\gamma \right) + \varepsilon^7 \varrho_F(\varepsilon; \mathbf{v}).$$

$$\frac{d^2}{dt^2}\left\{ \rho^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon_i v^\varepsilon_i \delta^\varepsilon dx^\varepsilon \right\} = \varepsilon^5 \frac{d^2}{dt^2}\left\{ \rho \int_\Omega u_3(\varepsilon)\, v_3 dx \right\} + \varepsilon^7 \varrho_D(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}).$$

So that the first equation in variational problem $(P^\varepsilon)^{anis}_{dyn}$ may be written as

$$\begin{aligned} D^t(\mathbf{u}(\varepsilon), \mathbf{v}) + B^\theta(\sigma(\varepsilon), \mathbf{v}) + 2C^\theta(\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) = F(\mathbf{v}) + \\ \varepsilon^2 R(\varepsilon; \sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}),\end{aligned}$$

where

$$\begin{aligned} R(\varepsilon; \sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) = \varrho_F(\varepsilon; \mathbf{v}) - \varrho_B(\varepsilon; \sigma(\varepsilon), \mathbf{v}) - \\ \varrho_C(\varepsilon; \sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) - \varrho_D(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}).\end{aligned}$$

Next, we have

$$\int_{\Omega^\varepsilon} (A\sigma^\varepsilon)_{ij} \tau^\varepsilon_{ij} \delta^\varepsilon dx^\varepsilon = \varepsilon^5 \int_\Omega c_{\alpha\beta\gamma\delta}(\varepsilon) \sigma_{\gamma\delta}(\varepsilon) \tau_{\alpha\beta} dx + \varepsilon^7 \varrho_A(\varepsilon; \sigma(\varepsilon), \tau),$$

$$\int_{\Omega^\varepsilon} \tau^\varepsilon_{ij} b^\varepsilon_{kj} \partial^\varepsilon_k u^\varepsilon_i \delta^\varepsilon dx^\varepsilon = \varepsilon^5 \int_\Omega \tau_{ij}(\varepsilon)\, \gamma^\theta_{ij}(\mathbf{u}(\varepsilon))\, dx + \varepsilon^7 \varrho_B(\varepsilon; \tau, \mathbf{u}(\varepsilon)),$$

$$\begin{aligned}\frac{1}{2}\int_{\Omega^\varepsilon} \tau^\varepsilon_{ij} b^\varepsilon_{ki} \partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj} \partial^\varepsilon_m u^\varepsilon_l \delta^\varepsilon dx^\varepsilon &= \frac{\varepsilon^5}{2} \int_\Omega \tau_{ij} \partial^\theta_i u_3(\varepsilon)\, \partial^\theta_j u_3(\varepsilon)\, dx \\ &+ \varepsilon^7 \varrho_C(\varepsilon; \tau, \mathbf{u}(\varepsilon), \mathbf{u}(\varepsilon)).\end{aligned}$$

Then the second equation in variational problem $(P^{\varepsilon})_{dyn}^{anis}$ may be also written as

$$A(\sigma(\varepsilon),\tau) - B^{\theta}(\tau,\mathbf{u}(\varepsilon)) - C^{\theta}(\tau,\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) = \varepsilon^2 S(\varepsilon;\sigma(\varepsilon),\mathbf{u}(\varepsilon),\tau),$$

where

$$S(\varepsilon;\sigma(\varepsilon),\mathbf{u}(\varepsilon),\tau) = \varrho_B(\varepsilon;\tau,u(\varepsilon)) + \varrho_C(\varepsilon;\tau,\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)) - \varrho_A(\varepsilon;\sigma(\varepsilon),\tau).$$

Now, note that, there exists a positive constant C such that, for all $\mathbf{u},\ \mathbf{v} \in \mathbf{V}(\Omega)$ and $\sigma,\ \tau \in \Sigma(\Omega)$

$$\sup_{0\le\varepsilon\le\varepsilon_0} \int_\Omega |\varrho_A(\varepsilon;\sigma,\tau)|dx \ \le\ C|\sigma|_{0,\Omega}|\tau|_{0,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} \int_\Omega |\varrho_B(\varepsilon;\tau,\mathbf{v})|dx \ \le\ C|\tau|_{0,\Omega}\|\mathbf{v}\|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} \int_\Omega |\varrho_C(\varepsilon;\tau,\mathbf{u},\mathbf{v})|dx \ \le\ C|\tau|_{0,\Omega}|\mathbf{u}|_{1,4,\Omega}|\mathbf{v}|_{1,4,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} \int_\Omega |\varrho_F(\varepsilon;\mathbf{v})|dx \ \le\ C\|\mathbf{v}\|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} \int_\Omega |\varrho_D(\varepsilon,\mathbf{u}(\varepsilon),\mathbf{v})|dx \ \le\ C\|\frac{\partial^2\mathbf{u}(\varepsilon)}{\partial t^2}\|_{-1,\frac{4}{3},\Omega}\|\mathbf{v}\|_{1,4,\Omega}.$$

We can also refer to [CP86] for more details. ∎

3.2.2 The limit three-dimensional problem

Assume that the scaled displacement-stress $(\mathbf{u}(\varepsilon),\sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(\mathbf{u}(\varepsilon),\sigma(\varepsilon)) = (\mathbf{u}^0,\sigma^0) + \varepsilon(\mathbf{u}^1,\sigma^1) + \varepsilon^2(\mathbf{u}^2,\sigma^2) + \cdots, \tag{3.5}$$

with
$\mathbf{u}^0 = (u_i^0) \in \mathbf{V}(\Omega)$, $\partial_3 u_3^0 \in C^0\left(\bar{\Omega}\right)$, $\mathbf{u}^p = (u_i^p) \in W^{1,4}(\Omega;\mathbb{R}^3)\ \ \forall p \ge 1$, $\sigma^p \in \Sigma(\Omega)\ \forall p \ge 0$,
and

$$c_{ijkl}(\varepsilon)(x) = c^0_{ijkl}(x) + \varepsilon c^1_{ijkl}(x) + \varepsilon^2 c^2_{ijkl}(x) + \cdots, \tag{3.6}$$

with

$c^0_{ijkl}(x) = c_{ijkl}(0)(x)$, $c^p_{ijkl} \in L^\infty(\Omega)$ $\forall p \geq 0$.

We also assume that when $\varepsilon \to 0$

$$\mathbf{p}(\varepsilon) \to \mathbf{p}^0 \text{ in } \mathbf{V}(\Omega),\ \mathbf{q}(\varepsilon) \to \mathbf{q}^0 \text{ in } L^2(\Omega;\mathbb{R}^3).$$

We substitute the formal asymptotic expansion (3.5)-(3.6) into the variational problem $(P(\varepsilon))^{anis}_{dyn}$, we obtain the following limit three-dimensional problem

Theorem 3.2 *The leading term $(\mathbf{u}^0, \sigma^0)$ satisfies the following variational problem:*

$$(P^0_1)^{anis}_{dyn}\begin{cases} \textit{Find } (\mathbf{u}^0,\sigma^0) \in \mathbf{V}(\Omega)\times\Sigma(\Omega)\ \forall t\in[0,T],\ \textit{such that,} \\ \int_\Omega \sigma^0_{i\alpha}\partial_i v_\alpha dx - \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_3 v_\alpha dx = \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma, \\ \forall v_\alpha \in V_\alpha(\Omega), \forall t\in\,]0,T[\,, \\ \frac{d^2}{dt^2}\left\{\rho\int_\Omega u^0_3 v_3 dx\right\} + \int_\Omega \sigma^0_{i3}\partial_i v_3 dx + \int_\Omega \sigma^0_{ij}\partial_i u^0_3\partial_j v_3 dx \\ -\int_\Omega \sigma^0_{\alpha3}\partial_\alpha\theta\partial_3 v_3 dx - \int_\Omega\{\sigma^0_{\alpha j}\partial_\alpha\theta\partial_3 u^0_3\partial_j v_3 + \sigma^0_{i\beta}\partial_i u^0_3\partial_\beta\theta\partial_3 v_3\}dx \\ +\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha\theta\partial_3 u^0_3\partial_\beta\theta\partial_3 v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 d\Gamma, \\ \forall v_3\in V_3(\Omega), \forall t\in\,]0,T[\,, \\ \int_\Omega c^0_{\alpha\beta\gamma\delta}\sigma^0_{\gamma\delta}\tau_{\alpha\beta}dx - \int_\Omega \tau_{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^0)dx - \frac{1}{2}\int_\Omega \tau_{\alpha\beta}\partial_\alpha u^0_3\partial_\beta u^0_3 dx \\ +\frac{1}{2}\int_\Omega \tau_{\alpha\beta}(\partial_\alpha\theta\partial_3 u^0_\beta + \partial_\beta\theta\partial_3 u^0_\alpha)dx \\ +\frac{1}{2}\int_\Omega \tau_{\alpha\beta}(\partial_\alpha\theta\partial_\beta u^0_3 + \partial_\beta\theta\partial_\alpha u^0_3)\partial_3 u^0_3 dx \\ -\frac{1}{2}\int_\Omega \tau_{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta(\partial_3 u^0_3)^2 dx = 0, \\ \forall(\tau_{\alpha\beta})\in L^2(\Omega;\mathbb{S}^2), \forall t\in\,]0,T[\,, \\ \int_\Omega \tau_{\alpha3}(\partial_\alpha u^0_3 + \partial_3 u^0_\alpha)dx + \int_\Omega \tau_{\alpha3}\partial_\alpha u^0_3\partial_3 u^0_3)dx \\ -\int_\Omega \tau_{\alpha3}\partial_\alpha\theta\partial_3 u^0_3 dx - \int_\Omega \tau_{\alpha3}\partial_\alpha\theta(\partial_3 u^0_3)^2 dx = 0, \\ \forall(\tau_{\alpha3})\in L^2(\Omega;\mathbb{R}^2), \forall t\in\,]0,T[\,, \\ \int_\Omega \tau_{33}\partial_3 u^0_3 dx + \frac{1}{2}\int_\Omega \tau_{33}(\partial_3 u^0_3)^2 dx = 0, \\ \forall\tau_{33}\in L^2(\Omega), \forall t\in\,]0,T[\,, \\ \mathbf{u}^0(x,0) = \mathbf{p}^0 \textit{ and } \frac{\partial\mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0 \textit{ in } \Omega, \end{cases}$$

where

$\gamma_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$.

Proof. Using technics of the asymptotic analysis method, we first replace $\mathbf{u}(\varepsilon)$, $\sigma(\varepsilon)$, and $c_{ijkl}(\varepsilon)(x)$ by their expansions (3.5)-(3.6) in the forms A, B^θ, C^θ, D^t and F. Next we equate to zero the terms which are independent of ε in $(P(\varepsilon))^{anis}_{dyn}$, then we show that $(\mathbf{u}^0, \sigma^0)$ satisfy $(P^0_1)^{anis}_{dyn}$. ■

Theorem 3.3 *The leading term* $\mathbf{u}^0$ *satisfies the following variational problem:*

$$(P^0_2)^{anis}_{dyn}\begin{cases} \textit{Find } \mathbf{u}^0 \in \mathbf{V}_{KL}(\Omega)\ \forall t \in [0,T], \textit{ such that,} \\ \frac{d^2}{dt^2}\left\{\rho\int_\Omega u^0_3 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u^0_3+\theta)\partial_\beta v_3 dx = \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\} d\gamma, \\ \forall v \in \mathbf{V}_{KL}(\Omega), \forall t \in \left]0,T\right[, \\ \mathbf{u}^0\,(x,0) = \mathbf{p}^0 \textit{ and } \frac{\partial \mathbf{u}^0}{\partial t}\,(x,0) = \mathbf{q}^0 \textit{ in } \Omega, \end{cases}$$

where

$$\begin{cases} \sigma^0_{\alpha\beta} = c^{0,-1}_{\alpha\beta\gamma\delta}(x)\bar{E}^0_{\gamma\delta}(\mathbf{u}^0), \\ (c^{0,-1}_{\alpha\beta\gamma\delta}) \textit{ is the inverse of } (c^0_{\alpha\beta\gamma\delta}), \\ \bar{E}^0_{\gamma\delta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\gamma u^0_\delta + \partial_\delta u^0_\gamma + \partial_\gamma\theta\partial_\delta u^0_3 + \partial_\delta\theta\partial_\gamma u^0_3 + \partial_\gamma u^0_3\partial_\delta u^0_3\right). \end{cases}$$

Proof.

The proof has been divided into 3 steps.

Step 1. The fifth equation in $(P^0_1)^{anis}_{dyn}$ gives

$$\partial_3 u^0_3(1 + \frac{1}{2}\partial_3 u^0_3) = 0,$$

so that

$$\partial_3 u^0_3 = 0 \text{ or } \partial_3 u^0_3 = -2.$$

Since the inclusion $H^3(\Omega) \hookrightarrow \mathcal{C}^1(\Omega)$ and $u^0_3 = 0$ on $\gamma_1 \times [-1,1]$, the solution $\partial_3 u^0_3 = -2$ is eliminated. Hence we obtain

$$\partial_3 u^0_3 = 0 \text{ in } \Omega. \tag{3.7}$$

Consequently, the fourth equation in $(P^0_1)^{anis}_{dyn}$ reduces to

$$\partial_\alpha u^0_3 + \partial_3 u^0_\alpha = 0 \text{ in } \Omega. \tag{3.8}$$

Step 2. Taking into account the equations (3.7)-(3.8), the third equation in $(P_1^0)_{dyn}^{anis}$ reduces to

$$c^0_{\alpha\beta\gamma\delta}\sigma^0_{\gamma\delta} - \gamma_{\alpha\beta}(\mathbf{u}^0) - \frac{1}{2}\partial_\alpha u_3^0 \partial_\beta u_3^0 - \frac{1}{2}(\partial_\alpha\theta\partial_\beta u_3^0 + \partial_\beta\theta\partial_\alpha u_3^0) = 0. \tag{3.9}$$

We observe that

$$\gamma_{\alpha\beta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha\right).$$

If $(c^{0,-1}_{\alpha\beta\gamma\delta})$ is the inverse of $(c^0_{\alpha\beta\gamma\delta})$, we show that

$$\sigma^0_{\alpha\beta} = c^{0,-1}_{\alpha\beta\gamma\delta}(x)\bar{E}^0_{\gamma\delta}(\mathbf{u}^0).$$

Note that

$$c^{0,-1}_{\alpha\beta\gamma\delta}(x) = a_{\alpha\beta\gamma\delta}(x) - a_{\alpha\beta i3}(x)i_{ij}(x)a_{j3\gamma\delta}(x),$$

where $i = (i_{ij})$ is the inverse of the matrix (a_{i3j3}).

Step 3. Taking into account the equation (3.7), we next find that the second equation in $(P_1^0)_{dyn}^{anis}$ reduce to

$$\begin{aligned}\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u_3^0 \partial_\beta v_3 dx \quad &= \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 d\Gamma&,\end{aligned} \tag{3.10}$$

From the first equation and the relation (3.8), we conclude that

$$\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx = \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx - \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma. \tag{3.11}$$

We replace the integral $\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx$ in equation (3.10) by their expression (3.11), we find that

$$\begin{aligned}\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u_3^0+\theta)\partial_\beta v_3 dx \quad &= \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma&.\end{aligned}$$

■

3.2.3 The limit two-dimensional problem

We use some technics employed by Raoult [Rao85], who assumed that the initial data $\varphi_3 = p_3^0$ and $\psi_3 = q_3^0$ are independent of x_3 and sufficiently smooth. We also assume that the initial data $p_\alpha^0 = \varphi_\alpha - x_3\partial_\alpha p_3^0$ and $q_\alpha^0 = \psi_\alpha - x_3\partial_\alpha q_3^0$, such that φ_α and ψ_α are independent of x_3 and sufficiently smooth.

First, we show in the next theorem that $(P_2^0)_{dyn}^{anis}$ is in a sense of two-dimensional problem posed over the two-dimensional domain $\bar{\omega}$.

To formulate this result, we define the following coefficients

$$C^0_{\alpha\beta\gamma\delta}(x_1,x_2) = \int_{-1}^{1} c^{0,-1}_{\alpha\beta\gamma\delta}(x)dx_3, \tag{3.12}$$

$$C^1_{\alpha\beta\gamma\delta}(x_1,x_2) = \int_{-1}^{1} x_3 c^{0,-1}_{\alpha\beta\gamma\delta}(x)dx_3, \tag{3.13}$$

$$C^2_{\alpha\beta\gamma\delta}(x_1,x_2) = \int_{-1}^{1} x_3^2 c^{0,-1}_{\alpha\beta\gamma\delta}(x)dx_3. \tag{3.14}$$

Moreover we define also the tensors $(\bar{N}^{anis}_{\alpha\beta})$ and $(m^{anis}_{\alpha\beta})$, associated to the Kirchhoff-Love displacement $\mathbf{u}^0$, by

$$\bar{N}^{anis}_{\alpha\beta}(\zeta) = C^0_{\alpha\beta\gamma\delta}\bar{E}^0_{\gamma\delta}(\zeta) + C^1_{\alpha\beta\gamma\delta}\Upsilon_{\gamma\delta}(\zeta_3), \tag{3.15}$$

$$m^{anis}_{\alpha\beta}(\zeta) = C^1_{\alpha\beta\gamma\delta}\bar{E}^0_{\gamma\delta}(\zeta) + C^2_{\alpha\beta\gamma\delta}\Upsilon_{\gamma\delta}(\zeta_3), \tag{3.16}$$

where
$\bar{E}^0_{\gamma\delta}(\zeta) = \frac{1}{2}(\partial_\gamma\zeta_\delta + \partial_\delta\zeta_\gamma + \partial_\gamma\theta\partial_\delta\zeta_3 + \partial_\delta\theta\partial_\gamma\zeta_3 + \partial_\gamma\zeta_3\partial_\delta\zeta_3)$,
$\Upsilon_{\gamma\delta}(\zeta_3) = -\partial_{\gamma\delta}\zeta_3$,

denote the components of the two-dimensional strain tensor and of the two-dimensional curvature tensor.

Theorem 3.4 *The leading term $\mathbf{u}^0 = (u_i^0)$ is of the form $u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ and $u_3^0 = \zeta_3$ with $\zeta = (\zeta_i) \in \mathbf{V}(\omega)\ \forall t \in [0,T]$. In addition, the field ζ satisfies the following limit scaled two-dimensional displacement problem:*

$$(P(\omega))^{anis}_{dyn}\begin{cases} \textit{Find } \zeta \in \mathbf{V}(\omega)\ \forall t \in [0,T], \textit{ such that,} \\ 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega \\ + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \forall\eta \in \mathbf{V}(\omega), \forall t \in]0,T[, \\ \zeta(.,0) = \varphi \textit{ and } \frac{\partial\zeta}{\partial t}(.,0) = \psi \textit{ in } \omega, \end{cases}$$

where

$p_3 = \int_{-1}^{1} f_3 dx_3 + g_3(.,+1) + g_3(.,-1)$.

Proof.

i) From $\mathbf{v} \in \mathbf{V}_{KL}(\Omega)$, by a standard argument due to Ciarlet (see,e.g., [Cia97, Theorem 1.4-4]), we get

$$u_\alpha^0 = \zeta_\alpha - x_3 \partial_\alpha \zeta_3 \text{ and } u_3^0 = \zeta_3 \text{ with } \zeta = (\zeta_i) \in \mathbf{V}(\omega).$$

ii) We observe that

$$\bar{E}_{\gamma\delta}^0(\mathbf{u}^0) = \bar{E}_{\gamma\delta}^0(\zeta) + x_3 \Upsilon_{\gamma\delta}(\zeta_3). \tag{3.17}$$

From the definition of $\sigma_{\alpha\beta}^0$ and (3.17), we conclude that

$$\begin{aligned}
\int_{-1}^{1} \sigma_{\alpha\beta}^0 dx_3 &= \int_{-1}^{1} c_{\alpha\beta\gamma\delta}^{0,-1}(x) \left[\bar{E}_{\gamma\delta}^0(\zeta) + x_3 \Upsilon_{\gamma\delta}(\zeta_3)\right] dx_3 \\
&= \left(\int_{-1}^{1} c_{\alpha\beta\gamma\delta}^{0,-1}(x) dx_3\right) \bar{E}_{\gamma\delta}^0(\zeta) + \left(\int_{-1}^{1} x_3 c_{\alpha\beta\gamma\delta}^{0,-1}(x) dx_3\right) \Upsilon_{\gamma\delta}(\zeta_3) \\
&= C_{\alpha\beta\gamma\delta}^0 \bar{E}_{\gamma\delta}^0(\zeta) + C_{\alpha\beta\gamma\delta}^1 \Upsilon_{\gamma\delta}(\zeta_3) \\
&= \bar{N}_{\alpha\beta}^{anis}(\zeta),
\end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{1} x_3 \sigma_{\alpha\beta}^0 dx_3 &= \int_{-1}^{1} x_3 c_{\alpha\beta\gamma\delta}^{0,-1}(x) \left[\bar{E}_{\gamma\delta}^0(\zeta) + x_3 \Upsilon_{\gamma\delta}(\zeta_3)\right] dx_3 \\
&= \left(\int_{-1}^{1} x_3 c_{\alpha\beta\gamma\delta}^{0,-1}(x) dx_3\right) \bar{E}_{\gamma\delta}^0(\zeta) + \left(\int_{-1}^{1} x_3^2 c_{\alpha\beta\gamma\delta}^{0,-1}(x) dx_3\right) \Upsilon_{\gamma\delta}(\zeta_3) \\
&= C_{\alpha\beta\gamma\delta}^1 \bar{E}_{\gamma\delta}^0(\zeta) + C_{\alpha\beta\gamma\delta}^2 \Upsilon_{\gamma\delta}(\zeta_3) \\
&= m_{\alpha\beta}^{anis}(\zeta),
\end{aligned}$$

iii) First we choose, in $(P_2^0)_{dyn}^{anis}$, $\mathbf{v} \in \mathbf{V}_{KL}(\Omega)$ with the components

$$v_\alpha(x) = -x_3 \partial_\alpha \eta_3(x_1, x_2), \ v_3(x) = \eta_3(x_1, x_2),$$

with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_1.

This choice shows that $(P_2^0)_{dyn}^{anis}$ reduce to

$$\frac{d^2}{dt^2}\left\{\rho \int_\Omega \zeta_3 \eta_3 dx\right\} - \int_\Omega x_3 \sigma_{\alpha\beta}^0 \partial_{\alpha\beta} \eta_3 dx + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha(\zeta_3^0 + \theta) \partial_\beta \eta_3 dx = \int_\Omega f_3 \eta_3 dx + \int_{\Gamma_+ \cup \Gamma_-} g_3 \eta_3 d\Gamma. \tag{3.18}$$

The second choice of $\mathbf{v} \in \mathbf{V}_{KL}(\Omega)$ is

$$v_\alpha(x) = \eta_\alpha(x_1, x_2),\ \ v_3(x) = 0,$$

with $\eta_\alpha \in H^1(\omega)$.

In this case shows that $(P_2^0)_{dyn}^{anis}$ reduce to

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma \tag{3.19}$$

Using Fubini's Formula: $\int_\Omega F dx = \int_\omega \left\{\int_{-1}^1 F dx_3\right\} d\omega$, we have

$$\frac{d^2}{dt^2}\{\rho\int_\Omega \zeta_3\eta_3 dx\} = 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega,$$

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx = -\int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega,$$

$$\begin{aligned}\int_\Omega f_3\eta_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3\eta_3 d\Gamma &= \int_\omega\{\int_{-1}^1 f_3 dx_3 + g_3(.,+1) + g_3(.,-1)\}\eta_3 d\omega \\ &= \int_\omega \mathrm{p}_3\eta_3 d\omega,\end{aligned}$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

Then

$$\begin{aligned}2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega \\ + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.\end{aligned}$$

■

Next, we write the two-dimensional boundary value problem as an equivalent boundary value problem $(\bar{P}(\omega))^{anis}_{dyn}$. Using Green's formulas and equating to zero all the factors of η_α, η_3, and $\partial_\nu\eta_3$ in their respective domains of integration, we obtain

Theorem 3.5 *Assume that the boundary γ is sufficiently smooth. Then any smooth solution $\zeta = (\zeta_i)$ of the variational problem $(P(\omega))^{anis}_{dyn}$ is also a solution of the following two-dimensional displacement problem:*

$$(\bar{P}(\omega))^{anis}_{dyn}\begin{cases} \textit{Find } ((\zeta_\alpha),\zeta_3) \in (H^1(\omega))^2 \times H^2(\omega)\ \forall t \in [0,T], \textit{ such that,} \\ 2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m^{anis}_{\alpha\beta} - \bar{N}^{anis}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3+\theta\right) = \mathrm{p}_3 \textit{ in } \omega\times]0,T[\,, \\ \partial_\beta \bar{N}^{anis}_{\alpha\beta} = 0 \textit{ in } \omega\times]0,T[\,, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \textit{ on } \gamma_1 \times]0,T[\,, \\ \bar{N}^{anis}_{\alpha\beta}\nu_\beta = 2h_\alpha \textit{ on } \gamma_1\times]0,T[\,, \\ m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[\,, \\ \partial_\alpha m^{anis}_{\alpha\beta}\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \textit{ on } \gamma_2\times]0,T[\,, \\ \bar{N}^{anis}_{\alpha\beta}\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[\,, \\ \zeta(.,0) = \varphi \textit{ and } \frac{\partial\zeta}{\partial t}(.,0) = \psi \textit{ in } \omega. \end{cases}$$

Proof.

Applying the Green formulas, we obtain

$$\begin{aligned} -\int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega &= \int_\gamma \left\{\left(\partial_\alpha m^{anis}_{\alpha\beta}\right)\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right)\right\}\eta_3 d\gamma \\ &- \int_\gamma m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma - \int_\omega \left(\partial_{\alpha\beta}m^{anis}_{\alpha\beta}\right)\eta_3 d\omega, \end{aligned}$$

$$\begin{aligned} \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\partial_\beta\eta_3 d\omega &= -\int_\omega \left\{\partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right)\right\}\eta_3 d\omega \\ &+ \int_\gamma \left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right)\nu_\beta\eta_3 d\gamma, \end{aligned}$$

$$\int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = -\int_\omega \left(\partial_\beta\bar{N}^{anis}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \bar{N}^{anis}_{\alpha\beta}\nu_\beta\eta_\alpha d\gamma.$$

Then

$$\begin{aligned} &\int_\omega \left[2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m^{anis}_{\alpha\beta} - \partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right) - p_3\right]\eta_3 d\omega \ - \\ &\int_\omega \left(\partial_\beta\bar{N}^{anis}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma\left(\bar{N}^{anis}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma \ + \\ &\int_{\gamma_2}\left\{\left[\partial_\alpha m^{anis}_{\alpha\beta} + \bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right]\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right)\right\}\eta_3 d\gamma = 0, \end{aligned}$$

for all $\eta = (\eta_\alpha, \eta_3) \in V(\omega)$, with the functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by

$$\widetilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0,T].$$

These equations imply that all the factors of η_α, η_3, and $\partial_\nu \eta_3$ vanish in their respective domains of integration. Then we get

$$2\rho \frac{\partial^2 \zeta_3}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}^{anis} - \partial_\beta \left(\bar{N}_{\alpha\beta}^{anis} \partial_\alpha (\zeta_3 + \theta) \right) = p_3 \text{ in } \omega \times]0,T[\,,$$

and

$$\partial_\beta \bar{N}_{\alpha\beta}^{anis} = 0 \text{ in } \omega \times]0,T[\,,$$

so that

$$\partial_\beta \left(\bar{N}_{\alpha\beta}^{anis} \partial_\alpha (\zeta_3 + \theta) \right) = \bar{N}_{\alpha\beta}^{anis} \partial_{\alpha\beta} (\zeta_3 + \theta) \text{ in } \omega \times]0,T[\,,$$

consequently

$$2\rho \frac{\partial^2 \zeta_3}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}^{anis} - \bar{N}_{\alpha\beta}^{anis} \partial_{\alpha\beta} (\zeta_3 + \theta) = p_3 \text{ in } \omega \times]0,T[\,.$$

For boundary conditions, we get

$$\bar{N}_{\alpha\beta}^{anis} \nu_\beta - 2\widetilde{h}_\alpha = 0 \text{ on } \gamma \times]0,T[\,,$$

thus

$$\bar{N}_{\alpha\beta}^{anis} \nu_\beta = 2h_\alpha \text{ on } \gamma_1 \times]0,T[\,,$$

and

$$\bar{N}_{\alpha\beta}^{anis} \nu_\beta = 0 \text{ on } \gamma_2 \times]0,T[\,.$$

We also get

$$m_{\alpha\beta}^{anis} \nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2 \times]0,T[\,,$$

and

$$\left[\partial_\alpha m_{\alpha\beta}^{anis} + \bar{N}_{\alpha\beta}^{anis} \partial_\alpha (\zeta_3 + \theta) \right] \nu_\beta + \partial_\tau \left(m_{\alpha\beta}^{anis} \nu_\alpha \tau_\beta \right) = 0 \text{ on } \gamma_2 \times]0,T[\,,$$

since $\bar{N}_{\alpha\beta}^{anis} \nu_\beta = 0$ on $\gamma_2 \times]0,T[$, we conclude that

$$\partial_\alpha m_{\alpha\beta}^{anis} \nu_\beta + \partial_\tau \left(m_{\alpha\beta}^{anis} \nu_\alpha \tau_\beta \right) = 0 \text{ on } \gamma_2 \times]0,T[\,.$$

■

3.3 Dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells

We now rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))^{anis}_{dyn}$ in the form of dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shell as follows:

Note that

$$\bar{N}^{anis}_{\alpha\beta}(\zeta) = N^1_{\alpha\beta}(\bar{\zeta}) + N^{2,\theta}_{\alpha\beta}(\zeta_3), \tag{3.20}$$

$$m^{anis}_{\alpha\beta}(\zeta) = m^1_{\alpha\beta}(\bar{\zeta}) + m^{2,\theta}_{\alpha\beta}(\zeta_3), \tag{3.21}$$

where

$$N^1_{\alpha\beta}(\bar{\zeta}) = C^0_{\alpha\beta\gamma\delta} e_{\gamma\delta}(\bar{\zeta}), \tag{3.22}$$

$$N^{2,\theta}_{\alpha\beta}(\zeta_3) = C^0_{\alpha\beta\gamma\delta}\bar{E}^{\theta}_{\gamma\delta}(\zeta_3) + C^1_{\alpha\beta\gamma\delta}\Upsilon_{\gamma\delta}(\zeta_3), \tag{3.23}$$

$$m^1_{\alpha\beta}(\bar{\zeta}) = C^1_{\alpha\beta\gamma\delta} e_{\gamma\delta}(\bar{\zeta}), \tag{3.24}$$

$$m^{2,\theta}_{\alpha\beta}(\zeta_3) = C^1_{\alpha\beta\gamma\delta}\bar{E}^{\theta}_{\gamma\delta}(\zeta_3) + C^2_{\alpha\beta\gamma\delta}\Upsilon_{\gamma\delta}(\zeta_3), \tag{3.25}$$

such that
$\bar{\zeta} = (\zeta_1, \zeta_2)$, $e_{\gamma\delta}(\bar{\zeta}) = \frac{1}{2}(\partial_\gamma\zeta_\delta + \partial_\delta\zeta_\gamma)$, $\bar{E}^{\theta}_{\gamma\delta}(\zeta_3) = \frac{1}{2}(\partial_\gamma\theta\partial_\delta\zeta_3 + \partial_\delta\theta\partial_\gamma\zeta_3 + \partial_\gamma\zeta_3\partial_\delta\zeta_3)$.

We assume that $C^0_{\alpha\beta\gamma\delta}$, $C^1_{\alpha\beta\gamma\delta}$ and $C^2_{\alpha\beta\gamma\delta}$ are smooth enough functions.

Theorem 3.6 *Assume that the set ω is simply-connected and that its boundary γ is sufficiently smooth. Let $\zeta = (\zeta_i)$ be a solution of $(\bar{P}(\omega))^{anis}_{dyn}$ with the regularity*
$\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^4(\omega)$ $\forall t \in [0,T]$.
Then

a) *The functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by*

$$\widetilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0,T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma = 0.$$

b) *Furthermore, there exists a function* $\Phi \in H^4(\omega)$*, uniquely defined by the relations* $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$*, such that*

$$\bar{N}_{11}^{anis} = 2\partial_{22}\Phi,\ \bar{N}_{12}^{anis} = \bar{N}_{21}^{anis} = -2\partial_{12}\Phi,\ \bar{N}_{22}^{anis} = 2\partial_{11}\Phi.$$

c) *Finally, the pair* $(\zeta_3, \Phi) \in H^4(\omega) \times H^4(\omega)\ \forall t \in [0,T]$*, satisfies the following problem*

$$(P)_{dyn}^{anis} \begin{cases} 2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}M_{\alpha\beta}^{anis}(\zeta_3,\Phi) = 2\left[\Phi, \zeta_3 + \theta\right] + \mathrm{p}_3 \ \textit{in}\ \omega \times]0,T[\,, \\ \Delta^2\Phi = \frac{1}{2}\mathfrak{L}(\zeta_3,\Phi)\ \textit{in}\ \omega \times]0,T[\,, \\ \zeta_3 = \partial_\nu\zeta_3 = 0\ \textit{on}\ \gamma_1\times]0,T[\,, \\ M_{\alpha\beta}^{anis}(\zeta_3,\Phi)\nu_\alpha\nu_\beta = 0\ \textit{on}\ \gamma_2\times]0,T[\,, \\ \partial_\alpha M_{\alpha\beta}^{anis}(\zeta_3,\Phi)\nu_\beta + \partial_\tau\left(M_{\alpha\beta}^{anis}(\zeta_3,\Phi)\nu_\alpha\tau_\beta\right) = 0\ \textit{on}\ \gamma_2 \times]0,T[\,, \\ \Phi = \Phi_0\ \textit{and}\ \partial_\nu\Phi = \Phi_1\ \textit{on}\ \gamma \times]0,T[\,, \\ \zeta_3\,(.,0) = \varphi_3\ \textit{and}\ \frac{\partial\zeta_3}{\partial t}\,(.,0) = \psi_3\ \textit{in}\ \omega, \end{cases}$$

where

$$\begin{cases} \Phi_0(y) = -y_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \\ \Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma,\ y = (y_1,y_2) \in \gamma, \\ [\Phi,\zeta] = \partial_{11}\Phi\partial_{22}\zeta + \partial_{22}\Phi\partial_{11}\zeta - 2\partial_{12}\Phi\partial_{12}\zeta, \\ M_{\alpha\beta}^{anis}(\zeta_3,\Phi) = \mathcal{F}_{\alpha\beta}^{\theta}(\zeta_3,\Phi) + m_{\alpha\beta}^{2,\theta}(\zeta_3), \\ \mathfrak{L}(\zeta_3,\Phi) = \Delta\left[C_{\alpha\alpha\gamma\delta}^{0}C_{\sigma\varsigma\gamma\delta}^{1,-1}\mathcal{F}_{\sigma\varsigma}^{\theta}(\zeta_3,\Phi) + N_{\alpha\alpha}^{2,\theta}(\zeta_3)\right], \\ \mathcal{F}_{\alpha\beta}^{\theta}(\zeta_3,\Phi) = C_{\alpha\beta\gamma\delta}^{1}[C_{11\gamma\delta}^{0,-1}(2\partial_{22}\Phi - N_{11}^{2,\theta}(\zeta_3)) + C_{22\gamma\delta}^{0,-1}(2\partial_{11}\Phi - N_{22}^{2,\theta}(\zeta_3)) + \\ 2C_{12\gamma\delta}^{0,-1}(-2\partial_{12}\Phi - N_{12}^{2,\theta}(\zeta_3))], \end{cases}$$

such that $C_{\alpha\beta\gamma\delta}^{0,-1}$ *and* $C_{\alpha\beta\gamma\delta}^{1,-1}$ *are the inverse of* $C_{\alpha\beta\gamma\delta}^{0}$ *and* $C_{\alpha\beta\gamma\delta}^{1}$*, respectively.*

Proof.

The proof is divided into three steps.

a) The smoothness of functions $C_{\alpha\beta\gamma\delta}^{0}$, $C_{\alpha\beta\gamma\delta}^{1}$ and the regularity of functions ζ_i imply that $\bar{N}_{\alpha\beta}^{anis} \in H^2(\omega)$ and $\widetilde{h}_\alpha \in H^{\frac{3}{2}}(\gamma)$.

The functions $\widetilde{h}_\alpha$ satisfy the compatibility conditions, to see this, we observe that, if we choose $\eta = (a_1 - bx_2, a_2 - bx_1, 0)$ for any constants a_1, a_2 and b in the variational problem $(P(\omega))^{anis}_{dyn}$, we obtain

$$a_\alpha \int_\gamma \widetilde{h}_\alpha d\gamma + b \int_\gamma (x_1 \widetilde{h}_2 - x_2 \widetilde{h}_1) d\gamma = 0. \tag{3.26}$$

b) Since the set ω is simply-connected and by using the generalized Poincaré theorem (see [Sch66, Theorem VI, p.59]), the equation $\partial_\beta \bar{N}^{anis}_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in D'(\omega)$, unique up to the addition of constants, such that $\bar{N}^{anis}_{1\alpha} = 2\partial_2 \psi_\alpha$, $\bar{N}^{anis}_{2\alpha} = -2\partial_1 \psi_\alpha$.

Since the equation $\bar{N}^{anis}_{12} = \bar{N}^{anis}_{21}$ implies that $\partial_\alpha \psi_\alpha = 0$. Another application of the same result shows that there exist a distribution $\Phi \in D'(\omega)$, unique up to the addition of polynomials of degree ≤ 1, such that $\psi_1 = \partial_2 \Phi$, $\psi_2 = -\partial_1 \Phi$, so that

$$\bar{N}^{anis}_{11} = 2\partial_{22}\Phi,\ \bar{N}^{anis}_{12} = \bar{N}^{anis}_{21} = -2\partial_{12}\Phi,\ \bar{N}^{anis}_{22} = 2\partial_{11}\Phi \text{ in } \omega. \tag{3.27}$$

The regularities of $\bar{N}^{anis}_{\alpha\beta} \in H^2(\omega)$ imply that $\Phi \in H^4(\omega)$. Then Φ is uniquely defined if we impose that $\Phi(0) = \partial_1 \Phi(0) = \partial_2 \Phi(0) = 0$.

c) **(i)** From $\bar{N}^{anis}_{\alpha\beta} \nu_\beta = 2\widetilde{h}_\alpha$ on γ, we obtain

$$\begin{aligned}
\widetilde{h}_1 &= \frac{1}{2} \bar{N}^{anis}_{1\beta} \nu_\beta \\
&= \frac{1}{2} \left(\nu_1 \bar{N}^{anis}_{11} + \nu_2 \bar{N}^{anis}_{12} \right) \\
&= \nu_1 \partial_{22}\Phi - \nu_2 \partial_{21}\Phi \\
&= \partial_\tau \left(\partial_2 \Phi \right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{h}_2 &= \frac{1}{2} \bar{N}^{anis}_{2\beta} \nu_\beta \\
&= \frac{1}{2} \left(\nu_1 \bar{N}^{anis}_{21} + \nu_2 \bar{N}^{anis}_{22} \right) \\
&= -\nu_1 \partial_{12}\Phi + \nu_2 \partial_{11}\Phi \\
&= -\partial_\tau \left(\partial_1 \Phi \right),
\end{aligned}$$

thus

$$\partial_1 \Phi\,(y) = -\int_{\gamma(y)} \widetilde{h}_2 d\gamma \quad et \quad \partial_2 \Phi\,(y) = \int_{\gamma(y)} \widetilde{h}_1 d\gamma, \tag{3.28}$$

for all $y \in \gamma$,

then

$$\begin{aligned} \partial_\nu \Phi(y) &= \nu_1(y)\,\partial_1\Phi(y) + \nu_2(y)\,\partial_2\Phi(y) \\ &= -\nu_1(y)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \nu_2(y)\int_{\gamma(y)} \widetilde{h}_1 d\gamma. \end{aligned} \tag{3.29}$$

So that

$$\partial_\nu \Phi(y) = \Phi_1 \text{ on } \gamma,$$

and

$$\begin{aligned} \partial_\tau \Phi(y) &= \tau_1(y)\,\partial_1\Phi(y) + \tau_2(y)\,\partial_2\Phi(y) \\ &= -\tau_1(y)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \tau_2(y)\int_{\gamma(y)} \widetilde{h}_1 d\gamma. \end{aligned} \tag{3.30}$$

Since

$$\partial_\tau \Phi(y) = \partial_\tau \Phi_0 \text{ and } \Phi(0) = \partial_\tau \Phi(0) = 0,$$

we conclude that

$$\Phi(y) = \Phi_0 \text{ on } \gamma.$$

(ii) We have

$$\begin{aligned} [\Phi, \zeta_3 + \theta] &= \partial_{11}\Phi\partial_{22}(\zeta_3+\theta) + \partial_{22}\Phi\partial_{11}(\zeta_3+\theta) - 2\partial_{12}\Phi\partial_{12}(\zeta_3+\theta) \\ &= \frac{1}{2}\left[\bar{N}_{22}^{anis}\partial_{22}(\zeta_3+\theta) + \bar{N}_{11}^{anis}\partial_{11}(\zeta_3+\theta) + 2\bar{N}_{12}^{anis}\partial_{12}(\zeta_3+\theta)\right] \\ &= \frac{1}{2}\bar{N}_{\alpha\beta}^{anis}\partial_{\alpha\beta}(\zeta_3+\theta), \end{aligned} \tag{3.31}$$

thus

$$\bar{N}_{\alpha\beta}^{anis}\partial_{\alpha\beta}(\zeta_3+\theta) = 2[\Phi, \zeta_3+\theta]. \tag{3.32}$$

From (3.22), we get

$$e_{\gamma\delta}(\bar{\zeta}) = C_{\alpha\beta\gamma\delta}^{0,-1} N_{\alpha\beta}^{1}(\bar{\zeta}), \tag{3.33}$$

using (3.24) and (3.27), we obtain

$$\begin{aligned} m^1_{\alpha\beta}(\bar{\zeta}) &= C^1_{\alpha\beta\gamma\delta}C^{0,-1}_{\sigma\varsigma\gamma\delta}N^1_{\sigma\varsigma}(\bar{\zeta}) \\ &= C^1_{\alpha\beta\gamma\delta}C^{0,-1}_{\sigma\varsigma\gamma\delta}(\bar{N}^{anis}_{\sigma\varsigma}(\zeta) - N^{2,\theta}_{\sigma\varsigma}(\zeta_3)) \\ &= C^1_{\alpha\beta\gamma\delta}[C^{0,-1}_{11\gamma\delta}(2\partial_{22}\Phi - N^{2,\theta}_{11}(\zeta_3)) \\ &+ C^{0,-1}_{22\gamma\delta}(2\partial_{11}\Phi - N^{2,\theta}_{22}(\zeta_3)) \\ &+ 2C^{0,-1}_{12\gamma\delta}(-2\partial_{12}\Phi - N^{2,\theta}_{12}(\zeta_3))] \\ &= \mathcal{F}^\theta_{\alpha\beta}(\zeta_3, \Phi), \end{aligned} \tag{3.34}$$

which yields

$$\begin{aligned} m^{anis}_{\alpha\beta}(\zeta) &= \mathcal{F}^\theta_{\alpha\beta}(\zeta_3, \Phi) + m^{2,\theta}_{\alpha\beta}(\zeta_3) \\ &= M^{anis}_{\alpha\beta}(\zeta_3, \Phi). \end{aligned} \tag{3.35}$$

Then, we deduce

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}M^{anis}_{\alpha\beta}(\zeta_3, \Phi) = 2\,[\Phi, \zeta_3 + \theta] + \mathrm{p}_3 \text{ in } \omega \times]0, T[\,. \tag{3.36}$$

(iii) Notice that

$$\begin{aligned} \Delta^2\Phi &= \Delta\,(\Delta\Phi) \\ &= \Delta\,(\partial_{\alpha\alpha}\Phi) \\ &= \frac{1}{2}\Delta\bar{N}^{anis}_{\alpha\alpha} \\ &= \frac{1}{2}\Delta[N^1_{\alpha\alpha}(\bar{\zeta}) + N^{2,\theta}_{\alpha\alpha}(\zeta_3)]. \end{aligned}$$

We have

$$N^1_{\alpha\alpha} = C^0_{\alpha\alpha\gamma\delta}e_{\gamma\delta}(\bar{\zeta}), \tag{3.37}$$

taking into account (3.24) and (3.34), we deduce that

$$e_{\gamma\delta}(\bar{\zeta}) = C^{1,-1}_{\alpha\beta\gamma\delta}\mathcal{F}^\theta_{\alpha\beta}(\zeta_3, \Phi), \tag{3.38}$$

thus

$$N^1_{\alpha\alpha} = C^0_{\alpha\alpha\gamma\delta}C^{1,-1}_{\sigma\varsigma\gamma\delta}\mathcal{F}^\theta_{\sigma\varsigma}(\zeta_3, \Phi). \tag{3.39}$$

So that

$$\begin{aligned} \Delta^2\Phi &= \frac{1}{2}\Delta[C^0_{\alpha\alpha\gamma\delta}C^{1,-1}_{\sigma\varsigma\gamma\delta}\mathcal{F}^\theta_{\sigma\varsigma}(\zeta_3, \Phi) + N^{2,\theta}_{\alpha\alpha}(\zeta_3)] \\ &= \frac{1}{2}\mathfrak{L}(\zeta_3, \Phi). \end{aligned} \tag{3.40}$$

■

3.4 Conclusion

An application of the technics from formal asymptotic analysis to the three-dimensional nonlinear system of elastodynamics corresponding to a shallow shell, with a specific class of boundary conditions of generalized Marguerre-von Kármán type, with nonhomogeneous anisotropic material, shows that the leading term of the expansion is characterized by a two-dimensional dynamical boundary value problem called the dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells, which depends on the Airy function Φ and the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell.

Part II

Problems with unilateral contact

Chapter 4

Asymptotic modeling of a Signorini problem of generalized Marguerre-von Kármán shallow shell

In the paper [CB08], Chacha and Bensayah have studied the asymptotic modeling of unilateral contact problem with Coulomb frictional between an elastic nonlinear von Kármán plate and a rigid obstacle. In this Chapter, we extend this study to the case of a shallow shell under generalized Marguerre-von Kármán conditions. This work was published in [BCG13].

4.1 Setting of the problem

We suppose that $\hat{\Omega}^\varepsilon$ is occupied by a nonlinear, elastic, homogeneous, isotropic body. In its natural configuration: a shallow shell of thickness 2ε whose Lamé's constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The shallow shell is supposed to be subjected to applied body forces of density $\hat{f}^\varepsilon \in (L^2(\hat{\Omega}^\varepsilon))^3$, its lower face $\hat{\Gamma}^\varepsilon_- = \Theta^\varepsilon(\Gamma^\varepsilon_-)$ subjected to a surface forces of density $\hat{g}^\varepsilon \in (L^2(\hat{\Gamma}^\varepsilon_-))^3$ such that $\hat{f}^\varepsilon_\alpha = \hat{g}^\varepsilon_\alpha = 0$ and to applied surface forces of von Kármán's type $\hat{h}^\varepsilon_\alpha \in L^2(\hat{\gamma}^\varepsilon_1)$ only on a portion $\Theta^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the remaining portion $\boldsymbol{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free. We suppose also that this shell is in unilateral contact with Coulomb friction (Λ its coefficient) at the upper face $\hat{\Gamma}^\varepsilon_+ = \Theta^\varepsilon(\Gamma^\varepsilon_+)$ against a rigid foundation, where $\hat{d}^\varepsilon (\geq 0)$ is the gap function defined on $\hat{\Gamma}^\varepsilon_+$ which describes the distance between the upper face and the foundation measured in the normal direction. We supposed, also that the system is in static case.

The problem consists of finding the displacement $\hat{\mathbf{u}}^\varepsilon$ and the force $\hat{\mathbf{G}}^\varepsilon$ which satisfy the problem:

$$(C.\hat{P}^\varepsilon)^{iso}_{sta,c} \begin{cases} -\hat{\partial}^\varepsilon_j(\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i) = \hat{f}^\varepsilon_i \text{ in } \hat{\Omega}^\varepsilon, \\ \begin{cases} \hat{u}^\varepsilon_\alpha \text{ independent of } \hat{x}^\varepsilon_3 \text{ and } \hat{u}^\varepsilon_3 = 0 \text{ on } \boldsymbol{\Theta}^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right), \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}^\varepsilon_{\alpha\beta} + \hat{\sigma}^\varepsilon_{k\beta}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_\alpha) \circ \boldsymbol{\Theta}^\varepsilon\}\nu_\beta dx^\varepsilon_3 = \hat{h}^\varepsilon_\alpha \circ \boldsymbol{\Theta}^\varepsilon \text{ on } \gamma_1, \end{cases} \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = 0 \text{ on } \gamma_2 \times [-\varepsilon, \varepsilon], \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = \hat{g}^\varepsilon_i \circ \Theta^\varepsilon \text{ on } \Gamma^\varepsilon_-, \\ \hat{u}^\varepsilon_N \leq \hat{d}^\varepsilon, \hat{G}^\varepsilon_N \leq 0, \hat{G}^\varepsilon_N(\hat{u}^\varepsilon_N - \hat{d}^\varepsilon) = 0 \; on \; \hat{\Gamma}^\varepsilon_+, \\ |\hat{\mathbf{G}}^\varepsilon_T| < \Lambda|\hat{G}^\varepsilon_N| \Rightarrow \hat{\mathbf{u}}^\varepsilon_T = 0 \; on \; \hat{\Gamma}^\varepsilon_+, \\ |\hat{\mathbf{G}}^\varepsilon_T| = \Lambda|\hat{G}^\varepsilon_N| \Rightarrow \exists\delta > 0, \, \hat{\mathbf{u}}^\varepsilon_T = -\delta\hat{\mathbf{G}}^\varepsilon_T \; on \; \hat{\Gamma}^\varepsilon_+, \end{cases}$$

where

$$\begin{cases} \hat{G}^\varepsilon_i = (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j, \\ \hat{G}^\varepsilon_N = \hat{G}^\varepsilon_i \hat{n}^\varepsilon_i, \hat{\mathbf{G}}^\varepsilon_T = \hat{\mathbf{G}}^\varepsilon - \hat{G}^\varepsilon_N \hat{n}^\varepsilon, \\ \hat{\sigma}^\varepsilon_{ij} = \lambda \hat{E}^\varepsilon_{pp}\left(\hat{\mathbf{u}}^\varepsilon\right)\delta_{ij} + 2\mu \hat{E}^\varepsilon_{ij}\left(\hat{\mathbf{u}}^\varepsilon\right), \\ \hat{E}^\varepsilon_{ij}\left(\hat{\mathbf{u}}^\varepsilon\right) = \dfrac{1}{2}(\hat{\partial}^\varepsilon_i \hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j \hat{u}^\varepsilon_i + \hat{\partial}^\varepsilon_i \hat{u}^\varepsilon_k \hat{\partial}^\varepsilon_j \hat{u}^\varepsilon_k). \end{cases}$$

Multiplying the system of equilibrium equations in $(C.\hat{P}^\varepsilon)^{iso}_{sta,c}$ by functions $\hat{v}^\varepsilon_i$ and integrating over the set $\hat{\Omega}^\varepsilon$, after that using the Green formula and the boundary conditions we obtain the following variational formulation of the problem $(C.\hat{P}^\varepsilon)^{iso}_{sta,c}$:

$$
(V.\hat{P}^\varepsilon)^{iso}_{sta,c}\begin{cases} Find\ (\hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{G}}^\varepsilon) \in \mathbf{K}(\hat{\Omega}^\varepsilon) \times (L^2(\hat{\Gamma}^\varepsilon_+))^3\ such\ that, \\ \hat{A}^\varepsilon(\hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon) = \hat{L}^\varepsilon(\hat{\mathbf{v}}^\varepsilon) + \int_{\hat{\gamma}_1} (\int_{-\varepsilon}^{\varepsilon} (\hat{v}^\varepsilon_\alpha \circ \Theta^\varepsilon) dx^\varepsilon_3) \hat{h}^\varepsilon_\alpha d\hat{\gamma} + \langle \hat{G}^\varepsilon_i, \hat{v}^\varepsilon_i \rangle\ \forall\ \hat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\hat{\Omega}^\varepsilon), \\ \langle \hat{G}^\varepsilon_N, \hat{v}^\varepsilon_N - \hat{u}^\varepsilon_N \rangle \geq 0\,,\ \forall \hat{\mathbf{v}}^\varepsilon \in \mathbf{K}(\hat{\Omega}^\varepsilon), \\ \langle \hat{\mathbf{G}}^\varepsilon_T, \hat{\mathbf{v}}^\varepsilon_T - \hat{\mathbf{u}}^\varepsilon_T \rangle + \langle \Lambda |\hat{G}^\varepsilon_N|, |\hat{\mathbf{v}}^\varepsilon_T| - |\hat{\mathbf{u}}^\varepsilon_T| \rangle \geq 0,\ \forall\ \hat{\mathbf{v}}^\varepsilon \in \mathbf{V}(\hat{\Omega}^\varepsilon), \end{cases}
$$

where

$$
\begin{cases} \hat{A}^\varepsilon(\hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj} \hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i) \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i d\hat{x}^\varepsilon, \\ \hat{L}^\varepsilon(\hat{\mathbf{v}}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_i \hat{v}^\varepsilon_i d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_-} \hat{g}^\varepsilon_i \hat{v}^\varepsilon_i d\hat{\Gamma}^\varepsilon, \\ \langle \hat{G}^\varepsilon_i, \hat{\phi}^\varepsilon_i \rangle = \int_{\hat{\Gamma}^\varepsilon_+} \hat{G}^\varepsilon_i \hat{\phi}^\varepsilon_i d\hat{\Gamma}^\varepsilon. \end{cases}
$$

In order to transform the problem $(V.\hat{P}^\varepsilon)^{iso}_{sta,c}$ into problem posed over the cylindrical domain Ω^ε, we use the one to one mapping $(\mathbf{\Theta}^\varepsilon)^{-1}$ and the relations (2.3) obtained from this transformation.

Then by a simple computation, we obtain

Proposition 4.1 *Suppose that* ε *is small enough. Then the variational problem* $(V.\hat{P}^\varepsilon)^{iso}_{sta,c}$ *is equivalent to the following variational problem :*

$$
(P^\varepsilon)^{iso}_{sta,c}\begin{cases} Find\ (\mathbf{u}^\varepsilon, \mathbf{G}^\varepsilon) \in \mathbf{K}(\Omega^\varepsilon) \times (L^2(\Gamma^\varepsilon_+))^3,\ such\ that, \\ A^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = L^\varepsilon(\mathbf{v}^\varepsilon) + \int_{\gamma_1} h^\varepsilon_\alpha \{\int_{-\varepsilon}^{\varepsilon} v^\varepsilon_\alpha dx^\varepsilon_3\} d\gamma + \langle G^\varepsilon_i, v^\varepsilon_i \rangle\,, \forall \mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon), \\ \langle G^\varepsilon_N, v^\varepsilon_N - u^\varepsilon_N \rangle \geq 0\,,\ \forall \mathbf{v}^\varepsilon \in \mathbf{K}(\Omega^\varepsilon), \\ \langle \mathbf{G}^\varepsilon_T, \mathbf{v}^\varepsilon_T - \mathbf{u}^\varepsilon_T \rangle + \langle \Lambda\, |G^\varepsilon_N|\,, |\mathbf{v}^\varepsilon_T| - |\mathbf{u}^\varepsilon_T| \rangle \geq 0,\ \forall\, \mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon), \end{cases}
$$

where

$$
\begin{cases} A^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{kj} \partial^\varepsilon_k v^\varepsilon_i \delta^\varepsilon dx^\varepsilon + \int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{ki} \partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj} \partial^\varepsilon_m v^\varepsilon_l \delta^\varepsilon dx^\varepsilon, \\ L^\varepsilon(\mathbf{v}^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon_-} g^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon, \langle G^\varepsilon_i, v^\varepsilon_i \rangle = \int_{\Gamma^\varepsilon_+} G^\varepsilon_i v^\varepsilon_i \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon, \\ u^\varepsilon_i = \hat{u}^\varepsilon_i \circ \Theta^\varepsilon, \sigma^\varepsilon_{ij} = \hat{\sigma}^\varepsilon_{ij} \circ \Theta^\varepsilon, G^\varepsilon_i = \hat{G}^\varepsilon_i \circ \Theta^\varepsilon, n^\varepsilon_i = \hat{n}^\varepsilon_i \circ \Theta^\varepsilon, f^\varepsilon_i = \hat{f}^\varepsilon_i \circ \Theta^\varepsilon\ ,\ g^\varepsilon_i = \hat{g}^\varepsilon_i \circ \Theta^\varepsilon, \\ h^\varepsilon_\alpha = \hat{h}^\varepsilon_\alpha \circ \Theta^\varepsilon,\ d^\varepsilon = \hat{d}^\varepsilon \circ \Theta^\varepsilon. \end{cases}
$$

4.2 Asymptotic study

4.2.1 The scaled problem

Let $\Omega = \omega\times] -1, +1[, \Gamma_\pm = \omega \times \{\pm 1\}, \Gamma_0 = \gamma \times [-1, +1]$ and $x = (x_i) \in \bar{\Omega}$ denote a generic point in the set $\bar{\Omega}$.

We now transform the domain Ω^ε having the thickness 2ε into fixed domain Ω independent of ε via the simple mapping: $\pi^\varepsilon : \Omega \to \Omega^\varepsilon$ where $x_\alpha = x_\alpha^\varepsilon, x_3^\varepsilon = \varepsilon x_3$ hence:

$$\pi^\varepsilon(\Omega) = \Omega^\varepsilon, \pi^\varepsilon(\Gamma_\pm) = \Gamma_\pm^\varepsilon, \pi^\varepsilon(\Gamma_0) = \Gamma_0^\varepsilon, \partial_\alpha^\varepsilon = \partial_\alpha, \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$$

We introduce the scaled displacement $\mathbf{u}(\varepsilon)$, test function $\mathbf{v}(\varepsilon)$ and stress tensor $\sigma(\varepsilon)$ for all $x^\varepsilon = \pi^\varepsilon(x)$ as follows:

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(\varepsilon)(x), v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(\varepsilon)(x) \\ \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon) = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)(x), \sigma_{\alpha 3}^\varepsilon(x^\varepsilon) = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x), \sigma_{33}^\varepsilon(x^\varepsilon) = \varepsilon^4 \sigma_{33}(\varepsilon)(x) \end{cases}$$

Noting that the unit normal $\hat{\mathbf{n}}^\varepsilon$ on $\hat{\Gamma}_+^\varepsilon$ reads $\hat{\mathbf{n}}^\varepsilon = (-\partial_1^\varepsilon\theta^\varepsilon + O(\varepsilon^3), -\partial_2^\varepsilon\theta^\varepsilon + O(\varepsilon^3), 1 + O(\varepsilon^2))$. If we pose $G_i = \sigma_{ij} n_j^\theta$ such that $\mathbf{n}^\theta = (-\partial_1\theta, -\partial_2\theta, 1)$ then a simple computation gives:

$$G_\alpha^\varepsilon = \varepsilon^3 G_\alpha + O(\varepsilon^5),$$
$$G_3^\varepsilon = \varepsilon^4 G_3 + \varepsilon^4 \sigma_{ij} n_j^\theta \partial_i^\theta u_3 + O(\varepsilon^6),$$

then

$$\mathbf{v}_T^\varepsilon = (\varepsilon^2(v_1 - v_3 n_1^\theta) + O(\varepsilon^4), \varepsilon^2(v_2 - v_3 n_2^\theta) + O(\varepsilon^4), O(\varepsilon^3)),$$
$$v_N^\varepsilon = \varepsilon v_N(\varepsilon), v_N(\varepsilon) = v_3 n_3^\theta + O(\varepsilon^2),$$

and

$$\mathbf{G}_T^\varepsilon = (\varepsilon^3 G_1 + O(\varepsilon^5), \varepsilon^3 G_2 + O(\varepsilon^5), \varepsilon^4(G_3 - G_i n_i^\theta) + O(\varepsilon^6)).$$

We also introduce the scalings: $f_3^\varepsilon = \varepsilon^3 f_3, g_3^\varepsilon = \varepsilon^4 g_3, h_\alpha^\varepsilon = \varepsilon^2 h_\alpha$ and $d^\varepsilon = \varepsilon d(\varepsilon)$ where f_3, g_3 and h_α supposed independent of ε.

By using the assumptions and notations above we obtain the result:

Proposition 4.2 *For ε small enough the scaled solution of the problem $(P^\varepsilon)_{sta,c}^{iso}$ solves the following problem:*

$$(P(\varepsilon))_{sta,c}^{iso} \begin{cases} Find\ (\mathbf{u}(\varepsilon), \mathbf{G}(\varepsilon)) \in \mathbf{K}(\varepsilon)(\Omega) \times (L^2(\Gamma_+))^3\ such\ that, \\ A^\theta(\mathbf{u}(\varepsilon), \mathbf{v}) = L(\mathbf{v}) + 2\int_{\gamma_1} h_\alpha v_\alpha dx_3 d\gamma + \langle G_i(\varepsilon), v_i \rangle + \int_{\Gamma_+} \sigma_{ij}(\varepsilon) n_j^\theta \partial_i^\theta u_3(\varepsilon) v_3 d\Gamma \\ +\varepsilon^2 r_1, \forall \mathbf{v} \in \mathbf{V}(\Omega), \\ \left\langle G_i(\varepsilon) n_i^\theta + \sigma_{ij}(\varepsilon) n_j^\theta \partial_i^\theta u_3(\varepsilon), v_3 - u_3(\varepsilon) \right\rangle + \varepsilon^2 r_2 \geq 0\,, \forall \mathbf{v} \in \mathbf{K}(\varepsilon)(\Omega), \\ \left\langle G_\alpha(\varepsilon), (v_\alpha - u_\alpha(\varepsilon)) - (v_3 - u_3(\varepsilon)) n_\alpha^\theta \right\rangle + \varepsilon r_3 \geq 0, \forall\, \mathbf{v} \in \mathbf{V}(\Omega), \end{cases}$$

where

$$\begin{cases} A^{\theta}(\mathbf{u}(\varepsilon),\mathbf{v}) = \int_{\Omega} \sigma_{ij}\left(\varepsilon\right) \gamma^{\theta}_{ij}(v) dx + \int_{\Omega} \sigma_{ij}\left(\varepsilon\right) \partial^{\theta}_{i} u_3\left(\varepsilon\right) \partial^{\theta}_{j} v_3 dx, \\ L(\mathbf{v}) = \int_{\Omega} f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma, \\ \langle G_i(\varepsilon), v_i\rangle = \int_{\Gamma_+} G_i(\varepsilon) v_i d\Gamma, \\ \partial^{\theta}_{\alpha} v = \partial_{\alpha} v - \partial_{\alpha}\theta\partial_3 v,\ \partial^{\theta}_{3} v = \partial_3 v,\ \gamma^{\theta}_{ij}(\mathbf{v}) = \frac{1}{2}\left(\partial^{\theta}_{i} v_j + \partial^{\theta}_{j} v_i\right), \end{cases}$$

r_i *are bounded functions.*

Proof. First, we infer from assumption $\theta^{\varepsilon}(x_1,x_2) = \varepsilon\theta(x_1,x_2)$ for all $(x_1,x_2) \in \bar{\omega}$ with $\theta \in \mathcal{C}^3(\bar{\omega})$ that, for $\varepsilon_0 > 0$ small enough,
$b^{\varepsilon}_{\alpha\beta}(x^{\varepsilon}) = \delta_{\alpha\beta} + \varepsilon^2 r_{\alpha\beta}(\varepsilon;x_1,x_2)$, $b^{\varepsilon}_{\alpha 3}(x^{\varepsilon}) = \varepsilon(\partial_{\alpha}\theta + \varepsilon^2 r_{\alpha 3}(\varepsilon;x_1,x_2))$, $b^{\varepsilon}_{3\beta}(x^{\varepsilon}) = -\varepsilon(\partial_{\beta}\theta + \varepsilon^2 r_{3\beta}(\varepsilon;x_1,x_2))$, $b^{\varepsilon}_{33}(x^{\varepsilon}) = 1 + \varepsilon^2 r_{33}(\varepsilon;x_1,x_2)$, $\delta^{\varepsilon}(x^{\varepsilon}) = 1 + \varepsilon^2 r_{\delta}(\varepsilon;x_1,x_2)$, for all $x^{\varepsilon} \in \bar{\Omega}^{\varepsilon}$, and $\beta^{\varepsilon}(x^{\varepsilon}) = 1 + \varepsilon^2 r_{\beta}(\varepsilon;x_1,x_2)$, for all $x^{\varepsilon} \in \Gamma^{\varepsilon}_+ \cup \Gamma^{\varepsilon}_+$, where the real-valued functions r_{ij}, r_{δ}, r_{β} are bounded. (For details, see [CP86, Theorem 3.1]).

Next, we insert the above equalities with the change of variables, we obtain,

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}_{ij} b^{\varepsilon}_{kj} \partial^{\varepsilon}_{k} v^{\varepsilon}_{i} \delta^{\varepsilon} dx^{\varepsilon} = \varepsilon^5 \int_{\Omega} \sigma_{ij}(\varepsilon) \gamma^{\theta}_{ij}(\mathbf{v}) dx + \varepsilon^7 \rho_1(\varepsilon;\sigma(\varepsilon),\mathbf{v}),$$

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}_{ij} b^{\varepsilon}_{ki} \partial^{\varepsilon}_{k} u^{\varepsilon}_{l} b^{\varepsilon}_{mj} \partial^{\varepsilon}_{m} v^{\varepsilon}_{l} \delta^{\varepsilon} dx^{\varepsilon} = \varepsilon^5 \int_{\Omega} \sigma_{ij}(\varepsilon) \partial^{\theta}_{i} u_3(\varepsilon) \partial^{\theta}_{j} v_3 dx + \varepsilon^7 \rho_2(\varepsilon;\sigma(\varepsilon),\mathbf{u}(\varepsilon),\mathbf{v}),$$

$$\int_{\Omega^{\varepsilon}} f^{\varepsilon}_{i} v^{\varepsilon}_{i} \delta^{\varepsilon} dx^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{-}} g^{\varepsilon}_{i} v^{\varepsilon}_{i} \delta^{\varepsilon} \beta^{\varepsilon} d\Gamma^{\varepsilon} = \varepsilon^5 (\int_{\Omega} f_i v_i dx + \int_{\Gamma_-} g_i v_i d\Gamma) + \varepsilon^7 \rho_3(\varepsilon;\mathbf{v}),$$

$$\langle G^{\varepsilon}_{i}, v^{\varepsilon}_{i}\rangle = \varepsilon^5 \langle G_i(\varepsilon), v_i\rangle + \varepsilon^7 \rho_4(\varepsilon;\mathbf{G}(\varepsilon),\mathbf{u}(\varepsilon),\mathbf{v}),$$

where there exists a constant c_1 such that, for all $\mathbf{u}(\varepsilon) \in \mathbf{K}(\varepsilon)(\Omega)$, $\mathbf{v} \in \mathbf{V}(\Omega)$, $\sigma(\varepsilon) \in \Sigma(\Omega)$ and $\mathbf{G}(\varepsilon) \in (L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0} |\rho_1\left(\varepsilon;\sigma(\varepsilon),\mathbf{v}\right)| \le c_1 |\sigma(\varepsilon)|_{0,\Omega} |\mathbf{v}|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} |\rho_2\left(\varepsilon;\sigma(\varepsilon),\mathbf{u}(\varepsilon),\mathbf{v}\right)| \le c_1 |\sigma(\varepsilon)|_{0,\Omega} |\mathbf{u}(\varepsilon)|_{1,4,\Omega} |\mathbf{v}|_{1,4,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} |\rho_3\left(\varepsilon;\mathbf{v}\right)| \le c_1 \|\mathbf{v}\|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0} |\rho_4(\varepsilon;\mathbf{G}(\varepsilon),\mathbf{u}(\varepsilon),\mathbf{v})| \le c_1(\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+} + \|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+} \|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+}) \|\mathbf{v}\|_{\frac{1}{2},\Gamma_+}.$$

Diving by ε^5 and combining the above estimates, we get

$$\begin{aligned}\sup_{0\le\varepsilon\le\varepsilon_0}|r_1| &\le c_1\left((1+|\sigma(\varepsilon)|_{0,\Omega}+|\sigma(\varepsilon)|_{0,\Omega}|\mathbf{u}(\varepsilon)|_{1,4,\Omega})\|\mathbf{v}\|_{1,4,\Omega}\right.\\ &+ \left.(\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+})\|\mathbf{v}\|_{\frac{1}{2},\Gamma_+}\right).\end{aligned}$$

For finding what the unilateral contact conditions become, we use the relations

$$\langle G_N^\varepsilon, v_N^\varepsilon - u_N^\varepsilon\rangle = \varepsilon^5\left\langle G_\alpha(\varepsilon)n_\alpha^\theta + G_3(\varepsilon), v_3-u_3(\varepsilon)\right\rangle + \varepsilon^7 r_2,$$

$$\langle \mathbf{G}_T^\varepsilon, \mathbf{v}_T^\varepsilon - \mathbf{u}_T^\varepsilon\rangle + \langle \Lambda\,|G_N^\varepsilon|\,, |\mathbf{v}_T^\varepsilon| - |\mathbf{u}_T^\varepsilon|\rangle = \varepsilon^5\left\langle G_\alpha(\varepsilon), (v_\alpha - u_\alpha(\varepsilon)) - (v_3-u_3(\varepsilon))n_\alpha^\theta\right\rangle + \varepsilon^6 r_3,$$

where there exists two constants c_2 and c_3 such that for all $\mathbf{u}(\varepsilon)\in\mathbf{K}(\varepsilon)(\Omega)$, $\mathbf{v}\in\mathbf{K}(\varepsilon)(\Omega)$ and $\mathbf{G}(\varepsilon)\in(L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0}|r_2| \le c_2(\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+})(\|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+}+\|\mathbf{v}\|_{\frac{1}{2},\Gamma_+}),$$

and for all $\mathbf{u}(\varepsilon)\in\mathbf{K}(\varepsilon)(\Omega)$, $\mathbf{v}\in\mathbf{V}(\Omega)$ and $\mathbf{G}(\varepsilon)\in(L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0}|r_3| \le c_3(\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|\mathbf{G}(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+})(\|\mathbf{u}(\varepsilon)\|_{\frac{1}{2},\Gamma_+}+\|\mathbf{v}\|_{\frac{1}{2},\Gamma_+}).$$

■

4.2.2 The two-dimensional problem

We assume that the scaled displacement-stress $(\mathbf{u}(\varepsilon),\sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(\mathbf{u}(\varepsilon),\sigma(\varepsilon)) = (\mathbf{u}^0,\sigma^0)+\varepsilon(\mathbf{u}^1,\sigma^1)+\varepsilon^2(\mathbf{u}^2,\sigma^2)+... \tag{4.1}$$

then

$$G_i(\varepsilon) = G_i^0+\varepsilon G_i^1+\varepsilon^2 G_i^2+...,\ \textit{with}\ G_i^k = \sigma_{ij}^k n_j^\theta.$$

Substituting expansion (4.1) into the scaled variational problem $(P(\varepsilon))^{iso}_{sta,c}$, we obtain:

Proposition 4.3 *Assume that* $\partial_3 u_3^0\in C^0\left(\bar\Omega\right)$ *then the leading term* $(\mathbf{u}^0,\sigma^0)$ *of the expansion* (4.1) *is a solution of the problem:*

$$(P_{KL})^{iso}_{sta,c}\begin{cases} \textit{Find}\ \ (\mathbf{u}^0,\sigma^0,G_3^0)\in(\mathbf{V}_{KL}(\Omega)\cap\mathbf{K}(\Omega))\times\Sigma(\Omega)\times L^2(\Gamma_+)\ \textit{such that}: \\ \int_\Omega\sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx+\int_\Omega\sigma^0_{\alpha\beta}\partial_\alpha(u_3^0+\theta)\partial_\beta v_3 dx = L(v)+\langle G_3^0,v_3\rangle+2\int_{\gamma_1}h_\alpha v_\alpha d\gamma, \\ \forall\mathbf{v}\in\mathbf{V}_{KL}(\Omega), \\ \langle G_3^0, v_3-u_3^0\rangle\ge 0, \forall\mathbf{v}\in\mathbf{K}(\Omega), \end{cases}$$

where

$$\begin{cases} \sigma^0_{\alpha\beta} = \frac{2\lambda\mu}{\lambda+2\mu}\bar{E}^0_{\sigma\sigma}(\mathbf{u}^0)\delta_{\alpha\beta} + 2\mu\bar{E}^0_{\alpha\beta}(\mathbf{u}^0), \\ \bar{E}^0_{\alpha\beta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha + \partial_\alpha u^0_3\partial_\beta u^0_3 + \partial_\alpha\theta\partial_\beta u^0_3 + \partial_\beta\theta\partial_\alpha u^0_3\right), \\ G^0_3 = -\sigma^0_{31}\partial_1\theta - \sigma^0_{32}\partial_2\theta + \sigma^0_{33}. \end{cases}$$

Proof. We introduce the formal series expansions of the scaled displacement and the scaled stresses into the variational problem $(P(\varepsilon))^{iso}_{sta,c}$ and cancel the successive powers of ε, until we can fully identify the leading term. ■

We deduce from the following Proposition that the leading term $(\mathbf{u}^0, \sigma^0)$ is characterized by an unilateral contact problem without friction.

Proposition 4.4 *If* $\mathbf{u}^0$ *is a solution of the problem* $(P_{KL})^{iso}_{sta,c}$ *such that* $u^0_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ *and* $u^0_3 = \zeta_3$*,* (ζ_α)*,* ζ_3 *sufficiently regular. Then* (ζ_α)*,* ζ_3 *verify the two-dimensional problem:*

$$(\bar{P}(\omega))^{iso}_{sta,c} \begin{cases} Find\ (\zeta_\alpha) \in (H^1(\omega))^2, \zeta_3 \in H^2(\omega), \zeta_3 \le d, G^0_3 \in L^2(\omega)\ such\ that \\ -\partial_{\alpha\beta}m_{\alpha\beta} - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3 + \theta) = h^0_3 + G^0_3\ in\ \omega \\ \partial_\beta\bar{N}_{\alpha\beta} = 0\ in\ \omega, \\ \zeta_3 = \partial_\nu\zeta_3 = 0\ on\ \gamma_1, \\ \bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha\ on\ \gamma_1 \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0\ on\ \gamma_2, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0\ on\ \gamma_2, \\ \bar{N}_{\alpha\beta}\nu_\beta = 0\ on\ \gamma_2 \\ G^0_3(d - \zeta_3) = 0\ in\ \omega, G^0_3 \le 0\ in\ \omega \end{cases}$$

where

$$\begin{cases} m_{\alpha\beta} = -\frac{1}{3}\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\}, \\ \bar{N}_{\alpha\beta} = 2\lambda^\star\bar{E}^0_{\gamma\gamma}(\zeta)\delta_{\alpha\beta} + 4\mu\bar{E}^0_{\alpha\beta}(\zeta),\ \lambda^\star = \frac{2\lambda\mu}{\lambda+2\mu}, \\ \bar{E}^0_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3 + \partial_\alpha\zeta_3\partial_\beta\zeta_3), \\ h^0_i = \int_{-1}^1 f_i dx_3 + g^-_i,\ g^-_i = g_i(x_1, x_2, -1). \end{cases}$$

Proof. The proof will be divided into 3 steps.

Step 1. First, we show that $(P_{KL})^{iso}_{sta,c}$ is in a sense a two-dimensional problem, posed over the middle surface $\bar{\omega}$ of the shell.

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega (h_3^0 + G_3^0)\eta_3 d\omega$$
$$+ 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \forall\eta \in \mathbf{V}(\omega).$$

It is known that $\mathbf{v} = (v_i) \in \mathbf{V}_{KL}(\Omega)$ if and only if there exists $\eta = (\eta_i) \in \mathbf{V}(\omega)$ such that $v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3$ and $v_3 = \eta_3$ (see [Cia97, Théorème 1.4-4]). The same proof works for Gratie [Gra02, Theorem 3]. In $(P_{KL})^{iso}_{sta,c}$, we take test-functions $\mathbf{v} = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$, with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu\eta_3 = 0$ on γ_1. Next, we take $\mathbf{v} = (\eta_1, \eta_2, 0)$, with $\eta_\alpha \in H^1(\omega)$. The first choice yields

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma + \left\langle G_3^0, \eta_3\right\rangle.$$

The second choice yields

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

Using Fubini's formula to the above integrals, we get

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx = -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega,$$

$$\int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma + \left\langle G_3^0, \eta_3\right\rangle = \int_\omega \{\int_{-1}^1 f_3 dx_3 + g_3(x_1,x_2,-1) + G_3^0(x_1,x_2,+1)\}\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma,$$

where

$$G_3^0(.,+1) = -\sigma^0_{31}(.,+1)\partial_1\theta - \sigma^0_{32}(.,+1)\partial_2\theta + \sigma^0_{33}(.,+1).$$

Step 2. Applying Green formulas, we obtain

$$\int_\omega \left[-\partial_{\alpha\beta}m_{\alpha\beta} - \partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right) - (h_3^0 + G_3^0)\right]\eta_3 d\omega -$$

$$\int_\omega \left(\partial_\beta\bar{N}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \left(\bar{N}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma +$$

$$\int_{\gamma_2} \left\{ \left[\partial_\alpha m_{\alpha\beta} + \bar{N}_{\alpha\beta} \partial_\alpha (\zeta_3 + \theta) \right] \nu_\beta + \partial_\tau (m_{\alpha\beta} \nu_\alpha \tau_\beta) \right\} \eta_3 d\gamma = 0,$$

for all $\eta = (\eta_\alpha, \eta_3) \in \mathbf{V}(\omega)$, and the functions $\widetilde{h}_\alpha : \gamma \to \mathbb{R}$ defined by :

$$\widetilde{h}_\alpha = h_\alpha \ \ on \ \ \gamma_1 \ and \ \widetilde{h}_\alpha = 0 \ \ on \ \ \gamma_2.$$

So that, all the factors of η_α, η_3, and $\partial_\nu \eta_3$ in the above integrals vanish in their respective domains of integration. (For more details we refer the reader to [Gra02, Theorem 5])

Step 3. It remains to prove the unilateral contact conditions, in this conditions of $(P_{KL})^{iso}_{sta,c}$, we try the test-functions $v = d$, and then to try $v = 2\zeta_3 - d$, with $\zeta_3 \in H^2(\omega)$, we obtain

$$G_3^0(d - \zeta_3) = 0 \ in \ \omega.$$

Taking into account

$$\left\langle G_3^0, \eta_3 - d \right\rangle \geq 0, \text{ for all } \eta \in \mathbf{K}(\Omega),$$

we obtain

$$G_3^0 \leq 0 \text{ in } \omega.$$

■

4.2.3 Computation of σ^0_{i3} in case $\gamma_1 = \gamma$

In the sequel, we compute the components σ^0_{i3}. In order to realize this, we suppose that $\gamma_1 = \gamma$ which the case of Marguerre-von Kármán conditions.

Computation of $\sigma^0_{\alpha 3}$

In the identification processus of factors of powers $\varepsilon^k, k = 0, 1, 2, 3, ...$, we obtain at the order ε^0, the equation

$$\begin{aligned} \int_\Omega \sigma^0_{ij} \gamma^\theta_{ij}(v)\, dx + \int_\Omega \sigma^0_{ij} \partial^\theta_i u^0_3 \partial^\theta_j v_3 dx &= \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma \\ &+ \int_{\Gamma_+} G^0_3 v_3 d\Gamma + 2 \int_\gamma h_\alpha \left\{ \int_{-1}^1 v_\alpha dx_3 \right\} d\gamma. \quad (4.2) \end{aligned}$$

The terms of left-hand side of the equation verify

$$\int_\Omega \sigma^0_{\alpha\beta}\gamma^\theta_{\alpha\beta}(v)\,dx = \frac{1}{2}\int_\Omega \sigma^0_{\alpha\beta}(\partial_\alpha v_\beta + \partial_\beta v_\alpha - \partial_\alpha\theta\partial_3 v_\beta - \partial_\beta\theta\partial_3 v_\alpha)dx,$$

$$\int_\Omega \sigma^0_{\alpha 3}\gamma^\theta_{\alpha 3}(v)\,dx = \frac{1}{2}\int_\Omega \sigma^0_{\alpha 3}(\partial_\alpha v_3 + \partial_3 v_\alpha - \partial_\alpha\theta\partial_3 v_3)dx,$$

$$\int_\Omega \sigma^0_{33}\gamma^\theta_{33}(v)\,dx = \int_\Omega \sigma^0_{33}\partial_3 v_3 dx,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial^\theta_\alpha u^0_3\partial^\theta_\beta v_3 dx = \int_\Omega \sigma^0_{\alpha\beta}(\partial_\alpha u^0_3 - \partial_\alpha\theta\partial_3 u^0_3)(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3)dx,$$

$$\int_\Omega \sigma^0_{\alpha 3}\partial^\theta_\alpha u^0_3\partial^\theta_3 v_3 dx = \int_\Omega \sigma^0_{\alpha 3}(\partial_\alpha u^0_3 - \partial_\alpha\theta\partial_3 u^0_3)\partial_3 v_3 dx,$$

$$\int_\Omega \sigma^0_{3\alpha}\partial^\theta_3 u^0_3\partial^\theta_\alpha v_3 dx = \int_\Omega \sigma^0_{3\alpha}\partial_3 u^0_3(\partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3)dx,$$

$$\int_\Omega \sigma^0_{33}\partial^\theta_3 u^0_3\partial^\theta_3 v_3 dx = \int_\Omega \sigma^0_{3\alpha}\partial_3 u^0_3\partial_3 v_3 dx.$$

The equation (4.2) with $v_3 = 0$ yields

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha v_\beta + \int_\Omega \sigma^0_{\alpha 3}\partial_3 v_\alpha - \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_3 v_\alpha = 2\int_\gamma h_\alpha \left\{\int_{-1}^{1} v_\alpha dx_3\right\} d\gamma, \tag{4.3}$$

for all $v_\alpha \in H^1(\Omega)$ independent of x_3 on Γ_0.
On other hand

$$\int_\gamma h_\alpha \left\{\int_{-1}^{1} v_\alpha dx_3\right\} d\gamma = \frac{1}{2}\int_\Omega \bar{N}_{\alpha\beta}\partial_\beta v_\alpha dx, \text{ for all } v_\alpha \text{ independent of } x_3 \text{ on } \Gamma_0,$$

then (4.3) is *formally* equivalent to the following boundary value problem

$$\begin{cases} \partial_3\sigma^0_{\alpha 3} = \partial_3\sigma^0_{\alpha\beta}\partial_\beta\theta - \partial_\beta\sigma^0_{\alpha\beta} \text{ in } \Omega, \\ \sigma^0_{\alpha 3} = \sigma^0_{\alpha\beta}(., +1)\partial_\beta\theta \text{ on } \Gamma_+, \\ \sigma^0_{\alpha 3} = \sigma^0_{\alpha\beta}(., -1)\partial_\beta\theta \text{ on } \Gamma_-. \end{cases} \tag{4.4}$$

Noting that $\sigma^0_{\alpha\beta} = \frac{1}{2}\bar{N}_{\alpha\beta} + \frac{3}{2}x_3 m_{\alpha\beta}$ and from the Proposition 4.4 that $\partial_\beta\bar{N}_{\alpha\beta} = 0$ which makes the compatibility condition $\int_{-1}^{+1}\partial_\beta\sigma^0_{\alpha\beta}dx_3 = 0$ satisfied. Then, the explicit expressions of $\sigma^0_{\alpha 3}$ are given by

$$\sigma^0_{\alpha 3} = \frac{3}{4}(1 - x_3^2)\partial_\beta m_{\alpha\beta} + \sigma^0_{\alpha\beta}\partial_\beta\theta.$$

Computation of σ^0_{33}

We take $v_\alpha = 0$ in the equation (4.2). As $\partial_3 u^0_3 = 0$, we get

$$\int_\Omega \sigma^0_{\alpha 3}(\partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3)dx + \int_\Omega \sigma^0_{33}\partial_3 v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u^0_3(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3)dx$$
$$+ \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha u^0_3\partial_3 v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\Gamma_+} G^0_3 v_3 d\Gamma,$$

thus, we see that it is *formally* equivalent to the following boundary condition problem

$$\begin{cases} -\partial_3\sigma^0_{33} = -\partial_3\sigma^0_{\alpha 3}\partial_\alpha\theta + \partial_\alpha\sigma^0_{\alpha 3} + \partial_\beta(\sigma^0_{\alpha\beta}\partial_\alpha\zeta_3) - \partial_3\sigma^0_{\alpha\beta}\partial_\alpha\zeta_3\partial_\beta\theta + \partial_3\sigma^0_{\alpha 3}\partial_\alpha\zeta_3 + f_3 \text{ in } \Omega, \\ \sigma^0_{33} = G^0_3 + \sigma^0_{\alpha\beta}(.,+1)\partial_\alpha\theta\partial_\beta\theta \text{ on } \Gamma_+, \\ \sigma^0_{33} = -g^-_3 + \sigma^0_{\alpha\beta}(.,-1)\partial_\alpha\theta\partial_\beta\theta \text{ on } \Gamma_-. \end{cases} \tag{4.5}$$

Such that G^0_3 verifies with ζ_3 on Γ_+ the condition

$$G^0_3(d-\zeta_3) = 0, G^0_3 \le 0.$$

Taking in account that

$$\int_{-1}^{x_3} \partial_3\sigma^0_{\alpha 3}\partial_\alpha\zeta_3 dx_3 = \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha\zeta_3 + \frac{3}{2}x_3 m_{\alpha\beta}\partial_\beta\theta\partial_\alpha\zeta_3 + \frac{3}{2}m_{\alpha\beta}\partial_\beta\theta\partial_\alpha\zeta_3,$$
$$\int_{-1}^{x_3} \partial_3\sigma^0_{\alpha 3}\partial_\alpha\theta dx_3 = \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha\theta + \frac{3}{2}x_3 m_{\alpha\beta}\partial_\beta\theta\partial_\alpha\theta + \frac{3}{2}m_{\alpha\beta}\partial_\beta\theta\partial_\alpha\theta,$$
$$\int_{-1}^{x_3} \partial_3\sigma^0_{\alpha\beta}\partial_\alpha\zeta_3\partial_\beta\theta dx_3 = \frac{3}{2}x_3 m_{\alpha\beta}\partial_\alpha\zeta_3\partial_\beta\theta + \frac{3}{2}m_{\alpha\beta}\partial_\alpha\zeta_3\partial_\beta\theta,$$
$$\int_{-1}^{x_3} \partial_3\sigma^0_{\alpha 3} dx_3 = \frac{1}{4}(3x_3 - x_3^3)\partial_{\alpha\beta}m_{\alpha\beta} + \frac{1}{2}x_3\bar{N}_{\alpha\beta}\partial_{\alpha\beta}\theta + \frac{3}{4}x_3^2\partial_\alpha m_{\alpha\beta}\partial_\beta\theta$$
$$+ \frac{3}{4}x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\theta + \frac{1}{2}\partial_{\alpha\beta}m_{\alpha\beta} - \frac{3}{4}\partial_\alpha m_{\alpha\beta}\partial_\beta\theta - \frac{3}{4}m_{\alpha\beta}\partial_{\alpha\beta}\theta$$
$$+ \frac{1}{2}\bar{N}_{\alpha\beta}\partial_{\alpha\beta}\theta,$$
$$\int_{-1}^{x_3} \partial_\beta(\sigma^0_{\alpha\beta}\partial_\alpha\zeta_3)dx_3 = \frac{1}{2}x_3\bar{N}_{\alpha\beta}\partial_{\alpha\beta}\zeta_3 + \frac{3}{4}x_3^2\partial_\beta m_{\alpha\beta}\partial_\alpha\zeta_3 + \frac{3}{4}x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\zeta_3$$
$$+ \frac{1}{2}\bar{N}_{\alpha\beta}\partial_{\alpha\beta}\zeta_3 - \frac{3}{4}\partial_\beta m_{\alpha\beta}\partial_\alpha\zeta_3 - \frac{3}{4}m_{\alpha\beta}\partial_{\alpha\beta}\zeta_3.$$

Then

$$\sigma^0_{33} = -\frac{1}{4}x_3(1-x_3^2)\partial_{\alpha\beta}m_{\alpha\beta} + \frac{3}{4}(1-x_3^2)m_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) + \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha\theta$$
$$+ \frac{1}{2}(1+x_3)\int_{-1}^{1} f_3 dy_3 - \int_{-1}^{x_3} f_3 dy_3 + \frac{1}{2}(1+x_3)G^0_3 - \frac{1}{2}(1-x_3)g^-_3 + \sigma^0_{\alpha 3}\partial_\alpha\theta.$$

4.3 Generalized Marguerre-von Kármán equations with Signorini conditions

We can rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))^{iso}_{sta,c}$ as generalized Marguerre-von Kármán equations with Signorini conditions which depends on the Airy function Φ, the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell and G^0_3 as follows:

Proposition 4.5 *Assume that the set ω is simply-connected and that its boundary γ is smooth enough, and let $\zeta = (\zeta_i)$ be a solution $(\bar{P}(\omega))^{iso}_{sta,c}$ with the regularity $\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^4(\omega)$. Then*

a) The functions $\widetilde{h}_\alpha$ are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions :

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma = 0.$$

b) Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$, such that

$$\bar{N}_{11} = 2\partial_{22}\Phi,\ \bar{N}_{12} = \bar{N}_{21=-2\partial_{12}\Phi,\ \bar{N}_{22}=2\partial_{11}\Phi.}$$

c) Finally, the triple $(\zeta_3, \Phi, G^0_3) \in H^4(\omega) \times H^4(\omega) \times L^2(\omega)$, satisfies the following problem

$$(P)^{iso}_{sta,c} \begin{cases} k\Delta^2\zeta_3 = 2\left[\Phi, \zeta_3 + \theta\right] + h^0_3 + G^0_3 \ \ in\ \omega, \\ \Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}\left[\zeta_3, \zeta_3 + 2\theta\right]\ in\ \omega, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \ \ on\ \gamma_1, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad on\ \gamma_2, \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0\ on\ \gamma_2, \\ \Phi = \Phi_0\ and\ \partial_\nu\Phi = \Phi_1\ on\ \gamma, \\ G^0_3(d - \zeta_3) = 0\ in\ \omega, G^0_3 \leq 0\ in\ \omega, \end{cases}$$

where

$$\begin{cases} k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu}, G_3^0 = -\sigma_{31}^0\partial_1\theta - \sigma_{32}^0\partial_2\theta + \sigma_{33}^0, \\ \Phi_0(y) = -y_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \\ \Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma, y = (y_1, y_2) \in \gamma, \\ [\Phi,\zeta] = \partial_{11}\Phi\partial_{22}\zeta + \partial_{22}\Phi\partial_{11}\zeta - 2\partial_{12}\Phi\partial_{12}\zeta. \end{cases}$$

Proof. The proof is divided into three steps.

Step 1. From the regularity of functions ζ_i imply that $\bar{N}_{\alpha\beta} \in H^2(\omega)$ and $\bar{N}_{\alpha\beta}\nu_\beta = 2\widetilde{h}_\alpha$ *on* γ. Hence the functions $\widetilde{h}_\alpha$ belong to the space $\in H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions (see [CG01, Theorem 4]).

Step 2. Since the set ω is simply-connected and by using the generalized Poincaré theorem (see [Sch66, Theorem VI, p.59], [CG01, Theorem 7]), the equation $\partial_\beta\bar{N}_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in D'(\omega)$, unique up to the addition of constants, such that $\bar{N}_{1\alpha} = 2\partial_2\psi_\alpha$, $\bar{N}_{2\alpha} = -2\partial_1\psi_\alpha$.

Since the equation $\bar{N}_{12} = \bar{N}_{21}$ implies that $\partial_\alpha\psi_\alpha = 0$. Another application of the same result shows that there exist a distribution $\Phi \in D'(\omega)$, unique up to the addition of polynomials of degree ≤ 1, such that $\psi_1 = \partial_2\Phi$, $\psi_2 = -\partial_1\Phi$, so that $\bar{N}_{11} = 2\partial_{22}\Phi$, $\bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi$, $\bar{N}_{22} = 2\partial_{11}\Phi$ *in* ω.

Step 3. Since $\bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3 + \theta) = 2[\Phi, \zeta_3 + \theta]$, we have

$$-\partial_{\alpha\beta}m_{\alpha\beta} = k\Delta^2\zeta_3 = 2\left[\Phi, \zeta_3 + \theta\right] + h_3^0 + G_3^0 \;\; in\ \omega.$$

Since $\Delta^2\Phi = \frac{1}{2}\Delta\bar{N}_{\alpha\alpha}$ and $\partial_{\alpha\beta}\bar{N}_{\alpha\beta} = 0$, so that

$$\Delta^2\Phi = -\frac{\mu\left(3\lambda + 2\mu\right)}{2\left(\lambda + \mu\right)}\left[\zeta_3, \zeta_3 + 2\theta\right] \;\; in\ \omega.$$

■

4.4 Conclusion

The result obtained in this Chapter is similar to that of [Pau02] and [CB08] that the leading term $\mathbf{u}^0$ of the asymptotic expansion of displacements field is characterized by two-dimensional problem without friction. Thus if we consider the work of Léger and Miara

[LM08] but with Coulomb friction, we affirm that we obtain the same result formally. At the end, we deduce that the displacement $\mathbf{u}^0$ is characterized by a two-dimensional problem without friction. Then, our three-dimensional Signorini problem with Coulomb friction offers toward a two-dimensional problem without friction. The loss of frictional densities in $(P_{KL})^{iso}_{sta,c}$ and $(\bar{P}(\omega))^{iso}_{sta,c}$ is due to the fact that the friction force behaves as $O(\varepsilon^3)$ whereas the pressure force behaves as $O(\varepsilon^4)$ therefore, at least formally, via the Coulomb law $|\hat{\mathbf{G}}^{\varepsilon}_T| \leq \Lambda|\hat{G}^{\varepsilon}_N|$, when ε tends towards 0 the friction force must be canceled. The question which stands here is how to involve the friction force in the lower dimensional problem and, in the absence of convergence and the existence of asymptotic expansion, is it possible to obtain an algorithm which allows us to compute the higher terms in the asymptotic expansion?

Chapter 5

Dynamical contact equations of generalized Marguerre-von Kármán shallow shells

In this Chapter, we extend formally the study of the fourth Chapter to the dynamical case. More precisely, we considered a three-dimensional dynamical model for a Signorini problem with Coulomb friction of nonlinearly elastic shallow shell with a specific class of boundary conditions of generalized Marguerre-von Kármán type, made of homogeneous isotropic material. To this end, we have justified the dynamical contact equations of generalized Marguerre-von Kármán shallow shells. Then, we establish the existence of solutions to these equations, using penalization method.

5.1 Asymptotic analysis of elastodynamic Signorini problem with Coulomb friction of generalized Marguerre-von Kármán shallow shell

5.1.1 Setting of the problem

Consider a nonlinearly elastodynamic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^{\varepsilon}$, with thickness 2ε, its constituting material is a Saint Venant-Kirchhoff material with Lamé constants $\lambda^{\varepsilon} > 0$ and $\mu^{\varepsilon} > 0$.

The shell is subjected to vertical body forces of density $(\hat{f}_i^{\varepsilon}) = (0, 0, \hat{f}_3^{\varepsilon})$ in its interior $\hat{\Omega}^{\varepsilon}$, its lower face $\hat{\Gamma}_-^{\varepsilon}$ subjected to a surface forces of density $(\hat{g}_i^{\varepsilon}) = (0, 0, \hat{g}_3^{\varepsilon})$. We suppose also that this shell is in unilateral contact with Coulomb friction at the upper face $\hat{\Gamma}_+^{\varepsilon}$ and Λ its frictional coefficient, such that $\hat{d}^{\varepsilon}$ is the gap function which describes the distance between the upper face and the rigid foundation measured in the normal direction. We suppose that $\hat{d}^{\varepsilon} \in L^{\infty}(\hat{\Gamma}_+^{\varepsilon})$, $\hat{d}^{\varepsilon} \geq 0$ and independent of time t. On the portion $\mathbf{\Theta}^{\varepsilon}(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^{\varepsilon}, \hat{h}_2^{\varepsilon}, 0)$, the remaining portion $\mathbf{\Theta}^{\varepsilon}(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

The unknowns displacement field $\hat{\mathbf{u}}^{\varepsilon} = (\hat{u}_i^{\varepsilon})(\hat{x}^{\varepsilon}, t)$, stress field $\hat{\sigma}^{\varepsilon} = (\hat{\sigma}_{ij}^{\varepsilon})(\hat{x}^{\varepsilon}, t)$ and the contact force $\hat{\mathbf{G}}^{\varepsilon}$ satisfies the following three-dimensional boundary value problem in Cartesian coordinates:

$$(C.\hat{P}^{\varepsilon})_{dyn,c}^{iso} \begin{cases} \hat{\rho}^{\varepsilon} \frac{\partial^2 \hat{u}_i^{\varepsilon}}{\partial t^2} - \hat{\partial}_j^{\varepsilon}(\hat{\sigma}_{ij}^{\varepsilon} + \hat{\sigma}_{kj}^{\varepsilon} \hat{\partial}_k^{\varepsilon} \hat{u}_i^{\varepsilon}) = \ \hat{f}_i^{\varepsilon} \text{ in } \hat{\Omega}^{\varepsilon} \times]0, +\infty[\,, \\ \begin{cases} \hat{u}_{\alpha}^{\varepsilon} \text{ independent of } \hat{x}_3^{\varepsilon} \text{ and } \hat{u}_3^{\varepsilon} = 0 \text{ on } \mathbf{\Theta}^{\varepsilon}\,(\gamma_1 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{(\hat{\sigma}_{\alpha\beta}^{\varepsilon} + \hat{\sigma}_{k\beta}^{\varepsilon} \hat{\partial}_k^{\varepsilon} \hat{u}_{\alpha}^{\varepsilon}) \circ \mathbf{\Theta}^{\varepsilon}\} \nu_{\beta} dx_3^{\varepsilon} = \hat{h}_{\alpha}^{\varepsilon} \circ \mathbf{\Theta}^{\varepsilon} \text{ on } \gamma_1 \times]0, +\infty[\,, \end{cases} \\ (\hat{\sigma}_{ij}^{\varepsilon} + \hat{\sigma}_{kj}^{\varepsilon} \hat{\partial}_k^{\varepsilon} \hat{u}_i^{\varepsilon}) \hat{n}_j^{\varepsilon} \circ \mathbf{\Theta}^{\varepsilon} = 0 \text{ on } (\gamma_2 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[\,, \\ (\hat{\sigma}_{ij}^{\varepsilon} + \hat{\sigma}_{kj}^{\varepsilon} \hat{\partial}_k^{\varepsilon} \hat{u}_i^{\varepsilon}) \hat{n}_j^{\varepsilon} \circ \mathbf{\Theta}^{\varepsilon} = \hat{g}_i^{\varepsilon} \circ \mathbf{\Theta}^{\varepsilon} \text{ on } \Gamma_-^{\varepsilon} \times]0, +\infty[\,, \\ \hat{u}_N^{\varepsilon} \leq \hat{d}^{\varepsilon}, \hat{G}_N^{\varepsilon} \leq 0, \hat{G}_N^{\varepsilon}(\hat{u}_N^{\varepsilon} - \hat{d}^{\varepsilon}) = 0 \; on \; \hat{\Gamma}_+^{\varepsilon} \times]0, +\infty[\,, \\ |\hat{\mathbf{G}}_T^{\varepsilon}| < \Lambda |\hat{G}_N^{\varepsilon}| \Rightarrow \frac{\partial \hat{\mathbf{u}}_T^{\varepsilon}}{\partial t} = 0 \; on \; \hat{\Gamma}_+^{\varepsilon} \times]0, +\infty[\,, \\ |\hat{\mathbf{G}}_T^{\varepsilon}| = \Lambda |\hat{G}_N^{\varepsilon}| \Rightarrow \exists \delta > 0, \; \frac{\partial \hat{\mathbf{u}}_T^{\varepsilon}}{\partial t} = -\delta \hat{\mathbf{G}}_T^{\varepsilon} \; on \; \hat{\Gamma}_+^{\varepsilon} \times]0, +\infty[\,, \\ \hat{\mathbf{u}}^{\varepsilon}\,(\hat{x}^{\varepsilon}, 0) = \hat{\mathbf{p}}^{\varepsilon}, \frac{\partial \hat{\mathbf{u}}^{\varepsilon}}{\partial t}\,(\hat{x}^{\varepsilon}, 0) = \hat{\mathbf{q}}^{\varepsilon} \text{ in } \hat{\Omega}^{\varepsilon}, \end{cases}$$

where

$$\begin{cases} \hat{\sigma}^{\varepsilon}_{ij} = \lambda^{\varepsilon} \hat{E}^{\varepsilon}_{pp} \left(\hat{\mathbf{u}}^{\varepsilon} \right) \delta_{ij} + 2\mu^{\varepsilon} \hat{E}^{\varepsilon}_{ij} \left(\hat{\mathbf{u}}^{\varepsilon} \right), \\ \hat{E}^{\varepsilon}_{ij} \left(\hat{\mathbf{u}}^{\varepsilon} \right) = \dfrac{1}{2} (\hat{\partial}^{\varepsilon}_i \hat{u}^{\varepsilon}_j + \hat{\partial}^{\varepsilon}_j \hat{u}^{\varepsilon}_i + \hat{\partial}^{\varepsilon}_i \hat{u}^{\varepsilon}_k \hat{\partial}^{\varepsilon}_j \hat{u}^{\varepsilon}_k), \\ \hat{u}^{\varepsilon}_N = \hat{\mathbf{u}}^{\varepsilon} \hat{\mathbf{n}}^{\varepsilon}, \hat{\mathbf{u}}^{\varepsilon}_T = \hat{\mathbf{u}}^{\varepsilon} - \hat{u}^{\varepsilon}_N \hat{\mathbf{n}}^{\varepsilon}, \\ \frac{\partial \hat{u}^{\varepsilon}_N}{\partial t} = \frac{\partial \hat{\mathbf{u}}^{\varepsilon}}{\partial t} \hat{\mathbf{n}}^{\varepsilon}, \ \frac{\partial \hat{\mathbf{u}}^{\varepsilon}_T}{\partial t} = \frac{\partial \hat{\mathbf{u}}^{\varepsilon}}{\partial t} - \frac{\partial \hat{u}^{\varepsilon}_N}{\partial t} \hat{\mathbf{n}}^{\varepsilon}, \\ \hat{G}^{\varepsilon}_N = \hat{\mathbf{G}}^{\varepsilon} \hat{\mathbf{n}}^{\varepsilon}, \hat{\mathbf{G}}^{\varepsilon}_T = \hat{\mathbf{G}}^{\varepsilon} - \hat{G}^{\varepsilon}_N \hat{\mathbf{n}}^{\varepsilon}, \\ \hat{\mathbf{p}}^{\varepsilon}, \hat{\mathbf{q}}^{\varepsilon} : \text{ the given initial data,} \\ \hat{\rho}^{\varepsilon} : \text{ the mass density.} \end{cases} \tag{5.1}$$

We recall that $W^{\frac{3}{4},4}(\partial\hat{\Omega}^{\varepsilon};.)$ denotes the space formed by the traces on $\partial\hat{\Omega}^{\varepsilon}$ of the functions in the space $W^{1,4}(\hat{\Omega}^{\varepsilon};.)$ and $W^{-\frac{3}{4},\frac{4}{3}}(\partial\hat{\Omega}^{\varepsilon};.)$ is the topological dual space of $W_0^{\frac{3}{4},4}(\partial\hat{\Omega}^{\varepsilon};.)$, for more details, we refer to Adams [Ada75].

We define

$$\bar{\mathcal{V}}(\hat{\Omega}^{\varepsilon}) = \left\{ \ \hat{\mathbf{v}}^{\varepsilon}_{|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\} \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3),$$

$$\bar{\mathcal{V}}_N(\hat{\Omega}^{\varepsilon}) = \left\{ \ \hat{v}^{\varepsilon}_{N|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\},$$

$$\bar{\mathcal{V}}_T(\hat{\Omega}^{\varepsilon}) = \left\{ \ \hat{\mathbf{v}}^{\varepsilon}_{T|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\},$$

$$\bar{\mathfrak{V}}(\hat{\Omega}^{\varepsilon}) = \left\{ \ \frac{\partial \hat{\mathbf{v}}^{\varepsilon}}{\partial t}_{|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\} \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3),$$

$$\bar{\mathfrak{V}}_N(\hat{\Omega}^{\varepsilon}) = \left\{ \ \frac{\partial \hat{v}^{\varepsilon}_N}{\partial t}_{|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\},$$

$$\bar{\mathfrak{V}}_T(\hat{\Omega}^{\varepsilon}) = \left\{ \ \frac{\partial \hat{\mathbf{v}}^{\varepsilon}_T}{\partial t}_{|_{\hat{\Gamma}^{\varepsilon}_{+}}} ; \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}) \ \right\}.$$

We suppose that $\hat{\Gamma}^{\varepsilon}_{+}$ smooth enough, such that

$$\bar{\mathcal{V}}_N(\hat{\Omega}^{\varepsilon}) \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}), \tag{5.2}$$

$$\bar{\mathcal{V}}_T(\hat{\Omega}^{\varepsilon}) \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3), \tag{5.3}$$

$$\bar{\mathfrak{V}}_N(\hat{\Omega}^{\varepsilon}) \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}), \tag{5.4}$$

$$\bar{\mathfrak{V}}_T(\hat{\Omega}^{\varepsilon}) \subset W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3), \tag{5.5}$$

$$\bar{\mathcal{V}}'_N(\hat{\Omega}^{\varepsilon}) \subset W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}), \tag{5.6}$$

$$\bar{\mathcal{V}}'_T(\hat{\Omega}^{\varepsilon}) \subset W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3), \tag{5.7}$$

$$\bar{\mathfrak{V}}'_N(\hat{\Omega}^{\varepsilon}) \subset W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}), \tag{5.8}$$

$$\bar{\mathfrak{V}}'_T(\hat{\Omega}^{\varepsilon}) \subset W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_{+}; \mathbb{R}^3), \tag{5.9}$$

where $\bar{\mathcal{V}}'_N(\hat{\Omega}^{\varepsilon})$, $\bar{\mathcal{V}}'_T(\hat{\Omega}^{\varepsilon})$, $\bar{\mathfrak{V}}'_N(\hat{\Omega}^{\varepsilon})$ and $\bar{\mathfrak{V}}'_T(\hat{\Omega}^{\varepsilon})$ are the topological dual spaces of $\bar{\mathcal{V}}_N(\hat{\Omega}^{\varepsilon})$, $\bar{\mathcal{V}}_T(\hat{\Omega}^{\varepsilon})$, $\bar{\mathfrak{V}}_N(\hat{\Omega}^{\varepsilon})$ and $\bar{\mathfrak{V}}_T(\hat{\Omega}^{\varepsilon})$ respectively.

For simplicity, we note that $W^{\frac{3}{4},4}(.;\mathbb{R}) = W^{\frac{3}{4},4}(.)$, $W^{\frac{3}{4},4}(.;\mathbb{R}^3) = \mathbf{W}^{\frac{3}{4},4}(.)$, $W^{-\frac{3}{4},\frac{4}{3}}(.;\mathbb{R}) = W^{-\frac{3}{4},\frac{4}{3}}(.)$ and $W^{-\frac{3}{4},\frac{4}{3}}(.;\mathbb{R}^3) = \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(.)$

First, we rewrite the above boundary value problem $(C.\hat{P}^{\varepsilon})^{iso}_{dyn,c}$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem:

$$(V.\hat{P}^{\varepsilon})^{iso}_{dyn,c}\begin{cases} \text{Find } (\hat{\mathbf{u}}^{\varepsilon}, \hat{\sigma}^{\varepsilon}, \hat{G}^{\varepsilon}_N, \hat{\mathbf{G}}^{\varepsilon}_T) \in \mathcal{K}(\hat{\Omega}^{\varepsilon}) \times \Sigma(\hat{\Omega}^{\varepsilon}) \times W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_+) \times \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_+) \\ \forall t \geq 0, \text{ such that,} \\ \hat{D}^{\varepsilon,t}(\hat{\mathbf{u}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon}) + \hat{B}^{\varepsilon,\theta}(\sigma(\varepsilon), \hat{\mathbf{v}}^{\varepsilon}) + 2\hat{C}^{\varepsilon,\theta}(\sigma(\varepsilon), \hat{\mathbf{u}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon}) = \hat{F}^{\varepsilon}(\hat{\mathbf{v}}^{\varepsilon}) \\ +\langle \hat{G}^{\varepsilon}_N, \hat{v}^{\varepsilon}_N\rangle + \langle \hat{\mathbf{G}}^{\varepsilon}_T, \hat{\mathbf{v}}^{\varepsilon}_T\rangle, \ \forall \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}), \forall t > 0, \\ \langle \hat{G}^{\varepsilon}_N, \hat{v}^{\varepsilon}_N - \hat{u}^{\varepsilon}_N\rangle \geq 0, \forall \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{K}(\hat{\Omega}^{\varepsilon}), \forall t > 0, \\ \langle \hat{\mathbf{G}}^{\varepsilon}_T, \hat{\mathbf{v}}^{\varepsilon}_T - \frac{\partial \hat{\mathbf{u}}^{\varepsilon}_T}{\partial t}\rangle + \langle \Lambda \left|\hat{G}^{\varepsilon}_N\right|, |\hat{\mathbf{v}}^{\varepsilon}_T| - \left|\frac{\partial \hat{\mathbf{u}}^{\varepsilon}_T}{\partial t}\right|\rangle \geq 0, \forall \hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon}), \forall t > 0, \\ \hat{\mathbf{u}}^{\varepsilon}(\hat{x}^{\varepsilon}, 0) = \hat{\mathbf{p}}^{\varepsilon}, \frac{\partial \hat{\mathbf{u}}^{\varepsilon}}{\partial t}(\hat{x}^{\varepsilon}, 0) = \hat{\mathbf{q}}^{\varepsilon} \text{ in } \hat{\Omega}^{\varepsilon}, \end{cases}$$

where

$$\begin{cases} \hat{D}^{\varepsilon,t}(\hat{\mathbf{u}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon}) = \frac{d^2}{dt^2}\left\{\hat{\rho}^{\varepsilon} \int_{\hat{\Omega}^{\varepsilon}} \hat{u}^{\varepsilon}_i \hat{v}^{\varepsilon}_i d\hat{x}^{\varepsilon}\right\}, \\ \hat{B}^{\varepsilon,\theta}(\sigma(\varepsilon), \hat{\mathbf{v}}^{\varepsilon}) = \int_{\hat{\Omega}^{\varepsilon}} \hat{\sigma}^{\varepsilon}_{ij} \hat{\gamma}^{\varepsilon}_{ij}(\hat{\mathbf{v}}^{\varepsilon}) d\hat{x}^{\varepsilon}, \\ \hat{C}^{\varepsilon,\theta}(\sigma(\varepsilon), \hat{\mathbf{u}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon}) = \frac{1}{2}\int_{\hat{\Omega}^{\varepsilon}} \hat{\sigma}^{\varepsilon}_{ij} \hat{\partial}^{\varepsilon}_j \hat{u}^{\varepsilon}_l \hat{\partial}^{\varepsilon}_j \hat{v}^{\varepsilon}_l d\hat{x}^{\varepsilon}, \\ \hat{F}^{\varepsilon}(\hat{\mathbf{v}}^{\varepsilon}) = \int_{\hat{\Omega}^{\varepsilon}} \hat{f}^{\varepsilon}_3 \hat{v}^{\varepsilon}_3 d\hat{x}^{\varepsilon} + \int_{\hat{\Gamma}^{\varepsilon}_-} \hat{g}^{\varepsilon}_3 \hat{v}^{\varepsilon}_3 d\hat{\Gamma}^{\varepsilon} + \int_{\hat{\gamma}^{\varepsilon}_1}\{\int_{-\varepsilon}^{\varepsilon}(\hat{v}^{\varepsilon}_{\alpha} \circ \mathbf{\Theta}^{\varepsilon}) dx^{\varepsilon}_3\}\hat{h}^{\varepsilon}_{\alpha} d\hat{\gamma}^{\varepsilon}. \end{cases}$$

$\langle .,. \rangle$ is the duality pairing between $W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^{\varepsilon}_+)$ and $W^{\frac{3}{4},4}(\hat{\Gamma}^{\varepsilon}_+)$ such that

$$\begin{aligned} \langle \hat{\mathbf{G}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon}\rangle &= \langle \hat{G}^{\varepsilon}_N, \hat{v}^{\varepsilon}_N\rangle + \langle \hat{\mathbf{G}}^{\varepsilon}_T, \hat{\mathbf{v}}^{\varepsilon}_T\rangle \\ &= \hat{D}^{\varepsilon,t}(\hat{\mathbf{u}}^{\varepsilon}, \mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon}) + \hat{B}^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon}) + 2\hat{C}^{\varepsilon,\theta}(\sigma(\varepsilon), \hat{\mathbf{u}}^{\varepsilon}, \mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon}) \\ &- \hat{F}^{\varepsilon}(\mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon}), \end{aligned} \tag{5.10}$$

for all $\hat{\mathbf{v}}^{\varepsilon} \in \mathbf{W}^{\frac{3}{4},4}_0(\hat{\Gamma}^{\varepsilon}_+)$,
where $\mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon} \in \mathcal{V}(\hat{\Omega}^{\varepsilon})$ is any extension of $\hat{\mathbf{v}}^{\varepsilon}$ such that $\mathbf{P}_1\hat{\mathbf{v}}^{\varepsilon} = 0$ on $\partial\hat{\Omega}^{\varepsilon}\backslash\hat{\Gamma}^{\varepsilon}_+$, and $\mathbf{W}^{\frac{3}{4},4}_0(\hat{\Gamma}^{\varepsilon}_+)$ is the subspace of traces of functions from $\mathbf{W}^{1,4}(\hat{\Omega}^{\varepsilon})$ vanishing on $\partial\hat{\Omega}^{\varepsilon}\backslash\hat{\Gamma}^{\varepsilon}_+$.

We note that

$$\begin{aligned} \langle \hat{G}^{\varepsilon}_N, \hat{v}^{\varepsilon}_N\rangle &= \langle \hat{\mathbf{G}}^{\varepsilon}, \hat{\mathbf{n}}^{\varepsilon}\hat{v}^{\varepsilon}_N\rangle \\ \langle \hat{\mathbf{G}}^{\varepsilon}_T, \hat{\mathbf{v}}^{\varepsilon}_T\rangle &= \langle \hat{\mathbf{G}}^{\varepsilon}, \hat{\mathbf{v}}^{\varepsilon} - \hat{\mathbf{n}}^{\varepsilon}\hat{v}^{\varepsilon}_N\rangle. \end{aligned}$$

In order to transform the problem $(V.\hat{P}^{\varepsilon})^{iso}_{dyn,c}$ into problem posed over the cylindrical domain Ω^{ε}, we use the one to one mapping $(\mathbf{\Theta}^{\varepsilon})^{-1}$ and the relations (2.3).

Let there be a given C^1-diffeomorphism $\mathbf{\Theta}^\varepsilon$ that satisfies the orientation-preserving condition. Then the variational problem $(V.\hat{P}^\varepsilon)^{iso}_{dyn,c}$ is equivalent to the following variational problem:

$$(P^\varepsilon)^{iso}_{dyn,c}\begin{cases}\text{Find } (\mathbf{u}^\varepsilon, \sigma^\varepsilon, G^\varepsilon_N, \mathbf{G}^\varepsilon_T) \in \mathcal{K}(\Omega^\varepsilon) \times \Sigma(\Omega^\varepsilon) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^\varepsilon_+) \times \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma^\varepsilon_+) \\ \forall t \geq 0, \text{ such that,} \\ D^{\varepsilon,t}(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) + B^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{v}^\varepsilon) + 2C^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = F^\varepsilon(\mathbf{v}^\varepsilon) \\ +\langle G^\varepsilon_N, v^\varepsilon_N\rangle + \langle \mathbf{G}^\varepsilon_T, \mathbf{v}^\varepsilon_T\rangle, \forall \mathbf{v}^\varepsilon \in \mathcal{V}(\Omega^\varepsilon), \forall t > 0, \\ \langle G^\varepsilon_N, v^\varepsilon_N - u^\varepsilon_N\rangle \geq 0, \forall \mathbf{v}^\varepsilon \in \mathcal{K}(\Omega^\varepsilon), \forall t > 0, \\ \langle \mathbf{G}^\varepsilon_T, \mathbf{v}^\varepsilon_T - \frac{\partial \mathbf{u}^\varepsilon_T}{\partial t}\rangle + \langle \Lambda\, |G^\varepsilon_N|, |\mathbf{v}^\varepsilon_T| - \left|\frac{\partial \mathbf{u}^\varepsilon_T}{\partial t}\right|\rangle \geq 0, \forall\, \mathbf{v}^\varepsilon \in \mathcal{V}(\Omega^\varepsilon), \forall t > 0, \\ \mathbf{u}^\varepsilon(x^\varepsilon, 0) = \mathbf{p}^\varepsilon, \frac{\partial \mathbf{u}^\varepsilon}{\partial t}(x^\varepsilon, 0) = \mathbf{q}^\varepsilon \text{ in } \Omega^\varepsilon,\end{cases}$$

where

$$\begin{cases} D^{\varepsilon,t}(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \frac{d^2}{dt^2}\left\{\rho^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon_i v^\varepsilon_i \delta^\varepsilon dx^\varepsilon\right\}, \\ B^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{v}^\varepsilon) = +\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{kj} \partial^\varepsilon_k v^\varepsilon_i \delta^\varepsilon dx^\varepsilon, \\ C^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = \frac{1}{2}\int_{\Omega^\varepsilon} \sigma^\varepsilon_{ij} b^\varepsilon_{ki} \partial^\varepsilon_k u^\varepsilon_l b^\varepsilon_{mj} \partial^\varepsilon_m v^\varepsilon_l \delta^\varepsilon dx^\varepsilon, \\ F^\varepsilon(\mathbf{v}^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon dx^\varepsilon + \int_{\Gamma^\varepsilon_-} g^\varepsilon_3 v^\varepsilon_3 \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon \\ + \int_{\gamma_1} h^\varepsilon_\alpha \{\int_{-\varepsilon}^{\varepsilon} v^\varepsilon_\alpha dx^\varepsilon_3\} d\gamma, \end{cases}$$

such that

$$\begin{cases} u^\varepsilon_i = \hat{u}^\varepsilon_i \circ \mathbf{\Theta}^\varepsilon, \;\; v^\varepsilon_i = \hat{v}^\varepsilon_i \circ \mathbf{\Theta}^\varepsilon, \;\; \sigma^\varepsilon_{ij} = \hat{\sigma}^\varepsilon_{ij} \circ \mathbf{\Theta}^\varepsilon, \;\; \tau^\varepsilon_{ij} = \hat{\tau}^\varepsilon_{ij} \circ \mathbf{\Theta}^\varepsilon, \\ f^\varepsilon_3 = \hat{f}^\varepsilon_3 \circ \mathbf{\Theta}^\varepsilon, \;\; g^\varepsilon_3 = \hat{g}^\varepsilon_3 \circ \mathbf{\Theta}^\varepsilon, \;\; h^\varepsilon_\alpha = \hat{h}^\varepsilon_\alpha \circ \mathbf{\Theta}^\varepsilon, \\ p^\varepsilon_i = \hat{p}^\varepsilon_i \circ \mathbf{\Theta}^\varepsilon, \;\; q^\varepsilon_i = \hat{q}^\varepsilon_i \circ \mathbf{\Theta}^\varepsilon, \\ d^\varepsilon = \hat{d}^\varepsilon \circ \mathbf{\Theta}^\varepsilon. \end{cases}$$

$\langle .,. \rangle$ is the duality pairing between $W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^\varepsilon_+)$ and $W^{\frac{3}{4},4}(\Gamma^\varepsilon_+)$ such that

$$\begin{aligned} \langle \mathbf{G}^\varepsilon, \mathbf{v}^\varepsilon\rangle &= \langle G^\varepsilon_N, v^\varepsilon_N\rangle + \langle \mathbf{G}^\varepsilon_T, \hat{\mathbf{v}}^\varepsilon_T\rangle \\ &= D^{\varepsilon,t}(\mathbf{u}^\varepsilon, \mathbf{P}_2\mathbf{v}^\varepsilon) + B^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{P}_2\mathbf{v}^\varepsilon) + 2C^{\varepsilon,\theta}(\sigma(\varepsilon), \mathbf{u}^\varepsilon, \mathbf{P}_2\mathbf{v}^\varepsilon) \\ &- F^\varepsilon(\mathbf{P}_2\mathbf{v}^\varepsilon), \end{aligned} \tag{5.11}$$

for all $\mathbf{v}^\varepsilon \in \mathbf{W}^{\frac{3}{4},4}_0(\Gamma^\varepsilon_+)$,
where $\mathbf{P}_2\mathbf{v}^\varepsilon \in \mathcal{V}(\Omega^\varepsilon)$ is any extension of $\mathbf{v}^\varepsilon$ such that $\mathbf{P}_2\mathbf{v}^\varepsilon = 0$ on $\partial\Omega^\varepsilon \backslash \Gamma^\varepsilon_+$, and $\mathbf{v}^\varepsilon \in \mathbf{W}^{\frac{3}{4},4}_0(\Gamma^\varepsilon_+)$ is the subspace of traces of functions from $\mathbf{v}^\varepsilon \in \mathbf{W}^{1,4}(\Omega^\varepsilon)$ vanishing on $\partial\Omega^\varepsilon \backslash \Gamma^\varepsilon_+$.

We define

$$\begin{aligned} \langle G^\varepsilon_N, v^\varepsilon_N\rangle &= \langle \mathbf{G}^\varepsilon, \mathbf{n}^\varepsilon v^\varepsilon_N\rangle \\ \langle \mathbf{G}^\varepsilon_T, \mathbf{v}^\varepsilon_T\rangle &= \langle \mathbf{G}^\varepsilon, \mathbf{v}^\varepsilon - \mathbf{n}^\varepsilon v^\varepsilon_N\rangle, \end{aligned}$$

such that $v^\varepsilon_N = \hat{v}^\varepsilon_N \circ \mathbf{\Theta}^\varepsilon$, $\mathbf{v}^\varepsilon_T = \hat{\mathbf{v}}^\varepsilon_T \circ \mathbf{\Theta}^\varepsilon$ and $\mathbf{n}^\varepsilon = \hat{\mathbf{n}}^\varepsilon \circ \mathbf{\Theta}^\varepsilon$.

5.1.2 Asymptotic analysis

The scaled three-dimensional problem

We use the same arguments as in Chapter 2 to transform $(P^\varepsilon)^{iso}_{dyn,c}$ into a problem posed over an open set independent of ε.

First, to the functions $\mathbf{u}^\varepsilon$, $\mathbf{v}^\varepsilon \in \mathcal{V}(\Omega^\varepsilon)$, $\sigma^\varepsilon \in \Sigma(\Omega^\varepsilon)$, $G^\varepsilon_N \in W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^\varepsilon_+)$ and $\mathbf{G}^\varepsilon_T \in \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma^\varepsilon_+)$, we associate the scaled functions $\mathbf{u}(\varepsilon)$, $\mathbf{v} \in \mathcal{V}(\Omega)$, $\sigma(\varepsilon) \in \Sigma(\Omega)$, $G_N(\varepsilon) \in W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)$ and $\mathbf{G}_T(\varepsilon) = (\mathbf{G}_{T_i}(\varepsilon)) \in \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)$ defined by

$$\begin{cases} u^\varepsilon_\alpha(x^\varepsilon,t) = \varepsilon^2 u_\alpha(\varepsilon)(x,t), u^\varepsilon_3(x^\varepsilon,t) = \varepsilon u_3(\varepsilon)(x,t), \\ v^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), v^\varepsilon_3(x^\varepsilon) = \varepsilon v_3(x), \\ \sigma^\varepsilon_{\alpha\beta}(x^\varepsilon,t) = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon)(x,t), \sigma^\varepsilon_{\alpha3}(x^\varepsilon,t) = \varepsilon^3\sigma_{\alpha3}(\varepsilon)(x,t), \\ \sigma^\varepsilon_{33}(x^\varepsilon,t) = \varepsilon^4\sigma_{33}(\varepsilon)(x,t), \\ \langle G^\varepsilon_{T_\alpha}, v_{T_\alpha}\rangle = \varepsilon^3\langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}\rangle, \langle G^\varepsilon_{T_3}, v_{T_3}\rangle = \varepsilon^4\langle G_{T_3}(\varepsilon), v_{T_3}\rangle, \\ \langle G^\varepsilon_N, v_N\rangle = \varepsilon^4\langle G_N(\varepsilon), v_N\rangle, \end{cases} \tag{5.12}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon$.

Next, we make the following assumptions: there exists constant $\lambda > 0$, $\mu > 0$, $\rho > 0$ and for some $T > 0$, the functions $f_3 \in L^2(0,T;L^2(\Omega))$, $g_3 \in L^2(0,T;L^2(\Gamma_-))$, $h_\alpha \in L^2(0,T;L^2(\gamma_1))$, $\theta \in C^3(\bar{\omega})$ independent of ε and $\mathbf{p}(\varepsilon) \in \mathcal{V}(\Omega)$, $\mathbf{q}(\varepsilon) \in L^2(\Omega;\mathbb{R}^3)$, $d(\varepsilon) \in L^\infty(\Gamma_+)$ such that

$$\begin{cases} \lambda^\varepsilon = \lambda, \mu^\varepsilon = \mu, \rho^\varepsilon = \varepsilon^2\rho, \\ f^\varepsilon_3(x^\varepsilon,t) = \varepsilon^3 f_3(x,t)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ g^\varepsilon_3(x^\varepsilon,t) = \varepsilon^4 g_3(x,t)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma^\varepsilon_-, \\ h^\varepsilon_\alpha(y_1,y_2,t) = \varepsilon^2 h_\alpha(y_1,y_2,t)\ \forall (y_1,y_2) \in \gamma_1, \\ \theta^\varepsilon(x_1,x_2) = \varepsilon\theta(x_1,x_2)\ \forall (x_1,x_2) \in \bar{\omega}, \\ p^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 p_\alpha(\varepsilon)(x)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ p^\varepsilon_3(x^\varepsilon) = \varepsilon p_3(\varepsilon)(x)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 q_\alpha(\varepsilon)(x)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q^\varepsilon_3(x^\varepsilon) = \varepsilon q_3(\varepsilon)(x)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ d^\varepsilon(x^\varepsilon) = \varepsilon d(\varepsilon)(x)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma^\varepsilon_+. \end{cases} \tag{5.13}$$

Noting that the unit normal $\hat{n}^\varepsilon$ on $\hat{\Gamma}^\varepsilon_+$ reads $\hat{n}^\varepsilon = (-\partial^\varepsilon_1\theta^\varepsilon + O(\varepsilon^3), -\partial^\varepsilon_2\theta^\varepsilon + O(\varepsilon^3), 1 + O(\varepsilon^2))$. Then a simple computation gives

$$\begin{cases} v^\varepsilon_N = \varepsilon v_N(\varepsilon), v_N(\varepsilon) = v_3 n^\theta_3 + O(\varepsilon^2), \\ \frac{\partial u^\varepsilon_N}{\partial t} = \varepsilon\frac{\partial u_N(\varepsilon)}{\partial t}, \frac{\partial u_N(\varepsilon)}{\partial t} = \frac{\partial u_3}{\partial t} n^\theta_3 + O(\varepsilon^2), \\ v^\varepsilon_{T_\alpha} = \varepsilon^2 v_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) = v_\alpha - v_3 n^\theta_\alpha + O(\varepsilon^2), \\ v^\varepsilon_{T_3} = \varepsilon^2 v_{T_3}(\varepsilon), v_{T_3}(\varepsilon) = O(\varepsilon), \\ \frac{\partial u^\varepsilon_{T_\alpha}}{\partial t} = \varepsilon^2\frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}, \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t} = \frac{\partial u_\alpha}{\partial t} - \frac{\partial u_3}{\partial t} n^\theta_\alpha + O(\varepsilon^2), \\ \frac{\partial u^\varepsilon_{T_3}}{\partial t} = \varepsilon^2\frac{\partial u_{T_3}(\varepsilon)}{\partial t}, \frac{\partial u_{T_3}(\varepsilon)}{\partial t} = O(\varepsilon), \end{cases} \tag{5.14}$$

where $n^{\theta} = (-\partial_1\theta, -\partial_2\theta, 1)$.

Using the relations (3.2) and (5.14), the scalings (5.12) and the assumptions (5.13), we obtain

Theorem 5.1 *The scaled fields* $(\mathbf{u}(\varepsilon), \sigma(\varepsilon), G_N(\varepsilon), \mathbf{G}_T(\varepsilon))$ *satisfies the following variational problem:*

$$(P(\varepsilon))^{iso}_{dyn,c}\begin{cases} \textit{Find } (\mathbf{u}(\varepsilon), \sigma(\varepsilon), G_N(\varepsilon), \mathbf{G}_T(\varepsilon)) \in \mathcal{K}(\varepsilon)(\Omega) \times \Sigma(\Omega) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\times \\ \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \forall t \in [0,T],\ \textit{such that,} \\ D^t(\mathbf{u}(\varepsilon), \mathbf{v}) + B^{\theta}(\sigma(\varepsilon), \mathbf{v}) + 2C^{\theta}(\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) = F(\mathbf{v}) + \\ \langle G_N(\varepsilon), v_N(\varepsilon)\rangle + \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon)\rangle + \\ \varepsilon^2\left[\langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon)\rangle + R(\varepsilon;\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v})\right], \forall \mathbf{v} \in \mathcal{V}(\Omega), \forall t \in]0,T[\,, \\ \langle G_N(\varepsilon), v_N(\varepsilon) - u_N(\varepsilon)\rangle \geq 0\,,\ \forall \mathbf{v} \in \mathcal{K}(\varepsilon)(\Omega), \forall t \in]0,T[\,, \\ \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) - \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle + \\ \varepsilon\left[\langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon) - \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), |v_T(\varepsilon)| - \left|\frac{\partial u_T(\varepsilon)}{\partial t}\right|\rangle\right] \geq 0, \\ \forall \mathbf{v} \in \mathcal{V}(\Omega), \forall t \in]0,T[\,, \\ \mathbf{u}(\varepsilon)(x,0) = \mathbf{p}(\varepsilon)\,, \frac{\partial \mathbf{u}(\varepsilon)}{\partial t}(x,0) = \mathbf{q}(\varepsilon)\ \textit{ in } \Omega, \end{cases}$$

where

$$\begin{cases} B^{\theta}(\sigma(\varepsilon), \mathbf{v}) = \int_\Omega \sigma_{ij}(\varepsilon)\,\gamma^{\theta}_{ij}(\mathbf{v})dx, \\ C^{\theta}(\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}) = \frac{1}{2}\int_\Omega \sigma_{ij}(\varepsilon)\,\partial^{\theta}_i u_3(\varepsilon)\,\partial^{\theta}_j v_3 dx, \\ D^t(\mathbf{u}(\varepsilon), \mathbf{v}) = \frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3(\varepsilon)\,v_3 dx\right\}, \\ F(\mathbf{v}) = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^{1} v_\alpha dx_3\}d\gamma, \end{cases}$$

such that $\gamma^{\theta}_{ij}(\mathbf{v}) = \frac{1}{2}\left(\partial^{\theta}_i v_j + \partial^{\theta}_j v_i\right)$ *and the remainders* R *and* S *are bounded.*

Proof.

We have

$$\int_{\Omega^\varepsilon} \sigma^{\varepsilon}_{ij} b^{\varepsilon}_{kj}\partial^{\varepsilon}_k v^{\varepsilon}_i \delta^{\varepsilon} dx^{\varepsilon} = \varepsilon^5\int_\Omega \sigma_{ij}(\varepsilon)\,\gamma^{\theta}_{ij}(\mathbf{v})\,dx + \varepsilon^7\varrho_B(\varepsilon;\sigma(\varepsilon), \mathbf{v}),$$

$$\begin{aligned} \int_{\Omega^\varepsilon} \sigma^{\varepsilon}_{ij} b^{\varepsilon}_{ki}\partial^{\varepsilon}_k u^{\varepsilon}_l b^{\varepsilon}_{mj}\partial^{\varepsilon}_m v^{\varepsilon}_l \delta^{\varepsilon} dx^{\varepsilon} &= \varepsilon^5\int_\Omega \sigma_{ij}(\varepsilon)\,\partial^{\theta}_i u_3(\varepsilon)\,\partial^{\theta}_j v_3 dx \\ &+ \varepsilon^7\varrho_C(\varepsilon;\sigma(\varepsilon), \mathbf{u}(\varepsilon), \mathbf{v}), \end{aligned}$$

$$\int_{\Omega^\varepsilon} f_3^\varepsilon v_3^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_3^\varepsilon v_3^\varepsilon \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon + \int_{\gamma_1} h_\alpha^\varepsilon \left\{ \int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon \right\} d\gamma \ =$$

$$\varepsilon^5 \left(\int_\Omega f_3 v_3 dx + \int_{\Gamma_+ \cup \Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha \left\{ \int_{-1}^{1} v_\alpha dx_3 \right\} d\gamma \right) + \varepsilon^7 \varrho_F \left(\varepsilon; \mathbf{v}\right).$$

$$\frac{d^2}{dt^2} \left\{ \rho^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon \right\} = \varepsilon^5 \frac{d^2}{dt^2} \left\{ \rho \int_\Omega u_3\left(\varepsilon\right) v_3 dx \right\} + \varepsilon^7 \varrho_D(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}).$$

From the relation (5.14) and the scalings (5.12), we get

$$\langle G_N^\varepsilon, v_N^\varepsilon \rangle = \varepsilon^5 \langle G_N(\varepsilon), v_N(\varepsilon) \rangle,$$

$$\langle \mathbf{G}_T^\varepsilon, \mathbf{v}_T^\varepsilon \rangle = \varepsilon^5 \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) \rangle + \varepsilon^7 \langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon) \rangle.$$

So that the first equation in variational problem $(P^\varepsilon)^{iso}_{dyn,c}$ may be written as

$$D^t\left(\mathbf{u}\left(\varepsilon\right), \mathbf{v}\right) + B^\theta\left(\sigma\left(\varepsilon\right), \mathbf{v}\right) + 2C^\theta\left(\sigma\left(\varepsilon\right), \mathbf{u}\left(\varepsilon\right), \mathbf{v}\right) = F\left(\mathbf{v}\right) +$$

$$\langle G_N(\varepsilon), v_N(\varepsilon) \rangle + \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) \rangle + \varepsilon^2 \left[\langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon) \rangle + R\left(\varepsilon; \sigma\left(\varepsilon\right), \mathbf{u}\left(\varepsilon\right), \mathbf{v}\right) \right],$$

where

$$R\left(\varepsilon; \sigma\left(\varepsilon\right), \mathbf{u}\left(\varepsilon\right), \mathbf{v}\right) = \varrho_F\left(\varepsilon; \mathbf{v}\right) - \varrho_B\left(\varepsilon; \sigma\left(\varepsilon\right), \mathbf{v}\right) -$$

$$\varrho_C\left(\varepsilon; \sigma\left(\varepsilon\right), \mathbf{u}\left(\varepsilon\right), \mathbf{v}\right) - \varrho_D(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}).$$

Now, note that, there exists a positive constant C such that, for all $\mathbf{u}, \ \mathbf{v} \in \mathcal{V}(\Omega)$ and $\sigma \in \Sigma(\Omega)$

$$\sup_{0 \le \varepsilon \le \varepsilon_0} \int_\Omega |\varrho_B(\varepsilon; \sigma, \mathbf{v})| dx \ \le \ C|\sigma|_{0,\Omega} |\mathbf{v}|_{1,\Omega},$$

$$\sup_{0 \le \varepsilon \le \varepsilon_0} \int_\Omega |\varrho_C(\varepsilon; \sigma, \mathbf{u}, \mathbf{v})| dx \ \le \ C|\sigma|_{0,\Omega} |\mathbf{u}|_{1,4,\Omega} |\mathbf{v}|_{1,4,\Omega},$$

$$\sup_{0 \le \varepsilon \le \varepsilon_0} \int_\Omega |\varrho_F(\varepsilon; \mathbf{v})| dx \ \le \ C\|\mathbf{v}\|_{1,\Omega},$$

$$\sup_{0 \le \varepsilon \le \varepsilon_0} \int_\Omega |\varrho_D(\varepsilon, \mathbf{u}, \mathbf{v})| dx \ \le \ C\|\frac{\partial^2 \mathbf{u}}{\partial t^2}\|_{-1,\frac{4}{3},\Omega} \|\mathbf{v}\|_{1,\Omega}.$$

For the unilateral contact conditions, we have

$$\langle G_N^\varepsilon, v_N^\varepsilon - u_N^\varepsilon \rangle \ = \ \varepsilon^5 \langle G_N(\varepsilon), v_N(\varepsilon) - u_N(\varepsilon) \rangle,$$

$$\langle \mathbf{G}_T^\varepsilon, \mathbf{v}_T^\varepsilon - \frac{\partial \mathbf{u}_T^\varepsilon}{\partial t} \rangle + \langle \Lambda |G_N^\varepsilon|, |\mathbf{v}_T^\varepsilon| - |\frac{\partial \mathbf{u}_T^\varepsilon}{\partial t}| \rangle = \varepsilon^5 \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) - \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t} \rangle$$

$$+\varepsilon^6 [\langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon) - \frac{\partial u_{T_3}(\varepsilon)}{\partial t} \rangle - \langle \Lambda G_N(\varepsilon), |\mathbf{v}_T(\varepsilon)| - |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}| \rangle],$$

■

The limit three-dimensional problem

Assume that the scaled fields $(\mathbf{u}(\varepsilon), \sigma(\varepsilon), G_N(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(\mathbf{u}(\varepsilon), \sigma(\varepsilon), G_N(\varepsilon)) = (\mathbf{u}^0, \sigma^0, G_N^0) + \varepsilon(\mathbf{u}^1, \sigma^1, G_N^1) + \varepsilon^2(\mathbf{u}^2, \sigma^2, G_N^2) + \cdots , \qquad (5.15)$$

with
$\mathbf{u}^0 = (u_i^0) \in \mathcal{V}(\Omega)$, $\partial_3 u_3^0 \in C^0\left(\bar{\Omega}\right)$, $\mathbf{u}^p = (u_i^p) \in W^{1,4}(\Omega;\mathbb{R}^3)\ \ \forall p \geq 1$,
$(\sigma^p, G_N^p) \in \Sigma(\Omega) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \ \forall p \geq 0$.

We also assume that when $\varepsilon \to 0$

$$\mathbf{p}(\varepsilon) \to \mathbf{p}^0 \text{ in } \mathbf{V}(\Omega), \qquad (5.16)$$

$$\mathbf{q}(\varepsilon) \to \mathbf{q}^0 \text{ in } L^2(\Omega;\mathbb{R}^3), \qquad (5.17)$$

$$d(\varepsilon) \to d \text{ in } L^\infty(\Gamma_+), \qquad (5.18)$$

$$\varepsilon G_N(\varepsilon) \to 0 \text{ in } W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+). \qquad (5.19)$$

We substitute the formal asymptotic expansion (5.15) into the variational problem $(P(\varepsilon))^{iso}_{dyn,c}$, we obtain the following limit three-dimensional problem

Theorem 5.2 *The leading term $(\mathbf{u}^0, \sigma^0, G_N^0)$ satisfies the following variational problem:*

$$(P_1^0)^{iso}_{dyn,c} \begin{cases} \textit{Find } (\mathbf{u}^0, \sigma^0, G_N^0) \in \mathcal{K}(\Omega) \times \Sigma(\Omega) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \forall t \in [0,T],\ \textit{such that,} \\ \int_\Omega \sigma^0_{i\alpha}\partial_i v_\alpha dx - \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_3 v_\alpha dx = \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma, \\ \forall v_\alpha \in \mathcal{V}_\alpha(\Omega), \forall t \in\,]0,T[\,, \\ \frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{i3}\partial_i v_3 dx + \int_\Omega \sigma^0_{ij}\partial_i u_3^0 \partial_j v_3 dx \\ - \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha\theta\partial_3 v_3 dx - \int_\Omega\{\sigma^0_{\alpha j}\partial_\alpha\theta\partial_3 u_3^0\partial_j v_3 + \sigma^0_{i\beta}\partial_i u_3^0\partial_\beta\theta\partial_3 v_3\}dx \\ + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha\theta\partial_3 u_3^0\partial_\beta\theta\partial_3 v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \langle G_N^0, v_3\rangle, \\ \forall v_3 \in \mathcal{V}_3(\Omega), \forall t \in\,]0,T[\,, \\ \langle G_N^0, v_3 - u_3^0\rangle \geq 0\,, \forall \mathbf{v} \in \mathcal{K}(\Omega), \forall t \in\,]0,T[\,, \\ \mathbf{u}^0(x,0) = \mathbf{p}^0,\ \frac{\partial \mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0\ \textit{in } \Omega. \end{cases}$$

Proof. First, in the last inequality in $(P(\varepsilon))^{iso}_{dyn,c}$, we take the test function $\mathbf{v}_T(\varepsilon) = 0$ after that $\mathbf{v}_T(\varepsilon) = 2\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}$, we obtain

$$\langle G_{T_\alpha}(\varepsilon), -\frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle + \varepsilon[\langle G_{T_3}(\varepsilon), -\frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), -|\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle] \geq 0, \quad (5.20)$$

$$\langle G_{T_\alpha}(\varepsilon), \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle + \varepsilon[\langle G_{T_3}(\varepsilon), \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle] \geq 0. \quad (5.21)$$

Then, we conclude that

$$\langle G_{T_\alpha}(\varepsilon), \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle = \varepsilon[\langle \Lambda G_N(\varepsilon), |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle - \langle G_{T_3}(\varepsilon), \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle], \quad (5.22)$$

From the (5.19) and since $\frac{\partial u_{T_3}(\varepsilon)}{\partial t} = O(\varepsilon)$, we obtain

$$G_{T_\alpha}(\varepsilon) = 0 \text{ in } \Gamma_+, \forall t \in]0, T[\,.$$

Next, using technics of the asymptotic analysis method, we first replace $\mathbf{u}(\varepsilon)$, $\sigma(\varepsilon)$ and $G_N(\varepsilon)$ by their expansions (5.15) in the forms B^θ, C^θ, D^t and F and we equate to zero the terms which are independent of ε in $(P(\varepsilon))^{iso}_{dyn,c}$. Then we show that $(\mathbf{u}^0, \sigma^0, G_N^0)$ satisfy $(P_1^0)^{iso}_{dyn,c}$. ■

Theorem 5.3 *The leading term $(\mathbf{u}^0, G_N^0)$ satisfies the following variational problem:*

$$(P_2^0)^{iso}_{dyn,c}\begin{cases} \text{Find } (\mathbf{u}^0, G_N^0) \in \mathcal{K}_{KL}(\Omega) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+) \ \forall t \in [0,T], \text{ such that,} \\ \frac{d^2}{dt^2}\left\{\rho \int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u_3^0 + \theta)\partial_\beta v_3 dx = \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\} d\gamma + \langle G_N^0, v_3\rangle, \\ \forall \mathbf{v} \in \mathcal{V}_{KL}(\Omega), \forall t \in]0,T[\,, \\ \langle G_N^0, v_3 - u_3^0\rangle \geq 0\,, \forall \mathbf{v} \in \mathcal{K}_{KL}(\Omega), \forall t \in]0,T[\,, \\ \mathbf{u}^0(x,0) = \mathbf{p}^0, \frac{\partial \mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0 \text{ in } \Omega, \end{cases}$$

where

$$\begin{cases} \sigma^0_{\alpha\beta} = \frac{2\lambda\mu}{\lambda+2\mu}\bar{E}^0_{\sigma\sigma}(\mathbf{u}^0)\delta_{\alpha\beta} + 2\mu\bar{E}^0_{\alpha\beta}(\mathbf{u}^0), \\ \bar{E}^0_{\alpha\beta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha + \partial_\alpha\theta\partial_\beta u^0_3 + \partial_\beta\theta\partial_\alpha u^0_3 + \partial_\alpha u^0_3\partial_\beta u^0_3\right). \end{cases}$$

Proof.

The proof has been divided into two steps

Step 1. The relation $\hat{\sigma}^{\varepsilon}_{ij} = \lambda^{\varepsilon}\hat{E}^{\varepsilon}_{pp}\left(\hat{\mathbf{u}}^{\varepsilon}\right)\delta_{ij} + 2\mu^{\varepsilon}\hat{E}^{\varepsilon}_{ij}\left(\hat{\mathbf{u}}^{\varepsilon}\right)$ give

$$\partial_3 u_3^0(1 + \frac{1}{2}\partial_3 u_3^0) = 0,$$

and

$$\partial_\alpha u_3^0 + \partial_3 u_\alpha^0 = 0 \text{ in } \Omega. \tag{5.23}$$

So that

$$\partial_3 u_3^0 = 0 \text{ or } \partial_3 u_3^0 = -2.$$

Since the inclusion $H^3(\Omega) \hookrightarrow \mathcal{C}^1(\Omega)$ and $u_3^0 = 0$ on $\gamma_1 \times [-1, 1]$, the solution $\partial_3 u_3^0 = -2$ is eliminated. Hence we are left with

$$\partial_3 u_3^0 = 0 \text{ in } \Omega. \tag{5.24}$$

Then, we conclude

$$\sigma^0_{\alpha\beta} = \frac{2\lambda\mu}{\lambda + 2\mu}\bar{E}^0_{\sigma\sigma}(\mathbf{u}^0)\delta_{\alpha\beta} + 2\mu\bar{E}^0_{\alpha\beta}(\mathbf{u}^0).$$

Step 2. Taking into account the equation (5.24), we next find that the second equation in $(P_1^0)^{iso}_{dyn,c}$ reduce to

$$\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u_3^0 \partial_\beta v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \langle G_N^0, v_3\rangle, \tag{5.25}$$

From the first equation and the relation (5.23), we conclude that

$$\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx = \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx - \int_{\gamma_1} h_\alpha\{\int_{-1}^{1} v_\alpha dx_3\}d\gamma. \tag{5.26}$$

We replace the integral $\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx$ in equation (5.25) by their expression (5.26), we find that

$$\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u_3^0 + \theta)\partial_\beta v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^{1} v_\alpha dx_3\}d\gamma + \langle G_N^0, v_3\rangle.$$

■

The limit two-dimensional problem

We use some technics employed by Raoult [Rao85], who assumed that the initial data $\varphi_3 = p_3^0$ and $\psi_3 = q_3^0$ are independent of x_3 and sufficiently smooth. We also assume that the initial data $p_\alpha^0 = \varphi_\alpha - x_3\partial_\alpha p_3^0$ and $q_\alpha^0 = \psi_\alpha - x_3\partial_\alpha q_3^0$, such that φ_α and ψ_α are independent of x_3 and sufficiently smooth.

First, we show in the next theorem that $(P_2^0)^{iso}_{dyn,c}$ is in a sense of two-dimensional problem posed over the two-dimensional domain $\bar{\omega}$.

Theorem 5.4 *The leading term $\mathbf{u}^0 = (u_i^0)$ is of the form $u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ and $u_3^0 = \zeta_3$ with $\zeta = (\zeta_i) \in \mathbf{V}(\omega)$ $\forall t \in [0,T]$. In addition, the field ζ satisfies the following limit scaled two-dimensional problem:*

$$
(P(\omega))^{iso}_{dyn,c}\begin{cases}
\textit{Find } (\zeta, f_c) \in \mathcal{K}(\omega)\times H^{-2}(\omega)\ \forall t\in[0,T],\ \textit{such that,}\\
2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega\\
+\int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma + \langle f_c,\eta_3\rangle, \forall\eta\in\mathbf{V}(\omega), \forall t\in]0,T[,\\
\langle f_c, \eta_3-\zeta_3\rangle \geq 0\,,\ \forall\eta\in\mathcal{K}(\omega), \forall t\in]0,T[,\\
\zeta(.,0)=\varphi,\ \frac{\partial\zeta}{\partial t}(.,0)=\psi\ \textit{in } \omega,
\end{cases}
$$

where

$$
\begin{cases}
m_{\alpha\beta}(\nabla^2\zeta_3) = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\},\\
\bar{N}_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu}\bar{E}^0_{\sigma\sigma}(\zeta)\,\delta_{\alpha\beta} + 4\mu\bar{E}^0_{\alpha\beta}(\zeta)\,,\\
\bar{E}^0_{\alpha\beta}(\zeta) = \frac{1}{2}\left(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3 + \partial_\alpha\zeta_3\partial_\beta\zeta_3\right),\\
\mathrm{p}_3 = \int_{-1}^{1} f_3 dx_3 + g_3(.,-1),\\
d = d(.,+1),\\
\langle f_c,\eta_3\rangle = \langle G_N^0, v_3\rangle.
\end{cases}
$$

Proof.

i) From $\mathbf{v}\in\mathcal{V}_{KL}(\Omega)$, by a standard argument due to Ciarlet (see,e.g., [Cia97, Theorem 1.4-4]), we get

$$u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3^0 = \zeta_3 \text{ with } \zeta = (\zeta_i)\in\mathbf{V}(\omega).$$

From the definition of $\sigma^0_{\alpha\beta}$, we conclude that

$$\int_{-1}^{1} \sigma^0_{\alpha\beta} dx_3 \quad = \quad \bar{N}_{\alpha\beta}(\zeta),$$

and

$$\int_{-1}^{1} x_3 \sigma^0_{\alpha\beta} dx_3 \quad = \quad m_{\alpha\beta}(\zeta).$$

ii) First we choose, in $(P_2^0)^{iso}_{dyn,c}$, $\mathbf{v} \in \mathcal{V}_{KL}(\Omega)$ with the components

$$v_\alpha(x) = -x_3 \partial_\alpha \eta_3(x_1, x_2), \ v_3(x) = \eta_3(x_1, x_2),$$

with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_1.

This choice shows that $(P_2^0)^{iso}_{dyn,c}$ reduce to

$$\frac{d^2}{dt^2}\left\{\rho \int_\Omega \zeta_3 \eta_3 dx\right\} - \int_\Omega x_3 \sigma^0_{\alpha\beta} \partial_{\alpha\beta} \eta_3 dx + \int_\Omega \sigma^0_{\alpha\beta} \partial_\alpha (\zeta^0_3 + \theta) \partial_\beta \eta_3 dx \quad = \quad \int_\Omega f_3 \eta_3 dx + \int_{\Gamma_-} g_3 \eta_3 d\Gamma + \langle f_c, \eta_3 \rangle. \tag{5.27}$$

The second choice of $\mathbf{v} \in \mathcal{V}_{KL}(\Omega)$ is

$$v_\alpha(x) = \eta_\alpha(x_1, x_2), \ v_3(x) = 0,$$

with $\eta_\alpha \in H^1(\omega)$.

In this case shows that $(P_2^0)^{iso}_{dyn,c}$ reduce to

$$\int_\Omega \sigma^0_{\alpha\beta} \partial_\beta \eta_\alpha dx = 2 \int_{\gamma_1} h_\alpha \eta_\alpha d\gamma \tag{5.28}$$

Using Fubini's Formula: $\int_\Omega F dx = \int_\omega \left\{\int_{-1}^1 F dx_3\right\} d\omega$, we have

$$\frac{d^2}{dt^2}\{\rho \int_\Omega \zeta_3 \eta_3 dx\} = 2\rho \int_\omega \frac{\partial^2 \zeta_3}{\partial t^2} \eta_3 d\omega,$$

$$\int_\Omega -x_3 \sigma^0_{\alpha\beta} \partial_{\alpha\beta} \eta_3 dx = -\int_\omega m_{\alpha\beta} \partial_{\alpha\beta} \eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta} \partial_\alpha (\zeta_3 + \theta) \partial_\beta \eta_3 dx = \int_\omega \bar{N}_{\alpha\beta} \partial_\alpha (\zeta_3 + \theta) \partial_\beta \eta_3 d\omega,$$

$$\int_{\Omega} f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma \;=\; \int_{\omega}\{\int_{-1}^{1} f_3 dx_3 + g_3(.,-1)\}\eta_3 d\omega$$
$$= \int_{\omega} \mathrm{p}_3\eta_3 d\omega,$$

$$\int_{\Omega}\sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = \int_{\omega}\bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

Then

$$2\rho\int_{\omega}\frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_{\omega} m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_{\omega}\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega$$
$$+\int_{\omega}\bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_{\omega}\mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma + \langle f_c,\eta_3\rangle.$$

■

Next, we write the two-dimensional boundary value problem as an equivalent boundary value problem $(\bar{P}(\omega))^{iso}_{dyn,c}$. Using Green's formulas and equating to zero all the factors of η_α, η_3, and $\partial_\nu\eta_3$ in their respective domains of integration, we obtain

Theorem 5.5 *Assume that the boundary γ is sufficiently smooth. Then any smooth solution $\zeta = (\zeta_i)$ of the variational problem $(P(\omega))^{iso}_{dyn,c}$ is also a solution of the following two-dimensional problem:*

$$(\bar{P}(\omega))^{iso}_{dyn,c}\left\{\begin{array}{l}
\textit{Find } ((\zeta_\alpha),\zeta_3,f_c)\in (H^1(\omega))^2\times H^2(\omega)\times H^{-2}(\omega)\ \forall t\in[0,T],\ \textit{such that,}\\
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = \mathrm{p}_3 + f_c \ \textit{in } \omega\times]0,T[,\\
\partial_\beta\bar{N}_{\alpha\beta} = 0 \ \textit{in } \omega\times]0,T[,\\
\zeta_3 = \partial_\nu\zeta_3 = 0 \ \textit{on } \gamma_1\times]0,T[,\\
\bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha \ \textit{on } \gamma_1\times]0,T[,\\
m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \ \textit{on } \gamma_2\times]0,T[,\\
\partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0 \ \textit{on } \gamma_2\times]0,T[,\\
\bar{N}_{\alpha\beta}\nu_\beta = 0 \ \textit{on } \gamma_2\times]0,T[,\\
\zeta_3\le d, f_c\le 0, f_c(\zeta_3-d) = 0 \ \textit{in } \omega\times]0,T[,\\
\zeta(.,0) = \varphi, \frac{\partial\zeta}{\partial t}(.,0) = \psi \ \textit{in } \omega.
\end{array}\right.$$

Proof.

Applying the Green formulas, we obtain

$$
\begin{aligned}
-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega &= \int_\gamma \{(\partial_\alpha m_{\alpha\beta})\nu_\beta + \partial_\tau (m_{\alpha\beta}\nu_\alpha\tau_\beta)\}\eta_3 d\gamma \\
&- \int_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma - \int_\omega (\partial_{\alpha\beta}m_{\alpha\beta})\eta_3 d\omega,
\end{aligned}
$$

$$
\begin{aligned}
\int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega &= -\int_\omega \left\{\partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right)\right\}\eta_3 d\omega \\
&+ \int_\gamma \left(\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right)\nu_\beta\eta_3 d\gamma,
\end{aligned}
$$

$$
\int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = -\int_\omega \left(\partial_\beta\bar{N}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \bar{N}_{\alpha\beta}\nu_\beta\eta_\alpha d\gamma.
$$

Then

$$
\begin{aligned}
&\int_\omega \left[2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right) - p_3\right]\eta_3 d\omega - \langle f_c,\eta_3\rangle \; - \\
&\int_\omega \left(\partial_\beta\bar{N}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \left(\bar{N}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma \; + \\
&\int_{\gamma_2}\left\{\left[\partial_\alpha m_{\alpha\beta} + \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right]\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta)\right\}\eta_3 d\gamma \; = \; 0,
\end{aligned}
$$

for all $\eta = (\eta_\alpha, \eta_3) \in V(\omega)$, with the functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by

$$
\widetilde{h}_\alpha = h_\alpha \;\text{ on }\; \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \;\text{ on }\; \gamma_2 \times [0,T].
$$

These equations imply that all the factors of η_α, η_3, and $\partial_\nu\eta_3$ vanish in their respective domains of integration. Then we get

$$
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right) = p_3 + f_c \text{ in } \omega\times\,]0,T[\,,
$$

and

$$
\partial_\beta\bar{N}_{\alpha\beta} = 0 \;\text{ in } \omega\times\,]0,T[\,,
$$

so that

$$
\partial_\beta\left(\bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\right) = \bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) \;\text{ in } \omega\times\,]0,T[\,,
$$

consequently

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta} - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3+\theta\right) = p_3 + f_c \text{ in } \omega \times \left]0,T\right[\,.$$

For boundary conditions, we get

$$\bar{N}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha = 0 \ \text{ on } \gamma \times \left]0,T\right[\,,$$

thus

$$\bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha \ \text{ on } \gamma_1 \times \left]0,T\right[\,,$$

and

$$\bar{N}_{\alpha\beta}\nu_\beta = 0 \ \text{ on } \gamma_2 \times \left]0,T\right[\,.$$

We also get

$$m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \ \text{ on } \gamma_2 \times \left]0,T\right[\,,$$

and

$$\left[\partial_\alpha m_{\alpha\beta} + \bar{N}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right]\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \ \text{ on } \gamma_2 \times \left]0,T\right[\,,$$

since $\bar{N}_{\alpha\beta}\nu_\beta = 0$ on $\gamma_2 \times]0,T[$, we conclude that

$$\partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2 \times \left]0,T\right[\,.$$

Finally, in the last inequality in $(P(\omega))^{iso}_{dyn,c}$, we take the test function $\eta_3 = d$ after that $\eta_3 = 2\zeta_3 - d$, we obtain

$$\langle f_c, d-\zeta_3\rangle \geq 0,$$

and

$$\langle f_c, \zeta_3 - d\rangle \geq 0.$$

Thus

$$\langle f_c, \zeta_3 - d\rangle = 0.$$

■

5.1.3 Dynamical contact equations of generalized Marguerre-von Kármán shallow shells

We now rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))^{iso}_{dyn,c}$ in the form of dynamical contact equations for a generalized Marguerre-von Kármán shallow shell as follows:

Theorem 5.6 *Assume that the set ω is simply-connected and that its boundary γ is sufficiently smooth. Let $\zeta = (\zeta_i)$ be a solution of $(\bar{P}(\omega))^{iso}_{dyn,c}$ with the regularity*

$\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^3(\omega)$ and $f_c \in H^{-1}(\omega)$ $\forall t \in [0,T]$.

Then

a) *The functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by*

$$\widetilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0,T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1) d\gamma = 0.$$

b) *Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$, such that*

$$\bar{N}_{11} = 2\partial_{22}\Phi,\ \bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi,\ \bar{N}_{22} = 2\partial_{11}\Phi.$$

c) *Finally, the triple $(\zeta_3, \Phi, f_c) \in (H^3(\omega) \cap \mathcal{K}) \times H^4(\omega) \times H^{-1}(\omega)$ $\forall t \in [0,T]$, satisfies the following problem*

$$
(P)^{iso}_{dyn,c}\begin{cases}
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\zeta_3) = 2\left[\Phi, \zeta_3+\theta\right] + \mathrm{p}_3 + f_c \ \ in\ \omega\times\,]0,T[\,,\\
\Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}[\zeta_3, \zeta_3+2\theta]\ \ in\ \omega\times\,]0,T[\,,\\
\zeta_3 = \partial_\nu\zeta_3 = 0\ \ on\ \gamma_1\times\,]0,T[\,,\\
m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\nu_\beta = 0\ \ on\ \gamma_2\times\,]0,T[\,,\\
\partial_\alpha m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\tau_\beta\right) = 0\ \ on\ \gamma_2\times\,]0,T[\,,\\
\Phi = \Phi_0, \partial_\nu\Phi = \Phi_1\ \ on\ \gamma\times\,]0,T[\,,\\
\zeta_3 \le d, f_c \le 0, f_c(\zeta_3-d) = 0\ \ in\ \omega\times\,]0,T[\,,\\
\zeta_3\,(.,0) = \varphi_3, \frac{\partial\zeta_3}{\partial t}\,(.,0) = \psi_3\ \ in\ \omega,
\end{cases}
$$

where

$$
\begin{cases}
-\partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\zeta_3) = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3,\\
\Phi_0(y) = -y_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma,\\
\Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma, y = (y_1,y_2)\in\gamma,\\
[\Phi,\zeta] = \partial_{11}\Phi\partial_{22}\zeta + \partial_{22}\Phi\partial_{11}\zeta - 2\partial_{12}\Phi\partial_{12}\zeta.
\end{cases}
$$

Proof.

The proof is divided into three steps.

a) The regularity of functions ζ_i imply that $\bar{N}_{\alpha\beta}\in H^2(\omega)$ and $\widetilde{h}_\alpha \in H^{\frac{3}{2}}(\gamma)$.

The functions $\widetilde{h}_\alpha$ satisfy the compatibility conditions, to see this, we observe that, if we choose $\eta = (a_1 - bx_2, a_2 - bx_1, 0)$ for any constants a_1, a_2 and b in the variational problem $(P(\omega))^{iso}_{dyn,c}$, we obtain

$$
a_\alpha\int_\gamma \widetilde{h}_\alpha d\gamma + b\int_\gamma (x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma = 0. \tag{5.29}
$$

b) Since the set ω is simply-connected and by using the generalized Poincaré theorem (see [Sch66, Theorem VI, p.59]), the equation $\partial_\beta\bar{N}_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in D'(\omega)$, unique up to the addition of constants, such that $\bar{N}_{1\alpha} = 2\partial_2\psi_\alpha,\ \bar{N}_{2\alpha} = -2\partial_1\psi_\alpha$.

Since the equation $\bar{N}_{12} = \bar{N}_{21}$ implies that $\partial_\alpha \psi_\alpha = 0$. Another application of the same result shows that there exist a distribution $\Phi \in D'(\omega)$, unique up to the addition of polynomials of degree ≤ 1, such that $\psi_1 = \partial_2 \Phi$, $\psi_2 = -\partial_1 \Phi$, so that $\bar{N}_{11} = 2\partial_{22}\Phi$, $\bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi$, $\bar{N}_{22} = 2\partial_{11}\Phi$ in ω.

The regularities of $\bar{N}_{\alpha\beta} \in H^2(\omega)$ imply that $\Phi \in H^4(\omega)$. Then Φ is uniquely defined if we impose that $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$.

c) (i) From $\bar{N}_{\alpha\beta}\nu_\beta = 2\widetilde{h}_\alpha$ on γ, we obtain

$$\begin{aligned} \widetilde{h}_1 &= \frac{1}{2}\bar{N}_{1\beta}\nu_\beta \\ &= \frac{1}{2}\left(\nu_1\bar{N}_{11} + \nu_2\bar{N}_{12}\right) \\ &= \nu_1\partial_{22}\Phi - \nu_2\partial_{21}\Phi \\ &= \partial_\tau\left(\partial_2\Phi\right), \end{aligned}$$

$$\begin{aligned} \widetilde{h}_2 &= \frac{1}{2}\bar{N}_{2\beta}\nu_\beta \\ &= \frac{1}{2}\left(\nu_1\bar{N}_{21} + \nu_2\bar{N}_{22}\right) \\ &= -\nu_1\partial_{12}\Phi + \nu_2\partial_{11}\Phi \\ &= -\partial_\tau\left(\partial_1\Phi\right), \end{aligned}$$

thus

$$\partial_1\Phi\left(y\right) = -\int_{\gamma(y)} \widetilde{h}_2 d\gamma \ \ et \ \ \partial_2\Phi\left(y\right) = \int_{\gamma(y)} \widetilde{h}_1 d\gamma, \tag{5.30}$$

for all $y \in \gamma$,

then

$$\begin{aligned} \partial_\nu\Phi\left(y\right) &= \nu_1\left(y\right)\partial_1\Phi\left(y\right) + \nu_2\left(y\right)\partial_2\Phi\left(y\right) \\ &= -\nu_1\left(y\right)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \nu_2\left(y\right)\int_{\gamma(y)} \widetilde{h}_1 d\gamma. \end{aligned} \tag{5.31}$$

So that

$$\partial_\nu\Phi(y) = \Phi_1 \text{ on } \gamma,$$

and

$$\begin{aligned}\partial_\tau \Phi(y) &= \tau_1(y)\,\partial_1\Phi(y) + \tau_2(y)\,\partial_2\Phi(y)\\ &= -\tau_1(y)\int_{\gamma(y)} \widetilde{h}_2 d\gamma + \tau_2(y)\int_{\gamma(y)} \widetilde{h}_1 d\gamma.\end{aligned} \tag{5.32}$$

Since

$$\partial_\tau \Phi(y) = \partial_\tau \Phi_0 \text{ and } \Phi(0) = \partial_\tau \Phi(0) = 0,$$

we conclude that

$$\Phi(y) = \Phi_0 \text{ on } \gamma.$$

(ii) We have

$$\begin{aligned}[\Phi, \zeta_3 + \theta] &= \partial_{11}\Phi\partial_{22}(\zeta_3+\theta) + \partial_{22}\Phi\partial_{11}(\zeta_3+\theta) - 2\partial_{12}\Phi\partial_{12}(\zeta_3+\theta)\\ &= \frac{1}{2}\left[\bar{N}_{22}\partial_{22}(\zeta_3+\theta) + \bar{N}_{11}\partial_{11}(\zeta_3+\theta) + 2\bar{N}_{12}\partial_{12}(\zeta_3+\theta)\right]\\ &= \frac{1}{2}\bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta),\end{aligned} \tag{5.33}$$

thus

$$\bar{N}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = 2[\Phi, \zeta_3+\theta]. \tag{5.34}$$

Then, we deduce

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\zeta_3) = 2[\Phi, \zeta_3+\theta] + \mathrm{p}_3 + f_c \text{ in } \omega\times]0,T[.$$

Notice that

$$\begin{aligned}\Delta^2\Phi &= \Delta(\Delta\Phi)\\ &= \Delta(\partial_{\alpha\alpha}\Phi)\\ &= \frac{1}{2}\Delta\bar{N}_{\alpha\alpha}\\ &= -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}[\zeta_3, \zeta_3+2\theta].\end{aligned} \tag{5.35}$$

■

5.2 Existence result for dynamical contact equations of generalized Marguerre-von Kármán shallow shells

Theorem 5.7 *Assume that the set ω is simply-connected and that its boundary γ is sufficiently smooth. Assume that the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ $\forall t \in [0,T]$ satisfy the compatibility conditions. Let $\chi \in H^2(\omega)$ be the unique solution in the sense of distributions of*

$$\begin{cases} \Delta^2\chi = 0 \ \textit{in}\ \omega, \\ \chi = \Phi_0 \ \textit{and}\ \partial_\nu\chi = \Phi_1 \ \textit{on}\ \gamma, \\ \Phi_0 \in H^{\frac{3}{2}}(\gamma), \Phi_1 \in H^{\frac{1}{2}}(\gamma) \end{cases} \tag{5.36}$$

and let

$$E = \frac{\mu\,(3\lambda+2\mu)}{\lambda+\mu},\ \xi = \sqrt{E}\zeta_3,\ \widetilde{\theta} = \sqrt{E}\theta,\ f = \sqrt{E}\mathrm{p}_3,\ \widetilde{f_c} = \sqrt{E}f_c,$$
$$\widetilde{d} = \sqrt{E}d,\ \widetilde{\Phi} = \Phi - \chi. \tag{5.37}$$

The triple $(\zeta_3, \Phi, f_c) \in (H^3(\omega) \cap \mathcal{K}) \times H^4(\omega) \times H^{-1}(\omega)$ $\forall t \in [0,T]$, satisfies the dynamical contact equations of generalized Marguerre-von Kármán shallow shells in the sense of distributions, if and only if, the triple $(\xi, \widetilde{\Phi}, \widetilde{f_c}) \in (H^3(\omega) \cap \widetilde{\mathcal{K}}) \times (H^4(\omega) \cap H_0^2(\omega)) \times H^{-1}(\omega)$ $\forall t \in [0,T]$, satisfies

$$(\mathcal{P})^{iso}_{dyn,c} \begin{cases} 2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\xi) = 2[\widetilde{\Phi}+\chi, \xi+\widetilde{\theta}] + f + \widetilde{f_c} \ \textit{in}\ \omega\times]0,T[, \\ \Delta^2\widetilde{\Phi} = -\frac{1}{2}[\xi, \xi+2\widetilde{\theta}] \ \textit{in}\ \omega\times]0,T[, \\ \xi = \partial_\nu\xi = 0 \ \textit{on}\ \gamma_1\times]0,T[, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \ \textit{on}\ \gamma_2\times]0,T[, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \ \textit{on}\ \gamma_2\times]0,T[, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \ \textit{on}\ \gamma\times]0,T[, \\ \xi \le \widetilde{d}, \widetilde{f_c} \le 0, \widetilde{f_c}(\xi-\widetilde{d}) = 0 \ \textit{in}\ \omega\times\,]0,T[\,, \\ \xi(.,0) = \xi_0(.) \ \textit{and}\ \frac{\partial\xi}{\partial t}(.,0) = \xi_1(.) \ \textit{in}\ \omega. \end{cases}$$

Proof.

By classical elliptic theory, there exists a unique function $\chi \in H^2(\omega)$ such that $\Delta^2\chi = 0$ in ω, $\chi = \Phi_0$ and $\partial_\nu \chi = \Phi_1$ on γ (see [Cia97, Theorem 5.6-1]).
Letting $\widetilde{\Phi} = \Phi - \chi$, we clearly have

$$\begin{cases} \Delta^2\widetilde{\Phi} = \Delta^2\Phi \text{ in } \omega \times \left]0, T\right[, \\ \widetilde{\Phi} = \partial_\nu \widetilde{\Phi} = 0 \text{ on } \gamma\times \left]0, T\right[. \end{cases}$$

Using the functions ξ, $\widetilde{\theta}$, f, $\widetilde{f_c}$, $\widetilde{d}$ and $\widetilde{\Phi}$ defined in (5.37), we then see that the scaled dynamical equations of generalized Marguerre-von Kármán shallow shells presented in Theorem 5.6 is equivalent to the scaled problem $(\mathcal{P})^{iso}_{dyn,c}$. ■

The asymptotic analysis carried out in the first part is purely formal. In what follows, we establish the existence of solutions to the dynamical contact equations of generalized Marguerre-von Kármán shallow shells. We use penalization method.

5.2.1 Penalized problem

For any $\epsilon > 0$ we define the following penalized problem, using

$$\widetilde{f_c} = -\epsilon^{-1}[\xi - \widetilde{d}]^+, \tag{5.38}$$

with $[.]^+ = \max\{., 0\}$.

$$(\mathcal{P}_\epsilon)^{iso}_{dyn,c} \begin{cases} 2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\xi) = 2[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}] + f - \epsilon^{-1}[\xi - \widetilde{d}]^+ \text{ in } \omega\times]0, T[, \\ \Delta^2\widetilde{\Phi} = -\frac{1}{2}[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega\times]0, T[, \\ \xi = \partial_\nu\xi = 0 \text{ on } \gamma_1\times]0, T[, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2\times]0, T[, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2\times]0, T[, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \text{ on } \gamma\times]0, T[, \\ \xi(., 0) = \xi_0(.) \text{ and } \frac{\partial\xi}{\partial t}(., 0) = \xi_1(.) \text{ in } \omega. \end{cases}$$

Theorem 5.8 *Assume $f \in L^2(0, T; L^2(\omega))$, $\xi_0 \in V(\omega)$ and $\xi_1 \in L^2(\omega)$. Then there exists a solution $(\xi, \widetilde{\Phi})$ to the problem $(\mathcal{P}_\epsilon)^{iso}_{dyn,c}$, such that*

$$\begin{cases} \xi \in L^\infty(0, T; V(\omega)), \\ \frac{\partial\xi}{\partial t} \in L^\infty(0, T; L^2(\omega)), \\ \widetilde{\Phi} \in L^\infty(0, T; H^2_0(\omega)). \end{cases} \tag{5.39}$$

Proof.

Denote by G_2 the inverse of Δ^2 with homogenous Dirichlet boundary condition in ω (the Green operator), and write

$$\widetilde{\Phi} = -\frac{1}{2} G_2 \left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[\,.$$

Then

$$2\rho \frac{\partial^2 \xi}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \xi\right) = 2\left[-\frac{1}{2} G_2\left[\xi, \xi + 2\widetilde{\theta}\right] + \chi, \xi + \widetilde{\theta}\right] + f - \epsilon^{-1}[\xi - \widetilde{d}]^+ \quad \text{in } \omega \times]0,T[\,.$$

From (5.39), we get

$$\left[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}\right] \in L^\infty(0,T; L^1(\omega)),$$

and, for the first equation in $(\mathcal{P}_\epsilon)^{iso}_{dyn,c}$, we have

$$\frac{\partial^2 \xi}{\partial t^2} \in L^\infty(0,T; H^{-1}(\omega)),$$

so that the initial conditions make sense.

Step 1: (Faedo-Galerkin approximation)

Let w_i, $i \geq 1$ denote an orthonormal basis of the Hilbert space $V(\omega)$ and let V_m denote, for each integer $m \geq 1$, the subspace of $V(\omega)$ spanned by the functions w_i, $1 \leq i \leq m$.

We construct the Faedo-Galerkin approximation $\xi_m(t)$ of a solution in the form

$$\xi_m(t) = \sum_{i=1}^{m} \alpha_{im}(t) w_i.$$

Thus, the function $\xi_m(t)$ is the solution of the approximate problem

$$(\mathcal{P}_m)^{iso}_{dyn,c} \begin{cases} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega - \int_\omega \partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right) w_j d\omega = \\ 2\int_\omega \left[-\frac{1}{2} G_2\left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta}\right] + \chi, \xi_m(t) + \widetilde{\theta}\right] w_j d\omega + \\ \int_\omega f w_j d\omega - \epsilon^{-1}\int_\omega [\xi_m(t) - \widetilde{d}]^+ w_j d\omega, 1 \leq j \leq m \text{ in } \omega \times]0,T[\,, \\ \xi_m(t) = \partial_\nu \xi_m(t) = 0 \text{ on } \gamma_1 \times]0,T[\,, \\ m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right)\nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \partial_\alpha m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\left(\nabla^2 \xi_m(t)\right)\nu_\alpha \tau_\beta\right) = 0 \text{ on } \gamma_2 \times]0,T[\,, \\ \xi_m(.,0) = \xi_{0m}(.) \text{ and } \frac{\partial \xi_m}{\partial t}(.,0) = \xi_{1m}(.) \text{ in } \omega, \end{cases}$$

and we have

$$\xi_{0m} \in V_m \text{ and } \xi_{0m} \to \xi_0 \text{ in } V(\omega),\ \xi_{1m} \in V_m \text{ and } \xi_{1m} \to \xi_1 \text{ in } L^2(\omega).$$

Now, define

$$\widetilde{\Phi}_m(t) = -\frac{1}{2} G_2 \left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta} \right] \text{ in } \omega \times]0,T[\,, \tag{5.40}$$

and note that

$$\Delta^2 \widetilde{\Phi}_m(t) = -\frac{1}{2} \left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta} \right] \text{ in } \omega \times]0,T[\,, \tag{5.41}$$

$$\widetilde{\Phi}_m(t) \in H_0^2(\omega), \tag{5.42}$$

so that we may rewrite the first equation of $(\mathcal{P}_m)^{iso}_{dyn,c}$ as

$$\begin{aligned} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega + a(\xi_m(t), w_j) - 2 \int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta} \right] w_j d\omega = \\ 2 \int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta} \right] w_j d\omega + \int_\omega f w_j d\omega - \epsilon^{-1} \int_\omega [\xi_m(t) - \widetilde{d}]^+ w_j d\omega \\ ,\ 1 \le j \le m \ \text{ in } \omega \times]0,T[\,, \end{aligned} \tag{5.43}$$

where the form a is defined by (2.15).

Step 2: (A priori estimates)

Multiplying by $\frac{d\alpha_{jm}(t)}{dt}$ on both sides of (5.43) and summing on the index j, we obtain

$$\begin{aligned} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} \frac{\partial \xi_m(t)}{\partial t} d\omega + a(\xi_m(t), \frac{\partial \xi_m(t)}{\partial t}) \\ -2 \int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta} \right] \frac{\partial \xi_m(t)}{\partial t} d\omega = 2 \int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta} \right] \frac{\partial \xi_m(t)}{\partial t} d\omega \\ + \int_\omega f \frac{\partial \xi_m(t)}{\partial t} d\omega - \epsilon^{-1} \int_\omega [\xi_m(t) - \widetilde{d}]^+ \frac{\partial \xi_m(t)}{\partial t} d\omega \text{ in } \omega \times]0,T[\,. \end{aligned} \tag{5.44}$$

Since we have

$$2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} \frac{\partial \xi_m(t)}{\partial t} d\omega = \rho \frac{d}{dt} \int_\omega |\frac{\partial \xi_m(t)}{\partial t}|^2 d\omega = \rho \frac{d}{dt} ||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega},$$

$$\begin{aligned} \int_\omega [\xi_m(t) - \widetilde{d}]^+ \frac{\partial \xi_m(t)}{\partial t} d\omega &= \frac{d}{2dt} \int_\omega |[\xi_m(t) - \widetilde{d}]^+|^2 d\omega \\ &= \frac{d}{2dt} ||[\xi_m(t) - \widetilde{d}]^+||^2_{0,\omega}, \end{aligned}$$

$$a(\xi_m(t), \xi_m(t)) \geq \alpha ||\xi_m(t)||^2_{V(\omega)},$$

$$a(\xi_m(t), \frac{\partial \xi_m(t)}{\partial t}) = \frac{d}{2dt} a(\xi_m(t), \xi_m(t)).$$

Using the same arguments as in the Section 2.2, we prove that

$$\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta}\right] \frac{\partial \xi_m(t)}{\partial t} d\omega = \int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right] \widetilde{\Phi}_m(t) d\omega,$$

which yields

$$-2 \int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right] \widetilde{\Phi}_m(t) d\omega = \frac{d}{dt} ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega},$$

and

$$\begin{aligned} 2 \int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta}\right] \frac{\partial \xi_m(t)}{\partial t} d\omega &= 2 \int_\omega \left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right] \chi d\omega \\ &= -2 \int_\omega \Delta^2 \frac{\partial \widetilde{\Phi}_m(t)}{\partial t} \chi d\omega \\ &= 0. \end{aligned}$$

Then (5.44) can be written as

$$\frac{d}{dt}\{\rho || \frac{\partial \xi_m(t)}{\partial t} ||^2_{0,\omega} + \frac{1}{2} a(\xi_m(t), \xi_m(t)) + ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2} ||[\xi_m(t) - \widetilde{d}]^+||^2_{0,\omega}\} = \int_\omega f \frac{\partial \xi_m(t)}{\partial t} d\omega,$$

which, by integration from 0 to t, yields

$$\int_0^t \frac{d}{d\tau}\{\rho || \frac{\partial \xi_m(\tau)}{\partial \tau} ||^2_{0,\omega} + \frac{1}{2} a(\xi_m(\tau), \xi_m(\tau)) + ||\Delta \widetilde{\Phi}_m(\tau)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2} ||[\xi_m(\tau) - \widetilde{d}]^+||^2_{0,\omega}\} d\tau = \int_0^t \{\int_\omega f \frac{\partial \xi_m(\tau)}{\partial \tau} d\omega\} d\tau.$$

Hence, there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$\rho || \frac{\partial \xi_m(t)}{\partial t} ||^2_{0,\omega} + \frac{\alpha}{2} ||\xi_m(t)||^2_{V(\omega)} + ||\Delta \widetilde{\Phi}_m(t)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2} ||[\xi_m(t) - \widetilde{d}]^+||^2_{0,\omega} \leq C_1 \int_0^t ||f||^2_{0,\omega} d\tau + C_2 \int_0^t || \frac{\partial \xi_m(\tau)}{\partial \tau} ||^2_{0,\omega} d\tau + \rho || \frac{\partial \xi_m(0)}{\partial t} ||^2_{0,\omega} + \frac{\alpha'}{2} ||\xi_m(0)||^2_{V(\omega)} + ||\Delta \widetilde{\Phi}_m(0)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2} ||[\xi_m(0) - \widetilde{d}]^+||^2_{0,\omega},$$

so that

$$\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(t)||^2_{V(\omega)} + ||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2}||[\xi_m(t) - \widetilde{d}]^+||^2_{0,\omega} \leq$$
$$C_1\int_0^t ||f||^2_{0,\omega}d\tau + C_2\int_0^t ||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega}d\tau + \rho||\xi_{1m}||^2_{0,\omega} + \frac{\alpha'}{2}||\xi_{0m}||^2_{V(\omega)} +$$
$$||\Delta\widetilde{\Phi}_m(0)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2}||[\xi_{0m} - \widetilde{d}]^+||^2_{0,\omega},$$

Since

$$\begin{aligned}\Delta^2\widetilde{\Phi}_m(0) &= -\frac{1}{2}\left[\xi_m(0), \xi_m(0) + 2\widetilde{\theta}\right] \\ &= -\frac{1}{2}\left[\xi_{0m}, \xi_{0m} + 2\widetilde{\theta}\right],\end{aligned}$$

then, there exists a constant $C_3 > 0$ such that

$$||\Delta\widetilde{\Phi}_m(0)||_{0,\omega} \leq C_3.$$

Thus, there exists a constant $C_4 > 0$ such that

$$\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(t)||^2_{V(\omega)} + ||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega} + \frac{\epsilon^{-1}}{2}||[\xi_m(t) - \widetilde{d}]^+||^2_{0,\omega} \leq$$
$$C_4 + C_2\int_0^t ||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega}d\tau, \qquad (5.45)$$

for all $t \in [0, T]$, which implies that $t_m = T$.

Then, via Gronwall's inequality, we conclude that

$$\xi_m(t) \in L^\infty(0, T; V(\omega)), \qquad (5.46)$$

$$\frac{\partial \xi_m(t)}{\partial t} \in L^\infty(0, T; L^2(\omega)), \qquad (5.47)$$

$$\widetilde{\Phi}_m(t) \in L^\infty(0, T; H^2_0(\omega)), \qquad (5.48)$$

$$\epsilon^{-1}[\xi_m(t) - \widetilde{d}]^+ \in L^\infty(0, T; L^2(\omega)). \qquad (5.49)$$

Step 3: (Passing to the limit)
From (5.46)-(5.48), we observe that there exists $\xi_n(t)$ and $\widetilde{\Phi}_n(t)$ such that (weak convergence is denoted $\rightharpoonup$)

$$\xi_n(t) \rightharpoonup \xi(t) \text{ in } L^\infty(0, T; V(\omega)) \text{ weak}*,$$

$$\frac{\partial \xi_n(t)}{\partial t} \rightharpoonup \frac{\partial \xi(t)}{\partial t} \text{ in } L^\infty(0,T;L^2(\omega)) \text{ weak}*,$$

$$\widetilde{\Phi}_n(t) \rightharpoonup \widetilde{\Phi}(t) \text{ in } L^\infty(0,T;H_0^2(\omega)) \text{ weak}*.$$

According to the Rellich-Kondrachoff theorem [LM68, Chap 1, Theorem 16.1], the compact imbedding of $H^2(\omega\times]0,T[)$ into $L^2(\omega\times]0,T[)$ implies that

$$\xi_n(t) \to \xi(t) \text{ in } L^2(\omega\times]0,T[). \tag{5.50}$$

Let ϕ_j, $1 \leq j \leq j_0$ be functions of $C^1([0,T])$ such that

$$\phi_j(T) = 0 \text{ and } \psi = \sum_{j=1}^{j_0} \phi_j \otimes w_j. \tag{5.51}$$

For $m = n > j_0$, we obtain

$$2\rho \int_\omega \frac{\partial^2 \xi_n(t)}{\partial t^2}\psi(t)d\omega + a(\xi_n(t),\psi(t)) - = 2\int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega$$
$$2\int_\omega \left[\chi, \xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega + \int_\omega f\psi(t)d\omega - \epsilon^{-1}\int_\omega [\xi_m(t)-\widetilde{d}]^+\psi(t)d\omega$$
$$\text{in } \omega\times]0,T[\,.$$

Thus,

$$2\rho\int_0^T \{\int_\omega \frac{\partial^2 \xi_n(t)}{\partial t^2}\psi(t)d\omega\}dt + \int_0^T a(\xi_n(t),\psi(t))dt$$
$$-2\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt = 2\int_0^T\{\int_\omega\left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt$$
$$+\int_0^T\{\int_\omega f\psi(t)d\omega\}dt - \epsilon^{-1}\int_0^T\{\int_\omega[\xi_m(t)-\widetilde{d}]^+\psi(t)d\omega\}dt \text{ in } \omega\times]0,T[\,,$$

and we have

$$\int_0^T\{\int_\omega \frac{\partial^2\xi_n(t)}{\partial t^2}\psi(t)d\omega\}dt = -\int_0^T\{\int_\omega \frac{\partial \xi_n(t)}{\partial t}\frac{\partial\psi(t)}{\partial t}d\omega\}dt$$
$$+\int_\omega \frac{\partial\xi_n(T)}{\partial t}\psi(T)d\omega - \int_\omega\frac{\partial\xi_n(0)}{\partial t}\psi(0)d\omega = -\int_0^T\{\int_\omega\frac{\partial\xi_n(t)}{\partial t}\frac{\partial\psi(t)}{\partial t}d\omega\}dt$$
$$-\int_\omega \xi_{1n}\psi(0)d\omega.$$

Since $\psi(T)=0$, we also obtain

$$-2\rho\int_0^T\{\int_\omega \frac{\partial \xi_n(t)}{\partial t}\frac{\partial \psi(t)}{\partial t}d\omega\}dt+\int_0^T a(\xi_n(t),\psi(t))dt \quad - \quad 2\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt \quad = \quad 2\int_0^T\{\int_\omega \left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt+\int_0^T\{\int_\omega f\psi(t)d\omega\}dt \quad - \quad \epsilon^{-1}\int_0^T\{\int_\omega[\xi_m(t)-\widetilde{d}]^+\psi(t)d\omega\}dt+2\rho\int_\omega \xi_{1n}\psi(0)d\omega \text{ in } \omega\times]0,T[. \tag{5.52}$$

From (5.42), we get

$$\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt=\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\psi(t)\right](\xi_n(t)+\widetilde{\theta})d\omega\}dt,$$

and we have

$$[\widetilde{\Phi}_n(t),\psi(t)]\rightharpoonup[\widetilde{\Phi}(t),\psi(t)] \text{ in } L^2(\omega\times]0,T[).$$

Then, because $\xi_n(t)\to\xi(t)$ in $L^2(\omega\times]0,T[)$, we obtain

$$\begin{aligned}\int_0^T\{\int_\omega \left[\widetilde{\Phi}_n(t),\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt &\to \int_0^T\{\int_\omega \left[\widetilde{\Phi}(t),\psi(t)\right](\xi(t)+\widetilde{\theta})d\omega\}dt\\ &= \int_0^T\{\int_\omega \left[\widetilde{\Phi}(t),\xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt.\end{aligned}$$

We have

$$[\chi,\xi_n(t)+\widetilde{\theta}]\rightharpoonup[\chi,\xi(t)+\widetilde{\theta}] \text{ in } L^2(\omega\times]0,T[),$$

thus

$$\int_0^T\{\int_\omega \left[\chi,\xi_n(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt \;\to\; \int_0^T\{\int_\omega \left[\chi,\xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt.$$

Then passing to the limit in (5.52), we obtain

$$-2\rho\int_0^T\{\int_\omega \frac{\partial \xi(t)}{\partial t}\frac{\partial \psi(t)}{\partial t}d\omega\}dt+\int_0^T a(\xi(t),\psi(t))dt \quad - \quad 2\int_0^T\{\int_\omega \left[\widetilde{\Phi}(t),\xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt \quad = \quad 2\int_0^T\{\int_\omega \left[\chi,\xi(t)+\widetilde{\theta}\right]\psi(t)d\omega\}dt+\int_0^T\{\int_\omega f\psi(t)d\omega\}dt \quad - \quad \epsilon^{-1}\int_0^T\{\int_\omega[\xi_m(t)-\widetilde{d}]^+\psi(t)d\omega+2\rho\int_\omega \xi_1\psi(0)d\omega \text{ in } \omega\times]0,T[, \tag{5.53}$$

for all ψ of the form (5.51).

Passing to the limit, we deduce that (5.53) holds for all $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ and $\psi(T) = 0$. Using the density of functions of the form (5.51) in the space of functions $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ with $\psi(T) = 0$ see [DL72, LM68].

Then $(\xi, \widetilde{\Phi})$ satisfies

$$2\rho \frac{\partial^2 \xi}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 \xi) = 2[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}] + f - \epsilon^{-1}[\xi - \widetilde{d}]^+ \text{ in } \omega \times]0,T[,$$

and

$$\frac{\partial \xi}{\partial t}(0) = \xi_1.$$

Taking into account (5.46) and (5.47), and applying [Lio69, Lemma 1.2], we deduce that

$$\xi_n(0) \rightharpoonup \xi(0) \text{ in } L^2(\omega),$$

and we obtain

$$\xi_n(0) = \xi_{0n} \rightarrow \xi_0 \text{ in } V(\omega),$$

with the consequence that

$$\xi(0) = \xi_0.$$

Finally, the same arguments as in Section 2.2, show that

$$\Delta^2 \widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[\,,$$

the proof of the existence of a solution to the penalized problem is complete. ■

5.2.2 Existence of solutions to the dynamical contact equations of generalized Marguerre-von Kármán shallow shells

Theorem 5.9 *Assume $f \in L^2(0,T;L^2(\omega))$, $\xi_0 \in V(\omega)$ and $\xi_1 \in L^2(\omega)$. Then there exists a solution $(\xi, \widetilde{\Phi}, \widetilde{f_c})$ to the problem $(\mathcal{P})^{iso}_{dyn,c}$, such that*

$$\begin{cases} \xi \in L^\infty(0,T;V(\omega)), \\ \frac{\partial \xi}{\partial t} \in L^\infty(0,T;L^2(\omega)), \\ \widetilde{\Phi} \in L^\infty(0,T;H_0^2(\omega)), \\ \widetilde{f_c} \in L^\infty(0,T;H^{-1}(\omega)). \end{cases} \tag{5.54}$$

Proof. For ξ_ϵ a solution of the penalized problem, we obtain

$$2\rho \frac{\partial^2 \xi_\epsilon(t)}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \xi_\epsilon(t)\right) - 2\left[-\frac{1}{2}G_2\left[\xi_\epsilon(t), \xi_\epsilon(t) + 2\widetilde{\theta}\right] + \chi, \xi_\epsilon(t) + \widetilde{\theta}\right] + \epsilon^{-1}[\xi_\epsilon(t) - \widetilde{d}]^+ = f \text{ in } \omega \times]0,T[,$$

where

$$\widetilde{\Phi}_\epsilon(t) = -\frac{1}{2}G_2\left[\xi_\epsilon(t), \xi_\epsilon(t) + 2\widetilde{\theta}\right] \text{ in } \omega \times]0,T[,$$
$$\widetilde{\Phi}_\epsilon(t) \in H_0^2(\omega).$$

Then, we get the following variational formulation

$$\int_0^T \{2\rho \int_\omega \frac{\partial^2 \xi_\epsilon(t)}{\partial t^2} z d\omega + a(\xi_\epsilon(t), z) - 2\int_\omega \left[-\frac{1}{2}G_2\left[\xi_\epsilon(t), \xi_\epsilon(t) + 2\widetilde{\theta}\right] + \chi, \xi_\epsilon(t) + \widetilde{\theta}\right] z d\omega + \int_\omega \epsilon^{-1}[\xi_\epsilon(t) - \widetilde{d}]^+ z d\omega\} dt = \int_0^T \{\int_\omega f z d\omega\} dt. \tag{5.55}$$

We put $z = -1$ with $z \in H_0^2(\omega)$ in (5.55) and by using [Cia97, Theorem 5.8-2], we have

$$\int_0^T \{2\rho \int_\omega \frac{\partial^2 \xi_\epsilon(t)}{\partial t^2} z d\omega + a(\xi_\epsilon(t), z) - 2\int_\omega \left[-\frac{1}{2}G_2\left[\xi_\epsilon(t), \xi_\epsilon(t) + 2\widetilde{\theta}\right] + \chi, z\right] (\xi_\epsilon(t) + \widetilde{\theta}) d\omega + \int_\omega \epsilon^{-1}[\xi_\epsilon(t) - \widetilde{d}]^+ z d\omega\} dt = \int_0^T \{\int_\omega f z d\omega\} dt.$$

Which yields

$$\begin{aligned} 0 &\leq \int_0^T \{\int_\omega \epsilon^{-1}[\xi_\epsilon(t) - \widetilde{d}]^+ d\omega\} dt \\ &= 2\rho \int_\omega (\xi_1 - \frac{\partial \xi_\epsilon(T)}{\partial t}) d\omega + \int_0^T \{\int_\omega f d\omega\} dt \\ &\leq C_5, \end{aligned} \tag{5.56}$$

where C_5 is independent of ϵ.

The estimates analogous to (5.45), we have

$$\rho || \frac{\partial \xi_\epsilon(t)}{\partial t} ||_{0,\omega}^2 + \frac{\alpha}{2} ||\xi_\epsilon(t)||_{V(\omega)}^2 + ||\Delta \widetilde{\Phi}_\epsilon(t)||_{0,\omega}^2 + \frac{\epsilon^{-1}}{2} ||[\xi_\epsilon(t) - \widetilde{d}]^+||_{0,\omega}^2 \leq C_4 + C_2 \int_0^t || \frac{\partial \xi_\epsilon(\tau)}{\partial \tau} ||_{0,\omega}^2 d\tau. \tag{5.57}$$

Then, we conclude that

$$\xi_\epsilon(t) \in L^\infty(0, T; V(\omega)), \tag{5.58}$$

$$\frac{\partial \xi_\epsilon(t)}{\partial t} \in L^\infty(0, T; L^2(\omega)), \tag{5.59}$$

$$\widetilde{\Phi}_\epsilon(t) \in L^\infty(0, T; H_0^2(\omega)), \tag{5.60}$$

$$\epsilon^{-1}[\xi_\epsilon(t) - \widetilde{d}]^+ \in L^\infty(0, T; L^2(\omega)), \tag{5.61}$$

$$\frac{\partial^2 \xi_\epsilon(t)}{\partial t^2} \in L^\infty(0, T; H^{-1}(\omega)), \tag{5.62}$$

where the dual estimate of the acceleration term has the form

$$|| \frac{\partial^2 \xi_\epsilon(t)}{\partial t^2} ||_{L^\infty(0,T;H^{-1}(\omega))} \leq C_\epsilon. \tag{5.63}$$

From (5.58)-(5.62), we observe that there exists a sequence $\epsilon_n \to 0$, $\xi_{\epsilon_n}(t)$ and $\widetilde{\Phi}_{\epsilon_n}(t)$ such that

$$\xi_{\epsilon_n}(t) \rightharpoonup \xi(t) \text{ in } L^\infty(0, T; V(\omega)) \text{ weak}*, \tag{5.64}$$

$$\frac{\partial \xi_{\epsilon_n}(t)}{\partial t} \rightharpoonup \frac{\partial \xi(t)}{\partial t} \text{ in } L^\infty(0, T; L^2(\omega)) \text{ weak}*, \tag{5.65}$$

$$\widetilde{\Phi}_{\epsilon_n}(t) \rightharpoonup \widetilde{\Phi}(t) \text{ in } L^\infty(0, T; H_0^2(\omega)) \text{ weak}*, \tag{5.66}$$

$$-\epsilon_n^{-1}[\xi_{\epsilon_n}(t) - \widetilde{d}]^+ \rightharpoonup \widetilde{f}_c \text{ in } L^\infty(0, T; H^{-1}(\omega)) \text{ weak}*, \tag{5.67}$$

$$\frac{\partial^2 \xi_{\epsilon_n}(t)}{\partial t^2} \rightharpoonup \frac{\partial^2 \xi(t)}{\partial t^2} \text{ in } L^\infty(0,T;H^{-1}(\omega)) \text{ weak} * . \tag{5.68}$$

According to the Rellich-Kondrachoff theorem, the compact imbedding of $H^2(\omega\times]0,T[)$ into $L^2(\omega\times]0,T[)$ implies that

$$\xi_{\epsilon_n}(t) \to \xi(t) \text{ in } L^2(\omega\times]0,T[). \tag{5.69}$$

For $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ and $\psi(T)=0$, we have

$$2\rho\int_\omega \frac{\partial^2 \xi_{\epsilon_n}(t)}{\partial t^2}\psi(t)d\omega + a(\xi_{\epsilon_n}(t),\psi(t)) - 2\int_\omega \left[\widetilde{\Phi}_{\epsilon_n}(t), \xi_{\epsilon_n}(t)+\widetilde{\theta}\right]\psi(t)d\omega =$$
$$2\int_\omega \left[\chi, \xi_{\epsilon_n}(t)+\widetilde{\theta}\right]\psi(t)d\omega + \int_\omega f\psi(t)d\omega - \epsilon^{-1}\int_\omega [\xi_{\epsilon_n}(t)-\widetilde{d}]^+\psi(t)d\omega \text{ in } \omega\times]0,T[\,.$$

Using the similar approach as in the penalized problem, we prove that

$$2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\xi) = 2[\widetilde{\Phi}+\chi, \xi+\widetilde{\theta}] + f + \widetilde{f}_c \text{ in } \omega\times]0,T[,$$

$$\Delta^2\widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi+2\widetilde{\theta}\right] \text{ in } \omega\times]0,T[\,,$$

$$\xi(0)=\xi_0,\ \frac{\partial\xi}{\partial t}(0)=\xi_1.$$

From the estimate (5.56) and the convergence (5.69), we obtain $\xi-\widetilde{d}\leq 0$ in $\omega\times]0,T[$.

Then, the convergence (5.67) yields that $\widetilde{f}_c \leq 0$ in $\omega\times]0,T[$ in the dual sense.

From (5.61) and since $[\xi_{\epsilon_n}(t)-\widetilde{d}]^+ \to 0$ in $L^2(\omega\times]0,T[)$, we get

$$\begin{aligned}\langle \widetilde{f}_c, \xi-\widetilde{d}\rangle &\to -\epsilon_n^{-1}\langle[\xi_{\epsilon_n}(t)-\widetilde{d}]^+, [\xi_{\epsilon_n}(t)-\widetilde{d}]^+\rangle \\ &= 0.\end{aligned}$$

■

5.3 Conclusion

An application of the technics from formal asymptotic analysis to the three-dimensional dynamical model for a Signorini problem with Coulomb friction of nonlinearly elastic shallow shell with a specific class of boundary conditions of generalized Marguerre-von

Kármán type, made of homogeneous isotropic material, shows that the leading term of the expansion is characterized by a two-dimensional frictionless dynamical contact boundary value problem called the dynamical contact equations of generalized Marguerre-von Kármán shallow shells, which depends on the Airy function Φ, the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell and contact force f_c.

The application of the penalization method to the dynamical contact equations of generalized Marguerre-von Kármán shallow shells, shows that there exists a solution to these equations.

Chapter 6

Asymptotic analysis of elastodynamic Signorini problem with Coulomb friction of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shell

In this Chapter, we extend formally the study of the fifth Chapter to nonhomogeneous anisotropic material. More precisely, we considered a three-dimensional dynamical model for a Signorini problem with Coulomb friction of nonlinearly elastic shallow shell with a specific class of boundary conditions of generalized Marguerre-von Kármán type, made of a general nonhomogeneous anisotropic material.

6.1 Setting of the problem

Consider a nonlinearly elastodynamic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε. We assume that the elastic material constituting the shell is nonhomogeneous and anisotropic, and that the reference configuration is a natural state.

The shell is subjected to vertical body forces of density $(\hat{f}_i^\varepsilon) = (0, 0, \hat{f}_3^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$, its lower face $\hat{\Gamma}_-^\varepsilon$ subjected to a surface forces of density $(\hat{g}_i^\varepsilon) = (0, 0, \hat{g}_3^\varepsilon)$. We suppose also that this shell is in unilateral contact with Coulomb friction at the upper face $\hat{\Gamma}_+^\varepsilon$ and Λ its frictional coefficient, such that $\hat{d}^\varepsilon$ is the gap function which describes the distance between the upper face and the rigid foundation measured in the normal direction. We suppose that $\hat{d}^\varepsilon \in L^\infty(\hat{\Gamma}_+^\varepsilon)$, $\hat{d}^\varepsilon \geq 0$ and independent of time t. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kármán type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

The unknowns displacement field $\hat{\mathbf{u}}^\varepsilon = (\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon, t)$, stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)(\hat{x}^\varepsilon, t)$ and the contact force $\hat{\mathbf{G}}^\varepsilon$ satisfies the following three-dimensional boundary value problem in Cartesian coordinates:

$$(C.\hat{P}^\varepsilon)^{anis}_{dyn,c} \begin{cases} \hat{\rho}^\varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} - \hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \ \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[, \\ \begin{cases} \hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon = 0 \text{ on } \mathbf{\Theta}^\varepsilon\,(\gamma_1 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[, \\ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_\alpha^\varepsilon) \circ \mathbf{\Theta}^\varepsilon\} \nu_\beta dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \gamma_1 \times]0, +\infty[, \end{cases} \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) \hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = 0 \text{ on } (\gamma_2 \times [-\varepsilon, \varepsilon]) \times]0, +\infty[, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) \hat{n}_j^\varepsilon \circ \mathbf{\Theta}^\varepsilon = \hat{g}_i^\varepsilon \circ \mathbf{\Theta}^\varepsilon \text{ on } \Gamma_-^\varepsilon \times]0, +\infty[, \\ (A\hat{\sigma}^\varepsilon)_{ij} = \hat{\gamma}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) + \frac{1}{2} \hat{\partial}_i^\varepsilon \hat{u}_l^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_l^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \times]0, +\infty[, \\ \hat{u}_N^\varepsilon \leq \hat{d}^\varepsilon, \hat{G}_N^\varepsilon \leq 0, \hat{G}_N^\varepsilon(\hat{u}_N^\varepsilon - \hat{d}^\varepsilon) = 0 \ on \ \hat{\Gamma}_+^\varepsilon \times]0, +\infty[, \\ |\hat{\mathbf{G}}_T^\varepsilon| < \Lambda |\hat{G}_N^\varepsilon| \Rightarrow \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = 0 \ on \ \hat{\Gamma}_+^\varepsilon \times]0, +\infty[, \\ |\hat{\mathbf{G}}_T^\varepsilon| = \Lambda |\hat{G}_N^\varepsilon| \Rightarrow \exists \delta > 0, \ \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = -\delta \hat{G}_T^\varepsilon \ on \ \hat{\Gamma}_+^\varepsilon \times]0, +\infty[, \\ \hat{\mathbf{u}}^\varepsilon\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{p}}^\varepsilon, \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}\,(\hat{x}^\varepsilon, 0) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \end{cases}$$

where

$$\begin{cases} \hat{\gamma}_{ij}^\varepsilon(\hat{\mathbf{u}}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon), \\ \hat{u}_N^\varepsilon = \hat{\mathbf{u}}^\varepsilon \hat{\mathbf{n}}^\varepsilon, \hat{\mathbf{u}}_T^\varepsilon = \hat{\mathbf{u}}^\varepsilon - \hat{u}_N^\varepsilon \hat{\mathbf{n}}^\varepsilon, \\ \frac{\partial \hat{u}_N^\varepsilon}{\partial t} = \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t} \hat{\mathbf{n}}^\varepsilon, \ \frac{\partial \hat{\mathbf{u}}_T^\varepsilon}{\partial t} = \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t} - \frac{\partial \hat{u}_N^\varepsilon}{\partial t} \hat{\mathbf{n}}^\varepsilon, \\ \hat{G}_N^\varepsilon = \hat{\mathbf{G}}^\varepsilon \hat{\mathbf{n}}^\varepsilon, \hat{\mathbf{G}}_T^\varepsilon = \hat{\mathbf{G}}^\varepsilon - \hat{G}_N^\varepsilon \hat{\mathbf{n}}^\varepsilon, \\ \hat{\mathbf{p}}^\varepsilon, \hat{\mathbf{q}}^\varepsilon : \text{ the given initial data}, \\ \hat{\rho}^\varepsilon : \text{ the mass density}. \end{cases} \tag{6.1}$$

The mapping A is defined by

$$(A\hat{\sigma}^\varepsilon)_{ij} = \hat{c}_{ijkl}^\varepsilon \hat{\sigma}_{kl}^\varepsilon,$$

where $\hat{C}^\varepsilon = (\hat{c}^\varepsilon_{ijkl})$ is the compliance tensor, we suppose that the associated rigidity tensor $\hat{A}^\varepsilon = (\hat{a}^\varepsilon_{ijkl})$ satisfy the following conditions

$$\begin{cases} \hat{a}^\varepsilon_{ijkl}(\hat{x}^\varepsilon) \in L^\infty(\hat{\Omega}^\varepsilon), \\ \hat{a}^\varepsilon_{ijkl} = \hat{a}^\varepsilon_{jikl} = \hat{a}^\varepsilon_{klij} = \hat{a}^\varepsilon_{klji} \\ \exists c > 0,\ \hat{a}^\varepsilon_{ijkl}\hat{\tau}^\varepsilon_{kl}\hat{\tau}^\varepsilon_{ij} \geq c\hat{\tau}^\varepsilon_{ij}\hat{\tau}^\varepsilon_{ij},\ \hat{\tau}^\varepsilon_{ij} = \hat{\tau}^\varepsilon_{ji}. \end{cases}$$

We suppose that $\hat{\Gamma}^\varepsilon_+$ smooth enough, such that the relations (5.2)-(5.9) are satisfied.

First, we rewrite the above boundary value problem $(C.\hat{P}^\varepsilon)^{anis}_{dyn,c}$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem:

$$(V.\hat{P}^\varepsilon)^{anis}_{dyn,c} \begin{cases} \text{Find } (\hat{\mathbf{u}}^\varepsilon, \hat{\sigma}^\varepsilon, \hat{G}^\varepsilon_N, \hat{\mathbf{G}}^\varepsilon_T) \in \mathcal{K}(\hat{\Omega}^\varepsilon) \times \Sigma(\hat{\Omega}^\varepsilon) \times W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^\varepsilon_+) \times \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^\varepsilon_+) \\ \ \forall t \geq 0,\ \text{such that,} \\ \hat{D}^{\varepsilon,t}\left(\hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon\right) + \hat{B}^{\varepsilon,\theta}\left(\sigma\left(\varepsilon\right), \hat{\mathbf{v}}^\varepsilon\right) + 2\hat{C}^{\varepsilon,\theta}\left(\sigma\left(\varepsilon\right), \hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon\right) = \hat{F}^\varepsilon\left(\hat{\mathbf{v}}^\varepsilon\right) \\ +\langle \hat{G}^\varepsilon_N, \hat{v}^\varepsilon_N\rangle + \langle \hat{\mathbf{G}}^\varepsilon_T, \hat{\mathbf{v}}^\varepsilon_T\rangle,\ \forall \hat{\mathbf{v}}^\varepsilon \in \mathcal{V}(\hat{\Omega}^\varepsilon), \forall t > 0, \\ \int_{\hat{\Omega}^\varepsilon} (A\hat{\sigma}^\varepsilon)_{ij}\hat{\tau}^\varepsilon_{ij} d\hat{x}^\varepsilon - \int_{\hat{\Omega}^\varepsilon} \hat{\tau}^\varepsilon_{ij}\hat{\gamma}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon) d\hat{x}^\varepsilon - \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\tau}^\varepsilon_{ij}\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l d\hat{x}^\varepsilon = 0, \\ \forall \hat{\tau}^\varepsilon \in \Sigma(\hat{\Omega}^\varepsilon), \forall t > 0, \\ \langle \hat{G}^\varepsilon_N, \hat{v}^\varepsilon_N - \hat{u}^\varepsilon_N\rangle \geq 0\,,\ \forall \hat{\mathbf{v}}^\varepsilon \in \mathcal{K}(\hat{\Omega}^\varepsilon), \forall t > 0, \\ \langle \hat{\mathbf{G}}^\varepsilon_T, \hat{\mathbf{v}}^\varepsilon_T - \frac{\partial \hat{\mathbf{u}}^\varepsilon_T}{\partial t}\rangle + \langle \Lambda \left|\hat{G}^\varepsilon_N\right|, |\hat{\mathbf{v}}^\varepsilon_T| - \left|\frac{\partial \hat{\mathbf{u}}^\varepsilon_T}{\partial t}\right|\rangle \geq 0,\ \forall\, \hat{\mathbf{v}}^\varepsilon \in \mathcal{V}(\hat{\Omega}^\varepsilon), \forall t > 0, \\ \hat{\mathbf{u}}^\varepsilon\left(\hat{x}^\varepsilon, 0\right) = \hat{\mathbf{p}}^\varepsilon,\ \frac{\partial \hat{\mathbf{u}}^\varepsilon}{\partial t}\left(\hat{x}^\varepsilon, 0\right) = \hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \end{cases}$$

where

$$\begin{cases} \hat{D}^{\varepsilon,t}\left(\hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon\right) = \frac{d^2}{dt^2}\left\{\hat{\rho}^\varepsilon \int_{\hat{\Omega}^\varepsilon} \hat{u}^\varepsilon_i \hat{v}^\varepsilon_i d\hat{x}^\varepsilon\right\}, \\ \hat{B}^{\varepsilon,\theta}\left(\sigma\left(\varepsilon\right), \hat{\mathbf{v}}^\varepsilon\right) = \int_{\hat{\Omega}^\varepsilon} \hat{\sigma}^\varepsilon_{ij}\hat{\gamma}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) d\hat{x}^\varepsilon, \\ \hat{C}^{\varepsilon,\theta}\left(\sigma\left(\varepsilon\right), \hat{\mathbf{u}}^\varepsilon, \hat{\mathbf{v}}^\varepsilon\right) = \frac{1}{2}\int_{\hat{\Omega}^\varepsilon} \hat{\sigma}^\varepsilon_{ij}\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_l\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_l d\hat{x}^\varepsilon, \\ \hat{F}^\varepsilon\left(\hat{\mathbf{v}}^\varepsilon\right) = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_3\hat{v}^\varepsilon_3 d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_-} \hat{g}^\varepsilon_3\hat{v}^\varepsilon_3 d\hat{\Gamma}^\varepsilon + \int_{\hat{\gamma}^\varepsilon_1}\{\int_{-\varepsilon}^{\varepsilon}\left(\hat{v}^\varepsilon_\alpha \circ \mathbf{\Theta}^\varepsilon\right) dx^\varepsilon_3\}\hat{h}^\varepsilon_\alpha d\hat{\gamma}^\varepsilon. \end{cases}$$

$\langle .,. \rangle$ is the duality pairing between $W^{-\frac{3}{4},\frac{4}{3}}(\hat{\Gamma}^\varepsilon_+)$ and $W^{\frac{3}{4},4}(\hat{\Gamma}^\varepsilon_+)$ defined by (5.10).

In order to transform the problem $(V.\hat{P}^\varepsilon)^{anis}_{dyn,c}$ into problem posed over the cylindrical domain Ω^ε, we use the one to one mapping $(\mathbf{\Theta}^\varepsilon)^{-1}$ and the relations (2.3).

Let there be a given C^1-diffeomorphism $\mathbf{\Theta}^\varepsilon$ that satisfies the orientation-preserving condition. Then the variational problem $(V.\hat{P}^\varepsilon)^{anis}_{dyn,c}$ is equivalent to the following variational problem:

$$(P^{\varepsilon})^{anis}_{dyn,c}\left\{\begin{array}{l}\text{Find } (\mathbf{u}^{\varepsilon},\sigma^{\varepsilon},G^{\varepsilon}_N,\mathbf{G}^{\varepsilon}_T)\in\mathcal{K}(\Omega^{\varepsilon})\times\Sigma(\Omega^{\varepsilon})\times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^{\varepsilon}_+)\times\mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma^{\varepsilon}_+)\\ \forall t\geq 0,\ \text{such that,}\\ D^{\varepsilon,t}(\mathbf{u}^{\varepsilon},\mathbf{v}^{\varepsilon})+B^{\varepsilon,\theta}(\sigma(\varepsilon),\mathbf{v}^{\varepsilon})+2C^{\varepsilon,\theta}(\sigma(\varepsilon),\mathbf{u}^{\varepsilon},\mathbf{v}^{\varepsilon})=F^{\varepsilon}(\mathbf{v}^{\varepsilon})\\ +\langle G^{\varepsilon}_N,v^{\varepsilon}_N\rangle+\langle\mathbf{G}^{\varepsilon}_T,\mathbf{v}^{\varepsilon}_T\rangle,\forall\mathbf{v}^{\varepsilon}\in\mathcal{V}(\Omega^{\varepsilon}),\forall t>0,\\ \int_{\Omega^{\varepsilon}}(A\sigma^{\varepsilon})_{ij}\tau^{\varepsilon}_{ij}\delta^{\varepsilon}dx^{\varepsilon}-\int_{\Omega^{\varepsilon}}\tau^{\varepsilon}_{ij}b^{\varepsilon}_{kj}\partial^{\varepsilon}_k u^{\varepsilon}_i\delta^{\varepsilon}dx^{\varepsilon}\\ -\frac{1}{2}\int_{\Omega^{\varepsilon}}\tau^{\varepsilon}_{ij}b^{\varepsilon}_{ki}\partial^{\varepsilon}_k u^{\varepsilon}_l b^{\varepsilon}_{mj}\partial^{\varepsilon}_m u^{\varepsilon}_l\delta^{\varepsilon}dx^{\varepsilon}=0\\ \forall\tau^{\varepsilon}\in\Sigma(\Omega^{\varepsilon}),\forall t>0,\\ \langle G^{\varepsilon}_N,v^{\varepsilon}_N-u^{\varepsilon}_N\rangle\geq 0\,,\ \forall\mathbf{v}^{\varepsilon}\in\mathcal{K}(\Omega^{\varepsilon}),\forall t>0,\\ \langle\mathbf{G}^{\varepsilon}_T,\mathbf{v}^{\varepsilon}_T-\frac{\partial\mathbf{u}^{\varepsilon}_T}{\partial t}\rangle+\langle\Lambda\left|G^{\varepsilon}_N\right|,\left|\mathbf{v}^{\varepsilon}_T\right|-\left|\frac{\partial\mathbf{u}^{\varepsilon}_T}{\partial t}\right|\rangle\geq 0,\ \forall\,\mathbf{v}^{\varepsilon}\in\mathcal{V}(\Omega^{\varepsilon}),\forall t>0,\\ \mathbf{u}^{\varepsilon}(x^{\varepsilon},0)=\mathbf{p}^{\varepsilon},\ \frac{\partial\mathbf{u}^{\varepsilon}}{\partial t}(x^{\varepsilon},0)=\mathbf{q}^{\varepsilon}\ \text{in } \Omega^{\varepsilon},\end{array}\right.$$

where

$$\left\{\begin{array}{l}D^{\varepsilon,t}(\mathbf{u}^{\varepsilon},\mathbf{v}^{\varepsilon})=\frac{d^2}{dt^2}\left\{\rho^{\varepsilon}\int_{\Omega^{\varepsilon}}u^{\varepsilon}_i v^{\varepsilon}_i\delta^{\varepsilon}dx^{\varepsilon}\right\},\\ B^{\varepsilon,\theta}(\sigma(\varepsilon),\mathbf{v}^{\varepsilon})=+\int_{\Omega^{\varepsilon}}\sigma^{\varepsilon}_{ij}b^{\varepsilon}_{kj}\partial^{\varepsilon}_k v^{\varepsilon}_i\delta^{\varepsilon}dx^{\varepsilon},\\ C^{\varepsilon,\theta}(\sigma(\varepsilon),\mathbf{u}^{\varepsilon},\mathbf{v}^{\varepsilon})=\frac{1}{2}\int_{\Omega^{\varepsilon}}\sigma^{\varepsilon}_{ij}b^{\varepsilon}_{ki}\partial^{\varepsilon}_k u^{\varepsilon}_l b^{\varepsilon}_{mj}\partial^{\varepsilon}_m v^{\varepsilon}_l\delta^{\varepsilon}dx^{\varepsilon},\\ F^{\varepsilon}(\mathbf{v}^{\varepsilon})=\int_{\Omega^{\varepsilon}}f^{\varepsilon}_3 v^{\varepsilon}_3\delta^{\varepsilon}dx^{\varepsilon}+\int_{\Gamma^{\varepsilon}_-}g^{\varepsilon}_3 v^{\varepsilon}_3\delta^{\varepsilon}\beta^{\varepsilon}d\Gamma^{\varepsilon}\\ +\int_{\gamma_1}h^{\varepsilon}_{\alpha}\{\int_{-\varepsilon}^{\varepsilon}v^{\varepsilon}_{\alpha}dx^{\varepsilon}_3\}d\gamma,\end{array}\right.$$

such that

$$\left\{\begin{array}{l}u^{\varepsilon}_i=\hat{u}^{\varepsilon}_i\circ\mathbf{\Theta}^{\varepsilon},\ \ v^{\varepsilon}_i=\hat{v}^{\varepsilon}_i\circ\mathbf{\Theta}^{\varepsilon},\ \sigma^{\varepsilon}_{ij}=\hat{\sigma}^{\varepsilon}_{ij}\circ\mathbf{\Theta}^{\varepsilon},\ \tau^{\varepsilon}_{ij}=\hat{\tau}^{\varepsilon}_{ij}\circ\mathbf{\Theta}^{\varepsilon},\\ (A\sigma^{\varepsilon})_{ij}=(A\hat{\sigma}^{\varepsilon})_{ij}\circ\mathbf{\Theta}^{\varepsilon},\ c^{\varepsilon}_{ijkl}=\hat{c}^{\varepsilon}_{ijkl}\circ\mathbf{\Theta}^{\varepsilon},\\ f^{\varepsilon}_3=\hat{f}^{\varepsilon}_3\circ\mathbf{\Theta}^{\varepsilon},\ g^{\varepsilon}_3=\hat{g}^{\varepsilon}_3\circ\mathbf{\Theta}^{\varepsilon},\ h^{\varepsilon}_{\alpha}=\hat{h}^{\varepsilon}_{\alpha}\circ\mathbf{\Theta}^{\varepsilon},\\ p^{\varepsilon}_i=\hat{p}^{\varepsilon}_i\circ\mathbf{\Theta}^{\varepsilon},\ q^{\varepsilon}_i=\hat{q}^{\varepsilon}_i\circ\mathbf{\Theta}^{\varepsilon},\\ d^{\varepsilon}=\hat{d}^{\varepsilon}\circ\mathbf{\Theta}^{\varepsilon}.\end{array}\right.$$

$\langle.,.\rangle$ is the duality pairing between $W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^{\varepsilon}_+)$ and $W^{\frac{3}{4},4}(\Gamma^{\varepsilon}_+)$ defined by (5.11).

6.2 Asymptotic analysis

6.2.1 The scaled three-dimensional problem

We use the same arguments as in Chapter 3 to transform $(P^{\varepsilon})^{iso}_{dyn,c}$ into a problem posed over an open set independent of ε.

First, to the functions $\mathbf{u}^{\varepsilon}$, $\mathbf{v}^{\varepsilon}\in\mathcal{V}(\Omega^{\varepsilon})$, σ^{ε}, $\tau^{\varepsilon}\in\Sigma(\Omega^{\varepsilon})$, $G^{\varepsilon}_N\in W^{-\frac{3}{4},\frac{4}{3}}(\Gamma^{\varepsilon}_+)$ and $\mathbf{G}^{\varepsilon}_T\in\mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma^{\varepsilon}_+)$, we associate the scaled functions $\mathbf{u}(\varepsilon)$, $\mathbf{v}\in\mathcal{V}(\Omega)$, $\sigma(\varepsilon)$, $\tau\in\Sigma(\Omega)$,

$G_N(\varepsilon) \in W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)$ and $\mathbf{G}_T(\varepsilon) = (\mathbf{G}_{T_i}(\varepsilon)) \in \mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)$ defined by

$$\begin{cases} u^\varepsilon_\alpha(x^\varepsilon, t) = \varepsilon^2 u_\alpha(\varepsilon)(x,t), u^\varepsilon_3(x^\varepsilon, t) = \varepsilon u_3(\varepsilon)(x,t), \\ v^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), v^\varepsilon_3(x^\varepsilon) = \varepsilon v_3(x), \\ \sigma^\varepsilon_{\alpha\beta}(x^\varepsilon, t) = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)(x,t), \sigma^\varepsilon_{\alpha 3}(x^\varepsilon, t) = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x,t), \\ \sigma^\varepsilon_{33}(x^\varepsilon, t) = \varepsilon^4 \sigma_{33}(\varepsilon)(x,t), \\ \tau^\varepsilon_{\alpha\beta}(x^\varepsilon) = \varepsilon^2 \tau_{\alpha\beta}(x), \tau^\varepsilon_{\alpha 3}(x^\varepsilon) = \varepsilon^3 \tau_{\alpha 3}(x), \\ \tau^\varepsilon_{33}(x^\varepsilon) = \varepsilon^4 \tau_{33}(x), \\ \langle G^\varepsilon_{T_\alpha}, v_{T_\alpha} \rangle = \varepsilon^3 \langle G_{T_\alpha}(\varepsilon), v_{T_\alpha} \rangle, \langle G^\varepsilon_{T_3}, v_{T_3} \rangle = \varepsilon^4 \langle G_{T_3}(\varepsilon), v_{T_3} \rangle, \\ \langle G^\varepsilon_N, v_N \rangle = \varepsilon^4 \langle G_N(\varepsilon), v_N \rangle, \end{cases} \tag{6.2}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon$.

Next, we make the following assumptions : there exists constant $\rho > 0$ and for some $T > 0$, the functions $f_3 \in L^2(0,T; L^2(\Omega))$, $g_3 \in L^2(0,T; L^2(\Gamma_-))$, $h_\alpha \in L^2(0,T; L^2(\gamma_1))$, $\theta \in C^3(\bar{\omega})$ independent of ε and $\mathbf{p}(\varepsilon) \in \mathcal{V}(\Omega)$, $\mathbf{q}(\varepsilon) \in L^2(\Omega; \mathbb{R}^3)$, $c_{ijkl}(\varepsilon) \in L^\infty(\Omega)$, $d(\varepsilon) \in L^\infty(\Gamma_+)$ such that

$$\begin{cases} \rho^\varepsilon = \varepsilon^2 \rho, \\ f^\varepsilon_3(x^\varepsilon, t) = \varepsilon^3 f_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ g^\varepsilon_3(x^\varepsilon, t) = \varepsilon^4 g_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma^\varepsilon_-, \\ h^\varepsilon_\alpha(y_1, y_2, t) = \varepsilon^2 h_\alpha(y_1, y_2, t) \ \forall (y_1, y_2) \in \gamma_1, \\ \theta^\varepsilon(x_1, x_2) = \varepsilon \theta(x_1, x_2) \ \forall (x_1, x_2) \in \bar{\omega}, \\ p^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 p_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ p^\varepsilon_3(x^\varepsilon) = \varepsilon p_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 q_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q^\varepsilon_3(x^\varepsilon) = \varepsilon q_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ c^\varepsilon_{ijkl}(x^\varepsilon) = c_{ijkl}(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ d^\varepsilon(x^\varepsilon) = \varepsilon d(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma^\varepsilon_+. \end{cases} \tag{6.3}$$

Using the relations (3.2) and (5.14), the scalings (6.2) and the assumptions (6.3), we obtain

Theorem 6.1 *The scaled fields* $(\mathbf{u}(\varepsilon), \sigma(\varepsilon), G_N(\varepsilon), \mathbf{G}_T(\varepsilon))$ *satisfies the following varia-*

tional problem:

$$
(P(\varepsilon))^{anis}_{dyn,c}\begin{cases}
\textit{Find } (\mathbf{u}(\varepsilon),\sigma(\varepsilon),G_N(\varepsilon),\mathbf{G}_T(\varepsilon))\in\mathcal{K}(\varepsilon)(\Omega)\times\Sigma(\Omega)\times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\times\\
\mathbf{W}^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \forall t\in[0,T],\ \textit{such that,}\\
D^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)+B^\theta\left(\sigma\left(\varepsilon\right),\mathbf{v}\right)+2C^\theta\left(\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=F\left(\mathbf{v}\right)+\\
\langle G_N(\varepsilon),v_N(\varepsilon)\rangle+\langle G_{T_\alpha}(\varepsilon),v_{T_\alpha}(\varepsilon)\rangle+\\
\varepsilon^2\left[\langle G_{T_3}(\varepsilon),v_{T_3}(\varepsilon)\rangle+R\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)\right],\forall\mathbf{v}\in\mathcal{V}(\Omega),\forall t\in\left]0,T\right[,\\
A(\sigma\left(\varepsilon\right),\tau)-B^\theta\left(\tau,\mathbf{u}\left(\varepsilon\right)\right)-C^\theta\left(\tau,\mathbf{u}\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right)\right)=\\
\varepsilon^2S\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\tau\right),\forall\tau\in\Sigma(\Omega),\forall t\in\left]0,T\right[,\\
\langle G_N(\varepsilon),v_N(\varepsilon)-u_N(\varepsilon)\rangle\geq 0\,,\ \forall\mathbf{v}\in\mathcal{K}(\varepsilon)(\Omega),\forall t\in\left]0,T\right[,\\
\langle G_{T_\alpha}(\varepsilon),v_{T_\alpha}(\varepsilon)-\frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle+\\
\varepsilon\left[\langle G_{T_3}(\varepsilon),v_{T_3}(\varepsilon)-\frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle-\langle\Lambda G_N(\varepsilon),|v_T(\varepsilon)|-\left|\frac{\partial u_T(\varepsilon)}{\partial t}\right|\rangle\right]\geq 0,\\
\forall\mathbf{v}\in\mathcal{V}(\Omega),\forall t\in\left]0,T\right[,\\
\mathbf{u}\left(\varepsilon\right)(x,0)=\mathbf{p}\left(\varepsilon\right),\frac{\partial\mathbf{u}(\varepsilon)}{\partial t}\left(x,0\right)=\mathbf{q}\left(\varepsilon\right)\ \textit{in }\Omega,
\end{cases}
$$

where

$$
\begin{cases}
A(\sigma\left(\varepsilon\right),\tau)=\int_\Omega c_{\alpha\beta\gamma\delta}(\varepsilon)\sigma_{\gamma\delta}(\varepsilon)\tau_{\alpha\beta}dx,\\
B^\theta\left(\tau\left(\varepsilon\right),\mathbf{v}\right)=\int_\Omega\tau_{ij}\left(\varepsilon\right)\gamma^\theta_{ij}(\mathbf{v})dx,\\
C^\theta\left(\tau\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=\frac{1}{2}\int_\Omega\tau_{ij}\left(\varepsilon\right)\partial^\theta_i u_3\left(\varepsilon\right)\partial^\theta_j v_3dx,\\
D^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3\left(\varepsilon\right)v_3dx\right\},\\
F\left(\mathbf{v}\right)=\int_\Omega f_3v_3dx+\int_{\Gamma_-}g_3v_3d\Gamma+\int_{\gamma_1}h_\alpha\{\int_{-1}^1v_\alpha dx_3\}d\gamma,\\
\gamma^\theta_{ij}(\mathbf{v})=\frac{1}{2}\left(\partial^\theta_iv_j+\partial^\theta_jv_i\right),
\end{cases}
$$

such that the remainders R and S are bounded.

Proof.

The same argument used in the proof of Theorem 5.1 the first equation in variational problem $(P^\varepsilon)^{anis}_{dyn,c}$ may be written as

$$
D^t\left(\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)+B^\theta\left(\sigma\left(\varepsilon\right),\mathbf{v}\right)+2C^\theta\left(\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=F\left(\mathbf{v}\right)+
$$
$$
\langle G_N(\varepsilon),v_N(\varepsilon)\rangle+\langle G_{T_\alpha}(\varepsilon),v_{T_\alpha}(\varepsilon)\rangle+\varepsilon^2\left[\langle G_{T_3}(\varepsilon),v_{T_3}(\varepsilon)\rangle+R\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)\right],
$$

where

$$R\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)=\varrho_{F}\left(\varepsilon;\mathbf{v}\right)-\varrho_{B}\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{v}\right)-$$
$$\varrho_{C}\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\mathbf{v}\right)-\varrho_{D}(\varepsilon;\mathbf{u}(\varepsilon),\mathbf{v}).$$

Next, we have

$$\int_{\Omega^{\varepsilon}}(A\sigma^{\varepsilon})_{ij}\tau_{ij}^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon} = \varepsilon^{5}\int_{\Omega}c_{\alpha\beta\gamma\delta}(\varepsilon)\sigma_{\gamma\delta}(\varepsilon)\tau_{\alpha\beta}dx+\varepsilon^{7}\varrho_{A}\left(\varepsilon;\sigma(\varepsilon),\tau\right),$$

$$\int_{\Omega^{\varepsilon}}\tau_{ij}^{\varepsilon}b_{kj}^{\varepsilon}\partial_{k}^{\varepsilon}u_{i}^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon} = \varepsilon^{5}\int_{\Omega}\tau_{ij}\left(\varepsilon\right)\gamma_{ij}^{\theta}\left(\mathbf{u}(\varepsilon)\right)dx+\varepsilon^{7}\varrho_{B}\left(\varepsilon;\tau,\mathbf{u}(\varepsilon)\right),$$

$$\frac{1}{2}\int_{\Omega^{\varepsilon}}\tau_{ij}^{\varepsilon}b_{ki}^{\varepsilon}\partial_{k}^{\varepsilon}u_{l}^{\varepsilon}b_{mj}^{\varepsilon}\partial_{m}^{\varepsilon}u_{l}^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon} = \frac{\varepsilon^{5}}{2}\int_{\Omega}\tau_{ij}\partial_{i}^{\theta}u_{3}\left(\varepsilon\right)\partial_{j}^{\theta}u_{3}\left(\varepsilon\right)dx + \varepsilon^{7}\varrho_{C}\left(\varepsilon;\tau,\mathbf{u}\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right)\right).$$

Then the second equation in variational problem $(P^{\varepsilon})_{dyn,c}^{anis}$ may be also written as

$$A(\sigma\left(\varepsilon\right),\tau)-B^{\theta}\left(\tau,\mathbf{u}\left(\varepsilon\right)\right)-C^{\theta}\left(\tau,\mathbf{u}\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right)\right)=\varepsilon^{2}S\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\tau\right),$$

where

$$S\left(\varepsilon;\sigma\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right),\tau\right)=\varrho_{B}\left(\varepsilon;\tau,u(\varepsilon)\right)+\varrho_{C}\left(\varepsilon;\tau,\mathbf{u}\left(\varepsilon\right),\mathbf{u}\left(\varepsilon\right)\right)-\varrho_{A}\left(\varepsilon;\sigma(\varepsilon),\tau\right).$$

Now, note that, there exists a positive constant C such that, for all $\mathbf{u},\ \mathbf{v}\in\mathcal{V}(\Omega)$ and $\sigma,\ \tau\in\Sigma(\Omega)$

$$\sup_{0\leq\varepsilon\leq\varepsilon_{0}}\int_{\Omega}|\varrho_{A}(\varepsilon;\sigma,\tau)|dx \leq C|\sigma|_{0,\Omega}|\tau|_{0,\Omega},$$

$$\sup_{0\leq\varepsilon\leq\varepsilon_{0}}\int_{\Omega}|\varrho_{B}(\varepsilon;\tau,\mathbf{v})|dx \leq C|\tau|_{0,\Omega}|\mathbf{v}|_{1,\Omega},$$

$$\sup_{0\leq\varepsilon\leq\varepsilon_{0}}\int_{\Omega}|\varrho_{C}(\varepsilon;\tau,\mathbf{u},\mathbf{v})|dx \leq C|\tau|_{0,\Omega}|\mathbf{u}|_{1,4,\Omega}|\mathbf{v}|_{1,4,\Omega},$$

$$\sup_{0\leq\varepsilon\leq\varepsilon_{0}}\int_{\Omega}|\varrho_{F}(\varepsilon;\mathbf{v})|dx \leq C\|\mathbf{v}\|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\int_\Omega |\varrho_D(\varepsilon,\mathbf{u}(\varepsilon),\mathbf{v})|dx \quad\le\quad C\|\frac{\partial^2\mathbf{u}(\varepsilon)}{\partial t^2}\|_{-1,\frac{4}{3},\Omega}\|\mathbf{v}\|_{1,\Omega}.$$

For the unilateral contact conditions, we have

$$\langle G_N^\varepsilon, v_N^\varepsilon - u_N^\varepsilon\rangle \quad=\quad \varepsilon^5\langle G_N(\varepsilon), v_N(\varepsilon)-u_N(\varepsilon)\rangle,$$

$$\langle \mathbf{G}_T^\varepsilon, \mathbf{v}_T^\varepsilon - \frac{\partial \mathbf{u}_T^\varepsilon}{\partial t}\rangle + \langle \Lambda|G_N^\varepsilon|, |\mathbf{v}_T^\varepsilon| - |\frac{\partial \mathbf{u}_T^\varepsilon}{\partial t}|\rangle = \varepsilon^5\langle G_{T_\alpha}(\varepsilon), v_{T_\alpha}(\varepsilon) - \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle$$
$$+\varepsilon^6[\langle G_{T_3}(\varepsilon), v_{T_3}(\varepsilon) - \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), |\mathbf{v}_T(\varepsilon)| - |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle],$$

■

6.2.2 The limit three-dimensional problem

Assume that the scaled fields $(\mathbf{u}(\varepsilon),\sigma(\varepsilon),G_N(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(\mathbf{u}(\varepsilon),\sigma(\varepsilon),G_N(\varepsilon)) = (\mathbf{u}^0,\sigma^0,G_N^0)+\varepsilon(\mathbf{u}^1,\sigma^1,G_N^1)+\varepsilon^2(\mathbf{u}^2,\sigma^2,G_N^2)+\cdots, \tag{6.4}$$

with
$\mathbf{u}^0=(u_i^0)\in\mathcal{V}(\Omega)$, $\partial_3 u_3^0\in C^0\left(\bar\Omega\right)$, $\mathbf{u}^p=(u_i^p)\in W^{1,4}(\Omega;\mathbb{R}^3)\ \ \forall p\ge 1$,
$(\sigma^p,G_N^p)\in\Sigma(\Omega)\times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \ \forall p\ge 0$,
and

$$c_{ijkl}(\varepsilon)(x) = c^0_{ijkl}(x)+\varepsilon c^1_{ijkl}(x)+\varepsilon^2c^2_{ijkl}(x)+\cdots, \tag{6.5}$$

with
$c^0_{ijkl}(x)=c_{ijkl}(0)(x)$, $c^p_{ijkl}\in L^\infty(\Omega)\ \ \forall p\ge 0$.
We also assume that when $\varepsilon\to 0$

$$\mathbf{p}(\varepsilon)\to\mathbf{p}^0 \text{ in } \mathbf{V}(\Omega), \tag{6.6}$$

$$\mathbf{q}(\varepsilon)\to\mathbf{q}^0 \text{ in } L^2(\Omega;\mathbb{R}^3), \tag{6.7}$$

$$d(\varepsilon)\to d \text{ in } L^\infty(\Gamma_+), \tag{6.8}$$

$$\varepsilon G_N(\varepsilon)\to 0 \text{ in } W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+). \tag{6.9}$$

We substitute the formal asymptotic expansion (6.4)-(6.5) into the variational problem $(P(\varepsilon))^{anis}_{dyn,c}$, we obtain the following limit three-dimensional problem

Theorem 6.2 *The leading term* $(\mathbf{u}^0, \sigma^0, G_N^0)$ *satisfies the following variational problem:*

$$
(P_1^0)_{dyn,c}^{anis}\begin{cases}
\textit{Find } (\mathbf{u}^0, \sigma^0, G_N^0) \in \mathcal{K}(\Omega)\times\Sigma(\Omega)\times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \forall t\in[0,T],\ \textit{such that,}\\
\int_\Omega \sigma^0_{i\alpha}\partial_i v_\alpha dx - \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_3 v_\alpha dx = \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma,\\
\forall v_\alpha \in \mathcal{V}_\alpha(\Omega), \forall t\in\,]0,T[\,,\\
\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{i3}\partial_i v_3 dx + \int_\Omega \sigma^0_{ij}\partial_i u_3^0\partial_j v_3 dx\\
-\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha\theta\partial_3 v_3 dx - \int_\Omega\{\sigma^0_{\alpha j}\partial_\alpha\theta\partial_3 u_3^0\partial_j v_3 + \sigma^0_{i\beta}\partial_i u_3^0\partial_\beta\theta\partial_3 v_3\}dx\\
+\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha\theta\partial_3 u_3^0\partial_\beta\theta\partial_3 v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \langle G_N^0, v_3\rangle,\\
\forall v_3\in\mathcal{V}_3(\Omega), \forall t\in\,]0,T[\,,\\
\int_\Omega c^0_{\alpha\beta\gamma\delta}\sigma^0_{\gamma\delta}\tau_{\alpha\beta}dx - \int_\Omega \tau_{\alpha\beta}\gamma_{\alpha\beta}(\mathbf{u}^0)dx - \frac{1}{2}\int_\Omega \tau_{\alpha\beta}\partial_\alpha u_3^0\partial_\beta u_3^0 dx\\
+\frac{1}{2}\int_\Omega \tau_{\alpha\beta}(\partial_\alpha\theta\partial_3 u_\beta^0 + \partial_\beta\theta\partial_3 u_\alpha^0)dx\\
+\frac{1}{2}\int_\Omega \tau_{\alpha\beta}(\partial_\alpha\theta\partial_\beta u_3^0 + \partial_\beta\theta\partial_\alpha u_3^0)\partial_3 u_3^0 dx\\
-\frac{1}{2}\int_\Omega \tau_{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta(\partial_3 u_3^0)^2 dx = 0,\\
\forall(\tau_{\alpha\beta})\in L^2(\Omega;\mathbb{S}^2), \forall t\in\,]0,T[\,,\\
\int_\Omega \tau_{\alpha 3}(\partial_\alpha u_3^0 + \partial_3 u_\alpha^0)dx + \int_\Omega \tau_{\alpha 3}\partial_\alpha u_3^0\partial_3 u_3^0)dx\\
-\int_\Omega \tau_{\alpha 3}\partial_\alpha\theta\partial_3 u_3^0 dx - \int_\Omega \tau_{\alpha 3}\partial_\alpha\theta(\partial_3 u_3^0)^2 dx = 0,\\
\forall(\tau_{\alpha 3})\in L^2(\Omega;\mathbb{R}^2), \forall t\in\,]0,T[\,,\\
\int_\Omega \tau_{33}\partial_3 u_3^0 dx + \frac{1}{2}\int_\Omega \tau_{33}(\partial_3 u_3^0)^2 dx = 0,\\
\forall \tau_{33}\in L^2(\Omega), \forall t\in\,]0,T[\,,\\
\langle G_N^0, v_3 - u_3^0\rangle \geq 0\,,\ \forall \mathbf{v}\in\mathcal{K}(\Omega), \forall t\in\,]0,T[\,,\\
\mathbf{u}^0(x,0) = \mathbf{p}^0,\ \frac{\partial \mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0\ \textit{ in } \Omega.
\end{cases}
$$

Proof. First, in the last inequality in $(P(\varepsilon))_{dyn,c}^{anis}$, we take the test function $\mathbf{v}_T(\varepsilon) = 0$ after that $\mathbf{v}_T(\varepsilon) = 2\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}$, we obtain

$$\langle G_{T_\alpha}(\varepsilon), -\frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle + \varepsilon[\langle G_{T_3}(\varepsilon), -\frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), -|\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle] \geq 0, \tag{6.10}$$

$$\langle G_{T_\alpha}(\varepsilon), \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle + \varepsilon[\langle G_{T_3}(\varepsilon), \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle - \langle \Lambda G_N(\varepsilon), |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle] \geq 0. \tag{6.11}$$

Then, we conclude that

$$\langle G_{T_\alpha}(\varepsilon), \frac{\partial u_{T_\alpha}(\varepsilon)}{\partial t}\rangle = \varepsilon[\langle \Lambda G_N(\varepsilon), |\frac{\partial \mathbf{u}_T(\varepsilon)}{\partial t}|\rangle - \langle G_{T_3}(\varepsilon), \frac{\partial u_{T_3}(\varepsilon)}{\partial t}\rangle], \tag{6.12}$$

From the (6.9) and since $\frac{\partial u_{T_3}(\varepsilon)}{\partial t} = O(\varepsilon)$, we obtain

$$G_{T_\alpha}(\varepsilon) = 0 \text{ in } \Gamma_+, \forall t \in \,]0,T[\,.$$

Next, using technics of the asymptotic analysis method, we first replace $\mathbf{u}(\varepsilon)$, $\sigma(\varepsilon)$, $G_N(\varepsilon)$ and $c_{ijkl}(\varepsilon)(x)$ by their expansions (6.4)-(6.5) in the forms A, B^θ, C^θ, D^t and F and we equate to zero the terms which are independent of ε in $(P(\varepsilon))^{anis}_{dyn,c}$. Then we show that $(\mathbf{u}^0, \sigma^0, G^0_N)$ satisfy $(P^0_1)^{anis}_{dyn,c}$. ■

Theorem 6.3 *The leading term* $(\mathbf{u}^0, G^0_N)$ *satisfies the following variational problem:*

$$(P^0_2)^{anis}_{dyn,c}\begin{cases} \textit{Find } (\mathbf{u}^0, G^0_N) \in \mathcal{K}_{KL}(\Omega) \times W^{-\frac{3}{4},\frac{4}{3}}(\Gamma_+)\ \forall t \in [0,T], \ \textit{such that,} \\ \frac{d^2}{dt^2}\left\{\rho \int_\Omega u^0_3 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u^0_3+\theta)\partial_\beta v_3 dx = \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma + \langle G^0_N, v_3\rangle, \\ \forall \mathbf{v} \in \mathcal{V}_{KL}(\Omega), \forall t \in \,]0,T[\,, \\ \langle G^0_N, v_3 - u^0_3\rangle \geq 0\,,\ \forall \mathbf{v} \in \mathcal{K}_{KL}(\Omega), \forall t \in \,]0,T[\,, \\ \mathbf{u}^0(x,0) = \mathbf{p}^0,\ \frac{\partial \mathbf{u}^0}{\partial t}(x,0) = \mathbf{q}^0 \ \textit{in } \Omega, \end{cases}$$

where

$$\begin{cases} \sigma^0_{\alpha\beta} = c^{0,-1}_{\alpha\beta\gamma\delta}(x)\bar{E}^0_{\gamma\delta}(\mathbf{u}^0), \\ (c^{0,-1}_{\alpha\beta\gamma\delta}) \textit{ is the inverse of } (c^0_{\alpha\beta\gamma\delta}), \\ \bar{E}^0_{\gamma\delta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\gamma u^0_\delta + \partial_\delta u^0_\gamma + \partial_\gamma\theta\partial_\delta u^0_3 + \partial_\delta\theta\partial_\gamma u^0_3 + \partial_\gamma u^0_3\partial_\delta u^0_3\right). \end{cases}$$

Proof.

The proof has been divided into 3 steps

Step 1. The fifth equation in $(P^0_1)^{anis}_{dyn,c}$ give

$$\partial_3 u^0_3(1 + \frac{1}{2}\partial_3 u^0_3) = 0,$$

so that

$$\partial_3 u^0_3 = 0 \text{ or } \partial_3 u^0_3 = -2.$$

Since the inclusion $H^3(\Omega) \hookrightarrow \mathcal{C}^1(\Omega)$ and $u^0_3 = 0$ on $\gamma_1 \times [-1,1]$, the solution $\partial_3 u^0_3 = -2$ is eliminated. Hence we obtain

$$\partial_3 u^0_3 = 0 \text{ in } \Omega. \tag{6.13}$$

Consequently, the fourth equation in (P_1^0) reduce to

$$\partial_\alpha u_3^0 + \partial_3 u_\alpha^0 = 0 \text{ in } \Omega. \tag{6.14}$$

Step 2. Taking into account the equation (6.13)-(6.14), the third equation in $(P_1^0)_{dyn,c}^{anis}$ reduce to

$$c^0_{\alpha\beta\gamma\delta}\sigma^0_{\gamma\delta} - \gamma_{\alpha\beta}(\mathbf{u}^0) - \frac{1}{2}\partial_\alpha u_3^0 \partial_\beta u_3^0 - \frac{1}{2}(\partial_\alpha \theta \partial_\beta u_3^0 + \partial_\beta \theta \partial_\alpha u_3^0) = 0. \tag{6.15}$$

We observe that

$$\gamma_{\alpha\beta}(\mathbf{u}^0) = \frac{1}{2}\left(\partial_\alpha u_\beta^0 + \partial_\beta u_\alpha^0\right).$$

If $(c^{0,-1}_{\alpha\beta\gamma\delta})$ is the inverse of $(c^0_{\alpha\beta\gamma\delta})$, we show that

$$\sigma^0_{\alpha\beta} = c^{0,-1}_{\alpha\beta\gamma\delta}(x)\bar{E}^0_{\gamma\delta}(\mathbf{u}^0).$$

Note that

$$c^{0,-1}_{\alpha\beta\gamma\delta}(x) = a_{\alpha\beta\gamma\delta}(x) - a_{\alpha\beta i3}(x) i_{ij}(x) a_{j3\gamma\delta}(x),$$

where $i = (i_{ij})$ is the inverse of the matrix (a_{i3j3}).

Step 3. Taking into account the equation (6.13), we next find that the second equation in $(P_1^0)_{dyn,c}^{anis}$ reduce to

$$\begin{aligned}\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u_3^0 \partial_\beta v_3 dx &= \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \langle G_N^0, v_3\rangle,\end{aligned} \tag{6.16}$$

From the first equation and the relation (6.14), we conclude that

$$\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx = \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_\alpha v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx - \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma. \tag{6.17}$$

We replace the integral $\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx$ in equation (6.16) by their expression (6.17), we find that

$$\begin{aligned}\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3^0 v_3 dx\right\} + \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(u_3^0+\theta)\partial_\beta v_3 dx &= \\ \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\gamma_1} h_\alpha\{\int_{-1}^1 v_\alpha dx_3\}d\gamma + \langle G_N^0, v_3\rangle.\end{aligned}$$

■

6.2.3 The limit two-dimensional problem

We show in the next theorem that $(P_2^0)_{dyn,c}^{anis}$ is in a sense of two-dimensional problem posed over the two-dimensional domain $\bar{\omega}$.

Theorem 6.4 *The leading term $\mathbf{u}^0 = (u_i^0)$ is of the form $u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ and $u_3^0 = \zeta_3$ with $\zeta = (\zeta_i) \in \mathbf{V}(\omega)\ \forall t \in [0,T]$. In addition, the field ζ satisfies the following limit scaled two-dimensional problem:*

$$(P(\omega))_{dyn,c}^{anis}\begin{cases} \textit{Find } (\zeta, f_c) \in \mathcal{K}(\omega) \times H^{-2}(\omega)\ \forall t \in [0,T], \textit{ such that,} \\ 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m_{\alpha\beta}^{anis}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}^{anis}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega \\ + \int_\omega \bar{N}_{\alpha\beta}^{anis}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma + \langle f_c, \eta_3\rangle, \forall\eta \in \mathbf{V}(\omega), \forall t \in]0,T[, \\ \langle f_c, \eta_3 - \zeta_3\rangle \geq 0\,,\ \forall\eta \in \mathcal{K}(\omega), \forall t \in]0,T[, \\ \zeta(.,0) = \varphi,\ \frac{\partial\zeta}{\partial t}(.,0) = \psi \textit{ in } \omega, \end{cases}$$

where

$\mathrm{p}_3 = \int_{-1}^{1} f_3 dx_3 + g_3(.,-1)$, $d = d(.,+1)$, $\langle f_c, \eta_3\rangle = \langle G_N^0, v_3\rangle$, $\bar{N}_{\alpha\beta}^{anis}$ *and* $m_{\alpha\beta}^{anis}$ *are defined by the relations (3.15) and (3.16).*

Proof.

i) From $\mathbf{v} \in \mathcal{V}_{KL}(\Omega)$, by a standard argument due to P.G Ciarlet (see,e.g., [Cia97, Theorem 1.4-4]), we get

$$u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3^0 = \zeta_3 \text{ with } \zeta = (\zeta_i) \in \mathbf{V}(\omega).$$

ii) We observe that

$$\bar{E}_{\gamma\delta}^0(\mathbf{u}^0) = \bar{E}_{\gamma\delta}^0(\zeta) + x_3\Upsilon_{\gamma\delta}(\zeta_3). \tag{6.18}$$

From the definition of $\sigma_{\alpha\beta}^0$ and (6.18), we conclude that

$$\begin{aligned} \int_{-1}^{1}\sigma_{\alpha\beta}^0 dx_3 &= \int_{-1}^{1} c_{\alpha\beta\gamma\delta}^{0,-1}(x)\left[\bar{E}_{\gamma\delta}^0(\zeta) + x_3\Upsilon_{\gamma\delta}(\zeta_3)\right]dx_3 \\ &= \left(\int_{-1}^{1} c_{\alpha\beta\gamma\delta}^{0,-1}(x)dx_3\right)\bar{E}_{\gamma\delta}^0(\zeta) + \left(\int_{-1}^{1} x_3 c_{\alpha\beta\gamma\delta}^{0,-1}(x)dx_3\right)\Upsilon_{\gamma\delta}(\zeta_3) \\ &= C_{\alpha\beta\gamma\delta}^0\bar{E}_{\gamma\delta}^0(\zeta) + C_{\alpha\beta\gamma\delta}^1\Upsilon_{\gamma\delta}(\zeta_3) \\ &= \bar{N}_{\alpha\beta}^{anis}(\zeta), \end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{1} x_3\sigma^0_{\alpha\beta}dx_3 &= \int_{-1}^{1} x_3 c^{0,-1}_{\alpha\beta\gamma\delta}(x)\left[\bar{E}^0_{\gamma\delta}(\zeta) + x_3\Upsilon_{\gamma\delta}(\zeta_3)\right]dx_3 \\
&= \left(\int_{-1}^{1} x_3 c^{0,-1}_{\alpha\beta\gamma\delta}(x)dx_3\right)\bar{E}^0_{\gamma\delta}(\zeta) + \left(\int_{-1}^{1} x_3^2 c^{0,-1}_{\alpha\beta\gamma\delta}(x)dx_3\right)\Upsilon_{\gamma\delta}(\zeta_3) \\
&= C^1_{\alpha\beta\gamma\delta}\bar{E}^0_{\gamma\delta}(\zeta) + C^2_{\alpha\beta\gamma\delta}\Upsilon_{\gamma\delta}(\zeta_3) \\
&= m^{anis}_{\alpha\beta}(\zeta),
\end{aligned}$$

iii) First we choose, in (P^0_2), $\mathbf{v} \in \mathcal{V}_{KL}(\Omega)$ with the components

$$v_\alpha(x) = -x_3\partial_\alpha\eta_3(x_1,x_2),\ v_3(x) = \eta_3(x_1,x_2),$$

with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu\eta_3 = 0$ on γ_1.

This choice shows that (P^0_2) reduce to

$$\frac{d^2}{dt^2}\left\{\rho\int_\Omega \zeta_3\eta_3 dx\right\} - \int_\Omega x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta^0_3+\theta)\partial_\beta\eta_3 dx = \int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma + \langle f_c,\eta_3\rangle. \tag{6.19}$$

The second choice of $\mathbf{v} \in \mathcal{V}_{KL}(\Omega)$ is

$$v_\alpha(x) = \eta_\alpha(x_1,x_2),\ v_3(x) = 0,$$

with $\eta_\alpha \in H^1(\omega)$.

In this case shows that $(P^0_2)^{anis}_{dyn,c}$ reduce to

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma \tag{6.20}$$

Using Fubini's Formula: $\int_\Omega F dx = \int_\omega \left\{\int_{-1}^1 F dx_3\right\} d\omega$, we have

$$\frac{d^2}{dt^2}\{\rho\int_\Omega \zeta_3\eta_3 dx\} = 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega,$$

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx = -\int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 dx = \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega,$$

$$\int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma = \int_\omega \{\int_{-1}^{1} f_3 dx_3 + g_3(.,-1)\}\eta_3 d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega,$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma.$$

Then

$$2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega + \int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma + \langle f_c, \eta_3\rangle.$$

■

Next, we write the two-dimensional boundary value problem as an equivalent boundary value problem $(\bar{P}(\omega))^{anis}_{dyn,c}$.

Theorem 6.5 *Assume that the boundary γ is sufficiently smooth. Then any smooth solution $\zeta = (\zeta_i)$ of the variational problem $(P(\omega))^{anis}_{dyn,c}$ is also a solution of the following two-dimensional displacement problem:*

$$(\bar{P}(\omega))^{anis}_{dyn,c}\left\{\begin{array}{l}
\textit{Find } ((\zeta_\alpha),\zeta_3,f_c) \in (H^1(\omega))^2\times H^2(\omega)\times H^{-2}(\omega)\ \forall t\in[0,T],\ \textit{such that,}\\
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m^{anis}_{\alpha\beta} - \bar{N}^{anis}_{\alpha\beta}\partial_{\alpha\beta}(\zeta_3+\theta) = \mathrm{p}_3 + f_c \textit{ in } \omega\times]0,T[,\\
\partial_\beta\bar{N}^{anis}_{\alpha\beta} = 0 \textit{ in } \omega\times]0,T[,\\
\zeta_3 = \partial_\nu\zeta_3 = 0 \textit{ on } \gamma_1\times]0,T[,\\
\bar{N}^{anis}_{\alpha\beta}\nu_\beta = 2h_\alpha \textit{ on } \gamma_1\times]0,T[,\\
m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[,\\
\partial_\alpha m^{anis}_{\alpha\beta}\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \textit{ on } \gamma_2\times]0,T[,\\
\bar{N}^{anis}_{\alpha\beta}\nu_\beta = 0 \textit{ on } \gamma_2\times]0,T[,\\
\zeta_3\le d, f_c\le 0, f_c(\zeta_3-d) = 0 \textit{ in } \omega\times]0,T[,\\
\zeta(.,0) = \varphi, \frac{\partial\zeta}{\partial t}(.,0) = \psi \textit{ in } \omega.
\end{array}\right.$$

Proof.

Applying the Green formulas, we obtain

$$
\begin{aligned}
-\int_\omega m^{anis}_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega &= \int_\gamma \left\{ \left(\partial_\alpha m^{anis}_{\alpha\beta}\right)\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right)\right\}\eta_3 d\gamma \\
&- \int_\gamma m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma - \int_\omega \left(\partial_{\alpha\beta}m^{anis}_{\alpha\beta}\right)\eta_3 d\omega,
\end{aligned}
$$

$$
\begin{aligned}
\int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\partial_\beta\eta_3 d\omega &= -\int_\omega \left\{\partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right)\right\}\eta_3 d\omega \\
&+ \int_\gamma \left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right)\nu_\beta\eta_3 d\gamma,
\end{aligned}
$$

$$
\int_\omega \bar{N}^{anis}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = -\int_\omega \left(\partial_\beta\bar{N}^{anis}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \bar{N}^{anis}_{\alpha\beta}\nu_\beta\eta_\alpha d\gamma.
$$

Then

$$
\begin{aligned}
&\int_\omega \left[2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m^{anis}_{\alpha\beta} - \partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right) - p_3\right]\eta_3 d\omega - \langle f_c,\eta_3\rangle \; - \\
&\int_\omega \left(\partial_\beta\bar{N}^{anis}_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \left(\bar{N}^{anis}_{\alpha\beta}\nu_\beta - 2\widetilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m^{anis}_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma \; + \\
&\int_{\gamma_2}\left\{\left[\partial_\alpha m^{anis}_{\alpha\beta} + \bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right]\nu_\beta + \partial_\tau\left(m^{anis}_{\alpha\beta}\nu_\alpha\tau_\beta\right)\right\}\eta_3 d\gamma = 0,
\end{aligned}
$$

for all $\eta = (\eta_\alpha, \eta_3) \in V(\omega)$, with the functions $\widetilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by

$$
\widetilde{h}_\alpha = h_\alpha \;\text{ on }\; \gamma_1 \times [0,T] \text{ and } \widetilde{h}_\alpha = 0 \;\text{ on }\; \gamma_2 \times [0,T].
$$

These equations imply that all the factors of η_α, η_3, and $\partial_\nu\eta_3$ vanish in their respective domains of integration. Then we get

$$
2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m^{anis}_{\alpha\beta} - \partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right) = p_3 + f_c \text{ in } \omega \times \left]0,T\right[,
$$

and

$$
\partial_\beta\bar{N}^{anis}_{\alpha\beta} = 0 \;\text{ in } \omega \times \left]0,T\right[,
$$

so that

$$
\partial_\beta\left(\bar{N}^{anis}_{\alpha\beta}\partial_\alpha\left(\zeta_3+\theta\right)\right) = \bar{N}^{anis}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3+\theta\right) \;\text{ in } \omega \times \left]0,T\right[,
$$

consequently

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}^{anis} - \bar{N}_{\alpha\beta}^{anis}\partial_{\alpha\beta}\left(\zeta_3+\theta\right) = p_3 + f_c \text{ in } \omega\times\left]0,T\right[.$$

For boundary conditions, we get

$$\bar{N}_{\alpha\beta}^{anis}\nu_\beta - 2\widetilde{h}_\alpha = 0 \text{ on } \gamma\times\left]0,T\right[,$$

thus

$$\bar{N}_{\alpha\beta}^{anis}\nu_\beta = 2h_\alpha \text{ on } \gamma_1\times\left]0,T\right[,$$

and

$$\bar{N}_{\alpha\beta}^{anis}\nu_\beta = 0 \text{ on } \gamma_2\times\left]0,T\right[.$$

We also get

$$m_{\alpha\beta}^{anis}\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2\times\left]0,T\right[,$$

and

$$\left[\partial_\alpha m_{\alpha\beta}^{anis} + \bar{N}_{\alpha\beta}^{anis}\partial_\alpha\left(\zeta_3+\theta\right)\right]\nu_\beta + \partial_\tau\left(m_{\alpha\beta}^{anis}\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2\times\left]0,T\right[,$$

since $\bar{N}_{\alpha\beta}^{anis}\nu_\beta = 0$ on $\gamma_2\times\left]0,T\right[$, we conclude that

$$\partial_\alpha m_{\alpha\beta}^{anis}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}^{anis}\nu_\alpha\tau_\beta\right) = 0 \text{ on } \gamma_2\times\left]0,T\right[.$$

Finally, in the last inequality in $(P(\omega))_{dyn,c}^{anis}$, we take the test function $\eta_3 = d$ after that $\eta_3 = 2\zeta_3 - d$, we obtain

$$\langle f_c, d-\zeta_3\rangle \geq 0,$$

and

$$\langle f_c, \zeta_3 - d\rangle \geq 0.$$

Thus

$$\langle f_c, \zeta_3 - d\rangle = 0.$$

■

6.3 Dynamical contact equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells

We now rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))^{anis}_{dyn,c}$ in the form of dynamical contact equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shell as follows:

Theorem 6.6 *Assume that the set ω is simply-connected and that its boundary γ is sufficiently smooth. Let $\zeta = (\zeta_i)$ be a solution of $(\bar{P}(\omega))^{anis}_{dyn,c}$ with the regularity*
$\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^3(\omega)$ and $f_c \in H^{-1}(\omega)$ $\forall t \in [0, T]$.
Then

a) *The functions $\widetilde{h}_\alpha : \gamma \times [0, T] \to \mathbb{R}$ defined by*

$$\widetilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0, T] \text{ and } \widetilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0, T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions

$$\int_\gamma \widetilde{h}_1 d\gamma = \int_\gamma \widetilde{h}_2 d\gamma = \int_\gamma (x_1 \widetilde{h}_2 - x_2 \widetilde{h}_1) d\gamma = 0.$$

b) *Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1 \Phi(0) = \partial_2 \Phi(0) = 0$, such that*

$$\bar{N}^{anis}_{11} = 2\partial_{22}\Phi, \ \bar{N}^{anis}_{12} = \bar{N}^{anis}_{21} = -2\partial_{12}\Phi, \ \bar{N}^{anis}_{22} = 2\partial_{11}\Phi.$$

c) *Finally, the triple $(\zeta_3, \Phi, f_c) \in (H^3(\omega) \cap \mathcal{K}) \times H^4(\omega) \times H^{-1}(\omega)$ $\forall t \in [0, T]$, satisfies the following problem*

$$
(P)^{anis}_{dyn,c}\begin{cases}
2\rho\frac{\partial^2\zeta_3}{\partial t^2}-\partial_{\alpha\beta}M^{anis}_{\alpha\beta}(\zeta_3,\Phi)=2\left[\Phi,\zeta_3+\theta\right]+\mathrm{p}_3+f_c \ \ in\ \omega\times\left]0,T\right[,\\
\Delta^2\Phi=\frac{1}{2}\mathfrak{L}(\zeta_3,\Phi)\ \ in\ \omega\times\left]0,T\right[,\\
\zeta_3=\partial_\nu\zeta_3=0\ \ on\ \gamma_1\times\left]0,T\right[,\\
M^{anis}_{\alpha\beta}(\zeta_3,\Phi)\nu_\alpha\nu_\beta=0\ \ on\ \gamma_2\times\left]0,T\right[,\\
\partial_\alpha M^{anis}_{\alpha\beta}(\zeta_3,\Phi)\nu_\beta+\partial_\tau\left(M^{anis}_{\alpha\beta}(\zeta_3,\Phi)\nu_\alpha\tau_\beta\right)=0\ \ on\ \gamma_2\times\left]0,T\right[,\\
\Phi=\Phi_0,\partial_\nu\Phi=\Phi_1\ \ on\ \gamma\times\left]0,T\right[,\\
\zeta_3\le d, f_c\le 0, f_c(\zeta_3-d)=0\ \ in\ \omega\times\left]0,T\right[,\\
\zeta_3\left(.,0\right)=\varphi_3,\frac{\partial\zeta_3}{\partial t}\left(.,0\right)=\psi_3\ \ in\ \omega,
\end{cases}
$$

where

$$
\begin{cases}
\Phi_0(y)=-y_1\int_{\gamma(y)}\widetilde{h}_2d\gamma+y_2\int_{\gamma(y)}\widetilde{h}_1d\gamma+\int_{\gamma(y)}(x_1\widetilde{h}_2-x_2\widetilde{h}_1)d\gamma,\\
\Phi_1(y)=-\nu_1\int_{\gamma(y)}\widetilde{h}_2d\gamma+\nu_2\int_{\gamma(y)}\widetilde{h}_1d\gamma,\ y=(y_1,y_2)\in\gamma,\\
[\Phi,\zeta]=\partial_{11}\Phi\partial_{22}\zeta+\partial_{22}\Phi\partial_{11}\zeta-2\partial_{12}\Phi\partial_{12}\zeta,\\
M^{anis}_{\alpha\beta}(\zeta_3,\Phi)=\mathcal{F}^\theta_{\alpha\beta}(\zeta_3,\Phi)+m^{2,\theta}_{\alpha\beta}(\zeta_3),\\
\mathfrak{L}(\zeta_3,\Phi)=\Delta\left[C^0_{\alpha\alpha\gamma\delta}C^{1,-1}_{\sigma\varsigma\gamma\delta}\mathcal{F}^\theta_{\sigma\varsigma}(\zeta_3,\Phi)+N^{2,\theta}_{\alpha\alpha}(\zeta_3)\right],\\
\mathcal{F}^\theta_{\alpha\beta}(\zeta_3,\Phi)=C^1_{\alpha\beta\gamma\delta}[C^{0,-1}_{11\gamma\delta}(2\partial_{22}\Phi-N^{2,\theta}_{11}(\zeta_3))+C^{0,-1}_{22\gamma\delta}(2\partial_{11}\Phi-N^{2,\theta}_{22}(\zeta_3))+\\
2C^{0,-1}_{12\gamma\delta}(-2\partial_{12}\Phi-N^{2,\theta}_{12}(\zeta_3))],
\end{cases}
$$

such that $m^{2,\theta}_{\alpha\beta}(\zeta_3)$, $N^{2,\theta}_{\alpha\beta}(\zeta_3)$ *are defined in Section 3.3 and* $C^{0,-1}_{\alpha\beta\gamma\delta}$, $C^{1,-1}_{\alpha\beta\gamma\delta}$ *are the inverse of* $C^0_{\alpha\beta\gamma\delta}$, $C^1_{\alpha\beta\gamma\delta}$ *respectively.*

Proof.

The proof is similar to that of Theorem 3.6. We prove that

a) The functions $\widetilde{h}_\alpha\in H^{\frac{3}{2}}(\gamma)$ satisfy the compatibility conditions.

b) The regularities of $\bar{N}^{anis}_{\alpha\beta}\in H^2(\omega)$ imply that $\Phi\in H^4(\omega)$. Then Φ is uniquely defined if we impose that $\Phi(0)=\partial_1\Phi(0)=\partial_2\Phi(0)=0$, such that

$\bar{N}^{anis}_{11}=2\partial_{22}\Phi,\ \bar{N}^{anis}_{12}=\bar{N}^{anis}_{21}=-2\partial_{12}\Phi,\ \bar{N}^{anis}_{22}=2\partial_{11}\Phi$ in ω.

c) (i) From $\bar{N}^{anis}_{\alpha\beta}\nu_\beta = 2\widetilde{h}_\alpha$ on γ, we obtain

$$\Phi = \Phi_0, \partial_\nu \Phi = \Phi_1 \text{ on } \gamma \times]0, T[\,.$$

(ii) We have

$$\bar{N}^{anis}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3 + \theta\right) = 2\left[\Phi, \zeta_3 + \theta\right],$$

and

$$\begin{aligned} m^{anis}_{\alpha\beta}(\zeta) &= \mathcal{F}^{\theta}_{\alpha\beta}(\zeta_3, \Phi) + m^{2,\theta}_{\alpha\beta}(\zeta_3) \\ &= M^{anis}_{\alpha\beta}(\zeta_3, \Phi). \end{aligned}$$

Then, we deduce

$$2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}M^{anis}_{\alpha\beta}(\zeta_3, \Phi) = 2\left[\Phi, \zeta_3 + \theta\right] + \mathrm{p}_3 + f_c \text{ in } \omega \times]0, T[\,.$$

(iii) Notice that

$$\begin{aligned} \Delta^2\Phi &= \frac{1}{2}\Delta[C^0_{\alpha\alpha\gamma\delta}C^{1,-1}_{\sigma\varsigma\gamma\delta}\mathcal{F}^{\theta}_{\sigma\varsigma}(\zeta_3, \Phi) + N^{2,\theta}_{\alpha\alpha}(\zeta_3)] \\ &= \frac{1}{2}\mathfrak{L}(\zeta_3, \Phi). \end{aligned} \tag{6.21}$$

■

6.4 Conclusion

An application of the technics from formal asymptotic analysis to the three-dimensional dynamical model for a Signorini problem with Coulomb friction of nonlinearly elastic shallow shell with a specific class of boundary conditions of generalized Marguerre-von Kármán type, made of a general nonhomogeneous anisotropic material, shows that the leading term of the expansion is characterized by a two-dimensional frictionless dynamical contact boundary value problem called the dynamical contact equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells, which depends on the Airy function Φ, the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell and contact force f_c.

Part III

Numerical approximations

Chapter 7

Finite element approximations of generalized Marguerre-von Kármán equations

This Chapter, make as apply the finite element method for approximating solutions to the generalized Marguerre-von Kármán equations, solving these equations amounts to solving a single discrete cubic operator equation. This work was published in [GC14].

7.1 Generalized Marguerre-von Kármán equations

Ciarlet and Gratie [CG06b] have shown that, the generalized Marguerre-von Kármán equations written as

$$
(P)^{iso}_{sta}\begin{cases}
-\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\xi) = [\Phi, \xi + \widetilde{\theta}] + f \text{ in } \omega, \\
\triangle^2\Phi = -[\xi, \xi + 2\widetilde{\theta}] \text{ in } \omega, \\
\xi = \partial_\nu \xi = 0 \text{ on } \gamma_1, \\
m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_2, \\
\partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \text{ on } \gamma_2, \\
\Phi = \Phi_0 \text{ and } \ \partial_\nu\Phi = \Phi_1 \text{ on } \gamma,
\end{cases}
$$

where

$$
\begin{cases}
m_{\alpha\beta}(\nabla^2\xi) = \ -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\triangle\xi\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\xi\right\}, \\
\Phi_0(y) = -\gamma_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \gamma_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\widetilde{h}_2 - x_2\widetilde{h}_1)d\gamma, \quad y \in \gamma, \\
\Phi_1(y) = -\nu_1\int_{\gamma(y)}\widetilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\widetilde{h}_1 d\gamma, \quad y \in \gamma, \\
[\Phi, \xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi.
\end{cases}
$$

The known functions $\widetilde{\theta}$ and f are, up to constant factors, the function that defines the middle surface of the shell and the resultant of the vertical forces acting on the shell. The functions Φ_0 and Φ_1 are known functions of the appropriately "scaled" density $(h_\alpha) : \gamma_1 \to \mathbb{R}^2$ of the resultant of the horizontal forces acting on the portion of the lateral face of the shell with γ_1 as its middle line and the functions $\widetilde{h}_\alpha \in L^2(\gamma)$ defined by $\widetilde{h}_\alpha = h_\alpha$ on γ_1, $\widetilde{h}_\alpha = 0$ on γ_2. The constants λ and μ are the Lamé constants of the material. The unknown $\xi : \bar{\omega} \to \mathbb{R}$ is, up to constant factors, the vertical component of the displacement field of the middle surface of the shell and the unknown $\Phi : \bar{\omega} \to \mathbb{R}$ is the Airy function.

7.2 The continuous cubic operator equation

Let us briefly recall some of the results obtained in [CG06b] concerning the properties of the continuous cubic operator equation.

Let $\widetilde{\chi} \in H^2(\omega)$ denote the unique solution in the sense of distribution to the boundary value problem:

$$\Delta^2\widetilde{\chi} = [\widetilde{\theta}, \widetilde{\theta}] \text{ in } \omega, \tag{7.1}$$

$$\widetilde{\chi} = \Phi_0 \text{ and } \partial_\nu\widetilde{\chi} = \Phi_1 \text{ on } \gamma. \tag{7.2}$$

Let $F \in V(\omega)$ denote the unique solution in the sense of distribution to the boundary value problem:

$$-\partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 F\right) = f \text{ in } \omega, \tag{7.3}$$

$$F = \partial_\nu F = 0 \text{ on } \gamma_1, \tag{7.4}$$

$$m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2, \tag{7.5}$$

$$\partial_\alpha m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\left(\nabla^2 F\right)\nu_\alpha \tau_\beta\right) = 0 \text{ on } \gamma_2. \tag{7.6}$$

Let the bilinear mapping:

$$B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega),$$

be defined as follows: for each pair $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution in the sense of distribution to the boundary value problem:

$$\Delta^2 B(\xi, \eta) = [\xi, \eta] \text{ in } \omega, \tag{7.7}$$

$$B(\xi, \eta) = \partial_\nu B(\xi, \eta) = 0 \text{ on } \gamma. \tag{7.8}$$

Let the second bilinear mapping:

$$\widetilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega),$$

be defined as follows: for each pair $(\Phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\widetilde{B}(\Phi, \xi) \in V(\omega)$ is the unique solution in the sense of distribution to the boundary value problem:

$$-\partial_{\alpha\beta} m_{\alpha\beta}\left(\nabla^2 \widetilde{B}(\Phi, \xi)\right) = [\Phi, \xi] \text{ in } \omega, \tag{7.9}$$

$$\widetilde{B}(\Phi, \xi) = \partial_\nu \widetilde{B}(\Phi, \xi) = 0 \text{ on } \gamma_1, \tag{7.10}$$

$$m_{\alpha\beta}\left(\nabla^2 \widetilde{B}(\Phi, \xi)\right)\nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2, \tag{7.11}$$

$$\partial_\alpha m_{\alpha\beta} \nu_\beta + \partial_\tau\left(m_{\alpha\beta}\left(\nabla^2 \widetilde{B}(\Phi, \xi)\right)\nu_\alpha \tau_\beta\right) = 0 \text{ on } \gamma_2. \tag{7.12}$$

First, Ciarlet and Gratie [CG06b] have shown that, the generalized Marguerre-von Kármán equations are reduced to a cubic operator equation, such that a pair $(\xi, \Phi) \in V(\omega) \times H^2(\omega)$ satisfies the generalized Marguerre-von Kármán equations in the sense of distributions if and only if the function $\widetilde{\xi} = (\widetilde{\theta} + \xi) \in V(\omega)$ satisfies the cubic operator equation:

$$\widetilde{C}(\widetilde{\xi}) + (I - \widetilde{L})\widetilde{\xi} - \widetilde{F} = 0, \tag{7.13}$$

and the Airy function $\Phi \in H^2(\omega)$ is given by

$$\Phi = \widetilde{\chi} - B(\widetilde{\xi}, \widetilde{\xi}), \tag{7.14}$$

where

$$\widetilde{F} = \widetilde{\theta} + F.$$

The cubic mapping

$$\widetilde{C} : V(\omega) \to V(\omega),$$

is defined by

$$\widetilde{C}(\eta) = \widetilde{B}(B(\eta, \eta), \eta).$$

The linear mapping

$$\widetilde{L} : V(\omega) \to V(\omega),$$

is defined by

$$\widetilde{L}\eta = \widetilde{B}(\widetilde{\chi}, \eta).$$

Noting that, finding the solution $\widetilde{\xi}$ of the above operator equation (7.13) is equivalent to solving the following variational problem:

$$(\mathcal{P})^{iso}_{sta} \begin{cases} \text{Find } \widetilde{\xi} \in V(\omega) \text{ such that,} \\ ((\widetilde{C}(\widetilde{\xi}) + (I - \widetilde{L})\widetilde{\xi} - \widetilde{F}, \eta)) = 0 \text{ for all } \eta \in V(\omega), \end{cases}$$

where $((.,.))$ is the inner-product on $V(\omega)$ defined by

$$((\zeta, \eta)) = -\int_\omega m_{\alpha\beta}(\nabla^2\zeta)\partial_{\alpha\beta}\eta d\omega,$$

and let $\|.\|$ denote the norm associated with the inner product $((.,.))$ which is equivalent to the norm $\|.\|_{H^2(\omega)}$ over the space $V(\omega)$.

Next, Ciarlet and Gratie [CG06b] have shown that, under the assumptions (ω is simply-connected, the functions $\widetilde{h}_\alpha$ satisfy natural compatibility conditions, and the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough), the generalized Marguerre-von Kármán equations have at least one solution $(\xi, \Phi) \in V(\omega) \times H^2(\omega)$ in the sense of distributions.

The cubic operator equation (7.13) generalizes an operator equation originally introduced by Berger [Ber67] and Berger and Fife [BF68], then used by Naumann [Nau74] and Ciarlet et al. [CGK07] for analyzing the von Kármán equations for a nonlinearly elastic plate.

7.3 The finite element method

7.3.1 The discrete cubic operator equation

Let ω be a bounded connected, open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary γ, we henceforth assume that γ is a polygon, so that $\bar{\omega}$ can be exactly covered by a regular family of triangulations.

Let $W_h \subset H^2(\omega)$, $V_h \subset V(\omega)$, $V_{0h} \subset H_0^2(\omega)$, be standard conforming finite element spaces associated with such a family, i.e. that satisfy the minimal conditions of [[Cia78], Theorem 6.1-7]. As usual, the parameter h denotes the greatest diameter of all the finite elements found in a given triangulation, strong and weak convergence are noted $\to$ and $\rightharpoonup$ respectively. All convergences are meant to hold as $h \to 0$.

Let $\widetilde{\chi}_h \in W_h$ denote standard finite element approximation of $\widetilde{\chi} \in H^2(\omega)$, which therefor satisfies

$$\|\widetilde{\chi}_h - \widetilde{\chi}\|_{H^2(\omega)} \to 0. \tag{7.15}$$

Let $F_h \in V_h$ denote the unique solution of the variational equations

$$-\int_\omega \partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2 F_h)\eta_h d\omega = \int_\omega f \eta_h d\omega \text{ for all } \eta_h \in V_h,$$

with satisfies

$$\|F_h - F\|_{H^2(\omega)} \to 0. \tag{7.16}$$

Let the bilinear mapping

$$B_h : H^2(\omega) \times H^2(\omega) \to V_{0h},$$

be defined as follows: for each pair $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B_h(\xi, \eta) \in V_{0h}$ is the unique solution of the variational equations,

$$\int_\omega \Delta B_h(\xi, \eta) \Delta \varsigma_h d\omega = \int_\omega [\xi, \eta] \varsigma_h d\omega \text{ for all } \varsigma_h \in V_{0h},$$

hence, for $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$ fixed,

$$\|B_h(\xi, \eta) - B(\xi, \eta)\|_{H^2(\omega)} \to 0. \tag{7.17}$$

Finally, let the another bilinear mapping

$$\widetilde{B}_h : H^2(\omega) \times H^2(\omega) \to V_h,$$

be defined as follows: for each pair $(\Phi,\xi)\in H^2(\omega)\times H^2(\omega)$, the function $\widetilde{B}_h(\Phi,\xi)\in V_h$ is the unique solution of the variational equations

$$-\int_\omega \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\widetilde{B}_h(\Phi,\xi))\eta_h d\omega=\int_\omega[\Phi,\xi]\eta_h d\omega \text{ for all } \eta_h\in V_h,$$

hence, for $(\Phi,\xi)\in H^2(\omega)\times H^2(\omega)$ fixed,

$$\|\widetilde{B}_h(\Phi,\xi)-\widetilde{B}(\Phi,\xi)\|_{H^2(\omega)}\to 0. \tag{7.18}$$

For each $h>0$, the discrete problem is then defined through the following theorem:

Theorem 7.1 *The discrete problem of generalized Marguerre-von kármán equations consists in finding* $(\widetilde{\xi}_h,\Phi_h)\in V_h\times W_h$*, such that* $\widetilde{\xi}_h$ *satisfies the discrete operator equation:*

$$\widetilde{C}_h(\widetilde{\xi}_h)+(I-\widetilde{L}_h)\widetilde{\xi}_h-\widetilde{F}_h=0 \text{ in } V_h, \tag{7.19}$$

and Φ_h *is given by*

$$\Phi_h=\widetilde{\chi}_h-B_h(\widetilde{\xi}_h,\widetilde{\xi}_h) \text{ in } W_h, \tag{7.20}$$

where the discrete cubic mapping $\widetilde{C}_h:V_h\to V_h$ *is defined by*

$$\widetilde{C}_h(\eta_h)=\widetilde{B}_h(B_h(\eta_h,\eta_h),\eta_h),$$

the linear mapping $\widetilde{L}_h:V_h\to V_h$ *is defined by*

$$\widetilde{L}_h\eta_h=\widetilde{B}_h(\widetilde{\chi}_h,\eta_h),$$

and

$$\widetilde{\xi}_h=\widetilde{\theta}+\xi_h \text{ and } \widetilde{F}_h=\widetilde{\theta}+F_h.$$

Proof. The discrete problem of generalized Marguerre-von kármán equations consists in finding $(\widetilde{\xi}_h,\Phi_h)\in V_h\times W_h$, such that $\widetilde{\xi}_h$ satisfies the variational equation

$$-\int_\omega \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2(\widetilde{\xi}_h-\widetilde{\theta}))\eta_h d\omega=\int_\omega([\Phi_h,\widetilde{\xi}_h]+f)\eta_h d\omega \text{ for all } \eta_h\in V_h, \tag{7.21}$$

and Φ_h satisfies the variational equation

$$\int_\omega \Delta^2\Phi_h.\vartheta_h = \int_\omega ([\widetilde{\theta},\widetilde{\theta}] - [\widetilde{\xi}_h,\widetilde{\xi}_h])\vartheta_h d\omega \text{ for all } \vartheta_h \in W_h. \tag{7.22}$$

By definition of the function $\widetilde{\chi}_h$ and the mapping B_h, (7.22) imply that

$$\Phi_h = \widetilde{\chi}_h - B_h(\widetilde{\xi}_h,\widetilde{\xi}_h) \text{ in } W_h.$$

By definition of the function $\widetilde{F}_h$ and the mapping $\widetilde{B}_h$, (7.21) imply that

$$\widetilde{\xi}_h - \widetilde{F}_h = \widetilde{B}_h(\Phi_h,\widetilde{\xi}_h) \text{ in } V_h. \tag{7.23}$$

Eliminating Φ_h between these two operator equations (7.20) and (7.23), yields the single operator equation

$$\widetilde{B}_h(B_h(\widetilde{\xi}_h,\widetilde{\xi}_h),\widetilde{\xi}_h) + \widetilde{\xi}_h - \widetilde{B}_h(\widetilde{\chi}_h,\widetilde{\xi}_h) - \widetilde{F}_h = 0 \text{ in } V_h.$$

Then, we conclude that $\widetilde{\xi}_h \in V_h$ is found by solving the discrete operator equation:

$$\widetilde{C}_h(\widetilde{\xi}_h) + (I - \widetilde{L}_h)\widetilde{\xi}_h - \widetilde{F}_h = 0 \text{ in } V_h,$$

■

Naturally, finding the solution $\widetilde{\xi}_h$ of the above discrete operator equation (7.19) is equivalent to solving the following discrete variational problem:

$$(\mathcal{P}_h)^{iso}_{sta}\begin{cases} \text{Find } \widetilde{\xi}_h \in V_h \text{ such that,} \\ ((\widetilde{C}_h(\widetilde{\xi}_h) + (I - \widetilde{L}_h)\widetilde{\xi}_h - \widetilde{F}_h,\eta_h)) = 0 \text{ for all } \eta_h \in V_h. \end{cases}$$

7.3.2 Convergence

We will need the following lemma:

Lemma 7.1 *The bilinear mapping B_h is sequentially compact, hence a fortiori continuous, in the sense that, if*

$$(\xi_h,\eta_h) \rightharpoonup (\xi,\eta) \in [H^2(\omega)]^2,$$

then

$$B_h(\xi_h,\eta_h) \to B_h(\xi,\eta) \in H^2_0(\omega).$$

Proof.

We define the following inner-product on $H_0^2(\omega)$

$$(\zeta,\varsigma)_\Delta = \int_\omega \Delta\zeta\Delta\varsigma d\omega,$$

and let $\|.\|_\Delta$ denote the norm over the space $H_0^2(\omega)$, which corresponds to the inner product $(.,.)_\Delta$.

From the definition of the mapping B_h, we get

$$(B_h(\xi,\eta),\varsigma)_\Delta = \int_\omega [\xi,\eta]\varsigma d\omega,$$

for all $(\xi,\eta,\varsigma) \in [H^2(\omega)]^2 \times H_0^2(\omega)$.

Then there exists a constant c_1 such that

$$\|B_h(\xi,\eta)\|_\Delta \leq c_1\|\xi\|_{W^{1,4}(\omega)}\|\eta\|_{W^{1,4}(\omega)}, \tag{7.24}$$

for all $(\xi,\eta) \in [H^2(\omega)]^2$.

Let $(\xi_h,\eta_h) \rightharpoonup (\xi,\eta) \in [H^2(\omega)]^2$, using the bilinearity of B_h, we have

$$B_h(\xi_h,\eta_h) - B_h(\xi,\eta) = B_h(\xi_h-\xi,\eta) + B_h(\xi,\eta_h-\eta) + B_h(\xi_h-\xi,\eta_h-\eta).$$

From (7.24), it follows that there exists a constant c_2 such that

$$\begin{aligned}\|B_h(\xi_h,\eta_h) - B_h(\xi,\eta)\|_\Delta \quad &\leq \quad c_2(\|\xi_h-\xi\|_{W^{1,4}(\omega)}\|\eta\|_{W^{1,4}(\omega)} + \|\xi\|_{W^{1,4}(\omega)}\|\eta_h-\eta\|_{W^{1,4}(\omega)} \\ &+ \quad \|\xi_h-\xi\|_{W^{1,4}(\omega)}\|\eta_h-\eta\|_{W^{1,4}(\omega)}).\end{aligned}$$

The compact imbedding of $H^2(\omega)$ into $W^{1,4}(\omega)$ implies that $B_h(\xi_h,\eta_h) \to B_h(\xi,\eta) \in H_0^2(\omega)$, for more details see the proof (part (iv)) of [Cia97, Theorem 5.8-2]. ■

Theorem 7.2 *Assume that ω is simply-connected, the functions h_α satisfy natural compatibility conditions, and their norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough. Then*

(a) *there exists a constant M such that, for each $h > 0$, the discrete variational problem $(\mathcal{P}_h)_{sta}^{iso}$ has at least one solution $\widetilde{\xi}_h \in V_h$ that satisfies $\|\widetilde{\xi}_h\| \leq M$.*

(b) *Let $(\widetilde{\xi}_h)_{h>0}$ be any subsequence that weakly converges in $H^2(\omega)$, let $\xi \in V(\omega)$ denote its limit and let the associated subsequence $(\Phi_h)_{h>0}$ be defined by (7.20). Then ξ is a solution of the variational problem $(\mathcal{P})_{sta}^{iso}$, and*

$$(\widetilde{\xi}_h,\Phi_h) \to (\widetilde{\xi},\Phi) \text{ in } H^2(\omega)\times H^2(\omega),$$

where Φ is defined by (7.14).

Proof.

(a) (i) The definitions of the mapping $\widetilde{B}_h$ and the function $\widetilde{\chi}_h$ imply that the linear mapping $\widetilde{L}_h$ is continuous, then there exists a constant c_3 such that

$$\|\widetilde{L}_h\|_{\mathcal{L}(V(\omega))} \leq c_3 \sum_\alpha \|h_\alpha\|_{L^2(\gamma_1)}. \tag{7.25}$$

(ii) From the definitions of the mappings B_h, $\widetilde{B}_h$ and $\widetilde{C}_h$, we conclude that, for any $\eta_h \in V_h$

$$\begin{aligned} ((\widetilde{C}_h(\eta_h), \eta_h)) &= ((\widetilde{B}_h(B_h(\eta_h, \eta_h), \eta_h)) \\ &= \int_\omega [B_h(\eta_h, \eta_h), \eta_h]\eta_h d\omega. \end{aligned}$$

Taking into account $B_h(\eta_h, \eta_h) \in H_0^2(\omega)$, then applying [Cia97, Theorem 5.8-2], we deduce that

$$\begin{aligned} ((\widetilde{C}_h(\eta_h), \eta_h)) &= \int_\omega [\eta_h, \eta_h] B_h(\eta_h, \eta_h) d\omega \\ &= \int_\omega \Delta B_h(\eta_h, \eta_h) \Delta B_h(\eta_h, \eta_h) d\omega \\ &= \|\Delta B_h(\eta_h, \eta_h)\|^2_{L^2(\omega)} \geq 0. \end{aligned}$$

So that, the discrete cubic operator $\widetilde{C}_h$ satisfies

$$((\widetilde{C}_h(\eta_h), \eta_h)) \geq 0 \text{ for all } \eta_h \in V_h. \tag{7.26}$$

(iii) Since the mapping B_h is sequentially compact (see Lemma 7.1), so that if $\widetilde{\xi}_h \in W_h$, be such that

$$\widetilde{\xi}_h \rightharpoonup \widetilde{\xi} \text{ in } H^2(\omega).$$

Then

$$B_h(\widetilde{\xi}_h, \widetilde{\xi}_h) \to B_h(\widetilde{\xi}, \widetilde{\xi}) \text{ in } H_0^2(\omega).$$

From (7.17), it follows that

$$B_h(\widetilde{\xi}_h, \widetilde{\xi}_h) \to B(\widetilde{\xi}, \widetilde{\xi}) \text{ in } H_0^2(\omega). \tag{7.27}$$

(iv) Let $\widetilde{\chi}_h \in W_h$, $\widetilde{\xi}_h \in W_h$, and $\eta_h \in W_h$ be such that

$$\widetilde{\chi}_h \to \widetilde{\chi} \text{ in } H^2(\omega), \widetilde{\xi}_h \rightharpoonup \widetilde{\xi} \text{ in } H^2(\omega), \eta_h \rightharpoonup \eta \text{ in } H^2(\omega), \tag{7.28}$$

then

$$((\widetilde{B}_h(\widetilde{\chi}_h, \widetilde{\xi}_h), \eta_h)) \to ((\widetilde{B}(\widetilde{\chi}, \widetilde{\xi}), \eta)). \tag{7.29}$$

To shown this, we have

$$\begin{aligned} ((\widetilde{B}_h(\widetilde{\chi}_h, \widetilde{\xi}_h), \eta_h)) - ((\widetilde{B}(\widetilde{\chi}, \widetilde{\xi}), \eta)) &= \int_\omega [\widetilde{\chi}_h, \widetilde{\xi}_h]\eta_h d\omega - \int_\omega [\widetilde{\chi}, \widetilde{\xi}]\eta d\omega \\ &= \int_\omega [\widetilde{\chi}_h, \widetilde{\xi}_h]\eta_h d\omega + \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h]\eta_h d\omega \\ &- \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h]\eta_h d\omega + \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h]\eta d\omega \\ &- \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h]\eta d\omega - \int_\omega [\widetilde{\chi}, \widetilde{\xi}]\eta d\omega \\ &= \int_\omega [\widetilde{\chi}_h - \widetilde{\chi}, \widetilde{\xi}_h]\eta_h d\omega \\ &+ \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h](\eta_h - \eta) d\omega \\ &+ \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h - \widetilde{\xi}]\eta d\omega. \end{aligned}$$

The compact imbedding of $H^2(\omega)$ into $\mathcal{C}^0(\omega)$, gives

$$\int_\omega [\widetilde{\chi}_h - \widetilde{\chi}, \widetilde{\xi}_h]\eta_h d\omega \leq c_4 \|\widetilde{\chi}_h - \widetilde{\chi}\|_{H^2(\omega)} \|\widetilde{\xi}_h\|_{H^2(\omega)} \|\eta_h\|_{H^2(\omega)}, \tag{7.30}$$

and

$$\begin{aligned} \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h](\eta_h - \eta) d\omega &\leq \|[\widetilde{\chi}, \widetilde{\xi}_h]\|_{L^1(\omega)} \|\eta_h - \eta\|_{C^0(\bar{\omega})}, \\ &\leq c_5 \|\widetilde{\chi}\|_{H^2(\omega)} \|\widetilde{\xi}_h\|_{H^2(\omega)} \|\eta_h - \eta\|_{C^0(\bar{\omega})}. \end{aligned} \tag{7.31}$$

Since $\widetilde{\chi}_h \to \widetilde{\chi}$ in $H^2(\omega)$ and $\widetilde{\xi}_h \rightharpoonup \widetilde{\xi}$ in $H^2(\omega)$, the inequality (7.30) and (7.31) imply that the first and second terms approach zero as $h \to 0$.

We have

$$\begin{aligned} \int_\omega [\widetilde{\chi}, \widetilde{\xi}_h - \widetilde{\xi}]\eta d\omega &= \int_\omega (\eta \partial_{11}\widetilde{\chi}\partial_{22}\widetilde{\xi}_h + \eta \partial_{22}\widetilde{\chi}\partial_{11}\widetilde{\xi}_h - 2\eta \partial_{12}\widetilde{\chi}\partial_{12}\widetilde{\xi}_h) d\omega \\ &- \int_\omega [\widetilde{\chi}, \widetilde{\xi}]\eta d\omega, \end{aligned}$$

since $\eta\partial_{\alpha\beta}\chi \in L^2(\omega)$ and $\partial_{\sigma\tau}\xi_h \rightharpoonup \partial_{\sigma\tau}\xi$ in $L^2(\omega)$, imply that the third terms approach zero as $h \to 0$ which implies (7.29).

(v) By adapting a compactness method due to Lions (see [Lio69, Chap. 1, Theorem 4.3]), we show that, if the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough, there exists a constant M independent of h, such that the discrete problem $(\mathcal{P}_h)^{iso}_{sta}$ has at least one solution $\widetilde{\xi}_h$ that satisfies $\|\widetilde{\xi}_h\| \leq M$.

To see this, let w_i^h, $1 \leq i \leq d(h)$ be a basis of V_h that is orthonormal with respect to the inner product $((.,.))$.

Let $\langle .,. \rangle$ and $|.|$ denote the Euclidean inner product and Euclidean norm in $\mathbb{R}^{d(h)}$ and let $\mathbf{X} = (X_i)_{i=1}^{d(h)}$ be any vector in $\mathbb{R}^{d(h)}$.

We define the mapping $\eta_h : \mathbb{R}^{d(h)} \to V_h$ by letting

$$\eta_h(\mathbf{X}) = \sum_{i=1}^{d(h)} X_i w_i^h \text{ for all } \mathbf{X} \in \mathbb{R}^{d(h)},$$

and we define the mapping $\mathbf{G}^h = (G_i^h) : \mathbb{R}^{d(h)} \to \mathbb{R}^{d(h)}$ by letting, for all $\mathbf{X} \in \mathbb{R}^{d(h)}$

$$(G_i^h)(\mathbf{X}) = ((\widetilde{C}_h(\eta_h(\mathbf{X})) + (I - \widetilde{L}_h)\eta_h(\mathbf{X}) - \widetilde{F}_h, w_i^h)) = 0, \quad 1 \leq i \leq d(h).$$

So that, for all $\mathbf{X} \in \mathbb{R}^{d(h)}$

$$\left\langle \mathbf{G}^h(\mathbf{X}), \mathbf{X} \right\rangle = ((\widetilde{C}_h(\eta_h(\mathbf{X})) + (I - \widetilde{L}_h)\eta_h(\mathbf{X}) - \widetilde{F}_h, \eta_h(\mathbf{X}))).$$

Since $\|\widetilde{F}_h\| \leq c_6 \|f\|_{L^2(\omega)}$ and the properties established in (i) and (ii) imply that

$$\begin{aligned} \left\langle \mathbf{G}^h(\mathbf{X}), \mathbf{X} \right\rangle &\geq (1 - \|\widetilde{L}_h\|_{\mathcal{L}(V(\omega))})|\mathbf{X}|^2 - \|\widetilde{F}_h\||\mathbf{X}| \\ &\geq (1 - c_3 \sum_\alpha \|h_\alpha\|_{L^2(\gamma_1)})|\mathbf{X}|^2 - c_6 \|f\|_{L^2(\omega)}|\mathbf{X}|. \end{aligned}$$

We assume that the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough, in the sense that $\sum_\alpha \|h_\alpha\|_{L^2(\gamma_1)} \leq c_3^{-1}$ and let

$$M = c_6(1 - c_3 \sum_\alpha \|h_\alpha\|_{L^2(\gamma_1)})^{-1} \|f\|_{L^2(\omega)},$$

thus

$$\left\langle \mathbf{G}^h(\mathbf{X}), \mathbf{X} \right\rangle \geq 0,$$

for all $\mathbf{X} \in \mathbb{R}^{d(h)}$, such that $|\mathbf{X}| = M$.

Hence a simple corollary to the Brouwer fixed point theorem (see [Lio69, Chap. 1, Lemma 4.3]) applied to the continuous mapping $\mathbf{G}^h$ (the continuity of $\mathbf{G}^h$ follows from that of the mappings $\widetilde{C}_h$ and $\widetilde{L}_h$) shows that, there exists at least one vector $\mathbf{X} \in \mathbb{R}^{d(h)}$ such that

$$\mathbf{G}^h(\mathbf{X}) = 0 \text{ and } \|\mathbf{X}\| \le M.$$

Equivalently, there thus exists at least one solution

$$\widetilde{\xi}_h = \sum_{i=1}^{d(h)} X_i w_i^h \in V_h,$$

to problem $(\mathcal{P}_h)^{iso}_{sta}$ such that $\|\widetilde{\xi}_h\| \le M$.

(b) (i) Since the sequence $(\widetilde{\xi}_h)_{h>0}$ found in (a)-(v) is bounded independently of h in the space $V(\omega)$, there exists a subsequence $(\widetilde{\xi}_h)_{h>0}$ and $\widetilde{\xi} \in V(\omega)$ such that

$$\widetilde{\xi}_h \rightharpoonup \widetilde{\xi} \text{ in } H^2(\omega). \tag{7.32}$$

Then $\widetilde{\xi}$ is a solution of the variational problem $(\mathcal{P})^{iso}_{sta}$. Too see this, given any $\eta \in V(\omega)$, there exists functions $\eta_h \in V_h$ such that

$$\eta_h \to \eta \text{ in } H^2(\omega),$$

so that, by part (a)-(v),

$$((\widetilde{C}_h(\widetilde{\xi}_h) + (I - \widetilde{L}_h)\widetilde{\xi}_h - \widetilde{F}_h, \eta_h)) = 0 \text{ for all } \eta_h \in V_h. \tag{7.33}$$

From (7.16), it follows that

$$((\widetilde{\xi}_h - \widetilde{F}_h, \eta_h)) \to ((\widetilde{\xi} - \widetilde{F}, \eta)).$$

By definition of the linear mapping $\widetilde{L}_h$, we know that

$$((\widetilde{L}_h\widetilde{\xi}_h, \eta_h)) = ((\widetilde{B}_h(\widetilde{\chi}_h, \xi_h), \eta_h)),$$

by part (a)-(iv), we know that $((\widetilde{B}_h(\widetilde{\chi}_h, \xi_h), \eta_h)) \to ((\widetilde{B}(\widetilde{\chi}, \xi), \eta))$, hence

$$((\widetilde{L}_h\widetilde{\xi}_h, \eta_h)) \to ((\widetilde{L}\widetilde{\xi}, \eta)),$$

By definition of the cubic mapping $\widetilde{C}_h$, we know that

$$((\widetilde{C}_h(\xi_h), \eta_h)) \;=\; \int_\omega [B_h(\xi_h, \xi_h), \xi_h]\eta_h d\omega.$$

by part (a)-(iii), we know that $B_h(\widetilde{\xi}_h, \widetilde{\xi}_h) \to B(\widetilde{\xi}, \widetilde{\xi})$ in $H_0^2(\omega)$, so that

$$[B_h(\xi_h, \xi_h), \xi_h] \;\to\; [B(\xi, \xi), \xi] \text{ in } L^1(\omega), \tag{7.34}$$

thus

$$((\widetilde{C}_h(\widetilde{\xi}_h), \eta_h)) \to ((\widetilde{C}(\widetilde{\xi}), \eta)).$$

Then passing to the limit as $h \to 0$ in (7.33), we obtain

$$((\widetilde{C}(\widetilde{\xi}) + (I - \widetilde{L})\widetilde{\xi} - \widetilde{F}, \eta)) = 0 \text{ for all } \eta \in V(\omega).$$

(ii) The subsequence $(\widetilde{\xi}_h)_{h>0}$ found in (b)-(i), satisfies strongly convergent

$$\widetilde{\xi}_h \to \widetilde{\xi} \text{ in } H^2(\omega). \tag{7.35}$$

To shown this, we let $\eta_h = \widetilde{\xi}_h$ in the variational equations of $(\mathcal{P}_h)^{iso}_{sta}$. Then

$$((\widetilde{C}_h(\widetilde{\xi}_h), \widetilde{\xi}_h)) + \|\widetilde{\xi}_h\|^2 - ((\widetilde{B}_h(\widetilde{\chi}_h, \widetilde{\xi}_h), \widetilde{\xi}_h)) - ((\widetilde{F}_h, \widetilde{\xi}_h)) = 0.$$

From (7.16), it follows that

$$((\widetilde{F}_h, \widetilde{\xi}_h)) \to ((\widetilde{F}, \widetilde{\xi})). \tag{7.36}$$

by part (a)-(iv), we conclude that

$$((\widetilde{B}_h(\widetilde{\chi}_h, \xi_h), \xi_h)) \to ((\widetilde{B}(\widetilde{\chi}, \xi), \xi)) \tag{7.37}$$

By definition of the cubic mapping $\widetilde{C}_h$, we know that

$$((\widetilde{C}_h(\xi_h), \xi_h)) \;=\; \int_\omega [B_h(\xi_h, \xi_h), \xi_h]\xi_h d\omega.$$

Using (7.34), we get

$$((\widetilde{C}_h(\widetilde{\xi}_h), \xi_h)) \to ((\widetilde{C}(\widetilde{\xi}), \xi)). \tag{7.38}$$

From $(7.36) - (7.38)$ and since $\widetilde{\xi}$ is a solution to the variational problem $(\mathcal{P})^{iso}_{sta}$, we deduce that

$$\|\widetilde{\xi}_h\|^2 \to \|\widetilde{\xi}\|^2,$$

which implies (7.35).

(iii) It remains to be shown

$$\Phi_h \to \Phi \text{ in } H^2(\omega). \tag{7.39}$$

From (7.20), we know that

$$\Phi_h = \widetilde{\chi}_h - B_h(\widetilde{\xi}_h, \widetilde{\xi}_h).$$

Since $\widetilde{\chi}_h \to \widetilde{\chi}$ in $H^2(\omega)$ and $B_h(\widetilde{\xi}_h, \widetilde{\xi}_h) \to B(\widetilde{\xi}, \widetilde{\xi})$, we obtain

$$(\widetilde{\chi}_h - B_h(\widetilde{\xi}_h, \widetilde{\xi}_h)) \to (\widetilde{\chi} - B(\widetilde{\xi}, \widetilde{\xi})),$$

which implies (7.39).

■

7.4 Conclusion

In this Chapter, we establish the convergence of a conforming finite element approximations to the generalized Marguerre-von Kármán equations. We first reduce the discrete problem of these equations to a single discrete cubic operator equation, whose unknown is the approximate of vertical displacement of the shallow shell. We next solve this discrete operator equation, by adapting a compactness method due to J.L. Lions and Brouwer's fixed point theorem. Then we establish the convergence of a conforming finite element approximations to these equations. Using weak regularity on solutions, but in order to get an error estimates it need more regularity.

Conclusions and perspectives

The major conclusions of these studies are:

Firstly, a mathematical justification of five new two-dimensional models in nonlinear shallow shells theory by asymptotic analysis method:

1. Dynamical equations of generalized Marguerre-von Kármán shallow shells.
2. Dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells.
3. Generalized Marguerre-von Kármán equations with Signorini conditions.
4. Dynamical contact equations of generalized Marguerre-von Kármán shallow shells.
5. Dynamical contact equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells.

Secondly, the existence of a solution to the two models 1 and 4.

Thirdly, the convergence of a conforming finite element approximations to the generalized Marguerre-von Kármán equations.

Notice that, in the case $\gamma = \gamma_1$, the previous models reduce to the classical Marguerre-von Kármán shallow shell models. If the function $\theta \equiv 0$ in $\bar{\omega}$, the shallow shell becomes a plate. Then the generalized Marguerre-von Kármán shallow shell models reduce to the generalized von Kármán plate models.

As future work, we plan to:

1. Derive an estimate for the error of the approximate solution to the generalized Marguerre-von Kármán equations, which obtained in last Chapter and numerically study for these equations.

2. Study the existence of a solution to the models 2, 3 and 5.

3. Study the numerical analysis for two models 1 and 4. It is a natural complement to our study, where we establish the existence of solutions to these models.

4. Extend these studies for shallow shell to general shell problems.

Bibliography

[AB99] G. Andreoiu-Banica. Justification of the Marguerre-von Kármán equations in curvilinear coordinates. *Asymptotic Analysis, 19:35–55*, 1999.

[Ada75] R.A. Adams. *Sobolev space.* Academic Press, New York, 1975.

[AMM11] H. Abels, M.G. Mora, and S. Müller. The time-dependent von Kármán plate equation as a limit of 3d nonlinear elasticity. *Calc. Var. Partial Differ Equ.,41(1):241–259*, 2011.

[BBBT02] F. Ben Belgacem, C. Bernardi, A. Belouza, and F. Taallah. On the obstacle problem for a Naghdi shell. *J. Elasticity, 103:1–13*, 2002.

[BCG13] A. Bensayah, D.A. Chacha, and A. Ghezal. Asymptotic modelling of a Signorini problem of generalized Marguerre-von Kármán shallow shells. *Applicable Analysis, 92(9):1848–1862*, 2013.

[Ber67] M.S. Berger. On von Kármán equations and the buckling of a thin elastic plate. I. The clamped plate. *Comm. Pure Appl. Math., 20:687–719*, 1967.

[BF68] M.S. Berger and P.C. Fife. On von Kármán equations and the buckling of a thin elastic plate. II. Plate with general edge conditions. *Comm. Pure Appl. Math., 21:227–241*, 1968.

[BGL91] H. Begehr, R.P. Gilbert, and C.Y. Lo. The two-dimensional nonlinear orthotropic plate. *J. Elasticity, 26:147–167*, 1991.

[BJ09] I. Bock and J. Jarušek. Solvability of dynamic contact problems for elastic von Kármán plates. *SIAM J. Math. Anal., 41(1):37–45*, 2009.

[Böh96] M. Böhm. Global well posedness of the dynamic von Kármán equations for generalized solutions. *Nonlinear Anal. Theory Methods Appl., 27(3):339–351*, 1996.

[Bre78] F. Brezzi. Finite element approximations of the von Kármán equations. *R.A.I.R.O., Analyse numérique, 12(4):303–312*, 1978.

[BRR80] F. Brezzi, J. Rappaz, and P.A. Raviart. Finite dimensional approximation of nonlinear problems, Part I: Branches of nonsingular solutions. *Numer. Math., 36:1–25*, 1980.

[BRR81] F. Brezzi, J. Rappaz, and P.A. Raviart. Finite dimensional approximation of nonlinear problems, Part III: Simple bifurcation points. *Numer. Math., 38:1–30*, 1981.

[CB08] D.A. Chacha and A. Bensayah. Asymptotic modeling of a Coulomb frictional Signorini problem for the von Kármán plates. *C. R. Mécanique, 336:846–850*, 2008.

[CG01] P.G. Ciarlet and L. Gratie. Generalized von Kármán equations. *J. Math. Pures Appl., 80(3):263–279*, 2001.

[CG06a] P.G. Ciarlet and L. Gratie. From the classical to the generalized von Kármán and Marguerre-von Kármán equations. *Computational and Applied Mathematics, 190:470–486*, 2006.

[CG06b] P.G. Ciarlet and L. Gratie. On the existence of solutions to the generalized Marguerre-von Kármán equations. *Math. Mech. Solids, 11:83–100*, 2006.

[CGB10] D.A. Chacha, A. Ghezal, and A. Bensayah. Modélisation asymptotique d'une coque peu-profonde de Marguerre-von Kármán généralisée dans le cas dynamique. *Revue ARIMA, 13:63–76*, 2010.

[CGB13] D.A. Chacha, A. Ghezal, and A. Bensayah. Existence result for a dynamical equations of generalized Marguerre-von Kármán shallow shells. *Journal Elasticity, 111:265–283*, 2013.

[CGK07] P.G. Ciarlet, L. Gratie, and S. Kesavan. On the generalized von Kármán equations and their approximation. *Journal of Mathematical Models and Methods in Applied Sciences, 17:617–633*, 2007.

[Cia78] P.G. Ciarlet. The finite element method for elliptic problems. *North-Holland*, 1978.

[Cia90] P.G. Ciarlet. *Plates and junctions in elastic multistructures*. Masson, 1990.

[Cia97] P.G. Ciarlet. *Mathematical elasticity, Theory of plates, vol. II.* North-Holland, Amsterdam, 1997.

[CL04] I. Chueshov and I. Lasiecka. Global attractors for von Kármán evolutions with a non-linear boundary dissipation. *J. Differential Equations, 198:196–221*, 2004.

[CL07] I. Chueshov and I. Lasiecka. Long-time dynamics of von Kármán semi-flows with non-linear boundary/interior damping. *J. Differential Equations, 233:42–86*, 2007.

[CM12] D.A. Chacha and M. Miloudi. Asymptotic analysis of nonlinearly elastic shells "mixed approach". *Asymptotic Analysis, 80:323–346*, 2012.

[Cou81] C.A. Coulomb. *Théorie des machines simples, en ayant égard au frottement de leurs parties et la raideur des cordages.* Prix double Acad. Sci., 1781.

[CP86] P.G. Ciarlet and J.C. Paumier. A justification of the Marguerre-von Kármán equations. *Comput. Mech., 1:177–202*, 1986.

[CSP95] D. Caillerie and E. Sanchez-Palencia. Elastic thin shells: asymptotic theory in the anisotropic and heterogeneous cases. *Math. Models Methods Appl. Sci., 5:473–496*, 1995.

[Des80] P. Destuynder. *Sur une Justification des Modèles de Plaques et de Coques par les Méthodes Asymptotiques.* Doctoral Dissertation, Université Pierre et Marie Curie, Paris, 1980.

[DJG06] G. Devdariani, R. Janjgava, and B. Gulua. Dirichlet problem for the Marguerre-von Kármán equations system. *Bulletin of TICMI, 10:23–27*, 2006.

[DL72] G. Duvaut and J.L. Lions. Les inéquations en mécanique et en physique. *Dunod, Paris*, 1972.

[EJK05] C. Eck, J. Jarušek, and M. Krbec. Unilateral contact problems in mechanics. variational methods and existence theorems. *Singapore*, 2005.

[GC14] A. Ghezal and D.A. Chacha. Convergence of finite element approximations for generalized Marguerre-von Kármán equations. *Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics 87, 97–106*, 2014.

[Gir98] P. Giroud. Analyse asymptotique de coques inhomogènes en élasticité linéarisée anisotrope. *C.R. Acad. Sci. Paris, Sér. I, 327:1011–1014*, 1998.

[Gra02] L. Gratie. Generalized Marguerre-von Kármán equations of a nonlinearly elastic shallow shell. *Applicable Anal., 81:1107–1126*, 2002.

[GV00] R.P. Gilbert and T.S. Vashakmadze. A two-dimensional nonlinear theory of anisotropic plates. *Mathematical and Computer Modelling, 32:855–875*, 2000.

[Kes79] S. Kesavan. La méthode de Kikuchi appliquée aux équations de von Kármán. *Numer. Math., 32:209–232*, 1979.

[Kes80] S. Kesavan. Une méthode d'éléments finis mixte pour les équations de von Kármán. *R.A.I.R.O., Analyse numérique, 14(2):149–173*, 1980.

[KO88] N. Kikuchi and J.T. Oden. Contact problems in elasticity: A study of variational inequalities and finite element methods. *SIAM, Philadelphia*, 1988.

[KR93] O. Kavian and B.P. Rao. Une remarque sur l'existence de solutions non nulles pour les équations de Marguerre-von Kárm *C.R.Acad.Sci. Pari Sér., 1317:1137–1142*, 1993.

[KS83] S. Kesavan and P.N. Srikanth. On the Dirichlet problem for the Marguerre equations. *Nonlinear Analysis. Theory, Methods & Applications, 7(2):209–216*, 1983.

[KS93] H. Koch and A. Stahel. Global existence of classical solutions to the dynamical von Kármán equations. *Math. Meth. Appl. Sci., 16:581–586*, 1993.

[LB09] F. Li and Y. Bai. Uniform decay rates for nonlinear viscoelastic Marguerre-von Kármán equations. *J. Math. Anal. Appl., 351:522–535*, 2009.

[Li09] F. Li. Global existence and uniqueness of weak solution to nonlinear viscoelastic full Marguerre-von Kármán shallow shell equations. *Acta Mathematica Sinica. English Series, 25(12):2133–2156*, 2009.

[Li10] F. Li. Limit behavior of the solution to nonlinear viscoelastic Marguerre-von Kármán shallow shell system. *J. Differential Equations, 249:1241–1257*, 2010.

[Lio69] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.

[LM68] J.L. Lions and E. Magenes. Problèmes aux limites non homogènes et applications, vol. i et ii. *Dunod, Paris*, 1968.

[LM05] A. Léger and B. Miara. On the multiplicity of solutions to Marguerre-von Kármán membrane equations. *J. Math. Pures Appl., 84:357–374*, 2005.

[LM08] A. Léger and B. Miara. Mathematical justification of the obstacle problem in the case of a shallow shell. *J. Elasticity, 90:241–257*, 2008.

[LM11] A. Léger and B. Miara. The obstacle problem for shallow shells: curvilinear approach. *International journal of numerical analysis and modeling, series B, 2(1):1–26*, 2011.

[Mar38] K. Marguerre. Zur theorie der gekrummten platte grosser formanderung. In *Proceedings. Fifth International Congress for Applied Mechanics, 93–101*, 1938.

[Miy76] T. Miyoshi. A mixed finite element method for the solution of von Kármán equations. *Numer. Math., 26:255–269*, 1976.

[Nau74] J. Naumann. An existence theorem for the von Kármán equations under the condition of free boundary. *Aplikace Mat., 19:17–27*, 1974.

[Pau02] J.C. Paumier. Le problème de Signorini dans la Théorie des plaques minces de Kirchhoff-Love. *C. R. Acad. Sci. Paris, Ser. I, 335:567–570*, 2002.

[PT96] J.P. Puel and M. Tucsnak. Global existence for the full von Kármán system. *Appl. Math. Optim., 34:139–160*, 1996.

[Rao85] A. Raoult. Construction d'un modèle d'évolution de plaques avec terme d'inertie de rotation. *Annali di Matematica Pura ed Applicata, 139:361–400*, 1985.

[Rao95] B.P. Rao. Marguerre-von Kármán equations and membrane model. *Nonlinear Analysis, 24:1131–1140*, 1995.

[Rei82] L. Reinhart. On the numerical analysis of the von Kármán equations: Mixed finite element approximation and continuation techniques. *Numer. Math., 39:371–404*, 1982.

[Sch66] L. Schwartz. *Théorie des distributions.* Hermann, Paris, 1966.

[SHSP92] J. Sanchez-Hubert and E. Sanchez-Palencia. *Introduction aux Méthodes Asymptotiques et à l'Homogénéisation. Application à la Mécanique des Milieux Gontinus.* Masson, Paris, 1992.

[Sig33] A. Signorini. *Sopra akune questioni di elastostatica.* Atti della Societa Italiana per il Progresso delle Scienze, 1933.

[TT97] D. Tataru and M. Tucsnak. On the Cauchy problem for the full von Kármán system. *Nonlinear differ. equ. appl., 4:325–340*, 1997.

[vK10] T. von Kármán. Festigkeitesprobleme in maschinenbau. *encyklopadie der mathematischen wissenschaften, Taubner, 4:311–385*, 1910.

[vKT39] T. von Kármán and H.S. Tsien. The buckling of spherical shells by external pressure. *J. Aero. Sci., 7:43–50*, 1939.

[Xia98] L.M. Xiao. Asymptotic analysis of dynamic problems for linearly elastic shells-justification of equations for dynamic membrane shells. *Asymptotic Analysis, 17:121–134*, 1998.

[Xia99] L.M. Xiao. Existence and uniqueness of solutions to the dynamic equations for Koiter shells. *Applied Mathematics and Mechanics, 20:801–806*, 1999.

[Xia01a] L.M. Xiao. Asymptotic analysis of dynamic problems for linearly elastic shells-justification of equations for dynamic flexural shells. *Chin. Ann. of Math., 22(1):13–22*, 2001.

[Xia01b] L.M. Xiao. Asymptotic analysis of dynamic problems for linearly elastic shells-justification of equations for dynamic Koiter shells. *Chin. Ann. of Math., 22(3):267–274*, 2001.

[Yan06] G. Yan. Asymptotic analysis of linearly elastodynamic shallow shells with variable thickness. *Asymptotic Analysis, 50:1–12*, 2006.

[Ye03] J. Ye. Asymptotic analysis of dynamic problem for linearly elastic generalized membrane shells. *Asymptotic Analysis, 36:47–62*, 2003.

Title: Asymptotic analysis of a Signorini problem with Coulomb friction for shallow shells. Dynamical case

Abstract: The objective of this thesis is to study the asymptotic modeling of three-dimensional problems of nonlinearly elastic shallow shells, in dynamical case, with and without unilateral contact. Also, to study the numerical approximation of the generalized Marguerre-von Kármán equations.

In the first Part, we consider a three-dimensional dynamical models for a nonlinearly elastic shallow shells with a specific class of boundary conditions of generalized Marguerre-von Kármán type, without unilateral contact. Using technics from asymptotic analysis, we justify two two-dimensional models. The first model in homogeneous and isotropic material case, called dynamical equations of generalized Marguerre-von Kármán shallow shells. The second one in nonhomogeneous and anisotropic material case, called dynamical equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells. In addition, we establish the existence of solution to the first model.

In the second Part, we extend the two models in first part, to a Signorini contact with Coulomb friction case. To this end, we justify the dynamical contact equations of generalized Marguerre-von Kármán shallow shells. Also, we establish the existence of solution to these equations. Next, we justify the dynamical contact equations of generalized nonhomogeneous anisotropic Marguerre-von Kármán shallow shells. In addition, we justify the contact equations of generalized Marguerre-von Kármán shallow shells, in static case.

In the third Part, we establish the convergence of a conforming finite element approximations to the generalized Marguerre-von Kármán equations.

Key words: nonlinear shallow shell theory, asymptotic analysis, dynamical problem, Signorini problem, Coulomb friction, Marguerre-von Kármán equations.

Titre: Analyse asymptotique du problème de Signorini avec frottement de Coulomb pour les coques peu-profondes. Cas dynamique

Résumé: L'objectif de cette thèse est d'étudier la modélisation asymptotique des coques peu-profondes non linéairement élastiques, dans le cas dynamique, avec et sans contact unilatéral. Aussi, d'étudier l'approximation numérique des équations de Marguerre-von Kármán généralisées.

Dans la première Partie, nous considérons des modèles tri-dimensionnels dynamiques pour les coques peu-profondes non linéairement élastiques avec une classe spécifique de conditions aux limites de type Marguerre-von Kármán généralisé, sans contact unilatéral. En utilisant les techniques de l'analyse asymptotique, nous justifions deux modèles bi-dimensionnels. Le premier modèle dans le cas d'un matériau homogène et isotrope, appelé les équations dynamiques des coques peu-profondes de Marguerre-von Kármán généralisées. Le second dans le cas d'un matériau non homogène et anisotrope, appelé les équations dynamiques des coques peu-profondes non homogènes anisotropes de Marguerre-von Kármán généralisées. En plus, nous établissons l'existence de solution pour le premier modèle.

Dans la deuxième Partie, nous étendons les deux modèles en première Partie, au cas contact de Signorini avec frottement de Coulomb. À cette fin, nous justifions les équations de contact dynamiques des coques peu-profondes de Marguerre-von Kármán généralisées. Aussi, nous établissons l'existence de solution à ces équations. Ensuite, nous justifions les équations de contact dynamiques des coques peu-profondes non homogènes anisotropes de Marguerre-von Kármán généralisées. En plus, nous justifions les équations de contact des coques peu-profondes de Marguerre-von Kármán généralisées, dans le cas statique.

Dans la troisième Partie, nous établissons la convergence d'une approximations par éléments finis conforme pour les équations de Marguerre-von Kármán généralisées.

Mots clés: théorie de coque peu-profonde non linéaire, analyse asymptotique, problème dynamique, problème de Signorini, frottement de Coulomb, équations de Marguerre-von Kármán.

العنوان: التحليل المقارب لمسألة سينيوريني مع احتكاك كولوم لهياكل ضعيفة الانحناء. حالة ديناميكية

ملخص: الهدف من هذه الأطروحة هو دراسة النمذجة المقاربة لهياكل ضعيفة الانحناء ذات مرونة ليست خطية، في حالة ديناميكية، مع وبدون اتصال من جانب واحد. كذلك دراسة التقريب العددي لمعادلات مارغار- فون كارمان المعممة.

في الجزء الاول، نعتبر نماذج ديناميكية ثلاثية الأبعاد لهياكل ضعيفة الانحناء ذات مرونة ليست خطية مع شروط حدية من نوع مارغار- فون كارمان معمم. بدون اتصال من جانب واحد. باستخدام تقنيات التحليل المقارب، بررنا نموذجين ثنائي الأبعاد. النموذج الأول في الحالة المتجانسة و موحدة الخواص المادية، تدعى المعادلات الديناميكية لهياكل ضعيفة الانحناء لمارغار- فون كارمان المعممة. الثاني في حالة ليست متجانسة و ليست موحدة الخواص المادية، تدعى المعادلات الديناميكية لهياكل ضعيفة الانحناء ليست متجانسة وليست موحدة الخواص لمارغار- فون كارمان المعممة، بالإضافة إلى ذلك أثبتنا وجود حل للنموذج الأول.

في الجزء الثاني، مددنا النموذجين في الجزء الأول، إلى حالة اتصال سينيوريني مع احتكاك كولوم. لتحقيق هذه الغاية، بررنا معادلات الاتصال الديناميكية لهياكل ضعيفة الانحناء لمارغار- فون كارمان المعممة. أيضا أثبتنا وجود حل لهذه المعادلات. ثم بررنا معادلات الاتصال الديناميكية لهياكل ضعيفة الانحناء ليست متجانسة وليست موحدة الخواص لمارغار- فون كارمان المعممة. بالإضافة إلى ذلك بررنا معادلات الاتصال لهياكل ضعيفة الانحناء لمارغار- فون كارمان المعممة، في حالة سكون.

في الجزء الثالث، أثبتنا تقارب تقريبية العناصر المنتهية المتسقة لمعادلات مارغار- فون كارمان المعممة.

الكلمات المفتاحية: نظرية هيكل ضعيف الانحناء ليس خطي، تحليل مقارب، مسألة ديناميكية، مسألة سينيوريني، احتكاك كولوم، معادلات مارغار- فون كارمان.