REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

Université des Frères Mentouri Constantine 1

Faculté des Sciences Exactes

Département de Mathématiques

N° d'ordre: .....72/DS/2020.

Série: ....05/MATHS/2020.

Thèse

Présentée pour l'obtention du diplôme de :

Doctorat en Sciences.

Thème:

Existence and Asymptotic profiles for a problem of wave equation with strong damping

Option: Mathématiques Appliquées

Présentée par :

Mr. **Kassah Laouar Lakhdar**

Membres du jury:

Président	Mr. Ali Hameida	M.C.A	Université Constantine 1
Rapporteur	Mr. Khaled Zennir	M.C.A.	Qassim University- KSA
Co-rapporteur de soutenance	Mr. Mamou Ameur	M.C.A .	Universite de Msila
Examinateur	Mr. Ellagoune Fatah	Prof.	Université de Guelma
Examinatrice	Mlle.Soraya Boughaba	Prof.	Université Constantine 1

Année universitaire 2019/2020

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE

MINISTERE DE L'ENSEIGNEMENT SUPERIEUR ET DE LA RECHERCHE SCIENTIFIQUE

Université des Frères Mentouri Constantine 1

Faculté des Sciences Exactes

Département de Mathématiques

N° d'ordre: 72/DS/2020 .

Série: 05/MATHS/2020.

Thèse

Présentée pour l'obtention du diplôme de :

Doctorat en Sciences.

Thème:

Existence et comportement asymptotique des quelques problèmes des ondes dissipatifs

Option: Mathématiques Appliquées

Présentée par :

Mr. **Kassah Laouar Lakhdar**

Membres du jury: 08/11/2020

Président	Mr. Ali Hameida	M.C.A	Université Constantine 1
Rapporteur	Mr. Khaled Zennir	M.C.A.	Qassim University- KSA
Co-rapporteur de soutenance	Mr. Mamou Ameur	M.C.A .	Universite de Msila
Examinateur	Mr. Ellagoune Fatah	Prof.	Université de Guelma
Examinatrice	Mlle.Soraya Boughaba	Prof.	Université Constantine 1

Année universitaire 2019/2020

Existence et comportement asymptotique des quelques problemes des ondes dissipatifs

Présenté en vue de l'obtention du diplôme de
Doctorat en sciences mathematiques
Universite de Constantine 1

Par: Kassah Laouar Lakhdar

Encadrement: Dr. Khaled Zennir

Acknowledgement

i express my heartiest gratitude to supervisor of my PHD thesis, Dr.khaled Zennir . *For this* thorough guidance and enormous help for each and every progression in my project*, and remerciment Dr Berkane Abdelhak*

I extend thanks to Dr. Amar Guesmia, 20 Aout 1955 University,and Dr. Zoheir Chebbel,for his important Contributions ,and remerciment to Prof.Dennech Mohamed ,and Dr.habbeche Abdellah (allah yarhamou). And Prof . Y.DONG and Dr. Guesmia Aissa; university de Metz, et Nancy (france) Big thanks to Dr. Ali Hameida, Constantine University for his helpful discussions and guidance. I would also like to thank the member of my doctorate degree committee, Dr .Mamou Ameur de m'sila university.Prof Soraya Boughaba, Constantine University 1,I would also Like to thank Prof.. Fatteh Ellagoune, Guelma University.My family, my brothers and sisters, my wife and my children without whose love and Support this project would have been a dream.

Dedication To my family

ملخص

في هذه الأطروحة ، نعتبر دراســة بعض المشــكلات الزائدية (المعادلات ونظام المعادلات) بوجود حدود اللزوجة المرنة تحت بعض الافتراضــات بشــأن البيانات الأولية وشــروط الحدود، وشـروط التخميد وشـروط المصـدر. وتركز الدراسـة على وجود الحلول والسـلوك التقاربي.

الكلمات المفتاحية: معادلة الموجة غير الخطية، اللزوجة، التخميد، النظام المزدوج، الوجود، معدل الاضمحلال، | نظام النقل، الذاكرة النشيطة ، التأخير المتغير.

Abstract

In this thesis, we consider the study of some hyperbolic problems (equations and system of equations) with the presence of a viscoelastic term under some assumptions on initial data and boundary conditions, conditions on damping and source terms. The focuss of the study is on the existence and asymptotic behavior of solutions

Keywords: Nonlinear wave equation, Time delay term, Decay rate, Multiplier method, $p-$Laplacian, Plate equation; Viscoelastic term; Delay term, Nonlinear time-varying delay, Emden-Fowler wave equation; blow-up.

Mathematics Subject Classification: 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20

Résumé

Dans cette these, on considere l'études théorique de quelques problemes de type hyperbolique (équations et systemes des équations) à terme viscoélastique sous quelques hypotheses sur les conditions initiale et au bord, des conditions sur les termes de dissipation, termes sources. Nous avons étudié l'existence et le comportement asymptotique de l'énergie des solutions.

Mots-clés: Equations des ondes nonlineaire, Terme de retard, Mémoire infinie, Méthode de Multiplicateyr, déccroissance polynôme, stabilité exponentielle, semigroupe.

Mathematics Subject Classification: 35L05, 35L20, 35L70, 35L71, 37B25, 35B35, 93D15, 93D20, 93C20

Contents

Introduction

Motivation

The problem of stabilization and control of PDEs play a pivotal role in the current paradigm of fundamental sciences. Evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only in many fields of mathematics, but also in other branches of science such as physics, mechanics and material science. For example, Navier-Stokes and Euler equations of fluid mechanics, nonlinear reaction-diffusion equations of heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrodinger equations of quantum mechanics and Cahn-Hilliard equations of material science, to name just a few, are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences.

The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behavior depends on time not only through the present time but also through their past history.

The problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero, they are several type of stabilization:

1. Strong stabilization: $E(t) \to 0, \text{as } \ t \to \infty.$
2. Uniform stabilization: $E(t) \leq Cexp(-\delta t), \forall t > 0, \ \ (C, \delta > 0).$
3. Polynomial stabilization: $E(t) \leq Ct^{-\delta}, \forall t > 0, \ \ (C, \delta > 0).$
4. Logarithmic stabilization: $E(t) \leq C(ln(t))^{-\delta}, \forall t > 0, \ \ (C, \delta > 0).$

In recent years, an increasing interest has been developed to study the dynamical behavior of several thermoelastic problems so as to describe the thermo-mechanical interactions in elastic materials. In the begining, people mainly considered the dynamical problems of classical thermoelastic systems, the $1-D$ linear model of wich is given as follows:

$$\begin{cases} u_{tt} - u_{xx} - b\theta_x = 0, \quad x \in (0, L), t > 0 \\ \\ \theta_t + \theta_{xx} + bu_{xt} = 0, \quad x \in (0, L), t > 0 \end{cases} \tag{1}$$

Where $u(x,t)$ denotes the deplacement of the rod at time t, and $\theta(x,t)$ is the temperature difference with respect to a fixed reference temperature. In 1960s,Dafermos in [93] discussed the existence of solution of the classical thermoelastic system and showed the asymptotic stability of the system under certain condition.Rivera further proved that the solution of this kind of thermoelastic system decays exponentially.
The classical thermoelasticity is mainly modeled based on the Fourier's law, in wich the speed of thermal propagation is infinite. This violates practical conditions, since the whole materials will not fell instantly at a sudden didturbance in some point(see[64]). In order to eliminate this paradox, Lord and Shulman in [72] employed the modified Fourier's law, proposed by Cattaneo (named Cattaneo's law), and developed what now is known as extended thermoelasticity. Based on this nonclassical thermoelastic theory, many nice results on large time behavior of the thermoelastic systems.
In 1990s, three thermoelastic theories, known as typeI, typeII and type III, respectively, were proposed by Green and Naghdi[61]. They developed their theories by introducing the thermal displacement τ satisfying the following equation.

$$\tau(.,t) = \int_0^t \theta(.,s)ds + \tau(.,0) \tag{2}$$

The typeI theory is consistent with the classical thermoelasticity. the type II is also named thermoelasticity without dissipation, that is, the energy is conservative. these two theories, type I and type II, are restricted cases of the type III given as follows.

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - (\beta\theta_x + k\tau_x)_x = 0. \end{cases} \tag{3}$$

When $k = 0$, the above system becomes(1), the so-called type I thermoelasticity (classical one), and when $b = 0$, the following thermoelastic system is obtained, named type II

$$\begin{cases} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - k\tau_{xx} = 0. \end{cases} \tag{4}$$

Based on these three types of thermoelasticity, there has been an extensive literature on the decay rate for thermoelastic systems in recent years. We refer for instance, ([117]) for the exponential decay and polynomial decay of multi-dimensional thermoelasticity of type III by observability estimates; [58] for the exponential decay for thermoelasticity of type II with porous damping based on frequency domain analysis; [60]) for the stability analysis of thermoelastic Timoshenko-type systems of type III by energy multiplier method; and for the spectral properties of thermoelasticity of type II and for the stability analysis of transmission problem between thermoelasticity and pure elasticity at the interfaces; and [69]for analyticity of solution of thermoelasticities.
From the above results on asymptotic behavior of the systems, we find that for the linear $1-D$ thermoelastic models of type I and type III, the thermal effects are all always strong enough to stabilize the system exponentially, while the one of type II is a conservative system in which there is no dissipation. Thus, an interesting issue is roused that whether or not the system can achieve exponential decay rate when mixing two of them (type I, type II, type III) together, that is, in one part of the domain we have a type of thermoelasticity, but in the other part of the domain, we have another type of thermoelasticity coupling with certain transmission condition at the interface. The dynamical behavior of this kind of transmission problem is difficult to analyze, since coupling exist not only between the therm and elasticity but also at the interface. Liu and Quintanilla in [59]considered the asymptotic behavior of the mixed type II and type III thermoelastic system. They proved that the system is lack of exponential decay rate bat achieves polynomial decay under certain condition. However, the sharpness of the polynomial decay rate for this kind of system is still unknown, wich is very tough issue due to the complex couplings.

Conserved and dissipated quantities

The notion of dissipative - of number, energy, mass, momentum - is a fundamental principle that can be used to derive many partial differential equations.
Any function, especially one with several independent variables, carries a huge amount of information. The questions we want to answer about PDEs are often simple, however. Complete knowledge of the details of an equation's solution are frequently unavailable, and would be overkill in any event. It is therefore useful to study coarse grained quantities that arise in PDEs in order to circumvent a complete analysis of these problems. Notice this philosophy has a long history in science: physicists and chemists like to talk about a system's energy or entropy, which can be understood without any intimate knowledge of the microscopic details.
For some solution of a PDE $u(x,t)$, we can define a coarse-grained quantity as a functional,

which is a mapping from u to the real numbers. For example,

$$\int_{\Omega} u dx, \qquad \int_{\Omega} u_x^2 dx, \qquad \int_{\Omega} u_{xx}^4 dx,$$

are all examples of functionals. It often happens that functionals represent quantities of physical interest mass, energy, momentum, etc. ? but such an interpretation is not essential for these objects to be useful.

Suppose E is some functional of $u(x,t)$ of the form

$$E[u] = \int_{\Omega} f(u, u_x, \dots) dx.$$

so that E depends on t, but not on the variable x which has been integrated out. There are two common properties which depend on the time evolution of E. If $E' = 0$, then E is called conserved. If $E' \leq 0$, then E is called dissipated.

Suppose u solves the wave equation and boundary conditions

$$u'' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional (essentially the sum of kinetic and potential energy)

$$E(t) = \frac{1}{2}\int_0^L u'^2 + u_x^2 dx,$$

is conserved. Indeed,

$$E'(t) = \int_0^L u'u'' + u_x u_x' dx = [u_x u']_0^L + \int_0^L u'u'' + u'u_{xx} dx = 0$$

where integration by parts and the boundary condition was used for the second equality. The fact that E remains the same for all t has profound qualitative implications. Any solution which has wave oscillations initially (so that the energy is positive) must continue to have oscillations for all time - they never die out, for example. Conversely, if the initial conditions are quiescent, so that $E = 0$, then this must happen forever. Notice we learn these things without ever finding a solution of the equation.

As another example, suppose u solves the diffusion equation

$$u' - u_{xx} = 0, \qquad u(0,t) = 0 = u(L,t).$$

Then the energy functional

$$E(t) = \frac{1}{2}\int_0^L u_x^2 dx,$$

is dissipated, since

$$E'(t) = \int_0^L u_x u_x' dx = -\int_0^L u' u_{xx} dx = -\int_0^L u_{xx}^2 dx < 0$$

where again integration by parts and the boundary condition was used.
We can interpret E as follows. The arclength of $x-$cross sections of u can be approximated for small u_x as

$$\int_0^L \sqrt{1+u_x^2}dx \equiv \int_0^L 1+\frac{1}{2}u_x^2 dx.$$

Since $E' \leq 0$, the approximate arclength must also diminish over time. This means the graph of $u(x,.)$ gradually becomes smoother, and oscillations die away. This statement will be made perfectly quantitative by solving the equation outright using separation of variables.

Overview of the dissertation and Target problems

The thesis divided in to four chapters beginning by a general introduction.

The First Chapter

This chapter summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this study. In the next four chapters, we develop our main results for nonlinear evolution problems of hyperbolic type

The second Chapter

In this chapter, We consider the nonlinear (in space and time) wave equation with delay term in the internal feedback. Under conditions on the delay term and the term without delay, we study the asymptotic behavior of solutions using the multiplier method and general weighted integral inequalities. This is published in [Kh. Zennir and L. L. Lakhdar, *Energy decay result for a nonlinear wave $p-$Laplace equation with a delay term*, MATHEMATICA APPLICANDA, Vol. 45(1) 2017, p. 65-80. doi: 10.14708/ma.v45i1.603]

The third Chapter

This chapter, an extensible viscoelastic plate equation with a nonlinear time-varying delay feedback and nonlinear source term is considered. Under suitable assumptions on relaxation function, nonlinear internal delay feedback and source term, we establish general decay of energy by using the multiplier method if the weight of weak dissipation and the delay satisfy $\mu_2 < \frac{\mu_1\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}$. This is published in [B. Feng, Kh. Zennir and

L. L. Lakhdar, *General decay of solutions to an extensible viscoelastic plate equation with a nonlinear time-varying delay feedback*, Bull. Malays. Math. Sci. Soc. (2018). https://doi.org/10.1007/s40840-018-0602-4]

The fourth Chater

The main contributions here are to show the lack of exponential stability and to prove that the t^{-1} is the sharp decay rate of problem (4.1). That is to show that for this types of materials, the dissipation produced by the weak-infinite memories are not strong enough to produce an exponential decay of the solution under usual/non usual conditions on the the relaxation functions's growth. This work extends the previous results by [3], [26] to the weak-viscoelasticities in two parts. In order to fill this gaps, we use an appropriate estimates.This is paper [L. L. LAKHDAR, KH. ZENNIR and S. Boulaaras, The sharp decay rate of thermoelastic transmission system with infinite memories, Rendiconti del Circolo Matematico di Palermo Series 2, https://doi.org/10.1007/s12215-019-00408-1, 2019]

CHAPITER 1

Preliminaries- Technical tools

The aim of this chapter is to recall the essential notions and results used throughout this work. First, we recall some definitions and results on Sobolev spaces and the spaces $L^p(0,T,X)$ and give the statement of some important theorems in the analysis of problems to be studied and eventually some notations used throughout this study.

1.1 Function Annalysis

Normed spaces, Banach spaces and their properties
Let V be linear space.

Definition 1.1 *A* non-negative, degree-1 homogeneous, subadditive functional $\|.\|_V$: $V \longrightarrow \mathbb{R}$ is called a norm if it vanishes only at 0, often, we will write briefly $\|.\|$ instead of $\|.\|_V$ if the following properties are satisfying respectively

$$\begin{cases} \|v\| \geq 0 \\ \|av\| = |a|\|v\| \\ \|u+v\| \leq \|u\| + \|v\| \\ \|v\| = 0 \longrightarrow v = 0. \end{cases}$$

for any $v \in V$ and $a \in \mathbb{R}$.
A linear space equipped with a norm is called a normed linear space. If the last $(i.e.\|v\|_v = 0 \longrightarrow v = 0)$is missing, we call such a functional a semi-norm.

Definition 1.2 A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.

Example of Banach spaces

1. $C[\alpha, \beta]$
   Let $[\alpha, \beta]$ be closed interval $-\infty \leq \alpha < \beta \leq \infty$. Let $C[\alpha, \beta]$ denote the set of all bounded continuous complex-valued functions $x(t)$ on $[\alpha, \beta]$ (If the interval is not bounded, we assume further that $x(t)$ is uniformly continuous). Define $x + y$ and αx by
   $$(x + y)(t) = x(x) + y(t)$$
   $$(\alpha x)(t) = \alpha.x(x)$$
   $C[\alpha, \beta]$ is a Banach space with the norm given by
   $$\|x\| = \sup_{t \in [\alpha, \beta]} |x(t)|.$$
   Converges in this metric is nothing but uniform convergence on the whole space.

2. $L^p(\alpha, \beta), (1 \leq p \leq \infty)$.
   This is the space of all real or complex valued Lebesgue functions f on the open interval (α, β) for which $|f(t)|^p$ is Lebesgue summable over (α, β); two functions f and g which are equal almost everywhere are considered to define the same vector of $L^p(\alpha, \beta)$. $L^p(\alpha, \beta)$ is a Banach space with the norm:
   $$\|f\| \Big(\int_\alpha^\beta |f(t)|^p dt \Big)^{1/p}.$$
   The fact that $\|Vert$ thus defined is a norm follows from Minkowski's inequality; the Riesz-Fischer theorem asserts the completeness of L^p.

3. $L^\infty(\alpha, \beta)$.
   This is the space of all measurable (complex valued) functions f on (α, β) which are essentially bounded, i.e., for every $f \in L^\infty(\alpha, \beta)$ there exists $a > 0$ such that $|f(t)| \leq a$ almost everywhere. Define $\|f\|$ to be the infinitum of such a. (Here also we identify two functions which are equal almost everywhere).

4. V' equipped with the norm $\|.\|_{V'}$ defined by
   $$\|u\|_{V'} = sup|u(x)| : \|x\| \leq 1,$$
   is also a Banach space.
   If V is a Banach space such that, for any
   $$v \in V, V \longrightarrow \mathbb{R} : u \longrightarrow \|u + v\|^2 - \|u - v\|^2,$$

is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by

$$(u, v) = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2.$$

Definition 1.3 Since u is linear we see that

$$u : V \longrightarrow V'',$$

is a linear isometry of V onto a closed subspace of V'' , we denote this by

$$V \longrightarrow V''.$$

Let V be a Banach space and $u \in V'$. Denote by

$$\phi_u : V \longrightarrow \mathbb{R}$$

$$x \longmapsto \phi_u(V),$$

when u covers V', we obtain a family of applications to $V \in \mathbb{R}$.

Definition 1.4 The weak topology on V, denoted by $\sigma(V, V')$, is the weakest topology on V for which every $(\phi_u)_{u \in V'}$ is continuous. We will define the third topology on V', the weak star topology, denoted by $\sigma(V', V)$. For all $x \in V$, denote by

$$\begin{array}{rcl} \phi_x : V' & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \phi_x(u) = < u, x >_{V',V} \end{array}$$

when x cover V, we obtain a family $(\phi_x)_{x \in V}$, of applications to V' in $\mathbb{R}$.

Theorem 1.1 *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \leq 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 1.1 *Every weakly y^* convergent sequence in V' must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Definition 1.5 Let V be a Banach space and let $(u_n)_{n\in N}$ be a sequence in V. Then u_n converges strongly to u in V if and only if

$$\lim_{t\longrightarrow\infty} \|u_n - u\|_V = 0$$

and this is denoted by $u_n \longrightarrow u$, or

$$\lim_{t\longrightarrow\infty} u_n = u$$

Remark 1.1 The weak convergence does not imply strong convergence in general

Example 1.1 We shall now show by an example that weak convergence does not imply strong convergence in general. Consider the sequence $\sin n\pi t$ in $L^2(0,1)$ (real). This sequence converges weakly to zero. Since, by the Riesz theorem, any linear functional is given by the scalar product with a function we have to show that

$$\int_0^t f(t)\sin n\pi t dt \to 0, \quad foreach \quad f \in L^2(0,1).$$

But By Bessel's inequality

$$\sum_{n=1}^{\infty} |\int_0^1 f(t)\sin n\pi t dt|^2 \le \int_0^1 |f(t)|^2 dt,$$

so $\int_0^t f(t)\sin n\pi t dt \to 0$ as $n \to \infty$. But $\sin n\pi t$ is not strongly convergent, since

$$\begin{aligned} \|\sin n\pi t - \sin m\pi t\|^2 &= \int_0^1 |\sin n\pi t - \sin m\pi t|^2 dt \\ &= 2 \quad for \quad n \neq m. \end{aligned}$$

Functional spaces

The $L^p(\Omega)$ spaces

Definition 1.6 Let $1 \le p \le \infty$; and let Ω be an open domain in $\mathbb{R}^n$; $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$; by:

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \quad is \quad measurable \quad and \int_\Omega |f|^p dx < \infty\}.$$

Notation 1.1 *For $p \in \mathbb{R}$ and $1 \le p \le \infty$, denote by:*

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}},$$

if $p = \infty$, we have

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f\, measurable \quad and\, \exists C \in \mathbb{R}_+, |f(x)| \le C \quad {}_{a.e}\}.$$

Theorem 1.2 *It is well known that $L^p(\Omega)$ equipped with the norm $\|.\|_p$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 1.2 In particular, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$< f, g >_{L^2(\Omega)}= \int_\Omega f(x).g(x)dx,$$

is a Hilbert space.

Theorem 1.3 *For $1 \leq p \leq \infty$, $L^p(\Omega)$ is a reflexive space.*

Definition 1.7 We define the function spaces of our problem and its norm as follows.

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n . \right\} \tag{1.1}$$

Note that $\mathcal{H}(\mathbb{R}^n)$ can be embedded continuously in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. The space $L^2_\rho(\mathbb{R}^n)$ we define to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{1.2}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

Sobolev spaces

Modern theory of differential equations is based on spaces of functions whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 1.1 *Let Ω be an open domain in $\mathbb{R}^n$, then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$if there exists a function $u \in L^p(\Omega)$ such that*

$$< T, \phi >= \int_\Omega u(x)\phi(x)dx, \forall \phi \in D(\Omega),$$

where $1 \leq p \leq \infty$, and it's well-known that u is unique.

Definition 1.8 Let $m \geq 2$ be an integer and let p be a real number with $1 \leq p < \infty$. we define by induction $W^{m,p}(\Omega)$ is the space of all $u \in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega), \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \forall i = 1, 2...N \right\}$$

Alternatively, these sets could also be introduced as

$$W^{m,p}(\Omega) = \Big\{ u \in L^p(\Omega),\ \forall \alpha \leq m, \exists v_\alpha \in L^p(\Omega) \textit{such that} \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi, \forall \varphi \in C^\alpha(\Omega$$

Theorem 1.4 *$W^{m,p}(\Omega)$ is a Banach space with its usual norm*

$$\|u\|_{W_{m,p}(\Omega)} = \sum_{\alpha < m} \|\partial^\alpha u\|_{L^p(\Omega)}, 1 \leq p < \infty \quad \forall u \in W^{m,p}(\Omega).$$

Notation 1.2 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Space $H^m(\Omega)$

Definition 1.9 When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$
and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ endowed with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{\alpha < m} (\|\partial^\alpha f\|)^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

which renders $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$< u, v >_{H^m(\Omega)} = \sum_{\alpha < m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Theorem 1.5 *1)$H^m(\Omega)$ endowed with inner product $< .,. >_{H^m(\Omega)}$ is a Hilbert space.*
2) If $m < m'$, $H^m(\Omega) \longrightarrow H^{m'}(\Omega)$, with continuous embedding.

Lemma 1.1 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω and we have*

$$D(\Omega) \longrightarrow H_0^m(\Omega) \longrightarrow L^2(\Omega) \longrightarrow H_0^{-m}(\Omega) \longrightarrow D'(\Omega)$$

1.2 Useful technical lemmas

Lemma 1.2 *For any* $v \in C^1\left(0, T, H^1(\mathbb{R}^n)\right)$ *we have*

$$\begin{aligned}
& -\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= & \ \frac{1}{2} \frac{d}{dt} \alpha(t) \left(g \circ A^{1/2} v\right)(t) \\
- & \ \frac{1}{2} \frac{d}{dt} \left[\alpha(t) \int_0^t g(s) \int_{\mathbb{R}^n} \left|A^{1/2} v(t)\right|^2 dx ds\right] \\
& -\frac{1}{2} \alpha(t) \left(g'^{1/2} v\right)(t) + \frac{1}{2} \alpha(t) g(t) \int_{\mathbb{R}^n} \left|A^{1/2} v(t)\right|^2 dx ds \\
& -\frac{1}{2} \alpha'(t) \left(g \circ A^{1/2} v\right)(t) + \frac{1}{2} \alpha'(t) \int_0^t g(s) ds \int_{\mathbb{R}^n} \left|A^{1/2} v(t)\right|^2 dx ds.
\end{aligned}$$

Proof.

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= & \ \alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(s) dx ds \\
= & \ \alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'(t) \left[A^{1/2} v(s) - A^{1/2} v(t)\right] dx ds \\
& +\alpha(t) \int_0^t g(t-s) \int_{\mathbb{R}^n} A^{1/2} v'^{1/2} v(t) dx ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s) A v(s) v'(t) ds dx \\
= & \ -\frac{1}{2} \alpha(t) \int_0^t g(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} \left|A^{1/2} v(s) - A^{1/2} v(t)\right|^2 dx ds \\
& +\alpha(t) \int_0^t g(s) \left(\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^n} \left|A^{1/2} v(t)\right|^2 dx\right) ds
\end{aligned}$$

which implies,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(t) \int_0^t g(t-s)Av(s)v'(t)dsdx \\
= \quad & -\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds\right] \\
&+\frac{1}{2}\frac{d}{dt}\left[\alpha(t)\int_0^t g(s)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds\right] \\
&+\frac{1}{2}\alpha(t)\int_0^t g'(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\
&-\frac{1}{2}\alpha(t)g(t)\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds. \\
&+\frac{1}{2}\alpha'(t)\int_0^t g(t-s)\int_{\mathbb{R}^n}\left|A^{1/2}v(s)-A^{1/2}v(t)\right|^2 dxds \\
&-\frac{1}{2}\alpha'(t)\int_0^s g(s)ds\int_{\mathbb{R}^n}\left|A^{1/2}v(t)\right|^2 dxds.
\end{aligned}$$

■

Lemma 1.3 *Let ρ satisfy (**A2**), then for any $u \in D(A^{1/2})$, we have*

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|A^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{1.3}$$

with

$$s = \frac{2n}{2n-qn+2q}, 2 \leq q \leq \frac{2n}{n-2}.$$

We define the function spaces of our problem and their norm as follows:

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\}$$

and the space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{1.4}$$

Lemma 1.4 *Let ρ satisfy (1.4), then for any $u \in \mathcal{H}(\mathbb{R}^n)$, for $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$ we have*

$$\|u\|_{L^p_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)}, \tag{1.5}$$

with $s = \frac{2n}{2n-pn+2p}, 2 \leq p \leq \frac{2n}{n-2}$.

the following Lemma concerning Logarithmic Sobolev inequality.

Lemma 1.5 *Let $u \in \mathcal{H}(\mathbb{R}^n)$ be any function and $c_1, c_2 > 0$ be any numbers. Then*

$$2\int_{\mathbb{R}^n} \rho(x)|u|^2 \ln\Big(\frac{|u|}{\|u\|^2_{L^2_\rho}}\Big)dx + n(1+c_1)\|u\|^2_{L^2_\rho}$$
$$\leq c_2\frac{\|\rho\|^2_{L^2}}{\pi}\|\nabla_x u\|^2_2$$

Some algebraic and integral inequalities

We give here some important integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Theorem 1.6 *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p < \infty$,then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \quad \leq \|f\|_{L^p(\Omega)}.\|g\|_{L^{p'}(\Omega)}$$

when $p = p' = 2$ one finds the Cauchy-Schwarz inequality.

Assume $f \in L^p(\Omega) \cap L^q(\Omega)$ then $f \in L^r(\Omega)$ for $r \in [p, q]$ and

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)}\|f\|^{1-\alpha}_{L^q(\Omega)},$$

with

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad for\, some \quad 0 \leq \alpha \leq 1.$$

Theorem 1.7 *Let a and b be strictly positive realities p and q such as, $\frac{1}{p}+\frac{1}{q} = 1$ and $1 < p < \infty$, we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The function f defined by:

$$f(x) = \frac{x^p}{p} - x$$

reached its minimum point $x = 1$ indeed :

$$y' = x^{p-1} \quad et \quad y'' = (p-1)x^{p-2} > 0$$

from where

$$f(ab^{1-q}) \geq f(1)$$

which gives

$$\frac{(ab^{1-q})^p}{p} - ab^{1-q} \geq \frac{1}{p} - 1 = -\frac{1}{q}$$

so that

$$\frac{a^p}{p} b^{(1-q)p} - ab^{1-q} + \frac{1}{q} \quad \geq 0$$

By dividing the two members by $b^{(1-q)p}$ we obtain :

$$\frac{a^p}{p} - ab^{(1-q)-p+pq} + \frac{b^q}{q} \geq 0$$

which yields

$$\frac{a^p}{p} - ab + \frac{b^q}{q} \geq 0$$

so that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

■

Remark 1.3 A simple case of Young's inequality is the inequality for $p = q = 2$:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

which also gives Young's inequality for all $\delta > 0$:

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2$$

Theorem 1.8 *(Young) Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$.*
Then for a.e.$x \in \mathbb{R}^N$ the function is integrable on $\mathbb{R}^n$ and we define:

$$(f * g) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

In addition

$$(f * g) \in L^p(\mathbb{R}^n)$$

and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

The following is an extension of Theorem 1.8.

Theorem 1.9 *(Young) Assume* $f \in L^1(\mathbb{R}^n)$ *and* $g \in L^p(\mathbb{R}^n$ *with* $1 \leq p \leq \infty$ *,* $1 \leq p \leq \infty$ *and* $x \in \mathbb{R}^N$ *the function is integrable on* $\mathbb{R}^n$ *and* $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$ *. Therefore*

$$(f * g) \in L^r(\mathbb{R}^n)$$

and

$$\|f * g\|_r \leq \|f\|_p \|g\|_p.$$

Remark 1.4 Young's inequality can sometimes be written in the form :

$$ab \leq \delta a^p + C(\delta) b^q, \qquad C(\delta) = \delta^{-\frac{1}{p-1}}$$

Holder's inequalities

Theorem 1.10 *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \leq p < \infty$*,Then* $fg \in L^1(\Omega)$ *and:*

$$\|fg\|_{L^1(\Omega)} \quad \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^{p'}(\Omega)},$$

when $p = p' = 2$*, we get the inequality of Cauchy-Schwartz inequality*

Corollary 1.2 *(Holder's inequality general form) Let* $f_1, f_2, ...f_k$ *be k functions such that,* $f_i \in L^{p_i}(\Omega)$*,* $1 \leq i \leq k$*, and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_k} \leq 1.$$

Then, the product $f_1, f_2, ...f_k \in L^p(\Omega)$ *and* $\|f_1 f_2 ... f_k\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_k\|_{p_k}$.

Lemma 1.6 *(Minkowski inequality) For* $1 \leq p \leq \infty$*, we have*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

Lemma 1.7 *(Cauchy-Schwarz inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$< x_1, x_2 > \leq \|x_1\| \|x_2\|.$$

The equality sign holds if and only if x_1 *and* x_2 *are dependent.*

Will give here some integral inequalities. These inequalities play an important role in applied mathematics and are also very useful in the next chapters.

Lemma 1.8 *let* $1 \leq p \leq r \leq q, \frac{1}{r} = \frac{\alpha}{p} + \frac{1}{q}$ *and* $1 \leq \alpha \leq 1$*. Then*

$$\|u\|_r \leq \|u\|_p^{\alpha} \|u\|_q^{1-\alpha}$$

1.3 Semi-group approach

Definition 1.10 Let $\{T_t\}_{t\geq 0}$ be a one-parameter family of linear operators on a Banach space X into itself satisfying the following conditions:

(1) $T_tT_s = T_{t+s}, T_0 = I, I$ denoting the identity operator on X (Semi -group property).

(2) $s - \lim\limits_{t\to t_0} T_t x = T_{t_0}x \leq 0$ and each $x \in X$ (strong continuity).

(3) there exists a real number $\beta \geq 0$ such that $\|T_t\| \leq e^{\beta t}$ for $t \geq 0$.

We call such a family $\{T_t\}$ a semi group of linear operators of normal type on the Banach space X, or simply a semi-group.

Definition 1.11 Let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be a (unbounded) linear operator. $\mathcal{A}$ is called dissipative if $\Re(\mathcal{A}v, v) \leqslant 0 \;\; \forall v \in D(\mathcal{A})$ the dissipative operator $\mathcal{A}$ is called m-dissipative if $\lambda I - \mathcal{A}$ is subjective for some $\lambda > 0$.

1.3.1 Some examples of semi-groups

I In $C[0, \infty]$ the space of bounded uniformly continuous functions on the closed interval $[0, \infty]$ define $\{T_t\}_{t\geq 0}$ by

$$(T_t x)(s) = x(t + s)(x \in C)$$

$\{T_t\}$ is a semi-group. Condition (1) is trivially verified. (2) follows from the uniform continuity of x, as

$$\|T_t x - T_{t_0}x\| = sup|x(t + s) - x(t_0 + s)|$$

Finally $\|T_t\| = 1$ and so (3) is satisfied with $\beta = 0$
In this example, we could replace $C[0, \infty]$ by $C[-\infty, \infty]$.

II On the space $C[0, \infty]$ (or $C[-\infty, \infty]$) define $\{T_t\}_{t\geq 0}$:

$$(T_t x)(s) = e^{\beta t}x(s)$$

Where β is a fixed non-negative number. Again (1)is trivial; for (2) we have

$$\|T_t\|x - T_{t_0}x\| = |e^{\beta t} - e^{\beta t_0}|sup|x(s)|$$

Trivially $\|T_t\| = e^{\beta t}$.

III Consider the space $C[-\infty, \infty]$

$$N_t(u) = \frac{1}{\sqrt{2\pi t}} e \frac{-u^2}{2t}, \quad t > 0.$$

The normal density define $T_{t t \geq 0}$ on $C[-\infty, \infty]$ by:

$$(T_t x)(s) = \begin{cases} \int\limits_{-\infty}^{+\infty} N_t(s-u)x(u)du, fort > 0 \\ x(s) fort = 0. \end{cases}$$

Each T_t is continuous:

$$\|T_t X\| \leq \|X\| \int\limits_{-\infty}^{+\infty} N_t(s-u)du = \|X\|, as \int\limits_{-\infty}^{+\infty} N_t(s-u)du = 1$$

Moreover it follows from this that condition (3) is valid with $\beta = 0$. By definition $T_0 = I$ and the semi-group property $T_t T_s = T_{t+s}$ is a consequence of the well -known formula concerning the Gaussian distribution.

$$\frac{1}{\sqrt{2\pi(t+t')}} e^{\frac{-u^2}{2(t+t')}} = \frac{1}{\sqrt{2\pi t}} \frac{1}{\sqrt{2\pi t'}} \int\limits_{-\infty}^{+\infty} e^{\frac{-(u-v)^2}{2t}} e^{\frac{-v^2}{2t'}} dv.$$

(Apply Fubini's theorem). To prove the strong continuity, consider $t, t_0 > 0$ with $t \neq t_0$. (The case $t_0 = 0$) is treated in a similar fashion. By definition

$$(T_t x)(s) - (T_{t_0} x)(s) = \int\limits_{-\infty}^{+\infty} (N_t(s-u)x(u) - N_{t_0}(s-u)x(u))du$$

The integral $\int\limits_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{\frac{-(s-u)}{2t}} x(u))du$ becomes, by the change of variable

$$\frac{s-u}{\sqrt{t}} = z, \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-z^2}{2}} X(s - \sqrt{t}z)dz$$

. Hence

$$(T_t x)(s) - (T_{t_0} x)(s) = \int\limits_{-\infty}^{+\infty} (N_1(z)x(s - \sqrt{t}z) - x(s - \sqrt{t_0}z)dz X(s)$$

being uniformly continuous on $-\infty, \infty$ for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that $|x(s_1) - x(s_2)|??$ whenever$|s_1 - s_2| > \delta$. Now, splitting the last integral

$$\|(T_t x)(s) - (T_{t_0} x)(s)\| = \int_{\|\sqrt{t}z - \sqrt{t_0}z\| \leq \delta} (N_1(z)\|x(s - \sqrt{t}z) - x(s - \sqrt{t_0}z)\|dz + \int_{\|\sqrt{t}z - \sqrt{t_0}z\| > \delta} N_1$$

$$\leq \varepsilon \int_{\|\sqrt{t}z-\sqrt{t_0}z\|\leq\delta} N_1(z)dz + 2\|x\| + \int_{\|\sqrt{t}z-\sqrt{t_0}z\|>\delta} N_1(z)dz\varepsilon$$

The second term on the right tends to 0 as $|t - t_0| > 0$, because the integral $\int\limits_{-\infty}^{+\infty} N_1(z)dz$ converges. Thus

$$\lim_{t\to 0} \sup \|(T_t x)(s) - (T_{t_0}x)(s)\| \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have proved the strong continuity at $t = t_0$ of T_t
In this example we can also replace $C[0,\infty]$ by $L_p[0,\infty], 0 < p < \infty$
Consider, for example $L_p[0,\infty]$ In this case

$$\|T_t x\| \leq \int\limits_{-\infty}^{+\infty} [\int\limits_{-\infty}^{+\infty} N_t(s-u)|X|ds]du \leq \|X\|$$

Applying Fubini's theorem.
As for the strong continuity, we have

$$\|(T_t x)(s) - (T_{t_0}x)(s)\| = \int\limits_{-\infty}^{+\infty} |\int\limits_{-\infty}^{+\infty} N_1(z)x(s-\sqrt{tz}) - x(s-\sqrt{t_0 z})dz|ds$$

$$\leq \int\limits_{-\infty}^{+\infty} N_1(z)[\int\limits_{-\infty}^{+\infty} |x(s-\sqrt{t}z) - x(s-\sqrt{t_0}z)|ds]dz$$

Since $N_1(z) \int\limits_{-\infty}^{+\infty} |x(s-\sqrt{t}z) - x(s-\sqrt{t_0}z)|ds \leq \|X\|N_1(z)$, we may apply Lebesgue's dominated convergence theorem. We then have

$$\lim_{t\to 0} \|(T_t x)(s) - (T_{t_0}x)(s)\| \int\limits_{-\infty}^{+\infty} N_1(z) \lim_{t\to 0}[\int\limits_{-\infty}^{+\infty} |x(s-\sqrt{t}z) - x(s-\sqrt{t_0}z)|ds]dz$$

by the continuity in mean of the Lebesgue integral.

IV Consider $C[-\infty,\infty]$. Let$\lambda > 0, \mu > 0$ Define $T_{tt\geq 0}$

$$(T_t x)(s) = e^{-\lambda t}\sum_{k=0}^{\infty} \frac{\lambda t^k}{k!} X(s-k\mu)$$

T_t is a semi-group. Strong continuity follows from:

$$\|(T_t x)(s) - (T_{t_0}x)(s)\| \leq \|X\||e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!} - e^{-\lambda t_0}\sum_{k=0}^{\infty}\frac{(\lambda t_0)^k}{k!}| = 0$$

(3) is satisfied with $\beta = 0$. To verify (1)

$$(T_\omega(T_t x))(s) = e^{-\lambda\omega} \sum_{l=0}^{\infty} \frac{(\lambda\omega)^l}{l!} [e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^{kk}}{k!} f(s - k\mu - 1\mu)]$$

$$= e^{-\lambda(\omega+t)} \sum_{p=0}^{\infty} \frac{1}{p!} [p \sum_{p=0}^{\infty} \frac{(\lambda\omega)(\lambda t)^{p-1}}{1!(p-1)!} f(s - p\mu)]$$

$$= e^{-\lambda(\omega+t)} \sum_{p=0}^{\infty} \frac{1}{p!} (\lambda\omega + \lambda t)^p f(\lambda + \lambda t)^p f(s - p\mu) = (T_{w+t} X)(s).$$

1.3.2 The infinitesimal generator of a semi-group

Definition 1.12 The infinitesimal generator A of a semi-group T_t is defined by:

$$Ax = s - \lim_{h \downarrow 0} h^{-1}(T_h - I)x$$

i.e, as the additive operator A whose domain is the set:

$$D(A) = \{x \backslash s - \lim_{h \downarrow 0} h^{-1}(T_h - I)x \ \ exists\} \quad \text{and for } x \in D(A)$$

$$Ax = s - \lim_{h \downarrow 0} h^{-1}(T_h - I)x$$

$D(A)$ is evidently non- empty; it contains at least zero. Actually $D(A)$ is larger. We prove the

Proposition 1.2 *$D(A)$ is dense in X (in the norm topology).*

Proof. Let$\varphi_n(s) = ne^{-ns}$. Introduce the linear operator C_{φ_n} defined by

$$C_{\varphi_n} x = \int_0^\infty \varphi_n(s) T_s x ds for x \in X and n > \beta,$$

the integral being taken in the sense of Riemann. (The ordinary procedure of defining the Riemann integral of a real or complex valued functions can be extended to a function with values in a Banach space, using the norm instead of absolute value). The convergence of the integral is a consequence of the strong continuity of T_s in s and the inequality,

$$\varphi_n(s) T_s x\| \leq ne^{(\beta-n)s} \|x\|$$

The operator C_{φ_n} is a linear operator whose norm satisfies the inequality

$$\|\varphi_n\| \leq n\int_0^\infty e^{(\beta-n)s}ds = \frac{1}{(1-\frac{\beta}{n})}.$$

We shall now show that $\Re(C(\varphi_n)) \subseteqq D(A), \Re(C(\varphi_n))$ denotes the range of $C(\varphi_n)$ for each $n > \beta$ and that for each $x \in X, s - \lim_{n\to\infty} \varphi_n x = x$ then $\cup\Re(C(\varphi_n))$ will be dense in X and a-portion $D(A)$ will be dense in X. We have

$$h^{-1}(T_h - I)C_{\varphi_n}x = h^{-1}\int_0^\infty \varphi_n(s)T_hT_sxds - h^{-1}\int_0^\infty \varphi_n(s)T_sxdx.$$

(The change of the order $T_h\int_0^\infty = \int_0^\infty T_h$ is justified, using the additivity and the continuity of $T_h$by approximating the integral by Riemann sums). Then

$$h^{-1}(T_h - I)C_{\varphi_n}x = h^{-1}\int_0^\infty \varphi_n(s)T_{h+s}xds - h^{-1}\int_0^\infty \varphi_n(s)T_sxdx$$

$$= h^{-1}\int_0^\infty \varphi_n(s-h)T_sxds - h^{-1}\int_0^\infty \varphi_n(s)T_sxdx$$

by a change of variable in the first integral.

$$= h^{-1}\int_0^\infty (\varphi_n(s-h) - \varphi_n(s))T_sxds$$

$$= h^{-1}\int_0^h \varphi_n(s)T_sxds$$

By the strong continuity of $\varphi_n(s)T_sx$ in s, the second term on the right converges strongly to $-\varphi_n(0)T_0x = -nx$ as $h \downarrow 0$

$$h^{-1}\int_h^\infty (\varphi_n(s-h) - \varphi_n(s))T_sxds$$

$$= \int_0^\infty -\varphi_n'(s)T_sxdx + \int_0^h \varphi_n'(s)T_sxds + \int_h^\infty (\varphi_n'(s) - \varphi_n'(s-\theta h))T_sxds$$

But $\int_0^h \varphi_n'(s)T_sxds \to 0$ as $h \downarrow 0$ and

$$\|\int_h^\infty (\varphi_n'(s) - \varphi_n'(s-\theta h))T_sxds\| \leq n^2\int_h^\infty |e^{-n(s-\theta h)} - e^{-ns}|e^{\beta x}\|x\|d$$

$$\leq n^2(e^{n\beta h} - 1)\int_h^\infty e^{s(n-\theta)}\|x\|ds \to 0$$

as $h \downarrow 0$ Thus we have proved that $\Re(C(\varphi_n)) \subseteqq D(A)$ and

$$AC(\varphi_n)x = n(C(\varphi_n) - I)x$$

As $\varphi'_n = -n\varphi_n$. Next, we show that $s - \lim_{n\to\infty} C(\varphi_n)(x) = x$ for each $x \in X$ We observe that $C(\varphi_n)x - x = \int_0^\infty ne^{-ns}T_s x ds - \int_0^\infty ne^{-ns}x ds$, as $\int_0^\infty ne^{-ns}ds = 1$

$$= n\int_0^\infty e^{-ns}[T_s x - x]ds$$

Approximating the integral by Riemann sums and using the triangle inequality we have

$$\|C(\varphi_n)x - x\| \le n\int_0^\infty e^{-ns}[T_s x - x]ds$$

$$= n\int_0^\delta e^{-ns}[T_s x - x]ds + n\int_\delta^\infty e^{-ns}[T_s x - x]ds$$

$$= I_1 + I_2$$

Given $\varepsilon > 0$, by strong continuity, we can choose a $\delta > 0$ such that $\|T_s x - x\| \le \varepsilon$ for $0 \le s \le \delta$; Then

$$I_1 \le n\varepsilon\int_0^\delta e^{-ns}ds + \le n\varepsilon\int_\delta^\infty e^{-ns}ds = \varepsilon$$

For a fixed $\delta > 0$, using the majorization condition in the definition of a semi-group,

$$I_2 \le n\int_\delta^\infty e^{-ns}(\beta s + 1)\|x\|ds = \|x\|[n\frac{e^{-ns}}{n}]_\delta^\infty - \|x\|[n\frac{e^{(n+\beta)s}}{-n}]_\delta^\infty$$

Each of the terms on the right tends to zero as $n \to \infty$. So $I_2 \le \varepsilon$ for $n \ge n_0$ Thus $C(\varphi_n)x \to x$ as $n \to \infty$. ∎

Remark 1.5 That $D(A)$ is dense in X can be proved more easily. But we need the considerations given in the above proof for later purpose.

Definition 1.13 For $x \in X$ define $D_t T_t x$ by

$$D_t T_t x = s - \lim_{h\to 0} h^{-1}(T_{t+h} - T_t)x$$

if the limit exists.

Proposition 1.3 *If$x \in D(A)$ then $x \in D(D_t)$ and $D_t T_t x = AT_t x = T_t Ax$*

Proof. If$x \in D(A)$ we have, since T_t is a linear operator,

$$T_t Ax = T_t s - \lim_{h\to 0} h^{-1}(T_h - I)x$$

$$= s - \lim_{h\to 0} h^{-1}(T_t T_h - T_t)x$$

$$= s - \lim_{h\to 0} h^{-1}(T_{t+h} - T_t)x$$

$$= s - \lim_{h\to 0} h^{-1}(T_h - I)T_t x$$
$$= AT_t x$$

Thus, If $x \in D(A)$ then $T_t x \in D(A)$ and $T_t Ax = AT_t x = s - \lim_{h\to 0} h^{-1}(T_{t+h} - T_t)x$ We have now proved that the strong right derivative of exists for $T_t x$ each $x \in D(A)$ We shall now show that the strong left derivative exists and is equal to the right derivative. For this, take any $f \in X^*$ For fixed $x, f(T_t x)$ is a continuous numerical function (real or complex - valued) on $t > 0$ By the above. has right derivative $\dfrac{d^+ f(T_t x)}{dt}$ and

$$\frac{d^+ f(T_t x)}{dt} = f(AT_t x) = f(T_t Ax)$$

But $f(T_t Ax)$ is a continuous function. It is well-known that if one of the Dini-derivatives of a numerical function is (finite and) continuous, then the function is differentiable (and the derivative, of course, is continuous).So $f(T_t x)$ is differentiable in t and

$$\int_0^t \frac{d^+ f(T_s x)}{ds} = \int_0^t f(T_s Ax)ds$$
$$= f(= \int_0^t Axds)$$

However, if every linear functional vanishes on an element $x \in X$ and $x = 0$ (by Hahn - Banach theorem). Consequently,

$$T_t x - x = \int_0^t T_s Axds$$

for each $x \in D(A)$ Since T_s is strongly continuous in s, it follows from this, that T_t is strongly derivable:

$$D_t T_t x = s - \lim_{h\to 0} h^{-1}(T_{t+h} - T_t)x$$
$$= s - \lim_{h\to 0} h^{-1} \int_t^{t+h} T_s Axds$$
$$= T_t Ax$$

. ■

Theorem 1.11 *$\mathcal{A}$ linear operator $\mathcal{A}$ is dissipative if and only if $\|(\lambda I - \mathcal{A})\|_X \geqslant \lambda\|X\|_X, \forall x \in D(\mathcal{A}), \lambda > 0$.*

Theorem 1.12 *$\mathcal{A}$ linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ generates a strongly continuous semigroup of contractions $(T(t))_{t\geq 0}$ on X if and only if $\mathcal{A}$ is m-dissipative,i.e., it satisfies:*

1- $\Re(\mathcal{A}v, v) \leqslant 0 \ \forall v \in D(\mathcal{A})$.

2- $\exists \lambda > 0, \lambda I - \mathcal{A}$ *is surjective.*

Theorem 1.13 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\zeta : \zeta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta|\to\infty}} \|(i\zeta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

CHAPITER 2

Energy decay result for a nonlinear wave $p-$Laplace equation with a delay term

We consider the nonlinear (in space and time) wave equation with delay term in the internal feedback. Under conditions on the delay term and the term without delay, we study the asymptotic behavior of solutions using the multiplier method and general weighted integral inequalities. This is published in [Kh. Zennir and L.Kassah.Laouar, *Energy decay result for a nonlinear wave $p-$Laplace equation with a delay term*, MATHEMATICA APPLICANDA, Vol. 45(1) 2017, p. 65-80. doi:

2.1 Introduction

It is well known that the $p-$Laplace equations is a degenerate equations in divergence form. It has been much studied during the last years and their results is by now rather developed, especially with delay. In the classical theory of the evolution equations several main parts of mathematics are joined in a fruitful way, it is very remarkable that the wave $p-$Laplace equation occupies a similar position, when it comes to nonlinear problems.

Here, we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation of the form

$$\begin{cases} \frac{d}{dt}(\phi_l(u')) - div(\phi_p(\nabla u)) + \mu_1 g(u'(x,t)) + \mu_2 g(u'(x,t-\tau)) = 0 \text{ in } \Omega\times]0,+\infty[, \\ u(x,t) = 0, \text{ on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \text{ in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau), \text{ in } \Omega\times]0,\tau[\end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable space. ∇ is the gradient operator such that $|\nabla u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2$, the function ϕ is defined by

$$\phi_x(y) = |y|^{x-2}y. \tag{2.2}$$

usually for $x \geq 2$.

For $p = 2$, when g is linear, it is well known that if $\mu_2 = 0$, that is, in the absence of a delay, the energy of problem (P) exponentially decays to zero (see for instance [37, 32, 41]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system (P) becomes unstable (see for instance [34]). In [34], the authors showed that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [42, 44, 43]). In [42] the authors examined the problem (P) with $p = 2$ and determined suitable relations between μ_1 and μ_2, for which stability or, alternatively, instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution will be unstable if $\mu_2 \geq \mu_1$. The main approach used in [42], is an observability inequality obtained by means of a Carleman estimate. The same results were shown if both the damping and the delay act in the boundary domain. We also recall the result by Xu, Yung and Li in [44], where the authors proved the same result as in [42] for the one-dimension space by adopting the spectral analysis approach.

When g is nonlinear and in the case $\mu_2 = 0, p = 2$, the problem of existence and energy decay have been previously studied by several authors (see [36, 38, 37, 27, 29]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of a global solution depends on the growth near zero of $g(s)$ as it was proved in [30], [37], [36], [38], [45], [46].

Our purpose here is to give energy decay estimates of solutions to the problem (P) for a nonlinear damping and a delay term, in the $p-$Laplace type to extend results obtained by A. Benaissa and *all* [Laboratory of ACEDPs, Djilali Liabes University, Sidi Bel Abbes, Algeria]. We use the multiplier method and some properties of convex functions.

2.2 Preliminaries and Notations

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this chapter are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$. We use familiar function spaces $W_0^{m,p}$.

First, let us assume the following hypotheses:

(H1) $g : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^0(\mathbb{R})$ such that there exist ϵ_1 (sufficiently small), c_1, $c_2, c_3, \alpha_1, \alpha_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0,\infty[)$ satisfying $H(0) = 0$, and H linear on $[0,\epsilon_1]$ or $(H'(0) = 0, H'' > 0$ on $]0,\epsilon_1])$, such that

$$c_1|s|^{l-1} \leq |g(s)| \leq c_2|s|^p, p \geq l-1 \quad \text{if} \quad |s| \geq \epsilon_1, \tag{2.3}$$

$$|s|^l + |g|^{1+\frac{1}{p}}(s) \leq H^{-1}(sg(s)) \quad \text{if } |s| \leq \epsilon_1, \tag{2.4}$$

$$|g'(s)| \leq c_3, \tag{2.5}$$

$$\alpha_1 \; sg(s) \leq G(s) \leq \alpha_2 \; sg(s), \tag{2.6}$$

where

$$G(s) = \int_0^s g(r)\,dr$$

(H2)

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{2.7}$$

We first state some Lemmas which will be needed later.

Lemma 2.1 (Sobolev–Poincaré's inequality) *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2, ..., p)$ or $2 \leq q \leq pn/(n-p)$ $(n \geq p+1)$. Then there is a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \leq c_* \|\nabla u\|_p \quad \textit{for} \quad u \in W_0^{1,p}(\Omega).$$

Lemma 2.2 ([35]) *Let $E:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing differentiable function and $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ a convex and increasing function such that $\Psi(0)=0$. Assume that*

$$\int_s^T \Psi(E(t))\,dt \le E(s) \quad \forall 0\le s\le T.$$

Then E satisfies the following estimate:

$$E(t)\le \psi^{-1}\left(h(t)+\psi(E(0))\right) \ \ \forall t\ge 0, \tag{2.8}$$

where $\psi(t)=\int_t^1 \frac{1}{\Psi(s)}\,ds$ for $t>0$, $h(t)=0$ for $0\le t\le \frac{E(0)}{\Psi(E(0))}$, and

$$h^{-1}(t)=t+\frac{\psi^{-1}\left(t+\psi(E(0))\right)}{\Psi\left(\psi^{-1}\left(t+\psi(E(0))\right)\right)} \ \ \forall t\ge \frac{E(0)}{\Psi(E(0))}.$$

Following the paper [42] , we introduce the new variable ρ and function

$$z(x,\rho,t)=u_t(x,t-\tau\rho), \quad x\in\Omega, \quad \rho\in(0,1), \quad t>0, \tag{2.9}$$

which satisfies

$$\tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 \quad \text{in } \Omega\times(0,1)\times(0,+\infty). \tag{2.10}$$

The original problem is rewritten with the help of the new function z. Thus, it becomes a system of two equations for two functions u and z, with an additional variable ρ:

$$\begin{cases} (\phi_l(u'))'-div(\phi_p(\nabla u))+\mu_1 g(u'(x,t))+\mu_2 g(z(x,1,t))=0 \text{ in } \Omega\times]0,+\infty[,\\ \tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0, \text{ in } \Omega\times]0,1[\times]0,+\infty[\\ u(x,t)=0, \text{ on } \partial\Omega\times[0,+\infty[,\\ z(x,0,t)=u'(x,t), \text{ on } \Omega\times[0,+\infty[,\\ u(x,0)=u_0(x)\ u'(x,0)=u_1(x), \text{ in } \Omega,\\ z(x,\rho,0)=f_0(x,-\rho\tau), \text{ in } \Omega\times]0,1[\,. \end{cases} \tag{2.11}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{2.12}$$

The energy of u at time t of the problem (2.11) is defined by

$$E(t)=\frac{l-1}{l}\|u'(t)\|_l^l+\frac{1}{p}\|\nabla_x u(t)\|_p^p+\xi\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx. \tag{2.13}$$

The first lemma proved here is the following energy estimate.

Lemma 2.3 *Let (u, z) be a solution of the problem (2.11). Then, the energy functional defined by (2.13) satisfies*

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u' g(u')\,dx \\ &- \left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx \\ &\leq 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.11) by u', integrating over Ω and using integration by parts, we get

$$\frac{d}{dt}\left(\frac{l-1}{l}\|u'\|_l^l + \frac{1}{p}\|\nabla_x u\|_p^p\right) + \mu_1\int_\Omega u'g(u')\,dx + \mu_2\int_\Omega u'g(z(x,1,t))\,dx = 0. \tag{2.15}$$

We multiply the second equation in (2.11) by $\xi g(z)$ and integrate the result over $\Omega\times(0,1)$ to obtain

$$\begin{aligned} \xi\int_\Omega\int_0^1 z'g(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_\Omega (G(z(x,1,t)) - G(z(x,0,t)))\,dx. \end{aligned} \tag{2.16}$$

Then

$$\xi\frac{d}{dt}\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_\Omega G(u')\,dx. \tag{2.17}$$

From (2.15), (2.17) and using the Young inequality we get

$$\begin{aligned} E'(t) &= -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx \\ &- \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx - \mu_2\int_\Omega u'(t)g(z(x,1,t))\,dx. \end{aligned} \tag{2.18}$$

Let us denote G^* to be the conjugate function of the convex function G, i.e., $G^*(s) = \sup_{t\in\mathbb{R}^+}(st - G(t))$. Then G^* is the Legendre transform of G which is given by (see Arnold [28, pp. 61–62] and Lasiecka [31, 33, 39])

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \quad \forall s \geq 0, \tag{2.19}$$

and satisfies the following inequality

$$st \leq G^*(s) + G(t) \quad \forall s, t \geq 0. \tag{2.20}$$

Then by the definition of G we get

$$G^*(s) = sg^{-1}(s) - G(g^{-1}(s)).$$

Hence

$$\begin{aligned} G^*(g(z(x,1,t))) &= z(x,1,t)g(z(x,1,t)) - G(z(x,1,t)) \\ &\leq (1-\alpha_1)z(x,1,t)g(z(x,1,t)). \end{aligned} \tag{2.21}$$

Making use of (2.18), (2.20) and (2.21), we have

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_\Omega u'g(u')\,dx - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx \\ &+ \mu_2\int_\Omega (G(u') + G^*(g(z(x,1,t))))\,dx \\ &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx - \frac{\xi}{\tau}\int_\Omega G(z(x,1,t))\,dx \\ &+ \mu_2\int_\Omega G^*(g(z(x,1,t)))\,dx. \end{aligned} \tag{2.22}$$

Using (2.6) and (2.12), we obtain

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_\Omega u'g(u')\,dx \\ &- \left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)g(z(x,1,t))\,dx \\ &\leq 0. \end{aligned} \tag{2.23}$$

2.3 Global existence

We are now ready to employ the Galerkin method to prove the global existence Theorem.

Theorem 2.1 *Let $(u_0, u_1, f_0) \in W^{2,p} \cap W_0^{1,p} \times W_0^{1,l}(\Omega) \times W_0^{1,l}(\Omega; W^{1,2}(0,1))$ and assume that the hypotheses* **(H1)–(H2)** *hold. Then the problem (P) admits a unique solution*

$$u \in L^\infty([0,\infty); W^{2,p} \cap W_0^{1,p}), \quad u' \in L^\infty([0,\infty); W_0^{1,l}), \tag{2.24}$$

Proof. (*of Theorem* 2.1.) Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we can extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2(\Omega\times[0,1])$ and denote Z_k the space generated by $\{\phi_1,\phi_2,\dots,\phi_k\}$.

We construct approximate solutions $(u_k,z_k)(k=1,2,3,\dots,)$ in the form

$$\begin{aligned} u_k(t) &= \textstyle\sum_{j=1}^k g_{jk}w_j,\\ z_k(t) &= \textstyle\sum_{j=1}^k h_{jk}\phi_j, \end{aligned}$$

where g_{ik} and h_{ik} $(j=1,2,\dots,m)$ are determined by the following ordinary differential equations:

$$\begin{cases} ((|u_k'(t)|^{l-2}u_k'(t))',w_j)+(|\nabla_x u_k(t)|^{p-2}\nabla_x u_k(t),\nabla_x w_j)+\mu_1(g(u_k'),w_j)\\ \qquad +\mu_1(g(z_k(\cdot,1)),w_j)=0, \quad 1\le j\le k,\\ z_n(x,0,t)=u_k'(x,t),\\ u_k(0)=u_{0k}=\sum_{j=1}^k(u_0,w_j)w_j\to u_0 \quad \text{in} \quad W^{2,p}\cap W_0^{1,p} \quad \text{as} \quad m\to+\infty,\\ u_k'(0)=u_{1k}=\sum_{j=1}^k(u_1,w_j)w_j\to u_1 \quad \text{in} \quad W_0^{1,l} \quad \text{as} \quad m\to+\infty, \end{cases} \tag{2.25}$$

and

$$(\tau z_{kt}+z_{k\rho},\phi_j)=0, 1\le j\le k, \tag{2.26}$$

$$z_k(\rho,0)=z_{0k}=\sum_{j=1}^k(f_0,\phi_j)\phi_j\to f_0 \ \text{ in } \ W_0^{1,l}(\Omega;W_0^{1,2}(0,1)) \ \text{ as } \ k\to+\infty. \tag{2.27}$$

By virtue of the theory of ordinary differential equations, and following the techniques due to Lions [40] in order to deal with the convergence of the nonlinear terms, the system (2.25)–(2.27) has a unique local solution which is extended to a maximal interval $[0,T_k[$.

Next, using a standard compactness argument for the limiting procedure, we obtain a priori estimates for the solution, so that it can be extended to global solution.

Since the sequences u_{0k},u_{1k} and z_{0k} converge, then from (2.14) we can find a positive constant C independent of k such that

$$E_k(t)+c_1\int_0^t u_k'(x,s)g(u_k'(x,s))ds+c_2\int_0^t z_k(x,1,s)g(z_k(x,1,s))ds\le C. \tag{2.28}$$

where

$$E_k(t)=\frac{l-1}{l}\|u_k'(t)\|_l^l+\frac{1}{p}\|\nabla_x u_k(t)\|_p^p+\xi\int_\Omega\int_0^1 G(z_k(x,\rho,t))\,d\rho\,dx\le C. \tag{2.29}$$

$$c_1=\mu_1-\frac{\xi\alpha_2}{\tau}-\mu_2\alpha_2 \text{ and } c_2=\frac{\xi\alpha_1}{\tau}-\mu_2(1-\alpha_1)$$

These estimates imply that the solution (u_k, z_k) exists globally in $[0, +\infty[$. For the regularity, differentiating (2.25), (2.26) with respect to t, multiplying by $g''_{jk}(t)$, $h'_{jk}(t)$ and summing over j from 1 to k, Taking the sum and using Cauchy-Schwarz, Young's inequalities and Gronwall's Lemma to conclude that

$$u'_k \text{ is bounded in } L^\infty(0, +\infty; W_0^{1,l}(\Omega)), \tag{2.30}$$

$$z'_k \text{ is bounded in } L^\infty(0, +\infty; W^{1,2}(\Omega \times (0,1))). \tag{2.31}$$

In the next estimate, replacing w_j by $-\Delta_p w_j$ in (2.25), multiplying the result by $g'_{jm}(t)$, summing over j from 1 to k and replacing ϕ_j by $-\Delta_p \phi_j$ in (2.26), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k. Using suitable calculus to conclude that

$$u_k \text{ is bounded in } L^\infty(0, +\infty; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)), \tag{2.32}$$

$$z_k \text{ is bounded in } L^\infty(0, +\infty; W_0^{1,l}(\Omega; W^{1,2}(0,1))). \tag{2.33}$$

Applying Dunford-Petti's theorem and Aubin-Lions' theorem [**40**], we can extract a subsequence (u_ν) of (u_k) such that

$$u'_\nu \to u' \text{ strongly in } \ L^l(\Omega).$$

Therefore,

$$u'_\nu \to u' \text{ a.e in } \Omega. \tag{2.34}$$

Similarly we obtain

$$z_\nu \to z \text{ a.e in } \Omega. \tag{2.35}$$

For the nonlinear terms, we need the next lemmas.

Lemma 2.4 *For each $T > 0$, $g(u'), g(z(x,1,t)) \in L^1(\Omega)$ and*

$$\|g(u')\|_{L^1(\Omega)}, \|g(z(x,1,t))\|_{L^1(\Omega)} \leq K_1,$$

where K_1 is a constant independent of t.

Proof. By (**H1**) we have

$$g(u'_k(x,t)) \to g(u'(x,t)) \quad \text{a.e. in} \quad \Omega,$$

$$0 \leq g(u'_k(x,t))u'_k(x,t) \to g(u'(x,t))u'(x,t) \quad \text{a.e. in} \quad \Omega.$$

Hence, by Fatou's lemma we have

$$\int_0^T \int_\Omega u'(x,t) g(u'(x,t))\, dx\, dt \leq K \quad \text{for} \quad T > 0. \tag{2.36}$$

By the Cauchy–Schwartz inequality and using (2.36), we have

$$\int_0^T \int_\Omega |g(u'(x,t))|\, dx\, dt \le c|\Omega|^{\frac{1}{2}} \left(\int_0^T \int_\Omega u' g(u')\, dx\, dt\right)^{\frac{1}{2}} \tag{2.37}$$

$$\le c|\Omega|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1. \tag{2.38}$$

■

Lemma 2.5 *$g(u'_k) \to g(u')$ in $L^1(\Omega \times (0,T))$ and $g(z_k) \to g(z)$ in $L^1(\Omega \times (0,T))$.*

Proof. Let $E \subset \Omega \times [0,T]$ and set

$$E_1 = \left\{(x,t) \in E;\ g(u'_k(x,t)) \le \frac{1}{\sqrt{|E|}}\right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbb{R} \ \text{ and } \ |g(s)| \ge r\}$,

$$\int_E |g(u'_k)|\, dxdt \le \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1} \int_{E_2} |u'_k g(u'_k)|\, dxdt.$$

We have $\sup_k \int_E |g(u'_k)|\, dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g(u'_k) \to g(u')$ in $L^1(\Omega \times (0,T))$, hence

$$g(u'_k) \to g(u') \quad \text{weak in} \quad L^2(\Omega).$$

Similarly, we have

$$g(z'_k) \to g(z') \quad \text{weak in} \quad L^2(\Omega),$$

and this implies that

$$\int_0^T \int_\Omega g(u'_k) v\, dx\, dt \to \int_0^T \int_\Omega g(u') v\, dx\, dt \ \text{ for all } \ v \in L^2(0,T; W_0^{1,l}), \tag{2.39}$$

$$\int_0^T \int_\Omega g(z_k) v\, dx\, dt \to \int_0^T \int_\Omega g(z) v\, dx\, dt \ \text{ for all } \ v \in L^2(0,T; W_0^{1,2}) \tag{2.40}$$

as $m \to +\infty$. It follows at once (2.39), (2.40) that for each fixed $v \in L^2(0,T; W_0^{1,l})$ and $w \in L^2(0,T; W_0^{1,2}(\Omega \times (0,1)))$

$$\int_0^T \int_\Omega ((|u'_k|^{l-2} u'_k)' - div(|\nabla_x u'_k|^{p-2} \nabla_x u'_k) + +\mu_1 g(u'_k) + \mu_2 g(z_k)) v\, dx\, dt$$
$$\to \int_0^T \int_\Omega ((|u'_k|^{l-2} u'_k)' - div(|\nabla_x u'_k|^{p-2} \nabla_x u'_k) + \mu_1 g(u') + \mu_2 g(z)) v\, dx\, dt$$
$$\int_0^T \int_0^1 \int_\Omega \left(\tau z'_k + \frac{\partial}{\partial \rho} z_k\right) w\, dx\, d\rho\, dt \to \int_0^T \int_0^1 \int_\Omega \left(\tau z' + \frac{\partial}{\partial \rho} z\right) w\, dx\, d\rho\, dt$$

as $m \to +\infty$. Hence

$$\int_0^T \int_\Omega ((|u'_k|^{l-2} u'_k)' - div(|\nabla_x u'_k|^{p-2} \nabla_x u'_k) + \mu_1 g(u') + \mu_2 g(z)) v\, dx\, dt = 0, \quad v \in L^2(0,T; W_0^{1,l})$$
$$\int_0^T \int_0^1 \int_\Omega (\tau u' + \frac{\partial}{\partial \rho} z) w\, dx\, d\rho\, dt = 0, \quad w \in L^2(0,T; W_0^{1,2}(\Omega \times (0,1))).$$

Thus the problem (P) admits a global weak solution u. ■ ■

2.4 Energy decay

The next main result reads as.

Theorem 2.2 *Let* $(u_0, u_1, f_0) \in W^{2,p} \cap W_0^{1,p} \times W_0^{1,l}(\Omega) \times W_0^{1,l}(\Omega; W^{1,2}(0,1))$ *and assume that the hypotheses* **(H1)–(H2)** *hold. Then, for some constants* ω, ϵ_0 *we have*

$$E(t) \leq \psi^{-1}\left(h(t) + \psi(E(0))\right) \quad \forall t > 0, \tag{2.41}$$

where $\psi(t) = \int_t^1 \frac{1}{\omega\varphi(\tau)} d\tau$ *for* $t > 0$, $h(t) = 0$ *for* $0 \leq t \leq \frac{E(0)}{\omega\varphi(E(0))}$,

$$h^{-1}(t) = t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\omega\varphi\left(\psi^{-1}\left(t + \psi(E(0))\right)\right)} \quad \forall t > 0,$$

$\varphi(s) = \{s$ ***if*** H ***is linear on*** $[0, \epsilon_1], sH'(\epsilon_0 s)$ ***if*** $H'(0) = 0$ ***and*** $H'' > 0$ ***on*** $]0, \epsilon_1].\}$

Proof. (*of Theorem* 2.2.) Multiplying the first equation of (2.11) by $\frac{\varphi(E)}{E}u$, we obtain

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E} \int_\Omega u\left((\phi_l(u'))' - div(\phi_p(\nabla u)) + \mu_1 g(u'(x,t)) + \mu_2 g(z(x,1,t))\right) dx\, dt \\
&= \left[\frac{\varphi(E)}{E} \int_\Omega u|u'|^{l-2}u' dx\right]_S^T - \int_S^T \left(\frac{\varphi(E)}{E}\right)' \int_\Omega u|u'|^{l-2}u' dx dt \\
&- \frac{3l-2}{l} \int_S^T \frac{\varphi(E)}{E} \int_\Omega u'^l\, dx\, dt + 2\int_S^T \frac{\varphi(E)}{E} \int_\Omega \left(\frac{l-1}{l}u'^l + \frac{1}{p}|\nabla u|^p\right) dx dt \\
&+ \mu_1 \int_S^T \frac{\varphi(E)}{E} \int_\Omega u g(u')\, dx\, dt + \mu_2 \int_S^T \frac{\varphi(E)}{E} \int_\Omega u g(z(x,1,t))\, dx dt \\
&+ (1 - \frac{2}{p}) \int_S^T \frac{\varphi(E)}{E} \int_\Omega |\nabla u|^p dx dt.
\end{aligned}$$

Similarly, we multiply the second equation of (2.11) by $\frac{\varphi(E)}{E}e^{-2\tau\rho}g(z(x,\rho,t))$, we have

$$\begin{aligned}
0 &= \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}g(z)(\tau z'+z_\rho)\,dxd\rho dt\\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T\\
&- \tau\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt\\
&+ \int_S^T \frac{\varphi(E)}{E}\int_\Omega\int_0^1\left(\frac{\partial}{\partial\rho}(e^{-2\tau\rho}G(z))+2\tau e^{-2\tau\rho}G(z)\right)dxd\rho dt\\
&= \left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 \tau e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T\\
&- \tau\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt\\
&+ \int_S^T \frac{\varphi(E)}{E}\int_\Omega(e^{-2\tau}G(z(x,1,t))-G(z(x,0,t)))\,dxdt\\
&+ 2\tau\int_S^T \frac{\varphi(E)}{E}\int_0^1\int_\Omega e^{-2\tau\rho}G(z)\,dxd\rho dt.
\end{aligned}$$

Since, $p\geq 2$, summing to obtain, for $A=\min\{2,2\tau e^{-2\tau}/2\xi\}$

$$\begin{aligned}
A\int_S^T\varphi(E)\,dt &\leq -\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T+\int_S^T(\frac{\varphi(E)}{E})'\int_\Omega u|u'|^{l-2}u'dxdt\\
&+ \frac{3l-2}{l}\int_S^T\frac{\varphi(E)}{E}\int_\Omega u'^l\,dxdt-\mu_1\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(u')\,dx\,dt\\
&- \mu_2\int_S^T\frac{\varphi(E)}{E}\int_\Omega ug(z(x,1,t))\,dxdt-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1\tau e^{-2\tau\rho}G(z)dxd\rho\right]_S^T\\
&+ \tau\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)dxd\rho dt\\
&- \int_S^T\frac{\varphi(E)}{E}\int_\Omega(e^{-2\tau}G(z(x,1,t))-G(z(x,0,t)))\,dxdt. \qquad (2.42)
\end{aligned}$$

Since E is non-increasing, using the Holder,Cauchy–Schwartz, Poincare and Young's inequalities with exponents $\frac{l}{l-1}, l$, to get

$$\begin{aligned}
-\left[\frac{\varphi(E)}{E}\int_\Omega u|u'|^{l-2}u'dx\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega u(S)|u'(S)|^{l-2}u'(S)dx\\
&- \frac{\varphi(E(T))}{E(T)}\int_\Omega u(T)|u'(T)|^{l-2}u'(T)dx\\
&\leq C\varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}\left|\int_S^T\left(\frac{\varphi(E)}{E}\right)'\int_\Omega u|u'|^{l-2}u'\,dxdt\right| &\leq c\int_S^T\left|\left(\frac{\varphi(E)}{E}\right)'\right|E\,dt\\ &\leq c\varphi(E(S)),\end{aligned}$$

$$\begin{aligned}-\left[\frac{\varphi(E)}{E}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho\right]_S^T &= \frac{\varphi(E(S))}{E(S)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,S))\,dxd\rho,\\ &- \frac{\varphi(E(T))}{E(T)}\int_\Omega\int_0^1 e^{-2\tau\rho}G(z(x,\rho,T))\,dxd\rho\\ &\leq C\varphi(E(S)),\end{aligned}$$

$$\begin{aligned}\int_S^T\left(\left(\frac{\varphi(E)}{E}\right)'\right)\int_\Omega\int_0^1 e^{-2\tau\rho}G(z)\,dxd\rho dt &\leq c\int_S^T\left(-\left(\frac{\varphi(E)}{E}\right)'\right)Edt\\ &\leq c\varphi(E(S)),\end{aligned}$$

$$\begin{aligned}\int_S^T\frac{\varphi(E)}{E}\int_\Omega e^{-2\tau}G((x,1,t))\,dxdt &\leq c\int_S^T\frac{\varphi(E)}{E}(-E')\,dt\\ &\leq c\varphi(E(S)),\end{aligned}$$

$$\begin{aligned}\int_S^T\frac{\varphi(E)}{E}\int_\Omega G(z(x,0,t))dxdt &= \int_S^T\frac{\varphi(E)}{E}\int_\Omega G(u'(x,t))\,dxdt\\ &\leq c\int_S^T\frac{\varphi(E)}{E}(-E')\,dt\\ &\leq c\varphi(E(S)),\end{aligned}$$

We conclude

$$\begin{aligned}A\int_S^T\varphi(E)dt &\leq c\varphi(E(S))+\mu_1\int_S^T\frac{\varphi(E)}{E}\int_\Omega|u||g(u')|\,dx\,dt\\ &+ \frac{3l-2}{l}\int_S^T\frac{\varphi(E)}{E}\int_\Omega u'^l dxdt+\mu_2\int_S^T\frac{\varphi(E)}{E}\int_\Omega|u||g(z(x,1,t))|dxdt\end{aligned}\tag{2.43}$$

In order to apply the results of Lemma 2.2, we estimate the terms of the right-hand side of (2.43) .

We distinguish two cases.

1. H **is linear on** $[0,\epsilon_1]$**.** We have $c_1|s|^{l-1}\leq|g(s)|\leq c_2|s|^p$ for all $s\in\mathbb{R}$, and then, using (2.6) and noting that $s\mapsto\frac{\varphi(E(s))}{E(s)}$ is non-increasing,

$$\int_S^T\frac{\varphi(E)}{E}\int_\Omega|u'|^l dxdt \;\leq\; c\int_S^T\frac{\varphi(E)}{E}\int_\Omega u'g(u')dxdt\leq c\varphi(E(S)),$$

Using the Poincaré, Young inequalities and the energy inequality from Lemma 2.3, we obtain, for all $\epsilon > 0$,

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E} \int_\Omega |ug(u')| dxdt &\leq \epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega u^{p+1} dxdt + c_\epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega g^{1+\frac{1}{p}}(u') dxdt \\
&\leq \epsilon c \int_S^T \varphi(E) dt + c_\epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega u' g(u') dxdt \\
&\leq \epsilon c \int_S^T \varphi(E) dt + c_\epsilon \varphi(E(S)),
\end{aligned}$$

$$\begin{aligned}
\int_S^T \frac{\varphi(E)}{E} \int_\Omega |ug(z(x,1,t))| dxdt &\leq \epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega u^{p+1} dxdt + c_\epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega g^{1+\frac{1}{p}}(z(x,1,t))d \\
&\leq \epsilon c \int_S^T \varphi(E) dt + c_\epsilon \int_S^T \frac{\varphi(E)}{E} \int_\Omega z(x,1,t) g(z(x,1,t)) dxdt \\
&\leq \epsilon c \int_S^T \varphi(E) dt + c_\epsilon \varphi(E(S)).
\end{aligned}$$

Inserting these two inequalities into (2.43), choosing $\epsilon > 0$ small enough, we deduce that

$$\int_S^T \varphi(E(t)) dt \leq c\varphi(E(S)).$$

Using Lemma 2.2 for E in the particular case where $\varphi(s) = s$, we deduce from (2.8) that

$$E(t) \leq ce^{-\omega t}.$$

2. $H'(0) = 0$ **and** $H'' > 0$ **on** $]0, \epsilon_1]$**.** For all $t \geq 0$, we consider the following partition of Ω

$$\begin{aligned}
\Omega_t^1 &= \{x \in \Omega : \, |u'| \geq \epsilon_1\}, & \Omega_t^2 &= \{x \in \Omega : \, |u'| \leq \epsilon_1\}, \\
\tilde{\Omega}_t^1 &= \{x \in \Omega : \, |z(x,1,t)| \geq \epsilon_1\}, & \tilde{\Omega}_t^2 &= \{x \in \Omega : \, |z(x,1,t)| \leq \epsilon_1\}.
\end{aligned}$$

Using (2.3), (2.6) and the fact that $s \mapsto \frac{\varphi(s)}{s}$ is non-decreasing, we obtain

$$c \int_S^T \frac{\varphi(E)}{E} \int_{\Omega_t^1} (|u'|^l + g^{1+\frac{1}{p}}(u')) dxdt \leq c \int_S^T \frac{\varphi(E)}{E} \int_\Omega u' g(u') dxdt \leq c\varphi(E(S)).$$

and

$$\int_S^T \frac{\varphi(E)}{E} \int_{\tilde{\Omega}_t^1} g^{1+\frac{1}{p}}(z(x,1,t)) dxdt \leq c \int_S^T \frac{\varphi(E)}{E} \int_\Omega z(x,1,t) g(z(x,1,t)) dxdt \leq c\varphi(E(S)).$$

On the other hand, since H is convex and increasing, H^{-1} is concave and increasing. Therefore (2.4) and the reversed Jensen's inequality for a concave function imply that

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l + g^{p/(p-1)}(u'))\,dxdt &\leq \int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2} H^{-1}(u'g(u'))\,dxdt \\ &\leq \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega u'g(u')dx\Big)dt \end{aligned} \tag{2.44}$$

Let us assume H^* to be the conjugate function of the convex function H, i.e., $H^*(s) = \sup_{t\in\mathbb{R}^+}(st - H(t))$. Then H^* is the Legendre transform of H, which is given by (see [28], [31, 33, 39])

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)] \quad \forall s \geq 0 \tag{2.45}$$

and satisfies the following inequality

$$st \leq H^*(s) + H(t) \quad \forall s,t \geq 0. \tag{2.46}$$

Due to our choice $\varphi(s) = sH'(\epsilon_0 s)$, we have

$$H^*\left(\frac{\varphi(s)}{s}\right) = \epsilon_0 sH'(\epsilon_0 s) - H(\epsilon_0 s) \leq \epsilon_0\varphi(s). \tag{2.47}$$

Making use of (2.44), (2.46) and (2.47), we have

$$\begin{aligned}\int_S^T \frac{\varphi(E)}{E}\int_{\Omega_t^2}(|u'|^l + g^{p/(p-1)}(u'))\,dxdt &\leq c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt + c\int_S^T\int_\Omega u'g(u')dt \\ &\leq \epsilon_0\int_S^T \varphi(E)dt + cE(S),\end{aligned}$$

$$\begin{aligned}&\int_S^T \frac{\varphi(E)}{E}\int_{\tilde{\Omega}_t^2}(|u'|^l + g^{p/(p-1)}(z(x,1,t)))\,dxdt \\ &\leq \int_S^T \frac{\varphi(E)}{E}\int_{\tilde{\Omega}_t^2} H^{-1}(z(x,1,t)g(z(x,1,t)))\,dxdt \\ &\leq \int_S^T \frac{\varphi(E)}{E}|\Omega|H^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega z(x,1,t)g(z(x,1,t))dx\Big)dt \\ &\leq c\int_S^T H^*\left(\frac{\varphi(E)}{E}\right)dt + c\int_S^T\int_\Omega z(x,1,t)g(z(x,1,t))dt \\ &\leq \epsilon_0\int_S^T \varphi(E)dt + cE(S).\end{aligned} \tag{2.48}$$

2.4. Energy decay

Then, choosing $\epsilon_0 > 0$ small enough and using (2.43), we obtain in both cases

$$\begin{aligned} \int_S^{+\infty} \varphi(E(t))dt &\leq c\Big(E(S) + \varphi(E(S))\Big) \\ &\leq c\left(1 + \frac{\varphi(E(S)}{E(S)}\right)E(S) \\ &\leq cE(S) \quad \forall S \geq 0. \end{aligned} \tag{2.49}$$

Using Lemma 2.2 in the particular case where $\Psi(s) = \omega\varphi(s)$, we deduce from (2.8) our estimate (2.41). The proof of Theorem 2.2 is now complete. ■

Remark 2.1 The nonlinearity in space and time makes our problem more general and very useful in the practical point of view.

CHAPTER 3

General decay of solutions to an extensible viscoelastic plate equation with a nonlinear time-varying delay feedback

An extensible viscoelastic plate equation with a nonlinear time-varying delay feedback and nonlinear source term is considered. Under suitable assumptions on relaxation function, nonlinear internal delay feedback and source term, we establish general decay of energy by using the multiplier method if the weight of weak dissipation and the delay satisfy $\mu_2 < \frac{\mu_1\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}$. This is published in [B. Feng, Kh. Zennir and L.Kassah.Laouar, *General decay of solutions to an extensible viscoelastic plate equation with a nonlinear time-varying delay feedback*, Bull. Malays. Math. Sci. Soc. (2018). https://doi.org/10.1007/s40840-018-0602-4]

3.1 Introduction

In this chapter, we consider the following plate equation:

$$u_{tt}(t) + \Delta^2 u(t) - M(\|\nabla u(t)\|^2)\Delta u(t) - \sigma(t)\int_0^t h(t-s)\Delta^2 u(s)ds$$
$$+\mu_1 g_1(u_t(t)) + \mu_2 g_2(u_t(t-\tau(t))) + f(u(t)) = 0, \quad (3.1)$$

defined in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with a sufficiently smooth boundary $\partial\Omega$. The function $u(x,t)$ is the transverse displacement of a plate filament. μ_1 and μ_2 are two positive constants and g_1, g_2 are two functions satisfying some assumptions. The function $\tau(t)$ represents the time-varying delay. The function $f(u)$ is source term.

To Eq. (3.1), we add the following initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3.2}$$

$$u_t(x,t) = f_0(x,t), \; x \in \Omega, \; t \in [-\tau(0), 0), \tag{3.3}$$

and simply supported boundary conditions:

$$u(x,t) = 0, \quad \Delta u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{3.4}$$

Up till now there are so many results to partial differential equations with time delay effects, and most of which are concerning with the global well-posedness of systems with time delay. The delay effects often arise in many practical problems. The presence of delay can be a source of instability and an arbitrarily small delay may destabilized a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been added. In recent years, the control of wave equation with time delay effects has become an active area of research. The general form of weak-viscoelastic wave equation with a nonlinear delayed term reads

$$u_{tt} - \Delta u + \sigma(t) \int_0^t h(t-s)\Delta u(s)ds + \mu_1 g_1(u_t) + \mu_2 g_2(u_t(t-\tau(t))) + f(u) = 0 \tag{3.5}$$

In absence of the viscoelastic term and source term, i.e, $h = f = 0$, if the functions $g_1(t) = g_2(t) = t$, the problem have been investigated by many authors, and have been proved the stability and instability under some suitable assumptions between μ_1 and μ_2. More precisely, the energy of the system is exponentially decay if $|\mu_2| < \mu_1$ for constant delay and $|\mu_2| < \sqrt{1-d}\mu_1$ for time-varying delay and the solution will be instable if $|\mu_2| \geq \mu_1$ for constant delay and $|\mu_2| \geq \sqrt{1-d}\mu_1$. See for example, Datko, Lagnese and Polis [55], Kafini, Messaoudi and Nicaise [63], Liu [69, 70], Nicaise and Pignotti [81, 79, 80], Nicaise, Valein and Fridman [82], Xu, Yung and Li [87], and the references therein. For the viscoelastic wave equation with linear time delay, when $\sigma(t) = 1$, one can find some results in Alabau-Boussouira, Nicaise and Pignotti [47], Kirane and Said-Houari [65], Dai and Yang [53], Liu and Zhang [71], Wu [86] and so on. For viscoelastic wave with nonlinear constant delay feedback (3.5), when $\sigma(t) = 1$ and $f(u) = 0$, Benaissa, Benguessoum and Messaoudi [50] studied the system. They proved the global existence of solutions by using Faode-Galerkin method and established the general decay of energy by using energy method under the assumption $\alpha_2\mu_2 < \alpha_1\mu_1$. Benaissa and Messaoudi [49] studied the following wave equation

$$u_{tt} - \Delta u + \mu_1\sigma(t)g_1(u_t) + \mu_2\sigma(t)g_2(u_t(t-\tau(t))) = 0,$$

and proved the global existence of solutions under a certain relation between the weight of the delay term in the feedback, the weight of the term without delay and the speed of the delay. In addition, they established the energy decay of energy. Recently, in [76], the authors considered a nonlinear viscoelastic wave equation with nonlinear time-varying delay,

$$(|u_t|^\gamma u_t)_t - \Delta u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1 \psi(u_t) + \mu_2 \psi(u_t(t-\tau(t))) = 0.$$

They proved the general decay of energy by using suitable Lyapunov functionals. For general decay of energy to viscoelastic wave equation without time delay term, we would like to refer the reader to Berrimi and Messaoudi [51], Messaoudi [75, 74], Messaoudi and Al-Gharabli [76], Messaoudi and Tartar [77, 78], Tatar [85] and so on for $\sigma(t) = 1$, and Messaoudi [73] for $\sigma(t) \neq$ constant.

For plate equation with time delay effects, we mention the work of Park [83]. The author studied the following plate equation of the form

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + \sigma(t)\int_0^t g(t-s)\Delta u(s)ds + a_0 u_t + a_1 u_t(t-\tau(t)) = 0,$$

and proved a general decay result of energy under the assumption $|a_1| < \sqrt{1-d}a_0$. Feng [57] investigated a plate equation with viscoelastic and a delay terms

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and proved the global existence of solutions with $|\mu_2| \leq \mu_1$ and general decay of energy under the assumption $|\mu_2| < \mu_1$. Yang [88] studied the following equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

Without any restriction on real numbers μ_1, μ_2, the author obtained the global existence of solutions. Under the assumption on $0 < |\mu_2| < \mu_1$, the author also proved the exponential stability of energy. Very recently, the present author considered a plate equation with past history, source term and time-varying delay

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-s)\Delta^2 u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau(t)) + f(u) = h(x),$$

and proved the global existence and continuous dependence of solutions for any real numbers μ_1, μ_2. Moreover, the author established the exponential decay of energy when $f(u) \neq 0$ under the assumption $|\mu_2| < \sqrt{1-d}\mu_1$ and when $f(u) = 0,\ \mu_1 = 0$

if the weight of delay $|\mu_2|$ is small, see [56]. Pignotti [84] considered the following second-order evolution equations

$$u_{tt} + Au - \int_0^\infty \mu(s)Au(t-s)ds + b(t)u_t(t-\tau) = 0,$$

and proved the stability of solutions if $b \in L^1(0,\infty)$ and the length of time intervals is very large. In addition, the author established the stability results for a problem with anti-damping.

It is remarkable that all above stability results on viscoelastic plate equation are only holds for $\sigma(t) = 1$ and linear time delay. To our best of knowledge, there is no stability result for plate equation with nonlinear time delay, nonlinear source term and $\sigma(t) \neq Const.$ So our main objective in the present work is to study the stability of solutions to problem (3.1)-(3.4), and establish general decay of energy of the problem from which the exponential decay and polynomial decay are only special cases. And hence our result extend some known results.

To deal with the time delay term, motivated by Nicaise and Pignotti [79], we introduce a new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad x \in \Omega, \quad \rho \in (0,1), \quad t > 0, \tag{3.6}$$

which gives us

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,\infty). \tag{3.7}$$

Then we can get a new system which is equivalent to (3.1)-(3.4):

$$\begin{aligned} u_{tt}(t) + \Delta^2 u(t) - M(\|\nabla u(t)\|^2)\Delta u(t) - \sigma(t)\int_0^t h(t-s)\Delta^2 u(s)ds& \\ +\mu_1 g_1(u_t(t)) + \mu_2 g_2(z(x,1,t)) + f(u(t)) = 0,& \qquad (3.8)\\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0,& \qquad (3.9)\end{aligned}$$

where $x \in \Omega, \rho \in (0,1)$ and $t > 0$, and the initial and boundary conditions are

$$\begin{cases} u(x,0) = u_0, \ u_t(x,0) = u_1, \quad x \in \Omega \\ z(x,0,\rho) = f_0(x, -\rho\tau(0)), \quad (x,\rho) \in \Omega \times (0,1), \\ u = \Delta u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(x,0,t) = u_t(x,t) \quad x \in \Omega, \quad t > 0. \end{cases} \tag{3.10}$$

3.2 Preliminaries and main results

In the following, we denote the standard notations of Lebesgue integral and Sobolev spaces by $L^q(\Omega)(1 \leq q \leq \infty)$ and $H^i(\Omega)(i = 1, 2, 3)$, respectively. The L^2-inner product is denoted by $(\cdot, \cdot)$ and $\| \cdot \|_B$ represents the norm in the space B. We use $\|u\|$ instead of $\|u\|_2$ when $q = 2$. The constants λ_1 and λ_2 are the embedding constants

$$\lambda_1 \|u\|^2 \leq \|\Delta u\|^2, \quad \lambda_2 \|\nabla u\|^2 \leq \|\Delta u\|^2,$$

for $u \in H^2 \cap H_0^1$.

Now we give some assumptions used here. The source term $f(u)$ is a nonlinear function that $f(0) = 0$, and

$$|f(u) - f(v)| \leq c_f(1 + |u|^p + |v|^p)|u - v|, \quad \forall\ u, v \in \mathbb{R}, \tag{3.1}$$

where c_f is a positive constant and

$$0 < p \leq \frac{4}{n-4} \text{ if } n \geq 5 \text{ and } p > 0 \text{ if } 1 \leq n \leq 4. \tag{3.2}$$

We assume further that

$$0 \leq F(u) \leq f(u)u, \quad \forall\ u \in \mathbb{R}, \tag{3.3}$$

where $F(u) = \int_0^u f(z)dz$. Assumptions (3.1) and (3.3) include the following nonlinear type

$$f(u) \approx |u|^p u + |u|^\alpha u, \quad 0 < \alpha < p.$$

Let $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone nondecreasing and C^1 function satisfying

$$zM(z) \geq \widehat{M}(z), \quad \widehat{M}(z) = \int_0^z M(s)ds. \tag{3.4}$$

With respect to the relaxation function h and the potential function σ, we assume
(H1) $h, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying for any $t \geq 0$,

$$h(0) > 0, \quad \int_0^\infty h(s)ds = l_0 < \infty, \quad \sigma(t) > 0,\ 1 - \sigma(t) \int_0^t h(s)ds \geq l > 0. \tag{3.5}$$

(H2) There exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t) > 0, \quad h'(t) \leq -\zeta(t)h(t) \quad \text{for } t \geq 0, \quad \lim_{t\to\infty} \frac{-\sigma'(t)}{\zeta(t)\sigma(t)} = 0. \tag{3.6}$$

Concerning with the time delay term $\tau(t)$, we assume
(D1)There exist two constants $\tau_0 > 0$ and $\tau_1 > 0$ such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall\, t > 0. \tag{3.7}$$

(D2)

$$\tau(t) \in W^{2,\infty}(0,T), \quad \text{and} \quad 0 < \tau'(t) \leq d < 1, \quad \forall\, T, t > 0. \tag{3.8}$$

For nonlinear functions g_1 and g_2, we assume that
(G1) The function $g_1 : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of class $C(\mathbb{R})$. And there exist two positive constants ϵ', c_1 and a convex and increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(R^+) \cap C^2([0,\infty))$ satisfying $H(0) = 0$, and H is linear on $[0,\epsilon']$ (or $H'(0) = 0$ and $H'' > 0$ on $[0,\epsilon']$), such that

$$|g_1(s)| \leq c_1|s|, \quad \text{if } |s| \geq \epsilon', \tag{3.9}$$

and

$$g_1^2(s) \leq H^{-1}(sg_1(s)), \quad \text{if } |s| \leq \epsilon'. \tag{3.10}$$

(G2) The function $g_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$. There exist three constants $c_2 > 0, 0 < \alpha_1 < 1$ and $\alpha_2 > 0$ such that

$$|g_2'(s)| \leq c_2, \tag{3.11}$$

and

$$\alpha_1 s g_2(s) \leq G_2(s) \leq \alpha_2 s g_1(s), \tag{3.12}$$

where

$$G_2(s) = \int_0^s g_2(r)dr.$$

The weight of weak dissipation and the delay satisfy

$$\mu_2 < \frac{\mu_1\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}. \tag{3.13}$$

Now we give the definition of weak solutions to problem (3.1)-(3.4): for given initial data $(u_0,u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (0,1))$, we say that a function $U = (u,u_t) \in C(\mathbb{R}^+, (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega))$ is a weak solution to the problem (3.1)-(3.4) if $U(0) = (u_0,u_1)$ and

$$(u_{tt},\omega) + (\Delta u, \Delta\omega) + (M(\|\nabla u\|^2)\nabla u, \nabla\omega) - \sigma(t)\int_0^t g(t-s)(\Delta u(s), \Delta\omega)ds$$
$$+\mu_1(g_1(u_t),\omega) + \mu_2(g_2(u_t(t-\tau(t))),\omega) + (f(u),\omega) = 0,$$

for all $\omega \in H^2(\Omega) \cap H_0^1(\Omega)$.

By using Faode-Galerkin approximation method, see, for example, [50, 57], we can prove the following theorem concerning the global existence of solutions.

Theorem 3.1 *Assume the assumptions* (3.1)-(3.13) *hold. Let* $(u_0, u_1, f_0) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega \times (0,1))$ *satisfying the compatibility condition* $f_0(\cdot, 0) = u_1$. *Then problem* (3.8)-(3.10) *has a weak solution* $(u, u_t) \in C(0,T; (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega))$ *such that for any* $T > 0$,

$$u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega))), \quad u_t \in L^\infty(0,T; L^2(\Omega)).$$

In the sequel we introduce the energy functional to problem (3.8)-(3.10) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(1 - \sigma(t)\int_0^t h(s)ds\right)\|\Delta u(t)\|^2 + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) \\ &\quad + \frac{1}{2}\sigma(t)(h \circ \Delta u)(t) + \xi\tau(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx + \int_\Omega F(u)dx \end{aligned} \tag{3.14}$$

where $\xi > 0$ is a constant satisfying

$$\frac{\mu_2(1-\alpha_1)}{(1-d)\alpha_1} < \xi < \frac{\mu_1 - \mu_2\alpha_2}{\alpha_2}, \tag{3.15}$$

and

$$(h \circ v) = \int_0^t h(t-s)\|v(t) - v(s)\|^2 ds.$$

Then the main result is the general decay of energy to problem (3.8)-(3.10) which is given in the following theorem.

Theorem 3.2 *Assume the assumptions* (3.1)-(3.13) *hold. Let* (u, u_t) *be the weak solutions of problem* (3.8)-(3.10) *with the initial data* $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$. *Then there exist two constants* $\beta > 0$ *and* $\gamma > 0$ *such that the energy* $E(t)$ *defined by* (3.14) *satisfies for ant* $t > 0$,

$$E(t) \le H_1^{-1}\left(\beta\int_0^t \sigma(s)\zeta(s)ds\right), \tag{3.16}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}ds,$$

and

$$H_2(t) = \begin{cases} t, & if\ H\ is\ linear\ on\ [0,\epsilon'], \\ tH'(\gamma t), & if\ H'(0) = 0\ and\ H'' > 0\ on\ [0,\epsilon']. \end{cases}$$

3.3 General decay

In this section, we shall establish the general decay of energy to problem (3.8)-(3.10). To prove Theorem 3.2, we will use some properties of convex functions introduced and developed by Arnold [48], Cavalcanti et al. [52], Daoulatli et al. [54], Lasiecka [66], Lasiecka and Doundykov [67] and Lasiecka and Tataru [68]. We need the following technical lemmas.

Lemma 3.1 *Under the assumption of Theorem 3.2, the energy functional defined by* (3.14) *is non-increasing and satisfies*

$$\begin{aligned} E'(t) \leq & -(\mu_1 - \xi\alpha_2 - \mu_2\alpha_2)\int_\Omega g_1(u_t)u_t dx \\ & -\Big[\xi(1-d)\alpha_1 - \mu_2 + \mu_2\alpha_1\Big]\int_\Omega g_2(z(x,1,t))z(x,1,t)dx \\ & -\frac{1}{2}\sigma'(t)\int_0^t h(s)ds\|\Delta u(t)\|^2 - \frac{1}{2}\sigma(t)h(t)\|\Delta u(t)\|^2 \\ & +\frac{1}{2}\sigma'(t)(h\circ\Delta u)(t) + \frac{1}{2}\sigma(t)(h'\circ\Delta u)(t). \end{aligned} \tag{3.1}$$

Proof. First by direct calculations, we have

$$\begin{aligned} \sigma(t)(h*u,u_t) = & -\frac{\sigma(t)}{2}h(t)\|u(t)\|^2 - \frac{d}{dt}\left[\frac{\sigma(t)}{2}(h\circ u)(t) - \frac{\sigma(t)}{2}\left(\int_0^t h(s)ds\right)\|u(t)\|^2\right] \\ & +\frac{\sigma(t)}{2}(h'\circ u)(t) + \frac{\sigma'(t)}{2}(h\circ u)(t) - \frac{\sigma'(t)}{2}\int_0^t h(s)ds\|u(t)\|^2, \end{aligned} \tag{3.2}$$

where

$$(h*u)(t) = \int_0^t g(t-s)u(s)ds.$$

Differentiating (3.14) and using (3.8) and (3.2), we shall get that

$$\begin{aligned}
E'(t) &= \int_\Omega u_t\Big(-\Delta^2 u + M(\|\nabla u\|^2)\Delta u + \sigma(t)\int_0^t h(t-s)\Delta^2 u(s)ds - \mu_1 g_1(u_t) \\
&\quad -\mu_2 g_2(z(x,1,t)) - f(u)\Big)dx + M(\|\nabla u\|^2)\int_\Omega \nabla u\cdot\nabla u_t dx + \int_\Omega \Delta u\cdot\Delta u_t dx \\
&\quad -\frac{1}{2}\sigma'(t)\int_0^t h(s)ds\|\Delta u\|^2 - \frac{1}{2}\sigma(t)h(t)\|\Delta u\|^2 + \frac{1}{2}\sigma'(t)(h\circ\Delta u) \\
&\quad +\frac{1}{2}\sigma(t)(h\circ\Delta u)_t + \left(1-\sigma(t)\int_0^t h(s)ds\right)\int_\Omega \Delta u\cdot\Delta u_t dx \\
&\quad +\frac{d}{dt}\left(\xi\tau(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx\right) \\
&= -\mu_1\int_\Omega g_1(u_t)u_t dx - \mu_2\int_\Omega g_2(z(x,1,t))u_t dx - \frac{1}{2}\sigma'(t)\int_0^t h(s)ds\|\Delta u\|^2 \\
&\quad -\frac{1}{2}\sigma(t)h(t)\|\Delta u\|^2 + \frac{1}{2}\sigma'(t)(h\circ\Delta u) + \frac{1}{2}\sigma(t)(h'\circ\Delta u) \\
&\quad +\frac{d}{dt}\left(\xi\tau(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx\right). \qquad (3.3)
\end{aligned}$$

Multiplying (3.9) by $\xi g_2(z(x,\rho,t))$ and integrating the result over $\Omega\times(0,1)$ with respect to x and ρ, we can obtain

$$\begin{aligned}
&\xi\tau(t)\int_\Omega\int_0^1 z_t(x,\rho,t)g_2(z(z,\rho,t))d\rho dx \\
&= -\xi\int_\Omega\int_0^1 (1-\tau'(t)\rho)z_\rho(x,\rho,t)g_2(z(x,\rho,t))d\rho dx \\
&= -\xi\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\Big[(1-\tau'(t)\rho)G_2(z(x,\rho,t))\Big]d\rho dx - \xi\tau'(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx \\
&= -\xi\int_\Omega\Big[(1-\tau'(t))G_2(z(x,1,t)) - G_2(u_t(t))\Big]dx - \xi\tau'(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx,
\end{aligned}$$

which gives us

$$\begin{aligned}
&\frac{d}{dt}\left(\xi\tau(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx\right) \\
&= \xi\tau'(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx + \xi\tau(t)\int_\Omega\int_0^1 z_t(x,\rho,t)g_2(z(z,\rho,t))d\rho dx \\
&= -\xi\int_\Omega(1-\tau'(t))G_2(z(x,1,t))dx + \xi\int_\Omega G_2(u_t(t))dx. \qquad (3.4)
\end{aligned}$$

Combining (3.3) with (3.4) and using (3.12), we have

$$\begin{aligned} E'(t) &\leq -(\mu_1 - \xi\alpha_2)\int_\Omega g_1(u_t)u_t dx - \mu_2 \int_\Omega g_2(z(x,1,t))u_t dx \\ &\quad -\xi\int_\Omega (1-\tau'(t))G_2(z(x,1,t))dx - \frac{1}{2}\sigma'(t)\int_0^t h(s)ds\|\Delta u\|^2 \\ &\quad -\frac{1}{2}\sigma(t)h(t)\|\Delta u\|^2 + \frac{1}{2}\sigma'(t)(h\circ\Delta u) + \frac{1}{2}\sigma(t)(h'\circ\Delta u). \end{aligned} \tag{3.5}$$

Following the same arguments as in Arnold [48], see also, Cavalcanti et al. [52], Daoulatli et al. [54], Lasiecka [66], Lasiecka and Doundykov [67] and Lasiecka and Tataru [68], we denote the conjugate function of the convex function G_2 by G_2^*, i.e., $G_2^*(s) = \sup_{t\in\mathbb{R}^+}(st - G_2(t))$. Then for any $s \geq 0$,

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)],$$

is the Legendre transform of G_2, which satisfies for any $s, t \geq 0$,

$$st \leq G_2^*(s) + G_2(t).$$

It follows from the definition of G_2 that

$$G_2^*(s) = sg_2^{-1}(s) - G_2(g_2^{-1}(s)),$$

which implies

$$G_2^*(z(x,1,t)) = g_2(z(x,1,t))z(x,1,t) - G_2(z(x,1,t)).$$

Hence

$$\begin{aligned} g_2(z(x,1,t))u_t &\leq G_2^*(g_2(z(x,1,t))) + G_2(u_t) \\ &= g_2(z(x,1,t))z(x,1,t) - G_2(z(x,1,t)) + G_2(u_t). \end{aligned} \tag{3.6}$$

Inserting (3.6) into (3.5) and using (3.8) and (3.12), we get

$$\begin{aligned} E'(t) &\leq -(\mu_1 - \xi\alpha_2)\int_\Omega g_1(u_t)u_t dx + \Big[\mu_2 - \xi(1-d)\alpha_1\Big]\int_\Omega g_2(z(x,1,t))z(x,1,t)dx \\ &\quad -\mu_2\int_\Omega G_2(z(x,1,t))dx + \mu_2\int_\Omega G_2(u_t)dx - \frac{1}{2}\sigma'(t)\int_0^t h(s)ds\|\Delta u\|^2 \\ &\quad -\frac{1}{2}\sigma(t)h(t)\|\Delta u\|^2 + \frac{1}{2}\sigma'(t)(h\circ\Delta u) + \frac{1}{2}\sigma(t)(h'\circ\Delta u). \end{aligned} \tag{3.7}$$

Therefore (3.1) follows from (3.7) and (3.12). The proof is complete. □

3.3. General decay

Lemma 3.2 *Under the assumptions in Theorem 3.2, then the functional $\phi(t)$ defined by*

$$\phi(t) = \int_\Omega u(t)u_t(t)dx, \tag{3.8}$$

satisfies that there exist positive constants C_1, C_2 and C_3 such that for any $t \geq 0$,

$$\begin{aligned}\phi'(t) \ &\leq \ \|u_t(t)\|^2 - \frac{l}{2}\|\Delta u(t)\|^2 - \widehat{M}(\|\nabla u(t)\|^2) + C_1\sigma(t)(h \circ \Delta u)(t) \\ &\quad + C_2\int_\Omega g_1^2(u_t(t))dx + C_3\int_\Omega g_2^2(z(x,1,t))dx. \end{aligned} \tag{3.9}$$

Proof. By using (3.8) and (3.4), we can get

$$\begin{aligned}\phi'(t) \ &= \ \|u_t\|^2 + \int_\Omega u(t)\cdot\Big(- \Delta^2 u + M(\|\nabla u\|^2)\Delta + \sigma(t)\int_0^t h(t-s)\Delta^2 u(s)ds \\ &\quad -\mu_1 g_1(u_t) - \mu_2 g_2(z(x,1,t)) - f(u)\Big)dx \\ &= \ \|u_t\|^2 - \|\Delta u\|^2 - M(\|\nabla u\|^2)\|\nabla u\|^2 \\ &\quad +\sigma(t)\int_\Omega \Delta u(t)\cdot\int_0^t h(t-s)\Delta u(s)dsdx - \mu_1\int_\Omega g(u_t)udx \\ &\quad -\mu_2\int_\Omega g_2(z(x,1,t)) - \int_\Omega f(u)udx \\ &\leq \ \|u_t\|^2 - \|\Delta u\|^2 - \widehat{M}(\|\nabla u\|^2) + \underbrace{\sigma(t)\int_\Omega \Delta u(t)\cdot\int_0^t h(t-s)\Delta u(s)dsdx}_{:=I_1} \\ &\quad \underbrace{-\mu_1\int_\Omega g(u_t)udx}_{:=I_2}\underbrace{-\mu_2\int_\Omega g_2(z(x,1,t))}_{:=I_3}. \end{aligned} \tag{3.10}$$

It follows from Young's inequality and Poincaré's inequality that for any $\varepsilon > 0$,

$$\begin{aligned} I_1 \ &= \ \sigma(t)\int_\Omega \Delta u(t)\cdot\int_0^t h(t-s)(\Delta u(s) - \Delta u(t))dsdx + \sigma(t)\int_0^t h(s)ds\|\Delta u(t)\|^2 \\ &\leq \ (1-l)\|\Delta u\|^2 + \varepsilon\|\Delta u\|^2 + \frac{\sigma^2(t)}{4\varepsilon}\int_\Omega\left(\int_0^t h(t-s)(\Delta u(s) - \Delta u(t))ds\right)^2 dx \\ &\leq \ (1-l+\varepsilon)\|\Delta u\|^2 + \frac{1-l}{4\varepsilon}\sigma(t)(h\circ\Delta u)(t), \end{aligned} \tag{3.11}$$

$$I_2 \leq \frac{\varepsilon}{\lambda_1}\|\Delta u\|^2 + \frac{\mu_1^2}{4\varepsilon}\int_\Omega g_1^2(u_t)dx, \tag{3.12}$$

$$I_3 \le \frac{\varepsilon}{\lambda_1}\|\Delta u\|^2 + \frac{\mu_2^2}{4\varepsilon}\int_\Omega g_2^2(z(x,1,t))dx. \tag{3.13}$$

Inserting (3.11)-(3.13) into (3.10), we shall see that for any $\varepsilon > 0$,

$$\begin{aligned}\phi'(t) \quad &\le \quad \|u_t(t)\|^2 - \left(l - \varepsilon - \frac{2\varepsilon}{\lambda_1}\right)\|\Delta u(t)\|^2 + \frac{1-l}{4\varepsilon}(h \circ \Delta u)(t) - \widehat{M}(\|\nabla u\|^2) \\ &+ \frac{\mu_1^2}{4\varepsilon}\int_\Omega g_1^2(u_t)dx + \frac{\mu_2^2}{4\varepsilon}\int_\Omega g_2^2(z(x,1,t))dx. \end{aligned} \tag{3.14}$$

Taking $\varepsilon > 0$ small enough in (3.14) such that

$$l - \varepsilon - \frac{2\varepsilon}{\lambda_1} > \frac{l}{2},$$

then we can obtain (3.9) with

$$C_1 = \frac{1-l}{4\varepsilon}, \quad C_2 = \frac{\mu_1^2}{4\varepsilon}, \quad C_3 = \frac{\mu_2^2}{4\varepsilon}.$$

The proof is hence complete. □

Lemma 3.3 *Under the assumptions in Theorem 3.2, then the functional $\psi(t)$ defined by*

$$\psi(t) = -\int_\Omega u(t)\int_0^t h(t-s)(u(t)-u(s))dsdx, \tag{3.15}$$

satisfies that there exist a positive constant C_4 such that for any $\delta > 0$,

$$\begin{aligned}\psi'(t) \quad &\le \quad \left[\delta(1-l)^2 + 3\delta\right]\|\Delta u(t)\|^2 - \left[\left(\int_0^t h(s)ds\right) - \delta\right]\|u_t(t)\|^2 + \delta\|g_1(u_t(t))\|^2 \\ &+ \delta\|g_2(z(x,1,t))\|^2 + C_4\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t) - \frac{h(0)}{4\delta\lambda_1}(h'\circ\Delta u)(t). \end{aligned} \tag{3.16}$$

Proof. Taking the derivative of $\psi(t)$ and using (3.8), we conclude that

$$\begin{aligned}
\psi'(t) &= \int_\Omega \Big(\Delta^2 u - M(\|\nabla u\|^2)\Delta u - \sigma(t)\int_0^t h(t-s)\Delta^2 u(s)ds \\
&\quad + \mu_1 g_1(u_t) + \mu_2 g_2(z(x,1,t)) + f(u)\Big)\int_0^t h(t-s)(u(t)-u(s))ds \\
&\quad - \int_\Omega u_t \int_0^t h'(t-s)(u(t)-u(s))dsdx - \int_0^t h(s)ds\|u_t\|^2 \\
&= \int_\Omega \Delta u(t)\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))dsdx \\
&\quad + M(\|\nabla u\|^2)\int_\Omega \nabla u(t)\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\quad - \sigma(t)\int_\Omega \left(\int_0^t h(t-s)\Delta u(s)ds\right)\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)dx \\
&\quad + \mu_1 \int_\Omega g_1(u_t)\int_0^t h(t-s)(u(t)-u(s))dsdx \\
&\quad + \mu_2 \int_\Omega g_2(z(x,1,t))\int_0^t h(t-s)(u(t)-u(s))dsdx \\
&\quad + \int_\Omega f(u)\int_0^t h(t-s)(u(t)-u(s))dsdx - \int_\Omega u_t \int_0^t h'(t-s)(u(t)-u(s))dsdx \\
&\quad - \int_0^t h(s)ds\|u_t\|^2 \\
&:= \sum_{i=1}^{7} J_i - \int_0^t h(s)ds\|u_t\|^2. \qquad (3.17)
\end{aligned}$$

By using Young's inequality and Poincaré's inequality, we can get for any $\delta > 0$,

$$J_1 \le \delta\|\Delta u\|^2 + \frac{1}{4\delta}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.18)$$

$$J_2 \le \delta\|\Delta u\|^2 + \frac{C'}{4\lambda_2\delta}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.19)$$

$$\begin{aligned}
J_3 &= \sigma(t)\int_\Omega\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)^2 dx \\
&\quad -\sigma(t)\int_\Omega\left(\int_0^t h(t-s)\Delta u(t)ds\right)\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)dx \\
&\leq \sigma(t)\left(\int_0^t h(s)ds\right)(h\circ\Delta u)+\delta\sigma^2(t)\left(\int_0^t h(s)ds\right)^2\|\Delta u\|^2 \\
&\quad +\frac{1}{4\delta}\int_\Omega\left(\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right)^2 dx \\
&\leq \delta(1-l)^2\|\Delta u\|^2+\left(\sigma(t)+\frac{1}{4\delta}\right)\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.20)
\end{aligned}$$

$$J_4 \leq \delta\|g_1(u_t)\|^2+\frac{\mu_1^2}{4\delta\lambda_1}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.21)$$

$$J_5 \leq \delta\|g_2(z(x,1,t))\|^2+\frac{\mu_2^2}{4\delta\lambda_1}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.22)$$

$$J_6 \leq \delta\|\Delta u\|^2+\frac{1}{4\delta\lambda_1}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t), \qquad (3.23)$$

$$\begin{aligned}
J_7 &\leq \delta\|u_t\|^2+\frac{1}{4\delta\lambda_1}\left(\int_0^t h'(s)ds\right)(h'\circ\Delta u)(t) \\
&\leq \delta\|u_t\|^2-\frac{h(0)}{4\delta\lambda_1}(h'\circ\Delta u)(t),
\end{aligned}$$

which, combining (3.18)-(3.23) with (3.17), gives us (3.16) with

$$C_4=\frac{1}{2\delta}+\frac{C'}{4\lambda_2\delta}+\sigma(0)+\frac{\mu_1^2}{4\delta\lambda_1}+\frac{\mu_2^2}{4\delta\lambda_1}+\frac{1}{4\delta\lambda_1},$$

where we used the fact $\sigma(t)<\sigma(0)$. Therefore the proof is complete. □

Lemma 3.4 *Under the assumptions in Theorem 3.2, we define the functional $I(t)$ by*

$$I(t)=\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho dx. \qquad (3.24)$$

Then we have

$$I'(t)\leq -2(1-d)I(t)-\frac{1-d}{\tau_1}e^{-2\tau_1}\int_\Omega G_2(z(x,1,t))dx+\frac{1}{\tau_0}\int_\Omega G_2(u_t(t))dx. \qquad (3.25)$$

3.3. General decay

Proof. Differentiating (3.24) with respect to t and noting the equation (3.9), we can get for any $t > 0$,

$$\begin{aligned}
I'(t) &= -2\rho\tau'(t)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho dx \\
&\quad +\int_\Omega\int_0^1 e^{-2\tau(t)\rho}g_2(z(x,\rho,t))z_t(x,\rho,t)d\rho dx \\
&= -2\rho\tau'(t)I(t) - \frac{1}{\tau(t)}\int_\Omega\int_0^1 e^{-2\tau(t)\rho}g_2(z(x,\rho,t))z_\rho(x,\rho,t)(1-\tau'(t)\rho)d\rho dx \\
&= -2\rho\tau'(t)I(t) - \frac{1}{\tau(t)}\int_\Omega\int_0^1 \left[\frac{\partial}{\partial\rho}\Big(e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\Big)\right. \\
&\quad \left. +2\tau(t)G_2(z(x,\rho,t))e^{-2\tau(t)\rho}\right](1-\tau'(t)\rho)d\rho dx \\
&= -2\rho\tau'(t)I(t) - \frac{1}{\tau(t)}\left[\int_\Omega\left(e^{-2\tau(t)\rho}G_2(z(x,\rho,t))(1-\tau'(t)\rho)\Big|_0^1\right.\right. \\
&\quad \left.\left. +\tau'(t)\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho\right)dx\right] \\
&\quad -2(1-\tau'(t)\rho)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho dx \\
&= -2\rho\tau'(t)I(t) - \frac{1-\tau'(t)}{\tau(t)}\int_\Omega e^{-2\tau(t)}G_2(z(x,1,t))dx + \frac{1}{\tau(t)}\int_\Omega G_2(u_t(t))dx \\
&\quad -\frac{\tau'(t)}{\tau(t)}I(t) - 2(1-\tau'(t)\rho)\int_\Omega\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))d\rho dx \\
&\le \left[-2\rho\tau'(t) - \frac{\tau'(t)}{\tau(t)} - 2(1-d)\right]I(t) - \frac{1-d}{\tau_1}e^{-2\tau_1}\int_\Omega G_2(z(x,1,t))dx \\
&\quad +\frac{1}{\tau_0}\int_\Omega G_2(u_t(t))dx \\
&\le -2(1-d)I(t) - \frac{1-d}{\tau_1}e^{-2\tau_1}\int_\Omega G_2(z(x,1,t))dx + \frac{1}{\tau_0}\int_\Omega G_2(u_t(t))dx.
\end{aligned}$$

Then the proof is done. □

Now we define a Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = ME(t) + \varepsilon_1\sigma(t)\phi(t) + \varepsilon_2\sigma(t)I(t) + \sigma(t)\psi(t), \tag{3.26}$$

where M, ε_1 and ε_2 are positive constants will be taken later.

Then we can get the following lemma.

Lemma 3.5 *For M large, there exist two positive constants β_1 and β_2 such that*

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t). \tag{3.27}$$

Proof. It is easy to verify that for any $t > 0$,

$$\begin{aligned}|\mathcal{L}(t) - ME(t)| &= |\varepsilon_1\sigma(t)\phi(t) + \varepsilon_2\sigma(t)I(t) + \sigma(t)\psi(t)| \\ &\leq \frac{\varepsilon_1}{2\lambda_1}\|\Delta u\|^2 + \left(\frac{\varepsilon_1}{2}\sigma(0) + \frac{\sigma(0)}{2}\right)\|u_t\|^2 \\ &\quad + \varepsilon_2\sigma(0)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx + \frac{1-l}{2}(h\circ\Delta u),\end{aligned}$$

which yields that there exists a constant $C^* > 0$ such that

$$|\mathcal{L}(t) - ME(t)| \leq C^*E(t).$$

Then we can obtain (3.27) for M large with $\beta_1 = M - C^*$ and $\beta_2 = M + C^*$. The proof is therefore complete. □

In the sequel we shall prove Theorem 3.2 combining above lemmas.

Proof of Theorem 3.2. First for any fixed $t_0 > 0$, we have for any $t \geq t_0$,

$$\int_0^t h(s)ds \geq \int_0^{t_0} h(s)ds := h_0.$$

By using Young's inequality and Poincaré's inequality, we arrive at

$$\begin{aligned}&\varepsilon_1\sigma'(t)\phi(t) + \varepsilon_2\sigma'(t)I(t) + \sigma'(t)\psi(t) \\ &\leq \frac{1+\varepsilon_1}{2}\sigma'(t)\|u_t(t)\|^2 + \frac{\varepsilon_1}{2\lambda_1}\sigma'(t)\|\Delta u(t)\|^2 + \frac{\sigma'(t)}{2\lambda_1}\left(\int_0^t h(s)ds\right)(h\circ\Delta u)(t) \\ &\quad + \varepsilon_2\sigma'(t)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx. \qquad (3.28)\end{aligned}$$

It follows from (3.1), (3.9), (3.16), (3.25) and (3.28) that for any $t > t_0$,

$$\begin{aligned}\mathcal{L}'(t) &= ME'(t) + \varepsilon_1\sigma(t)\phi'(t) + \varepsilon_2\sigma(t)I'(t) + \sigma(t)\psi'(t) \\ &\quad + \varepsilon_1\sigma'(t)\phi(t) + \varepsilon_2\sigma'(t)I(t) + \sigma'(t)\psi(t) \\ &\leq -\sigma(t)\left(h_0 - \delta - \varepsilon_1 + \frac{1+\varepsilon_1}{2}\frac{\sigma'(t)}{\sigma(t)}\right)\|u_t(t)\|^2 + \sigma(t)\left(\frac{M}{2} - \frac{h(0)}{4\delta\lambda_1}\right)(h'\circ\Delta u)(t) \\ &\quad - \sigma(t)\left[-\varepsilon_1\frac{\sigma'(t)}{2\lambda_1\sigma(t)} + \frac{l}{2}\varepsilon_1 - \left(\delta(1-l)^2 + 3\delta\right)\right]\|\Delta u(t)\|^2 \\ &\quad - 2\varepsilon_2(1-d)\sigma(t)I(t) - \varepsilon_1\sigma(t)\widehat{M}(\|\nabla u(t)\|^2) \\ &\quad + \sigma(t)\left[C_1\varepsilon_1\sigma(0) + C_4 l_0 + \frac{\sigma'(t)}{2\lambda_1\sigma(t)}l_0\right](h\circ\Delta u)(t)\end{aligned}$$

$$
\begin{aligned}
&-\left[MC+\frac{(1-d)\varepsilon_2}{\tau_1}e^{-2\tau_1}\alpha_1-\sigma(0)(\varepsilon_1 C_3+\delta)c_1\right]\int_\Omega g_2(z(x,1,t))z(x,1,t)dx\\
&-\left(MC-\frac{\varepsilon_2\alpha_2}{\tau_0}\right)\int_\Omega g_1(u_t(t))u_t(t)dx+\sigma(t)(\varepsilon_1 C_2+\delta)\|g_1(u_t(t))\|^2\\
&+\sigma(t)\left(\varepsilon_2\frac{\sigma'(t)}{\sigma(t)}\right)\int_\Omega\int_0^1 G_2(z(x,\rho,t))d\rho dx. \qquad (3.29)
\end{aligned}
$$

Now we first take $\varepsilon_1>0$ small enough so that

$$h_0-\varepsilon_1>\frac{1}{2}h_0.$$

For any fixed $\varepsilon_1>0$, we pick $\delta>0$ so small that

$$\frac{l}{2}\varepsilon_1-\left(\delta(1-l)^2+3\delta\right)>\frac{l}{4}\varepsilon_1,$$

and

$$h_0-\delta-\varepsilon_1>\frac{1}{4}h_0.$$

At last we choose $M>0$ large enough so that (3.27) hold, and further

$$\frac{M}{2}-\frac{h(0)}{4\delta\lambda_1}>0,\quad MC-\frac{\varepsilon_2\alpha_2}{\tau_0}>0,$$

and

$$MC+\frac{(1-d)\varepsilon_2}{\tau_1}e^{-2\tau_1}\alpha_1-\sigma(0)(\varepsilon_1 C_3+\delta)c_1>0.$$

Then from (3.29) and noting that the fact

$$\lim_{t\to+\infty}\frac{\sigma'(t)}{\sigma(t)}=0,$$

we can conclude that there exist three positive constant C_5, C_6 and C_7 such that for any $t\geq t_0$,

$$\mathcal{L}'(t)\leq -C_5\sigma(t)E(t)+C_6\sigma(t)(h\circ\Delta u)(t)+C_7\sigma(t)\|g_1(u_t(t))\|^2. \qquad (3.30)$$

In the sequel we shall exploit the same method as in [50] to deal with the last term in the right-hand side of (3.30). We define

$$\Omega^+=\{x\in\Omega:|u_t(t)|\geq\epsilon'\},\quad \Omega^+=\{x\in\Omega:|u_t(t)|\leq\epsilon'\}.$$

By using (3.9)-(3.10), we can obtain

$$\int_{\Omega^+}|g_1(u_t)|^2dx\leq\varsigma\int_{\Omega^+}u_t g_1(u_t)dx\leq-\varsigma E'(t), \qquad (3.31)$$

where $\varsigma > 0$ is a positive constant. Now we distinguish two cases.
Case 1. H is linear on $[0, \epsilon']$.
It is easy to get that there exists a positive $\varsigma' > 0$ such that $|g_1(s)| \le \varsigma'|s|$ for any $|s| \le \epsilon'$, then we have

$$\int_{\Omega^-} |g_1(u_t)|^2 dx \le \varsigma' \int_{\Omega^-} u_t g_1(u_t) dx \le -\varsigma' E'(t),$$

which, together with (3.30) and (3.31), gives us there exist a constant $\kappa_1 > 0$ such that

$$(\mathcal{L}(t) + \mu E(t))' \le -\kappa\sigma(t) H_2(E(t)) + C_6 \sigma(t)(h \circ \Delta u)(t), \tag{3.32}$$

where $\mu = C_8(\varsigma + \varsigma')$ and $C_8 = C_7\sigma(0)$.
Case 2. $H'(0) = 0$ and $H'' > 0$ on $[0, \epsilon']$.
Since the function H is convex and increasing, the function H^{-1} is concave and increasing. It follows from (3.10) and (3.1) that

$$\begin{aligned} \int_{\Omega^-} |g_1(u_t)|^2 dx &\le \int_{\Omega^-} H^{-1}(u_t g_1(u_t)) dx \\ &\le |\Omega| H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega^-} u_t g_1(u_t) dx \right) \\ &\le C H^{-1}(-C' E'(t)), \end{aligned} \tag{3.33}$$

where we used the reversed Jensen's inequality for concave function, and C, C' are two positive constants. Inserting into (3.31) and (3.33) into (3.30), we deduce that for any $t \ge t_0$,

$$\begin{aligned} (\mathcal{L}(t) + C_8 \varsigma E(t))' &\le -C_5 \sigma(t) E(t) + C_6 \sigma(t)(h \circ \Delta u)(t) \\ &\quad + \tilde{C}_7 \sigma(t) H^{-1}(-C' E'(t)). \end{aligned} \tag{3.34}$$

Let us denote the conjugate function of the convex function of H by H^*, i.e.,

$$H^*(s) = \sup_{t \in \mathbb{R}^+} (st - H(t)).$$

Then for any $s \ge 0$,

$$H^*(s) = s(H')^{-1}(s) - H\Big[(H')^{-1}(s)\Big], \tag{3.35}$$

is the Legendre transform and satisfies for any $s, t \ge 0$,

$$st \le H^*(s) + H(t). \tag{3.36}$$

By using (3.35) and nothing that the functions H and $(H')^{-1}$ are increasing and $H'(0) = 0$, we can obtain for any $s \geq 0$,

$$H^*(s) \leq s(H')^{-1}(s),$$

which, together with (3.34) and (3.36), implies for any ϵ_0 small enough,

$$\begin{aligned}
&\frac{d}{dt}\Big[H'(\epsilon_0 E(t))(\mathcal{L}(t) + C_8\varsigma E(t)) + \tilde{C}_7 C'\sigma(t)E(t)\Big] \\
&= \epsilon_0 E'(t)H''(\epsilon_0 E(t))(\mathcal{L}(t) + C_8\varsigma E(t)) + H'(\epsilon_0 E(t))(\mathcal{L}'(t) + C_8\varsigma E'(t)) \\
&\quad + \tilde{C}_7 C'\sigma'(t)E(t) + \tilde{C}_7 C'\sigma(t)E'(t) \\
&\leq -C_5\sigma(t)H'(\epsilon_0 E(t))E(t) + C_6\sigma(t)H'(\epsilon_0 E(t))(h \circ \Delta u)(t) \\
&\quad + \tilde{C}_7\sigma(t)H'(\epsilon_0 E(t))H^{-1}(-C'E'(t)) + \tilde{C}_7 C'\sigma'(t)E(t) + \tilde{C}_7 C'\sigma(t)E'(t) \\
&\leq -C_5\sigma(t)H'(\epsilon_0 E(t))E(t) + C_6\sigma(t)H'(\epsilon_0 E(t))(h \circ \Delta u)(t) + \tilde{C}_7\sigma(t)H^*(H'(\epsilon_0 E(t))) \\
&\leq -C_5\sigma(t)H'(\epsilon_0 E(t))E(t) + C_6\sigma(t)H'(\epsilon_0 E(t))(h \circ \Delta u)(t) + \tilde{C}_7\sigma(t)H'(\epsilon_0 E(t))\epsilon_0 E(t) \\
&\leq -\tilde{C}_5\sigma(t)H'(\epsilon_0 E(t))E(t) + C_6\sigma(t)H'(\epsilon_0 E(0))(h \circ \Delta u)(t) \\
&= -\tilde{C}_5\sigma(t)H_2(E(t)) + C_6\sigma(t)H'(\epsilon_0 E(0))(h \circ \Delta u)(t),
\end{aligned} \tag{3.37}$$

where we used the following fact

$$E'(t) \leq 0, \quad H''(t) \geq 0, \quad 0 \leq H'(\epsilon_0 E(t)) \leq H'(\epsilon_0 E(0)).$$

Let

$$\tilde{\mathcal{L}}(t) = \begin{cases} \mathcal{L}(t) + \mu E(t), & if\ H\ is\ linear\ on\ [0, \epsilon'], \\ H'(\epsilon_0 E(t))(\mathcal{L}(t) + C_8\varsigma E(t)) + \tilde{C}_7 C'\sigma(t)E(t), & \\ & if\ H'(0) = 0\ and\ H'' > 0\ on\ [0, \epsilon']. \end{cases} \tag{3.38}$$

Then it follows from (3.32) and (3.37) that for any $t \geq t_0$,

$$\tilde{\mathcal{L}}'(t) \leq -C_9\sigma(t)H_2(E(t)) + C_{10}\sigma(t)(h \circ \Delta u)(t), \tag{3.39}$$

where C_9 and C_{10} are positive constants. It is easy to verify that $\tilde{\mathcal{L}}(t)$ is equivalent to $E(t)$. Since $\zeta'(t) < 0$, then there exist a constant $\bar{\theta} > 0$ such that for any $t \geq t_0$,

$$\zeta(t)\tilde{\mathcal{L}}(t) + 2C_{10}E(t) \leq \bar{\theta}E(t). \tag{3.40}$$

Denote

$$\mathcal{E}(t) = \theta(\zeta(t)\tilde{\mathcal{L}}(t) + 2C_{10}E(t)), \quad \text{for } 0 < \theta < \frac{1}{\bar{\theta}}.$$

It follows from (3.6), (3.1) and (3.39)-(3.40) that for any $t \geq t_0$,

$$\begin{aligned}
\mathcal{E}'(t) &= \theta(\zeta'(t)\tilde{\mathcal{L}}(t) + \zeta(t)\tilde{\mathcal{L}}'(t) + 2C_{10}E'(t)) \\
&\leq -C_9\theta\sigma(t)\zeta(t)H_2(E(t)) + C_{10}\theta\sigma(t)\zeta(t)(h \circ \Delta u)(t) + 2C_{10}\theta E'(t) \\
&\leq -C_9\theta\sigma(t)\zeta(t)H_2(E(t)) - C_{10}\theta\sigma(t)(h' \circ \Delta u)(t) + 2C_{10}\theta E'(t) \\
&\leq -C_9\theta\sigma(t)\zeta(t)H_2(E(t)) \leq -C_9\theta\sigma(t)\zeta(t)H_2\left(\frac{1}{\bar{\bar{\theta}}}\Big(\zeta(t)\tilde{\mathcal{L}}(t) + 2C_{10}E(t)\Big)\right) \\
&\leq -C_9\theta\sigma(t)\zeta(t)H_2\Big((\zeta(t)\tilde{\mathcal{L}}(t) + 2C_{10}E(t))\Big) \\
&= -C_9\theta\sigma(t)\zeta(t)H_2(\mathcal{E}(t)),
\end{aligned}$$

which, noting that $H_1' = -\frac{1}{H_2}$, yields that for any $t \geq t_0$,

$$\mathcal{E}'(t)H_1'(\mathcal{E}(t)) \geq C_9\theta\sigma(t)\zeta(t). \tag{3.41}$$

Integrating (3.41) over (t_0, t), we have

$$H_1(\mathcal{E}(t)) \geq H_1(\mathcal{E}(t_0)) + C_9\theta\int_0^t \sigma(s)\zeta(s)ds - C_9\theta\int_0^{t_0}\sigma(s)\zeta(s)ds.$$

Taking $\theta > 0$ small enough so that

$$H_1(\mathcal{E}(t_0)) - C_9\theta\int_0^{t_0}\sigma(s)\zeta(s)ds > 0,$$

and noting that H_1^{-1} is decreasing, we deduce that

$$\begin{aligned}
\mathcal{E}(t) &\leq H_1^{-1}\left(H_1(\mathcal{E}(t_0)) + C_9\theta\int_0^t \sigma(s)\zeta(s)ds - C_9\theta\int_0^{t_0}\sigma(s)\zeta(s)ds\right) \\
&\leq H_1^{-1}\left(C_9\theta\int_0^t \sigma(s)\zeta(s)ds\right).
\end{aligned} \tag{3.42}$$

Then (3.16) follows from the equivalent of $E(t)$ and $\mathcal{E}(t)$ and by by renaming the constants. The proof of Theorem 3.2 is complete. □

CHAPTER 4

The sharp decay rate of thermoelastic transmission system with infinite memories

The main contributions here are to show the lack of exponential stability and to prove that the t^{-1} is the sharp decay rate of problem (4.1). That is to show that for this types of materials, the dissipation produced by the weak-infinite memories are not strong enough to produce an exponential decay of the solution under usual/non usual conditions on the the relaxation functions's growth. This work extends the previous results by [3], [26] to the weak-viscoelasticities in two parts. In order to fill this gaps, we use an appropriate estimates.This is paper [L.Kassah.Laouar, KH. ZENNIR and S. Boulaaras, The sharp decay rate of thermoelastic transmission system with infinite memories, Rendiconti del Circolo Matematico di Palermo Series 2, https://doi.org/10.1007/s12215-019-00408-1, 2019]

4.1 Introduction

The extent to which different types of decay rate of transmission system get our attention to study the role of the damped case, it is largely requested, especially with the rapid adoption of weak-viscoelasatic terms. While there is no novelty in the idea of inter-relation between the problems of thermoelastic and of viscoelastic in the transmission system with $1-d$ mixed type I and type II, nevertheless recent few works in only one part. This research addressed the needs of mathematical physics interests for the thermoelastic transmission problem with weak- infinite memories,

those which are acting in the first and second parts (see [3], [25], [26]).
The problem we will consider here, for $t > 0$, is

$$\begin{cases} \rho_1 u'' - a_1\Big(u_{xx} - \alpha_1(t)\int_{-\infty}^{t}\mu_1(t-s)u_{xx}(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L, 0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L, 0), \\ \rho_2 v'' - a_2\Big(v_{xx} - \alpha_2(t)\int_{-\infty}^{t}\mu_2(t-s)v_{xx}(s)ds\Big) + \beta_2 q_x = 0, & x \in (0, L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0, L), \\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \end{cases} \tag{4.1}$$

where u, v are the displacement of the system at time t in $(-L, 0)$ and $(0, L)$ and θ, q are respectively the temperature difference with respect to a fixed reference temperature, w_1, w_2 are the so-called thermal displacement, which satisfies

$$w_1(.,t) = \int_0^t \theta(.,s)ds + w_1(.,0)$$

and

$$w_2(.,t) = \int_0^t q(.,s)ds + w_2(.,0).$$

The parameters $a_1, a_2, \rho_1, \rho_2, \beta_1, \beta_2, c_1.c_2, k, l$ and $L < \infty$ are assumed to be positive constants. The functions $\mu_i, \alpha_i, i = 1, 2$ will be defined later.
The system (4.1) satisfies the Dirichlet boundary conditions:

$$\begin{cases} u(-L,t) = v(L,t) = 0, & t > 0, \\ \\ w_1(-L,t) = w_2(L,t) = 0, & t > 0, \end{cases} \tag{4.2}$$

and the following initial conditions:

$$\begin{cases} u(.,0) = u^0(x), u'(.,0) = u^1(x), w_1(.,0) = w_1^0(x), \theta(.,0) = \theta^0(x) \\ \\ v(.,0) = v^0(x), v'(.,0) = v^1(x), w_2(.,0) = w_2^0(x), q(.,0) = q^0(x). \end{cases} \tag{4.3}$$

We treat the infinite memories as Dafermos [94], [95] adding a new variables η_1, η_2 to the system which corresponds to the relative displacement history. Let us define the auxiliary variables

$$\eta_1 = \eta_1^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in (-L,0) \times \mathbb{R}^+.$$

and

$$\eta_2 = \eta_2^t(x,s) = v(x,t) - v(x,t-s), \quad (x,s) \in (0,L) \times \mathbb{R}^+.$$

By differentiation we have

$$\frac{d}{dt}\eta_1^t(x,s) = -\frac{d}{ds}\eta_1^t(x,s) + \frac{d}{dt}u(x,t), \quad (x,s) \in (-L,0) \times \mathbb{R}^+,$$

and

$$\frac{d}{dt}\eta_2^t(x,s) = -\frac{d}{ds}\eta_2^t(x,s) + \frac{d}{dt}v(x,t), \quad (x,s) \in (0,L) \times \mathbb{R}^+,$$

and we can take as initial condition ($t = 0$)

$$\eta_1^0(x,s) = u^0(x,0) - u^0(x,-s), \quad (x,s) \in (-L,0) \times \mathbb{R}^+.$$

and

$$\eta_2^0(x,s) = v^0(x,0) - v^0(x,-s), \quad (x,s) \in (0,L) \times \mathbb{R}^+.$$

Thus, the original weak-memories terms can be rewritten as

$$\begin{aligned}\alpha_1(t)\int_{-\infty}^{t}\mu_1(t-s)u_{xx}(s)ds &= \alpha_1(t)\int_0^\infty \mu_1(s)u_{xx}(t-s)ds \\ &= \alpha_1(t)\left(\int_0^\infty \mu_1(t)dt\right)u_{xx} - \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds.\end{aligned}$$

and

$$\begin{aligned}\alpha_2(t)\int_{-\infty}^{t}\mu_2(t-s)v_{xx}(s)ds &= \int_0^\infty \mu_2(s)v_{xx}(t-s)ds \\ &= \alpha_2(t)\left(\int_0^\infty \mu_2(t)dt\right)v_{xx} - \alpha_2(t)\int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds.\end{aligned}$$

The problem (4.1) is transformed into the system

$$\begin{cases} \rho_1 u'' - a_1\Big(\mu_{01}u_{xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds\Big) + \beta_1\theta_x = 0, & x \in (-L,0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L,0), \\ \rho_2 v'' - a_2\Big(\mu_{02}v_{xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds\Big) + \beta_2 q_x = 0, & x \in (0,L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0,L), \\ \frac{d}{dt}\eta_1^t(x,s) + \frac{d}{ds}\eta_1^t(x,s) - \frac{d}{dt}u(x,t) = 0, & x \in (-L,0), \\ \frac{d}{dt}\eta_2^t(x,s) + \frac{d}{ds}\eta_2^t(x,s) - \frac{d}{dt}v(x,t) = 0, & x \in (0,L), \\ \\ u(0,t) = v(0,t), \\ \theta(0,t) = q(0,t), \\ w_1(0,t) = w_2(0,t), \\ l\theta_x(0,t) = k w_{2,x}(0,t), \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), \\ \eta_1^0(x,s) = u^0(x,0) - u^0(x,-s), s > 0 \\ \eta_2^0(x,s) = v^0(x,0) - v^0(x,-s), s > 0 \end{cases} \tag{4.4}$$

where $\mu_{0i} = 1 - \alpha_i(t)\int_0^\infty \mu_i(t)dt, \qquad i = 1,2$. Studies included polynomial decay of solution for a thermoelastic transmission problem with $1-d$ mixed type I and type II with weak- infinite memories acting in the first and second parts.

4.2 Previous Results

The stability of various transmission problems on thermoelasticity have been considered [1], [2], [6], [7] [12], [14], [15], [20], [21], [22] and [24]. Very recentely, in [14], the authors treated a related problem, concerning the stabilization of a nonlinear rotating disk-beam system with localized thermal effect in the following

$$\begin{cases} y'' + y_{xxxx} = w^2(t)y, \qquad x \in (\xi, 1), \qquad 1 > 0, \\ y'' + y_{xxxx} + \alpha\theta_{xx} = w^2(t)y, \qquad x \in (0, \xi), \qquad 1 > 0, \\ \theta' - \theta_{xx} - \alpha y'_{xx} = 0, \qquad x \in (0, \xi), \qquad 1 > 0, \\ \frac{d}{dt}\Big\{w(t)\Big[I_d + \int_0^1 y^2 dx\Big]\Big\} = \Gamma(t), t > 0, \\ y(0,t) = y_x(0,t) = y_{xx}(1,t) = y_{xxx}(1,t) = \theta(0,t) = 0, t > 0, \\ y(x,0) = y_0(x), y'(x,0) = y_1(x), \theta(x,0) = \theta_0, w(0) = w_0. \end{cases} \tag{4.5}$$

The stabilization problem of a rotating disk-beam system with localized thermal effect and torque control is considered. Using only torque control, the authors proved that the system can be stabilized exponentially under certain condition on angular velocity, no matter how small the part with thermal effect of the beam is. The exponential stability is mainly proved by the resolvent estimate.
The transmission problem to hyperbolic equations was studied by Dautray and Lions [11] where the existence and regularity of solutions for the linear problem have been proved. In [21], the authors considered the transmission problem of viscoelastic waves

$$\begin{cases} \rho_1 u'' - \alpha_1 u_{xx} = 0, \qquad x \in (0, L_0), \\ \\ \rho_2 v'' - \alpha_2 v_{xx} + \int_0^t g(t-s)v_{xx}(s)ds = 0, \quad x \in (L_0, L), \end{cases} \tag{4.6}$$

satisfying boundary conditions and initial conditions. The authors studied the wave propagations over materials consisting of elastic and viscoelastic components. They showed that the viscoelastic part produce exponential decay of the solution. In [18], the authors investigated a 1D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero. Marzocchi et al [19] studied a multidimensional linear thermoelastic transmission problem. An existence and regularity result has been proved. When the solution is supposed to be spherically symmetric,

the authors established an exponential decay result similar to [18]. Next, Rivera and *all* [22], considered a transmission problem in thermoelasticity with memory. As time goes to infinity, they showed the exponential decay of the solution in case of radially symmetric situations. We must mention the pioneer work by Rivera and *all* in [15] where a semilinear transmission problem for a coupling of an elastic and a thermoelastic material is considered. The heat conduction is modeled by Cattaneo's law removing the physical paradox of infinite propagation speed of signals. The damped, totally hyperbolic system is shown to be exponentially stable. In 2009, Mesaoudi and *all* [20] proposed and studied a 1D linear thermoelastic transmission problem, where the heat conduction is described by the theories of Green and Naghdi. By using the energy method, they proved that the thermal effect is strong enough to produce an exponential stability of the solution.
The earliest result in this direction was established by [25], where the dynamical behavior of the system is described by

$$\begin{cases} \rho_1 u_1'' - a_1 u_{1,xx} + \beta_1 \theta_{1,x} = 0, & x \in (-1,0), \\ c_1 \tau_1'' - b\theta_{1,xx} + \beta_1 u_{1,x}' = 0, & x \in (-1,0), \\ \rho_2 u_2'' - a_2 u_{2,xx} + \beta_2 \theta_{2,x} = 0, & x \in (0,1), \\ c_2 \tau_2'' - k\tau_{2,xx} + \beta_2 u_{2,x}' = 0, & x \in (0,1), \end{cases} \tag{4.7}$$

the system consists of two kinds of thermoelastic components, one is of type I, another one is of type II. Under certain transmission conditions, these two components are coupled at the interface. The authors proved that the system is lack of exponential decay rate and further obtain the sharp polynomial decay rate. Our paper is devoted to showing that our system can achieve polynomial decay rate, where it can't be faster than t^{-1}. That is, our main result here is to show that for this types of materials the dissipation produced by the weak-viscoelasticities are not strong enough to produce an exponential decay of the solution, for any conditions on the nature for decay of μ_i, i.e. the decay rate of our system is independent of those of μ_i.

4.3 Research aims, approaches used

The potential functions α_1, α_2 are positive non-increasing functions defined on $\mathbb{R}^+$. We assume that the relaxation function $\mu_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$\begin{cases} 1-\alpha_i(t)\int_0^t \mu_i(s)ds \geq l > 0, i=1,2 \quad \forall t \in \mathbb{R}^+ \\ \\ \mu_i(0) > 0, \qquad \int_0^\infty \mu_i(s)ds < \infty \end{cases} \tag{4.8}$$

We denote by $\mathcal{A}$ the unbounded operator in an appropriate Hilbert state space given in (4.11).

Let

$$V^k(0,L) = \{h \in H^k(0,L); h(L) = 0\}$$

and

$$V^k(-L,0) = \{h \in H^k(-L,0); h(-L) = 0\}.$$

Let

$$\begin{aligned} \mathcal{H} &= V^1(-L,0) \times L^2(-L,0) \times L^2_{\mu_1}(\mathbb{R}^+, V^1(-L,0)) \times L^2(-L,0) \times \\ &\times V^1(0,L) \times L^2(0,L) \times L^2_{\mu_2}(\mathbb{R}^+, V^1(0,L)) \times V^1(0,L) \times L^2(0,L), \end{aligned} \tag{4.9}$$

equipped, for $(u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\eta_1^t}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{\eta_2^t}, \tilde{w}_2, \tilde{q}) \in \mathcal{H}$ endowed with the inner product

$$\Big\langle (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q), (\tilde{u}, \tilde{u^1}, \tilde{\eta_1^t}, \tilde{\theta}, \tilde{v}, \tilde{v^1}, \tilde{\eta_2^t}, \tilde{w}_2, \tilde{q}) \Big\rangle_{\mathcal{H}}$$

$$=$$

$$\int_{-L}^0 \Big[a_1 \Big(\mu_{01} u_x \overline{\tilde{u}_x} + \alpha_1(t) \int_0^\infty \mu_1(s)(\eta_{1,x}^t(s) \tilde{\eta^t}_{1,x}(s))ds \Big) + \rho_1 u^1 \overline{\tilde{u^1}} + c_1 \theta \overline{\tilde{\theta}} \Big] dx$$

$$+ \int_0^L \Big[a_2 \Big(\mu_{02} v_x \overline{\tilde{v}_x} + \alpha_2(t) \int_0^\infty \mu_2(s)(\eta_{2,x}^t(s) \tilde{\eta^t}_{2,x}(s))ds \Big) + \rho_2 v^1 \overline{\tilde{v^1}} + k w_{2,x} \overline{\tilde{w_{2,x}}} + c_2 q_x \overline{\tilde{q}_x} \Big] dx.$$

with domain

$$\mathcal{D}(\mathcal{A}) = (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q) \in \mathcal{H} : \begin{cases} u, \theta \in H^2(-L,0), u^1 \in H^1(-L,0), \\ \eta_1^t \in \mathcal{L}_{\mu_1}, \int_0^\infty \mu_1(s)\eta_{1,x}^t(s)ds \in H^1(-L,0) \\ v \in H^2(0,L), v^1, q \in H^1(0,L), w_2 \in H^2(0,L), \\ \eta_2^t \in \mathcal{L}_{\mu_2}, \int_0^\infty \mu_2(s)\eta_{2,x}^t(s)ds \in H^1(0,L) \\ \theta(-L) = q(L) = 0, l\theta_x(0) = kw_{2,x}(0) \\ a_1\mu_0 u_x(0) - \beta_1\theta(0) = a_2 v_x(0) - \beta_2 q(0) \\ u(0) = v(0), \theta(0) = q(0), \end{cases} \quad (4.1$$

where

$$\mathcal{L}_{\mu_1} = \{\eta_1^t \in L^2_{\mu_1}(\mathbb{R}^+, V^1(-L,0)), \partial\eta_1^t/\partial s \in L_{\mu_1}, \eta_1^t(0) = 0\}$$

and

$$\mathcal{L}_{\mu_2} = \{\eta_2^t \in L^2_{\mu_2}(\mathbb{R}^+, V^1(0,L)), \partial\eta_2^t/\partial s \in L_{\mu_2}, \eta_2^t(0) = 0\},$$

and the linear operator given by

$$\mathcal{A}\begin{pmatrix} u \\ u^1 \\ \eta_1^t \\ \theta \\ v \\ v^1 \\ \eta_2^t \\ w_2 \\ q \end{pmatrix} = \begin{pmatrix} u^1 \\ \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds\Big) - \beta_1\theta_x\Big) \\ \frac{\partial\eta_1^t}{\partial s} - u^1 \\ c_1^{-1}\Big(- \beta_1 u_x^1 + l\theta_{xx}\Big) \\ v^1 \\ \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds\Big) - \beta_2 q_x\Big) \\ \frac{\partial\eta_2^t}{\partial s} - v^1 \\ q \\ c_2^{-1}\Big(- \beta_2 v_x^1 + kw_{2,xx}\Big) \end{pmatrix} \quad (4.11)$$

The spaces $L^2_{\mu_1}(\mathbb{R}^+, V^1(-L,0))$ and $L^2_{\mu_2}(\mathbb{R}^+, V^1(0,L))$ are the weighted spaces with respect to the measure μ_1, μ_2 defined by

$$L^2_{\mu_1}(\mathbb{R}^+, V^1(-L,0)) = \Big\{\eta_1^t : \mathbb{R}^+ \to V^1(-L,0), \int_0^\infty \mu_1(s)\int_{-L}^0 |\eta_{1x}^t(s)|^2 dxds < \infty\Big\}$$

and

$$L^2_{\mu_2}(\mathbb{R}^+, V^1(0,L)) = \Big\{\eta_2^t : \mathbb{R}^+ \to V^1(0,L), \int_0^\infty \mu_2(s)\int_0^L |\eta_{2x}^t(s)|^2 dxds < \infty\Big\}$$

4.3. Research aims, approaches used

endowed with the inner product

$$\langle \nu_1, \nu_2 \rangle_{L^2_{\mu_1}(\mathbb{R}^+, V^1(-L,0))} = \int_0^\infty \mu_1(s) \langle \nu_{1x}(s), \nu_{2x}(s) \rangle ds,$$

and

$$\langle \nu_1, \nu_2 \rangle_{L^2_{\mu_2}(\mathbb{R}^+, V^1(0,L))} = \int_0^\infty \mu_2(s) \langle \nu_{1x}(s), \nu_{2x}(s) \rangle ds.$$

We need to prove that the operator $\mathcal{A}$ is dissipative in $\mathcal{H}$, to prepare the material in order to use the semi-group approach.

Lemma 4.1 *Let $\mathcal{A}$ and $\mathcal{H}$ be given in (4.9) and (4.11). Then, the operator $\mathcal{A}$ is dissipative in $\mathcal{H}$.*

Proof. Let $W = (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q)^T$, then it is note hard to see that

$$\begin{aligned}
\mathcal{R}(\mathcal{A}W, W)_{\mathcal{H}} &= \mathcal{R}\Big(\int_{-L}^{0} \rho_1^{-1} a_1 \Big(\mu_{01} u_x^1 \overline{u_x} + \alpha_1(t) \int_0^\infty \mu_1(s)(\eta_{1,x}^t(s)\tilde{\eta^t}_{1,x}(s))ds\Big) dx \\
&+ \int_{-L}^{0} \rho_1^{-1} \Big(a_1\mu_{01} u_x - \beta_1\theta\Big)_x \overline{u^1} - a_1\alpha_1(t) \int_0^\infty \mu_1(s)(\eta_{1,x}^t(s)\tilde{\eta^t}_{1,x}(s))dsdx \\
&+ \int_{-L}^{0} c_1^{-1}\Big(l\theta_{xx} - \beta_1 u_x'\Big)\overline{\theta} dx \\
&+ \int_0^L \rho_2^{-1} a_2 \Big(\mu_{02} v_x^1 \overline{v_x} + \alpha_2(t) \int_0^\infty \mu_2(s)(\eta_{2,x}^t(s)\tilde{\eta^t}_{2,x}(s))ds\Big) dx \\
&+ \int_0^L \rho_2^{-1}\Big(a_2\mu_{02} v_x - \beta_2 q\Big)_x \overline{v^1} - a_2\alpha_2(t) \int_0^\infty \mu_2(s)(\eta_{2,x}^t(s)\tilde{\eta^t}_{2,x}(s))dsdx \\
&+ \int_0^L \Big(kw_{2,xx} - \beta_2 v_x^1\Big)\overline{q_x} dx + \int_0^L kq_x\overline{w_{2,x}} dx\Big) \\
&= \mathcal{R}\Big(\int_{-L}^{0} \rho_1^{-1} a_1 \Big(\mu_{01} u_x^1 \overline{u_x} + \alpha_1(t) \int_0^\infty \mu_1(s)(\eta_{1,x}^t(s)\tilde{\eta^t}_{1,x}(s))ds\Big) dx \\
&- \int_{-L}^{0} \rho_1^{-1}\Big(a_1\mu_{01} u_x\Big)\overline{u_x^1} + a_1\alpha_1(t) \int_0^\infty \mu_1(s)(\eta_{1,x}^t(s)\tilde{\eta^t}_{1,x}(s))dsdx \\
&+ \int_{-L}^{0} c_1^{-1} l\theta_{xx}\overline{\theta} dx + \int_{-L}^{0} c_1^{-1}\beta_1 u^1\overline{\theta_x} dx - \int_{-L}^{0} \rho_1^{-1}\beta_1\theta_x\overline{u_x^1} dx \\
&+ \int_0^L \rho_2^{-1} a_2 \Big(\mu_{02} v_x^1 \overline{v_x} + \alpha_2(t) \int_0^\infty \mu_2(s)(\eta_{2,x}^t(s)\tilde{\eta^t}_{2,x}(s))ds\Big) dx \\
&- \int_0^L \rho_2^{-1} a_2\mu_{02} v_x\overline{v_x^1} + a_2\alpha_2(t) \int_0^\infty \mu_2(s)\langle\eta_{2,x}^t(s), \tilde{\eta^t}_{2,x}(s)\rangle dsdx \\
&+ \int_0^L \Big(kw_{2,xx} - \beta_2 v_x^1\Big)\overline{q_x} dx + \int_0^L kq_x\overline{w_{2,x}} dx - \int_0^L \rho_2^{-1}\beta_2 q_x\overline{v^1} dx\Big) \\
&= \mathcal{R}\Big(\int_{-L}^{0} c_1^{-1} l\theta_{xx}\overline{\theta} dx + \int_0^L kw_{2,xx}\overline{q_x} dx + \int_0^L kq_x\overline{w_{2,x}} dx\Big) \\
&= -c_1^{-1} l \int_{-L}^{0} |\theta_x|^2 dx \\
&\leq 0. \qquad (4.12)
\end{aligned}$$

Then, (4.12) means that the operator $\mathcal{A}$ is dissipative in $\mathcal{H}$. ■ In the next Theorem, we shall prove that the operator (4.11) generates a C_0 semigroup of contractions on

$\mathcal{H}$.

Theorem 4.1 *Let $\mathcal{A}$ and $\mathcal{H}$ be given in (4.9) and (4.11). Then, $\mathcal{A}$ generates a C_0 semi-group $S(t)$ of contractions on $\mathcal{H}$.*

Proof. For any

$$F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H},$$

the equation

$$\mathcal{A}W = F$$

has a unique solution $W = (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q)^T \in \mathcal{D}(\mathcal{A})$ satisfying the transmission and boundary conditions. Then, using Lemma 4.1 and Sobolev embedding theorem, one obtains $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. Therefore, the Lumer-Phillips theorem (see [23]) gives the result. This completes the proof. ■

4.4 Lack of Exponential Stability

For $\mathcal{U} = (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q)^T$, the problem (4.4) can then be reformulated under the abstract from

$$\mathcal{U}' = \mathcal{A}\mathcal{U}, \tag{4.13}$$

where $\mathcal{U}(0) = (u^0, u^1, \eta_1^0, \theta^0, v^0, v^1, \eta_2^0, w_2^0, q^0)^T \in \mathcal{H}$ is given.
The following is the well-known Gearhart-Herbst-Pruss-Huang theorem for dissipative systems. We will use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [16] and Huang [13]

Theorem 4.2 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\zeta : \zeta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta|\to\infty}} \|(i\zeta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Following the techniques in [30], it is easy to check that $(\mathcal{H}, \|.\|_{\mathcal{H}})$ is a Hilbert space. In this section we prove the lack of exponential decay using Theorem 4.2, that is we show that there exists a sequence of values h_m such that

$$\|(ih_m I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{4.14}$$

It is equivalent to prove that there exist a sequence of data $F_m \in \mathcal{H}$ and a sequence of real numbers $h_m \in \mathbb{R}$, with $\|F_m\|_{\mathcal{H}} \leq 1$ such that

$$\|(ih_m I - \mathcal{A})^{-1} F_m\|_{\mathcal{H}} = \|U_m\|^2_{\mathcal{H}} \to \infty. \tag{4.15}$$

Theorem 4.3 *Assume that the kernel satisfying hypothesis (4.8). The semi group $S(t)$ on $\mathcal{H}$ is not exponentially stable.*

Proof. We will find a sequence of bounded functions

$$F_m = (f_{1,m}, f_{2,m}, f_{3,m}, f_{4,m}, f_{5,m}, f_{6,m}, f_{7,m}, f_{8,m}, f_{9,m})^T \in \mathcal{H}, h \in \mathbb{R},$$

for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. We consider the spectral equation

$$ihU_m - \mathcal{A}U_m = F_m.$$

and show that the corresponding solution U_m is not bounded when F_m is bounded in $\mathcal{H}$. Rewriting the spectral equation in term of its components, we get

$$\begin{cases} ihu - u^1 = f_{1m} \\ ih\rho_1 u^1 - \Big(a_1\Big(\mu_{01}u_{xx} + \alpha_1(1)\int_0^\infty \mu_1(s)\eta^t_{1,xx}(s)ds\Big) - \beta_1\theta_x\Big) = \rho_1 f_{2m} \\ ihc_1\theta - \Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = c_1 f_{3m} \\ ihv - v^1 = f_{4m} \\ ih\rho_2 v^1 - \Big(a_2\Big(\mu_{02}v_{xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta^t_{2,xx}(s)ds\Big) - \beta_2 q_x\Big) = \rho_2 f_{5m} \\ ihw_2 - q = f_{6m} \\ ihc_2 q - \Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = c_2 f_{7m} \\ ih\eta^t_1 - u^1 + \eta^t_{1,s} = f_{8m} \\ ih\eta^t_2 - v^1 + \eta^t_{2,s} = f_{9m} \end{cases} \tag{4.16}$$

We prove that there exists a sequence of real numbers h_m so that (4.16) verified Let us consider $f_{1m} = f_{4m} = f_{6m} = f_{8m} = f_{9m} = 0$ and using the equations to eliminate the terms u^1, v^1 and chose $f_{2m} = f_{3m} = f_{5m} = f_{6m} = \lambda_m$ to obtain $u^1 = ihu$,

$v^1 = ihv$ and $q = ihw_2$. Then, system (4.16) becomes

$$\begin{cases} -h^2u - \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{2,xx}(s)ds\Big) - \beta_1\theta_x\Big) = \lambda_m \\ ih\theta - c_1^{-1}\Big(- \beta_1 u^1_x + l\theta_{xx}\Big) = \lambda_m \\ -h^2v - \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta^t_{2,xx}(s)ds\Big) - \beta_2 ihw_{2,x}\Big) = \lambda_m \\ -h^2w_2 - c_2^{-1}\Big(- \beta_2 v^1_x + kw_{2,xx}\Big) = \lambda_m \\ ih\eta^t_1 - ihu + \eta^t_{1,s} = 0 \\ ih\eta^t_2 - ihv + \eta^t_{2,s} = 0 \end{cases} \tag{4.17}$$

We look for solutions of the form

$$u = a\lambda_m, v = b\lambda_m, \theta = c\lambda_m, w_2 = d\lambda_m, u^1 = e\lambda_m,$$

$$v^1 = f\lambda_m, \eta^t_1(x,s) = \gamma(s)_1\lambda_m, \eta^t_2(x,s) = \gamma_2(s)\lambda_m$$

with $a, b, c, d, e, f \in \mathbb{C}$ and $\gamma_1(s), \gamma_2(s)$ depend on h and will be determined explicitly in what follows. From (4.17), we get a, b, c, d, e and f satisfy

$$\begin{cases} -h^2a - \rho_1^{-1}\Big(a_1h_m\Big(\mu_{01}a + \alpha_1(t)\int_0^\infty \mu_1(s)\gamma_1(s)ds\Big) - \beta_1 ch\Big) = 1, \\ ihc - c_1^{-1}\Big(- \beta_1 e + lh_m c\Big) = 1, \\ -h^2b - \rho_2^{-1}\Big(a_2h_m\Big(\mu_{02}b + \alpha_2(t)\int_0^\infty \mu_2(s)\gamma_2(s)ds\Big) - \beta_2 ihd\Big) = 1, \\ ihd - c_2^{-1}\Big(- \beta_2 f + kh_m d\Big) = 1, \\ \gamma_{1,s} + ih\gamma_1 - iha = 0. \\ \gamma_{2,s} + ih\gamma_2 - ihb = 0. \end{cases} \tag{4.18}$$

From $(4.18)_5$ and $(4.18)_6$ we get

$$\gamma_1(s) = a - ae^{-ihs}, \tag{4.19}$$

and

$$\gamma_2(s) = b - be^{-ihs}. \tag{4.20}$$

Then, from (4.19), (4.20) we have

$$\begin{aligned} \alpha_1(t)\int_0^\infty \mu_1(s)\gamma_1(s)\,ds &= \alpha_1(t)\int_0^\infty \mu_1(s)(a - ae^{-ihs})ds \\ &= a\alpha_1(t)\int_0^\infty \mu_1(s)ds - a\alpha_1(t)\int_0^\infty \mu_1(s)ae^{-ihs}ds \\ &= a(1-\mu_{01}) - a\alpha_1(t)\int_0^\infty \mu_1(s)e^{-ihs}\,ds. \end{aligned} \tag{4.21}$$

and

$$\alpha_2(t)\int_0^\infty \mu_2(s)\gamma_2(s)\,ds = b(1-\mu_{02}) - b\alpha_2(t)\int_0^\infty \mu_2(s)e^{-ihs}\,ds. \qquad (4.22)$$

Now, we would like to find the parameters constants. To this end, choosing

$$c_1 ih = h_m l, \qquad c_2 ih = kh_m, \qquad (4.23)$$

using equations $(4.18)_2$ and $(4.18)_4$, we obtain

$$e = \frac{c_1}{\beta_1}, \qquad (4.24)$$

$$f = \frac{c_2}{\beta_2}. \qquad (4.25)$$

By equations $(4.18)_1$ and $(4.18)_3$, we have

$$c = \frac{1}{(-h^2-\rho_1^{-1}h_m a_1\mu_{01})}\Big(1+\rho_1^{-1}h_m a_1\alpha_1(t)\int_0^\infty \mu_1(s)\gamma_1(s)ds - \rho_1^{-1}h_m\beta_1 c\Big),$$

$$d = \frac{1}{(-h^2-\rho_2^{-1}h_m a_2\mu_{02})}\Big(1+\rho_2^{-1}h_m a_2\alpha_1(t)\int_0^\infty \mu_2(s)\gamma_2(s)ds - \rho_1^{-1}h_m\beta_2 c\Big).$$

Recalling from (4.24), (4.25) that

$$\begin{aligned} u^1+v^1 &= e\lambda_m + f\lambda_m \\ &= \frac{c_1}{\beta_1}\lambda_m + \frac{c_2}{\beta_2}\lambda_m, \end{aligned}$$

we get

$$\|u^1\|_2^2 + \|v^1\|_2^2 = \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big]h_m^2.$$

Therefore we have

$$\begin{aligned} \lim_{m\to\infty}\|U_m\|_{\mathcal{H}}^2 &\geq \lim_{m\to\infty}[\|u^1\|_2^2 + \|v^1\|_2^2] \\ &= \lim_{m\to\infty}\Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big]h_m^2 \\ &= +\infty \end{aligned}$$

which completes the proof. ■

4.5 Polynomial Stability

Lemma 4.2 *[5], [17], [25] For some constant $C>0$, a C_0 semigroup $S(t)=e^{t\mathcal{A}}$ of contractions on a Hilbert space satisfies*

$$\|S(t)W_0\| \leq Ct^{-1}\|W_0\|_{\mathcal{D}(\mathcal{A})}, \forall W_0 \in \mathcal{D}(\mathcal{A}), t\to\infty,$$

if and only if the following conditions hold

1. $\rho(\mathcal{A}) \supseteq i\mathbb{R}$
2. $\lim_{\zeta\to\infty} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty$

Our next main result reads as follows.

Theorem 4.4 *Assume that (4.8) hold. Then* t^{-1} *is the sharp decay rate. Therefore, the decay rate of the system cannot be faster than* t^{-1}.

Proof. 1. Using proof by contradiction. For this purpose, we assume that there exists $\tilde{\lambda} = i\tilde{\xi} \in \delta(\mathcal{A}), \tilde{\delta} \in \mathbb{R}, \tilde{\delta} \neq 0$ on the imaginary axis and $\tilde{W} = (u, u^1, \eta_1^t, \theta, v, v^1, \eta_2^t, w_2, q) \in \mathcal{D}(\mathcal{A})$ is the eigenvector corresponding to $\tilde{\lambda}$. Then,

$$\tilde{\lambda}u = u^1, \tag{4.26}$$

$$\tilde{\lambda}u^1 = \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,xx}^t(s)ds\Big) - \beta_1\theta_x\Big), \tag{4.27}$$

$$\tilde{\lambda}u^1 = \partial\eta_1^t/\partial s \tag{4.28}$$

$$\tilde{\lambda}\theta = c_1^{-1}\Big(-\beta_1 u_x^1 + l\theta_{xx}\Big), \tag{4.29}$$

$$\tilde{\lambda}v = v^1, \tag{4.30}$$

$$\tilde{\lambda}v^1 = \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta_{2,xx}^t(s)ds\Big) - \beta_2 q_x\Big), \tag{4.31}$$

$$\tilde{\lambda}v^1 = \partial\eta_2^t/\partial s \tag{4.32}$$

$$\tilde{\lambda}w_2 = q, \tag{4.33}$$

$$\tilde{\lambda}q = c_2^{-1}\Big(-\beta_2 v_x^1 + kw_{2,xx}\Big). \tag{4.34}$$

Since $\mathcal{A}$ is dissipative by Lemma 4.1, we have

$$\mathcal{R}(\mathcal{A}\tilde{W}, \tilde{W}) = -c_1^{-1}l\int_{-L}^0 |\theta_x|^2 dx = 0,$$

which yields $\theta_x = \theta_{xx} = 0$, then by (4.29), we have $u_x^1 = 0$, then $u = u^1 = 0$. Hence $(u, u^1, w_2, q) = 0$ which contradicts the fact that $\tilde{W} = 0$ is an eigenvector. This completes the proof of the first point.

2. We would now show that

$$\lim_{\zeta\to\infty} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty \tag{4.35}$$

We prove that there at least exists a sequence

$$V_n = (u_n, u_n^1, \eta_{1,n}^t, \theta_n, v_n, v_n^1, \eta_{2,n}^t, w_{2,n}, q_n) \in \mathcal{D}(\mathcal{A}),$$

with $\|V_n\|_{\mathcal{H}} = 1$, and a sequence $\zeta_n \in \mathbb{R}$ with $\zeta_n \to \infty$ such that

$$\lim_{n\to\infty} \zeta_n \|(i\zeta_n I - \mathcal{A})V_n\|_{\mathcal{H}} = 0$$

or

$$\zeta_n(i\zeta_n u_n - u_n^1) \;\to 0, \quad in \quad H^1(-L,$$

$$\zeta_n\Big(i\zeta_n u_n^1 - \rho_1^{-1}\Big(a_1\Big(\mu_{01}u_{n,xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big) - \beta_1\theta_{n,x}\Big)\Big) \;\to 0, \quad in \quad L^2(-L,$$

$$\zeta_n\Big(i\zeta_n\theta_n - c_1^{-1}\Big(-\beta_1 u_{n,x}^1 + l\theta_{n,xx}\Big)\Big) \;\to 0, \quad in \quad L^2(-L$$

$$\zeta_n(i\zeta_n v_n - v_n^1) \;\to 0, \quad in \quad H^1(0, L$$

$$\zeta_n\Big(i\zeta_n v_n^1 - \rho_2^{-1}\Big(a_2\Big(\mu_{02}v_{n,xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta_{2,n,xx}^t(s)ds\Big) - \beta_2 q_{n,x}\Big)\Big) \;\to 0, \quad in \quad L^2(0, L$$

$$\zeta_n(i\zeta_n w_{2,n} - q_n) \;\to 0, \quad in \quad H^1(0, L$$

$$\zeta_n\Big(i\zeta_n q_n - c_2^{-1}\Big(-\beta_2 v_{n,x}^1 + kw_{2,n,xx}\Big)\Big) \;\to 0, \quad in \quad L^2(0,$$

$$ih\eta_1^t - u_{1,n}^1 + \eta_{1,s}^t = 0$$

$$ih\eta_2^t - v_{1,n}^1 + \eta_{2,s}^t = 0$$

Noting that

$$Re\langle\zeta_n(i\zeta_n - \mathcal{A})V_n, V_n\rangle_{\mathcal{H}} = \zeta_n\|\sqrt{l} \quad \theta_{n,x}\|_{L^2}^2 \to 0.$$

Then

$$\sqrt{\zeta_n} \quad \theta_{n,x} \to 0, \quad in \quad L^2(-L, 0). \tag{4.45}$$

4.5. Polynomial Stability

By Poincaré's inequality, we get

$$\sqrt{\zeta_n} \quad \theta_n \to 0, \quad in \quad L^2(-L,0). \tag{4.46}$$

Thanks to the Gagliardo-Nirenberg inequality, we have

$$\|\sqrt{\zeta_n} \quad \theta_n\|_{L^\infty} \le C_1\sqrt{\|\sqrt{\zeta_n} \quad \theta_{n,x}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n} \quad \theta_n\|_{L^2}} + C_2\|\sqrt{\zeta_n} \quad \theta_n\|_{L^2}. \tag{4.47}$$

Thus,

$$\sqrt{\zeta_n} \quad \theta_n(0) \to 0. \tag{4.48}$$

From (4.36), we have $\beta_1(i\zeta_n)^{-1}u^1_{n,x}$ is bounded in $L^2(-L,0)$. By (4.38) we have the boundedness of $(i\zeta_n)^{-1}\theta_{n,xx}$ in $L^2(-L,0)$.
Using again the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}\|_{L^\infty} &\le d_1\sqrt{\|(\zeta_n)^{-1} \quad \theta_{n,xx}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n} \quad \theta_{n,x}\|_{L^2}} + d_2\|\Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}\|_{L^2}\\ &\to 0.\end{aligned}$$

which gives

$$\Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}(-L) \to 0, \qquad \Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}(0) \to 0. \tag{4.49}$$

Multiplying (4.37) by $p(x)u_{n,x}$ in $L^2 - norm$ for $p(x) \in C^1[-L,0]$, to get

$$\begin{aligned}&-\zeta_n^2\langle u_n, p(x)u_{n,x}\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle \mu_{01}u_{n,xx}, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\ &-\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\ &+\rho_1^{-1}\beta_1\langle\theta_{n,x}, p(x)u_{n,x}\rangle_{L^2(-L,0)} \to 0.\end{aligned} \tag{4.50}$$

Integration by parts gives

$$-\zeta_n^2\langle u_n, p(x)u_{n,x}\rangle_{L^2(-L,0)} = \zeta_n^2 p(-L)|u_n(-L)|^2 - \zeta_n^2 p(0)|u_n(0)|^2 + \zeta_n^2\Big\langle p_x(x)u_n, u_n\Big\rangle_{L^2(-L,0)}$$

$$\begin{aligned}-\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)} &= -\rho_1^{-1}a_1\mu_{01}p(0)|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01}p(-L)|u_{n,x}(-L)|^2\\ &+ \rho_1^{-1}a_1\mu_{01}\Big\langle p_x(x)u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}\end{aligned}$$

and

$$-\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, p(x)u_{n,x}\Big\rangle_{L^2(-L,0)} \tag{4.51}$$

$$= -\rho_1^{-1}a_1 p(0)\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds u_{n,x}(0)$$

$$+\rho_1^{-1}a_1 p(-L)\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds u_{n,x}(-L)$$

$$+\rho_1^{-1}a_1\Big\langle p_x(x)\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}.$$

Since

$$\rho_1^{-1}\beta_1\langle\theta_{n,x}, p(x)u_{n,x}\rangle_{L^2(-L,0)} \to 0,$$

then by the above integrations, for $p(x) = x \in C^1[-L,0]$, Eq.(4.50) takes the form

$$-\zeta_n^2|u_n(-L)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\mu_{01}|u_{n,x}(-L)|^2 + \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds u_{n,x}(-L)$$
$$+ \quad \rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \tag{4.52}$$

and hence $u_{n,x}(-L)$ and $\zeta_n u_n(-L)$ are bounded.
Similarly, taking $p(x) = x + L \in C^1[-L,0]$, Eq.(4.50) takes the form

$$-\zeta_n^2|u_n(0)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\mu_{01}|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds u_{n,x}(0)$$
$$+ \quad \rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{4.53}$$

Then, we get boundedness of $\zeta_n u_n(0)$ and $u_{n,x}(0)$.
Multiplying (4.38) by $u_{n,x}$ and taking the integration to get since $\zeta_n > 0$,

$$i\zeta_n\Big\langle\theta_n, u_{n,x}\Big\rangle_{L^2(-L,0)} + c_1^{-1}\beta_1\Big\langle u_{1,n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} - c_1^{-1}l\Big\langle\theta_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.$$

By (4.46), we have after dividing by $i\sqrt{\zeta_n}$

$$i\zeta_n\Big\langle\theta_n, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0$$

Integrating by part to get

$$l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)} - \theta_{n,x}(0)\overline{u_{n,x}(0)}\Big) + l\Big\langle\sqrt{\zeta_n}\theta_{n,x}, (i\zeta_n)^{-1}u_{n,xx}\Big\rangle_{L^2(-L,0)}$$
$$+\beta_1\sqrt{\zeta_n}\Big\langle u_{1,n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \tag{4.54}$$

By(4.49)and the boundedness of $u_{n,x}(-L)$ and $u_{n,x}(0)$, we have

$$l(i\sqrt{\zeta_n})^{-1}\Big(\theta_{n,x}(-L)\overline{u_{n,x}(-L)} - \theta_{n,x}(0)\overline{u_{n,x}(0)}\Big) \to 0$$

Moreover, from (4.37), we obtain that $(i\zeta_n)^{-1}u_{n,xx}$ is bounded in $L^2(-L,0)$, thus

$$l(\sqrt{\zeta_n}\theta_{n,x}, (i\zeta_n)^{-1}u_{n,xx}) \to 0$$

Hence by(4.54),we get

$$\sqrt{\sqrt{\zeta_n}}\quad u_{n,x} \to 0, \qquad in \qquad L^2(-L,0) \tag{4.55}$$

thanks to the Poincaré inequality, we have

$$\sqrt{\sqrt{\zeta_n}}\quad u_n \to 0, \quad in \qquad L^2(-L,0) \tag{4.56}$$

By (4.55),(4.56) and Galiardo-Nirenberg inequality, we get

$$\sqrt{\sqrt{\zeta_n}}\quad u_n(0) \to 0 \tag{4.57}$$

From (4.37) and (4.45) and since $\zeta_n > 0$, we have

$$i\zeta_n u_{1,n} - \rho_1^{-1}a_1\Big(\mu_{01}u_{n,xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds\Big) \to 0, \qquad in \quad L^2(-L,0) \tag{4.58}$$

Multiplying the above by u_n, we get

$$i\zeta_n\Big\langle u_{1,n}, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\Big\langle\Big(\mu_{01}u_{n,xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds\Big), u_n\Big\rangle_{L^2(-L,0)} \to 0.$$

Integrating by part, we get

$$-\Big\langle u_{1,n}, u_{1,n}\Big\rangle_{L^2(-L,0)}$$
$$-\rho_1^{-1}a_1\mu_{01}u_{n,x}(0)\overline{u_n(0)} + \rho_1^{-1}a_1\mu_{01}u_{n,x}(-L)\overline{u_n(-L)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$
$$+\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds\overline{u_n(0)} - \rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds\overline{u_n(-L)}$$
$$+\rho_1^{-1}a_1\Big\langle\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.$$

Since $u_{n,x}(0), u_{n,x}(-L)$ are bounded, by (4.55) and $u_n(-L) \to 0, u_n(0) \to 0$, we have

$$u_{1,n}, \zeta_n u_n \to 0, \quad in \quad L^2(-L,0). \tag{4.59}$$

Multiplying (4.37) by $(x+L)u_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re\Big[-\Big\langle \zeta_n^2 u_{1,n}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\Big\langle\Big(\mu_{01}u_{n,xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big), (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \\
&= -\zeta_n^2|u_n(0)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds u_{n,x}(0) + \rho_1^{-1}a_1\alpha_1(t)\Big\langle\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)}
\end{aligned}$$

Hence by (4.55)and (4.59)

$$\zeta_n u_n(0), u_{n,x}(0) \to 0 \tag{4.60}$$

Now, multiplying (4.37) by $xu_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re\Big[-\Big\langle \zeta_n^2 u_n^1, xu_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\Big\langle\Big(\mu_{01}u_{n,xx} + \alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds\Big), xu_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \\
&= -\zeta_n^2|u_n(-L)|^2 + \zeta_n^2\Big\langle u_n, u_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}|u_{n,x}(-L)|^2 + \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds u_{n,x}(-L) \\
&+\rho_1^{-1}a_1\alpha_1(t)\Big\langle\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0,
\end{aligned} \tag{4.61}$$

Then

$$\zeta_n u_n(-L), u_{n,x}(-L) \to 0. \tag{4.62}$$

Taking again Eq.(4.37), multiplying by u_n, we have

$$\begin{aligned}
&\sqrt{\zeta_n}\Big\langle i\zeta_n u_n^1, u_n\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle\theta_{n,x}, u_n\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_{01}\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\alpha_1(t)\Big\langle\int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_n\Big\rangle_{L^2(-L,0)} \to 0,
\end{aligned} \tag{4.63}$$

By (4.55) and (4.60), we have

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_{01}\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}u_{n,x}(0)\overline{u_n(0)} + \rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}u_{n,x}(-L)\overline{u_n(-L)}$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0$$

and

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, u_n\Big\rangle_{L^2(-L,0)}$$

$$= -\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)ds\overline{u_n(0)}$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(-L,s)ds\overline{u_n(-L)}$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \quad (4.64)$$

Thus by(4.64) and (4.45), we have

$$\sqrt{\sqrt{\zeta_n}}u_n^1 \to 0, \quad in \qquad L^2(-L,0). \quad (4.65)$$

Multiplying (4.37) by $(x+L)u_{n,x}$, we have

$$\Big\langle i\sqrt{\zeta_n}\zeta_n u_n^1, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_{01}\Big\langle u_{n,xx}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}\sqrt{\zeta_n}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,xx}(s)ds, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \quad (4.66)$$

Integrating by parts and using (4.45)and the boundedness of $u_{n,x}$ in $L^2(-L,0)$, we get

$$-\sqrt{\zeta_n}|u_n^1(0)|^2 + \sqrt{\zeta_n}\Big\langle u_n^1, u_n^1\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}|u_{n,x}(0)|^2$$

$$+\rho_1^{-1}a_1\mu_{01}\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)}$$

$$-\rho_1^{-1}a_1\sqrt{\zeta_n}\alpha_1(t)\int_0^\infty \mu_1(s)\eta^t_{1,n,x}(0,s)dsu_{n,x}(0)$$

$$-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_{-L}^0 \alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta^t_{1,n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \quad (4.67)$$

Thus by (4.55)and (4.65)

$$\sqrt{\sqrt{\zeta_n}}u_n^1(0), \sqrt{\sqrt{\zeta_n}}u_{n,x}(0) \to 0 \tag{4.68}$$

Multiplication of (4.58) by $u_{n,x}$ yields

$$\begin{aligned} & i\zeta_n\Big\langle u_n^1, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} \\ & -\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0, \end{aligned} \tag{4.69}$$

Due to (4.60)and (4.62), we get

$$\begin{aligned} & -\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, u_{n,x}\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \\ & = \frac{1}{2}(-\rho_1^{-1}a_1\mu_{01})|u_{n,x}(0)|^2 + \rho_1^{-1}a_1\mu_{01})|u_{n,x}(-L)|^2 \\ & -\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds u_{n,x}(0) + \rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds u_{n,x}(-L) \\ & +\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \end{aligned} \tag{4.70}$$

Thus, it follows from (4.69)that

$$(i\zeta_n u_n^1, u_{n,x}) \to 0 \tag{4.71}$$

Taking the product of (4.58) with θ_n, yields

$$\begin{aligned} & i\zeta_n\Big\langle u_{1,n}, \theta_n\Big\rangle_{L^2(-L,0)} - \rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, \theta_n\Big\rangle_{L^2(-L,0)} \\ & -\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, \theta_n\Big\rangle_{L^2(-L,0)} \to 0, \quad in \quad L^2(-L,0), \end{aligned} \tag{4.72}$$

Due to (4.45),(4.48)and (4.60)

$$\begin{aligned} & -\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,xx}, \theta_n\Big\rangle_{L^2(-L,0)} \\ & = -\rho_1^{-1}a_1\mu_{01}u_{n,x}(0)\overline{\theta_n(0)} + \rho_1^{-1}a_1\mu_{01}u_{n,x}(-L)\overline{\theta_n(-L)} \\ & +\rho_1^{-1}a_1\mu_{01}\Big\langle u_{n,x}, \theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0 \end{aligned} \tag{4.73}$$

and

$$\begin{aligned} & -\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,xx}^t(s)ds, \theta_n\Big\rangle_{L^2(-L,0)} \\ & = -\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(0,s)ds\overline{\theta_n(0)} \\ & +\rho_1^{-1}a_1\alpha_1(t)\int_0^\infty \mu_1(s)\eta_{1,n,x}^t(-L,s)ds\overline{\theta_n(-L)} \\ & +\rho_1^{-1}a_1\alpha_1(t)\Big\langle \int_0^\infty \mu_1(s)\eta_{1,n,x}^t(s)ds, \theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \end{aligned} \tag{4.74}$$

Then from (4.72) that

$$i\zeta_n\left\langle u_n^1,\theta_n\right\rangle_{L^2(-L,0)} \to 0 \tag{4.75}$$

Multiplying (4.38)by u_n^1, we have

$$\left\langle i\zeta_n\theta_n,u_n^1\right\rangle_{L^2(-L,0)} - c_1^{-1}l\left\langle\theta_{n,xx},u_n^1\right\rangle_{L^2(-L,0)} + c_1^{-1}\beta_1\left\langle u_{n,x}^1,u_n^1\right\rangle_{L^2(-L,0)} \to 0 \tag{4.76}$$

by (4.71), (4.75) we have

$$\left\langle\theta_{n,xx},u_n^1\right\rangle_{L^2(-L,0)} \to 0 \tag{4.77}$$

Integrating by part

$$\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} - \left\langle\theta_{n,x},u_{n,x}^1\right\rangle_{L^2(-L,0)} \to 0 \tag{4.78}$$

Due to (4.49)and (4.68), we get

$$\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} \to 0. \tag{4.79}$$

From (4.78) we have

$$\left\langle\theta_{n,x},u_{n,x}^1\right\rangle_{L^2(-L,0)} \to 0. \tag{4.80}$$

Multiplying (4.38) by $(x+L)\theta_{n,x}$, integrating to get

$$\Re\Big[\left\langle i\zeta_n\theta_n,(x+L)\theta_{n,x}\right\rangle_{L^2(-L,0)} - c_1^{-1}\left\langle(l\theta_{n,xx}-\beta_1 u_{n,x}^1),(x+L)\theta_{n,x}\right\rangle_{L^2(-L,0)}\Big] \to 0 \tag{4.81}$$

By (4.45) and (4.46), we get

$$\left\langle i\zeta_n\theta_n,(x+L)\theta_{n,x}\right\rangle_{L^2(-L,0)} \to 0 \tag{4.82}$$

Thus by (4.81) and (4.45), we have

$$-c_1^{-1}l\theta_{n,x}(0)\overline{\theta_{n,x}(0)} + 2\Re[c_1^{-1}\beta_1(u_{n,x}^1,(x+L)\theta_{n,x})] \to 0 \tag{4.83}$$

Then by (4.80), we get

$$\theta_{n,x}(0) \to 0 \tag{4.84}$$

Hence, by (4.70),(4.60),(4.48)and (4.84), we have

$$u_{n,x}(0),u_n(0),\theta_n(0),\theta_{n,x}(0) \to 0 \tag{4.85}$$

On the hand taking the product of (4.42) with $(x-L)w_{2,n,x}$, yields

$$\begin{aligned}\Re\Big[i\zeta_n\left\langle q_n,(x-L)w_{2,n,x}\right\rangle_{L^2(0,L)} + c_2^{-1}\beta_2\left\langle v_{n,x}^1,(x-L)w_{2,n,x}\right\rangle_{L^2(0,L)}\\ -c_2^{-1}k\left\langle w_{2,n,xx},(x-L)w_{2,n,x}\right\rangle_{L^2(0,L)}\Big] \to 0,\end{aligned} \tag{4.86}$$

Using the transmission conditions in (4.1), we get

$$(q_n, q_n) + c_2^{-1}k(w_{2,n,x}, w_{2,n,x}) - 2\Re\Big[c_2^{-1}\beta_2\Big\langle v_{n,x}, (x-L)q_{n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0. \quad (4.87)$$

Taking the product of (4.40) with $(x-L)v_{n,x}$ to obtain

$$i\zeta_n\Big\langle v_n^1, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)} - \rho_2^{-1}a_2\Big\langle \mu_{02}v_{n,xx} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta^t_{2,n,xx}(s)ds, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)}$$
$$+\rho_2^{-1}\beta_2\Big\langle q_{n,x}, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)} \to 0, \quad (4.88)$$

Integrating (4.88)by parts we have

$$\Big\langle v_n^1, v_n^1\Big\rangle_{L^2(0,L)} + \rho_2^{-1}a_2\Big\langle \mu_{02}v_{n,x} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta^t_{2,n,x}(s)ds, v_{n,x}\Big\rangle_{L^2(0,L)}$$
$$+2\Re\Big[\rho_2^{-1}\beta_2\Big\langle q_{n,x}, (x-L)q_{n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0 \quad (4.89)$$

Thus by (4.87)and (4.89)

$$a_2\Big\langle \mu_{02}v_{n,x} + \alpha_2(t)\int_0^\infty \mu_2(s)\eta^t_{2,n,x}(s)ds, v_{n,x}\Big\rangle_{L^2(0,L)} + \Big\langle \rho_2 v_n^1, v_n^1\Big\rangle_{L^2(0,L)}$$
$$+c_2\Big\langle q_n, q_n\Big\rangle_{L^2(0,L)} + k\Big\langle w_{2,n,x}, w_{2,n,x}\Big\rangle_{L^2(0,L)} \to 0 \quad (4.90)$$

Then,

$$v_{n,x}, v_n^1, w_{2,n,x}, q_n \to 0, \qquad in \qquad L^2(0,L) \quad (4.91)$$

Thus (4.91) together with (4.46), (4.59) and (4.91), we obtain

$$V_n = (u_n, u_n^1, \theta_n, v_n, v_n^1, w_{2,n}, q_n)^T \to 0 \quad (4.92)$$

which contradicts $\|V_n\| = 1$ therefore, (4.35) holds.

■

Bibliography

[1] R.G.C. Almeida and M.L. Santos, Lack of exponential decay of a coupled system of wave equations with memory, Nonlinear Analysis: Real World Applications, 12 (2011), 1023-1032.

[2] A. Moncef, On uniform decay of a nonsimple thermoelastic bar with memory, NJ. Math. Anal. Appl. 402 no. 2(2013) pp 745-757.

[3] M. Bayoud, Kh. Zennir and H. Sissaoui, Transmission problem with $1-D$ mixed type in thermoelasticity and infinite memory, Applied Sciences, Vol.20, (2018) pp 18-35.

[4] A. Beniani, A. Benaissa and Kh. Zennir, Polynomial decay of solutions to the Cauchy problem for a Petrowsky-Petrowsky system in $\mathbb{R}^n$, Acta. Appl. Math., Vo 146, (1) (2016) pp 67-79.

[5] A. Borichev , Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, Math. Ann., 347, (2010) pp 455-478.

[6] G. Claudio; P. Vittorino, Stability of abstract linear thermoelastic systems with memory, Math. Models Methods Appl. Sci. 11 no. 4 (2001) pp 627-644.

[7] G. Claudio; N. M. Grazia; P. Vittorino, Exponential stability in linear heat conduction with memory: a semigroup approach, Commun. Appl. Anal., 5 no. 1 (2001) pp 121-133.

[8] C. M. Dafermos, An abstract Volterra equation with applications to linear viscoelasticity, J. Diff. Equations, 7 (1970), pp 554-569.

[9] C. M. Dafermos, On the existence and the asymptotic stability of solution to the equations of linear thermoelasticity, Arch. Ration. Mech. Anal., 29, (1968) pp 241-271.

[10] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37(1970) pp 297-308.

[11] R. Dautray and J. L. Lions, Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques, Vol. 1, Masson, Paris, 1984.

[12] C. Juan V. Bravo, Jaime E. Munoz Rivera, The transmission problem to thermoelastic plate of hyperbolic type, IMA Journal of Applied Mathematics, Volume 74, Issue 6, 1 (2009), pp 950?962.

[13] F. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert space, Ann. Differential. Equations, 1 (1) (1985) pp 43-56.

[14] Huan Geng, Zhong-Jie Han, Jing Wang, Gen-Qi Xu, Stabilization of a nonlinear rotating disk-beam system with localized thermal effect, Nonlinear Dyn, 93: (2018) pp 785-799.

[15] H. D. Fernandez Sare, Jaime E. Munoz Rivera and Reinhard Racke, Stability for a Transmission Problem in thermoelasticity with Second Sound, J. Thermal Stresses Vol. 31, Iss. 12 (2008) pp 1170-1189.

[16] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, Trans. AMS, 236 (1978) pp 385-394.

[17] Z. Liu, B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, Z. Angew. Math. Phys., 56 (2005) pp 630-644.

[18] A. Marzocchi, Jaime E. Munoz Rivera and M. G. Naso, Asymptotic behavior and exponential stability for a transmission problem in thermoelasticity, Math. Methods Appl. Sci., 25 (2002) pp 955-980.

[19] A. Marzocchi, Jaime E. Munoz Rivera and M. G. Naso, Transmission problem in thermoelasticity with symmetry, IMA J. Appl. Math., 68 (2002) pp 23-46.

[20] S. A. Mesaoudi, B. Said-Houari, Energy decay in a transmission problem in thermoelasticity of type III, IMA Journal of Applied Mathematics, 74 (2009) pp 344?360.

[21] Jaime E. Munoz Rivera and H.P. Oquendo, The Transmission Problem of Viscoelastic Waves, Acta Appl Math, 62: 1 (2000) pp 1-21.

[22] Jaime E. Munoz Rivera and M. G. Naso, About Asymptotic Behavior for a Transmission Problem in Hyperbolic thermoelasticity, Acta Appl Math, 99: 1(2007) pp 1-21.

[23] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin (1983).

[24] C. A. Raposo, W. D. Bastosand J. A. J. Avila, A Transmission Problem for Euler-Bernoulli beam with Kelvin-Voigt Damping, Applied Mathematics and Information Sciences? An International Journal, 5(1)(2011) pp 17-28 .

[25] J. Wang, Z.J. Han and G.Q. Xu, Energy decay rate of transmission problem between thermoelasticity of type I and type II, Z. Angew. Math. Phys., (2017) 68:65, Doi.org/10.1007/s00033-017-0807-7.

[26] Kh. Zennir, B. Feng, One spatial variable thermoelastic transmission problem inviscoelasticity located in the second part. Math Meth Appl Sci., (2018) 1?12. DOI: 10.1002/mma.5201

[27] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl. Math. Optim.* **51**:1 (2005), 61–105.

[28] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Translated from the Russian by K. Vogtmann and A. Weinstein. Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.

[29] A. Benaissa and A. Guesmia, Energy decay for wave equations of ϕ-Laplacian type with weakly nonlinear dissipation, *Electron. J. Differential Equations* **2008**, No. 109, pp. 1-22,

[30] A. Beniani , A. Benaissa and Kh. Zennir , *Polynomial Decay of Solutions to the Cauchy Problem for a Petrovsky-Petrovsky System in* $\mathbf{R}^n$, Acta. Appl. Math. **146** (2016), pp. 67-79.

[31] M. M. Cavalcanti, V. D. Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *J. Differential Equations* **236**:2 (2007), 407–459.

[32] G. Chen, Control and stabilization for the wave equation in a bounded domain, *SIAM J. Control Optim.* **17**:1 (1979), 66–81.

[33] M. Daoulatli, I. Lasiecka, and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions, *Discrete Contin. Dyn. Syst. Ser. S* **2**:1 (2009), .67–94.

[34] R. Datko, J. Lagnese, and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.* **24**:1 (1986), 152–156.

[35] M. Eller, J. E. Lagnese, and S. Nicaise, Decay rates for solutions of a Maxwell system with nonlinear boundary damping, *Comput. Appl. Math.* **21**:1 (2002), 135–165.,

[36] A. Haraux, Two remarks on hyperbolic dissipative problems, in: *Nonlinear partial differential equations and their applications.* College de France seminar,

Vol. VII (Paris, 1983–1984), 6, pp. 161–179, Res. Notes in Math., 122, Pitman, Boston, MA, 1985.

[37] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.

[38] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differential Integral Equations* **6**:3 (1993), 507-533.

[39] I. Lasiecka and D. Toundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms, *Nonlinear Anal.* **64**:8 (2006), 1757–1797.

[40] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969.

[41] M. Nakao, Decay of solutions of some nonlinear evolution equations, *J. Math. Anal. Appl.* **60**:2 (1977), 542–549.

[42] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.* **45**:5 (2006), 1561–1585.

[43] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differential Integral Equations* **21**:9-10 (2008), 935-958.

[44] C. Q. Xu, S. P. Yung, and L. K. Li, Stabilization of wave systems with input delay in the boundary control, *ESAIM Control Optim. Calc. Var.* **12**:4 (2006), 770–785.

[45] Kh. Zennir, *General decay of solutions for damped wave equation of Kirchhoff type with density in* $\mathbb{R}^n$. Ann Univ Ferrara, Vol. **61**, (2015) 381-394.

[46] S. Zitouni, and Kh. Zennir, *On the existence and decay of solution for viscoelastic wave equation with nonlinear source in weighted spaces.* Rend. Circ. Mat. Palermo, II. Ser, 2016.

[47] F. Alabau-Boussouira, S. Nicaise and C. Pignotti, Exponential stability of the wave equation with memory and time delay. In: New prospects in direct, inverse and control problems for evolution equations. Springer INdAM Series, **10**(2014), 1-22.

[48] V. I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1989.

[49] A. Benaissa and S. A. Messaoudi, Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks, J. Math. Phys., **53**(2012), 123514, 19pp.

[50] A. Benaissa, A. Benguessoum and S. A. Messaoudi, Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback, Int. J. Dyna. Sys. Differ. Equa., **5**(1)(2014), 1-26.

[51] S. Berrimi and S. A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, Nonlinear Anal., **64** (2006), 2314-2331.

[52] M. M. Cavalcanti, V. D. Cavalcanti and I. Lasiecka, ¡°Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, J. Differ. Equations, (2007)236, 407-459.

[53] Q. Dai and Z. Yang, Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay. Z. Angew. Math. Phys., **65**(2014), 885-903.

[54] M. Daoulatli, I. Lasiecka and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions, Discrete Conti. Dyna. Syst., **2**(2009), 67-95.

[55] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM J. Control Optim., **24**(1986), 152-156.

[56] B. Feng, Well-posedness and exponential stability for a plate equation with time-varying delay and past history. Z. Angew. Math. Phys., **68**(2017), DOI 10.1007/s00033-016-0753-9

[57] B. Feng, Global well-posedness and stability for a viscoelastic plate equation with a time delay. Math. Probl. Eng., **2015**(2015), 585021, 10pp.

[58] Leseduarte, M.C., Magana, A., Quintanilla, R.: On the time decay of solutions in porous- thermo-elasticity of type II. Discrete Contin. Dyn. Syst. Ser. B13, 375-391 (2010)

[59] Liu, Z.Y., Quintanilla, R.: Energy decay rate of a mixed type II and type III thermoelastic system. Discrete Contin. Dyn. Syst. Ser. B14, 1433?1444 (2010)

[60] Marzocchi, A., Munoz Rivera, J.E., Naso, M.G.: Transmission problem in thermoelasticity with symmetry. IMA J. Appl. Math.68, 23?46 (2002)

[61] Green A. E. and Naghdi P. M. On undamped heat waves in an elastic solid. J. Therm. Stress., 15, 253-264 (1992).

[62] Liu, Z., Quintanilla, R.: Analyticity of solutions in type III thermoelastic plates. IMA J. Appl. Math.75, 356-365 (2010)

[63] M. Kafini, S. A. Messaoudi and S. Nicaise, A blow-up result in a nonlinear abstract evolution system with delay. NoDEA: Nonlinear Differ. Equ. Appl., (2016), DOI 10.1007/s00030-016-0371-4

[64] Ignaczak, J., Ostoja-Starzewski, M.: Thermoelasticity with Finite Wave Speeds, Oxford mathematical Monographs. Oxford University Press, New York (2010)

[65] M. Kirane and B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay. Z. Angew. Math. Phys., **62**(2011), 1065-1082.

[66] I. Lasiecka, Mathematical Control Theory of Coupled PDE's, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, PA, 2002.

[67] I. Lasiecka and D. Doundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and a nonlinear source, Nonlinear Anal., **64**(2006), 1757-1797.

[68] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, Differ. Int. Equations, **6**(1993), 507-533.

[69] W. J. Liu, General decay of the solution for viscoelastic wave equation with a time-varying delay term in the internal feedback. J. Math. Phys., **54**(2013), 043504.

[70] W. J. Liu, General decay rate estimate for the energy of a weak viscoelastic equation with an internal time-varying delay term. Taiwanese J. Math., **17**(2013), 2101-2115.

[71] G. Liu and H. Zhang, Well-posedness for a class of wave equation with past history and a delay. Z. Angew. Math. Phys., **67**(2016), DOI 10.1007/s00033-015-0593-z

[72] Lord, H.W., Shulman, Y.: A generalized dynamical theory of thermoelasticity. J. Mech. Phys. Solids 15, 299-309 (1967).

[73] S. A. Messaoudi, General decay of solutions of a weak viscoelastic equation, Arab. J. Sci. Eng., **36**(2011), 1569-1579.

[74] S. A. Messaoudi, General decay of solutions of a viscoelastic equation, J. Math. Anal. Appl., **341**(2008), 1457-1467.

[75] S. A. Messaoudi, General decay of solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal., **69** (2008), 2589-2598.

[76] S. A. Messaoudi and M. M. Al-Gharabli, A general decay result of a viscoelastic equation with past history and boundary feedback, Z. Angew. Math. Phys., **66**(2015), 1519-1528.

[77] S. A. Messaoudi and N. -E. Tartar, Global existence and asymptotic behavior for a nonlinear viscoelastic equation, Math. Meth. Sci. Res. J., **7**(4) (2003), 136-149.

[78] S. A. Messaoudi and N. -E. Tartar, Exponential and polynomial decay for a quasilinear viscoelastic equation, Nonlinear Anal., **68** (2008), 785-793.

[79] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim., **45**(2006), 1561-1585.

[80] S. Nicaise and C. Pignotti, Intetior feedback stabilization of wave equations with time dependent delay. Electron. J. Differ. Equ., **2011**(2011)(41), 1-20.

[81] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay. Differ. Int. Equ., **21**(2008), 935-958.

[82] S. Nicaise, J. Valein and E. Fridman, Stabilization of the heat and the wave equations with boundary time-varying delays. Discrete Conti. Dyna. Sys.- Series S, **2**(2009), 559-581.

[83] S. H. Park, Decay rate estimates for a weak viscoelastic beam equation with time-varying delay. Appl. Math. Lett., **31**(2014), 46-51.

[84] C. Pignotti, Stability for second-order evolution equations with memory and switching time-delay. J. Dyn. Diff. Equat., (2016), DOI 10.1007/s10884-016-9545-3

[85] N. -E. Tatar, Arbitrary decays in linear viscoelasticity, J. Math. Phys., 52(2011), 013502, 12pp.

[86] S. T. Wu, Asymptotic behavior for a viscoelastic wave equation with a delay term, Taiwanese J. Math., **17**(2013), 765-784.

[87] G. Xu, S. Yung and L. Li, Stabilization of wave systems with input delay in the boundary control. ESAIM Control Optim. Calc. Var., **12**(2006), 770-785.

[88] Z. Yang, Existence and energy decay of solutions for the Euler-Bernoulli viscoelastic equation with a delay. Z. Angew. Math. Phys., **66**(2015), 727-745.

[89] A. Benaissa, D. Ouchenane and Kh. Zennir, *Blow up of positive initial-energy solutions to systems of nonlinear wave equations with degenerate damping and source terms*, Nonlinear studies. Vol. 19, No. 4, pp. 523-535, 2012.

[90] F. E. Browder; *On non-linear wave equations.* M.Z. 80. pp. 249-264 (1962).

[91] M.M. Cavalcanti, L.H. Fatori and T.F. Ma; *Attractors for wave equations with degenerate memory*, J. Diff. Eq., 260 (2016), pp. 56-83.

[92] S. Chandrasekhar; *Introduction to the Study of Stellar Structure*, Chap. 4. Dover, New York, 1957

[93] C.M. Dafermos. An abstract Volterra equation with applications to linear viscoelasticity. J. Diff. Equations, 7 (1970), 554-569.

[94] Dafermos C. M., On the existence and the asymptotic stability of solution to the equations of linear thermoelasticity. Arch. Ration. Mech. Anal., 29, (1968) 241-271.

[95] C. M. Dafermos, H.P. Oquendo, Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. 37(1970), 297-308.

[96] Conti, Graffi, G. Sansone; *Qualitative Methods in the Theory of Nonlinear Vibrations*, Proc. Internat. Sympos. Nonlinear Vibrations, vol. II, 1961, pp. 172-189.

[97] R. Emden, Gaskugeln; *Anwendungen der mechanischen Warmetheorie auf Kosmologie und meteorologische Probleme*, B. G.Teubner, Leipzig, Germany 1907.

[98] R. H. Fowler; *The form near infinity of real, continuous solutions of a certain differential equation of the second order*, Quart. J. Math., 45 (1914), pp. 289-350.

[99] R. H. Fowler; *The solution of Emden's and similar differential equations*, Monthly Notices Roy. Astro. Soc., 91 (1930), pp. 63-91.

[100] R. H. Fowler; *Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations*, Proc. London Math. Soc., 13 (1914), pp. 341-371.

[101] R. H. Fowler; *Further studies of Emden's and similar differential equations*, Quart. J. Math., 2 (1931), pp. 259-288.

[102] R. Glassey; *Finite-time blow-up for solutions of nonlinear wave equations.* M. Z. 177 (1981), pp. 323-340.

[103] F. John; *Blow-up for quasilinear wave equations in three space dimensions.* Comm.Pure. Appl. Math. 36 (1981) pp. 29-51.

[104] M. L. J. Hautus; *Uniformly asymptotic formulas for the Emden-Fowler differential equation*, J. Math. Anal. Appl., 30 (1970), pp. 680-694.

[105] S. Klainerman; *Global existence for nonlinear wave equations.* Comm.Pure Appl. Math. 33 (1980), pp. 43-101.

[106] S. Klainerman, G. Ponce; *Global, small amplitude solutions to nonlinear evolution equations.* Comm. Pure Appl. Math. 36 (1983), pp. 133-141.

[107] M. R. Li; *Nonexistence of global solutions of Emden-Fowler type semilinear wave equations with non-positive energy.* Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 93, pp. 1–10.

[108] M. R. Li; *Estimates for the life-span of the solutions of some semilinear wave equations.* ACPAA. Vol. 7 (2008), No. 2, pp. 417-432.

[109] M. R. Li; *Existence and uniqueness of local weak solutions for the Emden–Fowler wave equation in one dimension*, Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 145, pp. 1–10.

[110] D. Ouchenane, Kh. Zennir and M. Bayoud, *Global nonexistence of solutions for a system of nonlinear viscoelastic wave equations with degenerate damping and source terms*, Ukrainian Mathematical Journal, Vol. 65, No. 7(2013), 723-739.

[111] R. Racke; *Lectures on nonlinear Evolution Equations: Initial Value Problems.* Aspects of Math. Braunschweig Wiesbaden Vieweg(1992).

[112] A. Ritter; *Untersuchungen über die Höhe der Atmosphäre und die Konstitution gasformiger Weltkörper, 18 articles*, Wiedemann Annalen der Physik, 5-20, pp. 1878-1883.

[113] W. A. Strauss; *Nonlinear Wave Equations*, AMS Providence(1989). Dimensions. J. Differential Equations 52 (1984), pp.378-406.

[114] T. Sideris; *Nonexistence of global solutions to semilinear wave equations in high dimensions.* J. Differential Equations 52(1982). pp. 303-345.

[115] W. Thompson (Lord Kelvin); *On the convective equilibrium of temperature in the atmosphere*, Manchester Philos. Soc. Proc., 2 (1860-62), pp.170-176; reprint, Math. and Phys. Papers by Lord Kelvin, 3 (1890), pp. 255-260.

[116] Kh. Zennir and A. Guesmia; *Existence of solutions to nonlinear kth-order coupled Klein-Gordon equations with nonlinear sources and memory term.* Applied Mathematics E-Notes, 15(2015), 121-136.

[117] Zhang, X., Zuazua, E.: Decay of solutions of the system of thermoelasticity of type III. Commun. Contemp. Math. 5, 25-83 (2003)

[118] Kh. Zennir and S. Zitouni; *On the absence of solutions to damped system of nonlinear wave equations of Kirchhoff-type*. Vladikavkaz Mathematical Journal, 17(4), (2015), 44-58.