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Contents

Introduction

Mathematical modeling of physical phenomena and biological processes often leads to nonlocal problems for partial differential equations.

Recently, nonlocal boundary value problems for parabolic and hyperbolic equations with an integral condition on the lateral boundary have been actively studied, integral conditions appear in cases where, for example, a direct measurement of physical quantities is impossible, but their averaged values are known. Such situations occur in studying plasma processes [1], heat conduction [4, 81, 89], certain manufacturing processes [88], moisture transfer in porous media [119], inverse problems [120], as well as problems in mathematical biology [53] and demography [85].

Apparently, one of the first papers treating problems with integral conditions is [81], in which for the one-dimensional heat equation the unique solvability of the mixed problem with Dirichlet conditions on part of the boundary and an integral condition was established. In [89] this result was extended to the general equation of parabolic type. The development of the theory of nonlocal problems for differential equations is proceeding vigorously. Gushchin and Mikhailov (see [55] and the bibliography therein) studied the solvability of nonlocal problems for second order elliptic equations in which the values of the solution on the boundary are related to the values at the interior points by means of some operator, which in particular, can be an integral one. In [57] Paneyakh studied a class of nonlocal conditions such that the values of the solution at a point of the boundary are expressed in terms of the integral of the solution with respect to the measure corresponding to this point and obtained constraints under which there exists a unique classical solution of the problem under study. A number of papers dealt with the disposition of the spectrum of operators arising from nonlocal problems for ordinary [51] and partial differential equations [56].

The investigations on nonlocal problems with integral conditions for hyperbolic equations have appeared. Mixed problems in which one or both boundary conditions were replaced by integral ones were studied in [9, 62]. The unique solvability of a problem having as data only integral conditions was established in [91]. It should be noted that the classical solution of a problem in such a setting, one that can be described as the integral analog of the Goursat problem, was obtained for the simplest equation $u_{xy} = 0$ in [122].

This dissertation investigates the use of the Rothe discretization in time method in solving evolution problems with integrodifferential equations and a nonclassical boundary conditions. Since 1930, various classical types of initial boundary value problems have been investigated by many authors using this method, it's developed and applied to linear as well as nonlinear evolution equations by Rektorys [105], Bouziani [39], Kartsatos and Zigler [50], Nečas [80], Bahuguna and Raghavendra [59], and others. It consists in replacing the time derivatives in an evolution equation by the corresponding difference quotients giving rise to a system of time-independent operator equations. An approximate solution to the evolution equation is defined in terms of the solutions of these time-independent systems. After proving a priori estimates for the approximate solution, the convergence of the approximate solution to the unique solution of the evolution equation is established. We remark that the application of Rothe method to nonlocal problems in Chapters 3 and 4 is made possible thanks to the use of the so-called Bouziani space, first introduced by Bouziani Abd-Elfattah, see, for instance, [9, 19, 98].

This research began (Chapter 2) with the study of the following problem (with a quasilinear hyperbolic equations):

$$\frac{\partial^2 v}{\partial t^2} + \alpha^2 \Delta^{2k} v + \beta^2 \Delta^{2k} \frac{\partial v}{\partial t} = g\left(x, t, v, \frac{\partial v}{\partial t}\right), \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

$$v(0,x) = \varphi_1'(x), \quad \frac{\partial}{\partial t} v(0,x) = \varphi_2'(x), \quad x \in \Omega, \tag{2}$$

$$v_{|\Gamma\times(0,T)} = \psi_1, \quad \Delta^i v_{|\Gamma\times(0,T)} = \psi_i', \quad \Delta^{k+i} v_{|\Gamma\times(0,T)} = \psi_{i+2}, \quad i = 0, ..., k-1. \tag{3}$$

where v is an unknown function, $\varphi_1', \varphi_2', \psi_i, \psi_i'$ and g are a given functions supposed to be sufficiently regular and T is a positive constant. The present chapter can be considered as a generalization of the problems studied in my magister thesis in the way that the conditions are nonhomogenious and the considered equation is a $2k$-dimensional one. It appears in various fields of physics and engineering sciences, for example, in the study of transverse and longitudinal oscillations of a viscoelastic bar. Along a different line, the case of linear hyperbolic equation with homogeneous boundary and initial conditions was considered by Gaiduk [87] and Bouziani [21], in [87] the author proved, with the aid of the method of contour integral, while, in [21] the author use a functional analysis method based on an energy inequality to prove the existence and uniqueness of the solution.

In the next chapter, we deal with a class of semilinear parabolic integrodifferential equations (T is a positive constant and Ω is a bounded open domain in $\mathbb{R}^{\mathbf{n}}$ with a Lipschitz boundary Γ):

$$\frac{\partial v}{\partial t}(x,t)-\frac{\partial^2 v}{\partial x^2}(x,t)=\int_0^t a(t-s)\,k'(s,v(x,s))\,ds+g(x,t), \tag{4}$$

$$\int_0^1 v(x,t)dx=E(t), \qquad \int_0^1 xv(x,t)dx=G(t), \quad t\in[0,T], \tag{5}$$

$$v(x,0)=V_0(x), \quad x\in(0,1), \tag{6}$$

where v is an unknown function, $E,\ G,\ V_0,\ k'$ and a are a given functions supposed to be sufficiently regular and T is a positive constant. The linear case of this problem, i.e.$\int_0^t a(t-s)\,k'(s,v(x,s))\,ds=0$, appears for instance in the modeling of the quasi-static flexure of a thermoelastic rod (see [28]) and has been studied, firstly, with a more general second-order parabolic equation or a 2m-parabolic equation in [18, 28] by means of the energy-integrals method and, secondly, via the Rothe method [98]. For other models, we refer the reader, for instance, to [58-61], and references therein.

The purpose of the last chapter is to study the solvability of the following problem

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^3 v}{\partial t\partial x^2}=\int_0^t a(t-s)\,k'(s,v)\,ds+g\left(t,v,\frac{\partial v}{\partial t}\right), \tag{7}$$

$$\int_0^1 v(x,t)dx=E(t), \qquad \int_0^1 xv(x,t)dx=G(t), \quad t\in[0,T], \tag{8}$$

$$v(x,0)=V_0(x), \qquad \frac{\partial}{\partial t}v(x,0)=W_0(x), \qquad x\in(0,1), \tag{9}$$

where a, k', g, V_0, W_0, E and G are sufficiently regular given functions of the indicated variables and T is a positive constant.

Problems of this type were first introduced in [35], in which the first author proved the well-posedness of certain linear hyperbolic equations with integral conditions. Later, similar problems with equations

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} = g\left(t, v, \frac{\partial v}{\partial t}\right),$$

and

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \eta^2 \frac{\partial^3 v}{\partial t \partial x^2} = g\left(t, v, \frac{\partial v}{\partial t}\right),$$

have been studied in [59] by using the Rothe method and in [9, 17, 24, 30, 35, 62, 90, 91, 109] by other methods, as energetic method, the Schauder fixed point theorem, Galerkin method, and the theory of characteristics, other kinds of nonlinear integral perturbations have been investigated by Bahuguna and Raghavendra [59] for nonlinear parabolic and hyperbolic problems.

Chapter 1

Background

In the course of this thesis, we will work in the standard functional spaces of the types $C(I, X)$, $C^{0,1}(I, X)$, and $L^{\infty}(I, X)$, where X is a Banach space, the main properties of which can be found in [73].Our analysis requires also the use of the space of functions which are square integrable in the Bochner sense, i.e. Bochner integrable and satisfying

$$\int_I \|y(t)\|_H^2 \, dt < +\infty,$$

denotes this space by $L^2(I, H)$. A primitive function

$$Y(t) = \int_0^t y(\tau) \, d\tau,$$

to the function $y(t)$ can be defined, in this case, on the basis of the Riesz theorem, it can be shown that

Lemma 1 *If H is a Hilbert space and $y \in L^2(I, H)$, then, the function*

$$Y(t) = \int_0^t y(\tau) \, d\tau,$$

possesses the following properties

i) Y *is a continuous abstract function in the interval* I, *i.e.*

$$Y \in C(I, H).$$

ii) Y *is absolutely continuous in* I, *i.e.*

$$Y \in AC(I, H).$$

iii) Y *is strongly differentiable a.e. in* I, *we write*

$$Y'(t) = y(t) \text{ in } L^2(I, H).$$

iv) If Y *is integrable and* $g \in H$, *then*

$$(Y(t), g)_H = \int_0^t (y(\tau), g)_H d\tau,$$

holds for all $t \in I$.

v) Moreover, we have

$$Y(0) = 0 \text{ in } H.$$

Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the usual inner product and the corresponding norm respectively in $L_2(0,1)$.We define on $C_0(0,1)$ (the vector space of continuous functions with compact support in $(0,1)$) the bilinear form given by

$$((u, v)) = \int_0^1 \Im_x u \Im_x v dx, \tag{1.1}$$

where

$$\Im_x u = \int_0^x u(\zeta, .)\, d\zeta. \tag{1.2}$$

The bilinear form (1.1) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2 *We denote by* $B_2^1(0,1)$ *a completion of* $C_0(0,1)$ *for the scalar product* (2.1), *which is denoted* $(\cdot,\cdot)_B$, *called the Bouziani space or the space of square integrable primitive functions on* $(0,1)$. *By the norm of function* u *from* $B_2^1(0,1)$, *we understand the nonnegative number:*

$$\|u\|_B = \sqrt{(u,u)_B} = \|\Im_x u\| . \tag{1.3}$$

For $u \in L_2(0,1)$, we have the elementary inequality

$$\|u\|_B \leq \frac{1}{\sqrt{2}} \|u\| . \tag{1.4}$$

We denote by $L_2(0,T;B_2^1(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,v)_{L_2(0,T;B_2^1(0,1))} = \int_0^T (u(.,t), v(.,t))_B dt.$$

Since the space $B_2^1(0,1)$ is a Hilbert space, it can be shown that $L_2(0,T;B_2^1(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq\tau\leq T} \|u(.,\tau)\|_B , \tag{1.5}$$

is denoted $C(0,T;B_2^1(0,1))$.

Definition 3 *the set of all* $u \in L^2(\Omega)$ *such that* $\Delta^i u \in L^2(\Omega)$, $\quad i = \overline{1,k}$ *equipped with the norm*

$$\|u\|_H = \left(\|u\|^2 + \left\|\Delta^k u\right\|^2\right)^{\frac{1}{2}},$$

associated to the inner product

$$(u,v)_H = (u,v) + \left(\Delta^k u, \Delta^k v\right),$$

is called the space $H\left(\Delta^k, \Omega\right)$. *Clearly,* $H\left(\Delta^k, \Omega\right)$ *is a Hilbert space for* $(.,.)_H$.

The nature of the boundary conditions in our problems suggests to introduce the following spaces

$$V = \left\{v \in H\left(\Delta^k, \Omega\right);\ v \equiv 0 \text{ and } \Delta^i v \equiv 0, \quad i = 1, ..., k-1 \text{ over } \Gamma\right\}, \tag{1.6}$$

for the first problem and for the second and the last

$$W = \left\{v \in L_2(0,1);\ \int_0^1 v(x)\,dx = \int_0^1 xv(x)\,dx = 0\right\}, \tag{1.7}$$

which are clearly Hilbert spaces for $(\cdot,\cdot)$. Strong or weak convergence is denoted by $\rightarrow$ or $\rightharpoonup$, respectively. The letter C will stand for generic positive constant which may be different in the same discussion.

Lemma 4 (*Gronwall's Lemma*) (a_1) *Let* $x(t) \geq 0,\ h(t),\ y(t)$ *be real integrable functions on the interval* $[a,b]$. *If*

$$y(t) \leq h(t) + \int_a^t x(s)\,y(s)ds, \qquad \forall t \in (a,b), \tag{1.8}$$

then

$$y(t) \leq h(t) + \int_a^t h(s)\,x(s)\exp\left(\int_s^t x(\tau)\,d\tau\right)ds, \qquad \forall t \in (0,T). \tag{1.9}$$

In particular, if $x(t) \equiv C$ *is a constant and* $h(t)$ *is nondecreasing, then*

$$y(t) \leq h(t)e^{c(t-a)}, \qquad \forall t \in (0,T). \tag{1.10}$$

(a_2) *Let* $\{a_i\}_i$ *be a sequence of real nonnegative numbers satisfying*

$$a_i \leq A + Bh\sum_{k=1}^{i-1} a_k, \quad \forall i = 1, 2, ..., \tag{1.11}$$

where A, B and h are positive constants. Then

$$a_i \leq A \exp\left[B\left(i-1\right)h\right], \tag{1.12}$$

takes place for all $i = 1, 2, ...$

(a_3) *If*

$$a_i \leqslant A + Bh \sum_{k=1}^{i} a_k, \quad \forall i = 1, 2, ...,$$

with $h < \frac{1}{b}$, then

$$a_i \leqslant \frac{A}{1-Bh} \exp\left(\frac{B(i-1)h}{1-Bh}\right), \quad \forall i = 1, 2, ... \tag{1.13}$$

Proof. (a_1) Define

$$v\left(s\right) = \exp\left(-\int_a^s x\left(r\right) dr\right) \int_a^s x\left(r\right) y\left(r\right) dr, \qquad s \in I.$$

Using the product rule, the chain rule, the derivative of the exponential function and the fundamental theorem of calculus, we obtain for the derivative

$$v'\left(t\right) = \left(\underbrace{y\left(s\right) - \int_a^s x\left(r\right) y\left(r\right) dr}_{\leq h(s)}\right) - x\left(r\right) \exp\left(-\int_a^s x\left(r\right) dr\right), \qquad s \in I,$$

where we used the assumed integral inequality for the upper estimate. Since x and the exponential are non-negative, this gives an upper estimate for the derivative of v. Since $v(a) = 0$, integration of this inequality from a to t gives

$$v\left(t\right) \leq \int_a^t h\left(s\right) x\left(s\right) \exp\left(-\int_a^s x\left(r\right) dr\right) ds.$$

Using the definition of $v(t)$ for the first step, and then this inequality and the functional equation of the exponential function, we obtain

$$\begin{aligned}\int_a^t x(s)\,y(s)\,ds &= \exp\left(\int_a^t x(r)\,dr\right)v(t)\\ &\leq \int_a^t h(s)\,x(s)\exp\left(\int_a^t x(r)\,dr-\int_a^s x(r)\,dr\right)ds.\end{aligned}$$

Substituting this result into the assumed integral inequality gives Grönwall's inequality.

If the function h is non-decreasing, then part (a_1), the fact $h(s)\leq h(t)$, and the fundamental theorem of calculus imply that

$$\begin{aligned}y(t) &\leq h(t)+\left(-h(t)\exp\left(\int_s^t x(r)\,dr\right)\right)\Bigg|_{s=a}^{s=t}\\ &= h(t)\exp\left(\int_s^t x(r)\,dr\right), \qquad t\in I.\end{aligned}$$

To prove assertion (a_2), we rewrite the assumed inequality in the form

$$a_i\leq\overline{A_i}+\overline{L}\sum_{j=1}^{i-1}ha_i,\ \text{where } \overline{A_i}=\frac{A_i}{1-Lh},\ \ \overline{L}=L_h=\frac{L}{1-Lh},\qquad i=1,\ \dots\ ,$$

from this inequality we successively deduce

$$a_1\leq\overline{A_1},\ a_2\leq\overline{A_2}\left(1+\overline{L}h\right),\ \dots,\ a_i\leq\overline{A_i}\left(1+\overline{L}h\right)^{i-1}.$$

Hence and from

$$\left(1+\overline{L}h\right)^{i-1}=\left[\left(1+\overline{L}h\right)^{h^{-1}}\right]^{(i-1)h}\leq\exp\left((i-1)\,h\overline{L}\right),$$

we obtain assertion (a_2). ■

Lemma 5 *Let $\{u^j\}_j$ be a sequence such that $u^j\in L^2(\Omega)$, $\forall j\in\mathbb{N}^*$, hence*

$$\left(u^j-u^{j-1},u^j\right)=\frac{1}{2}\left\|u^j\right\|^2+\frac{1}{2}\left\|u^j-u^{j-1}\right\|^2-\frac{1}{2}\left\|u^{j-1}\right\|^2,$$

holds for all $j \in \mathbb{N}^*$.

The following lemma plays a crucial rôle, especially in chapters 3 and 4:

Lemma 6 *Let* $V,\ Y$ *be two Hilbert spaces with* $v \hookrightarrow Y$. *If* $u^n \to u$ *in* $C(I,Y)$ *and the estimates*

$$\begin{aligned} \left\|\tilde{u}_n(t)\right\|_v &\leqslant c, \quad \textit{for all} \ \ t \in I, \\ \left\|\frac{du^{(n)}}{dt}(t)\right\|_Y &\leqslant c, \quad \textit{for a.e. } t \in I, \end{aligned}$$

hold for all $n \geqslant n_0 > 0$, *then*

(i) $u \in L^\infty(I,V) \cap C^{0,1}(I,Y)$;

(ii) u *is differentiable a.e. in* I *and* $\dfrac{du}{dt} \in L^\infty(I,Y)$;

(iii) $u^n(t) \rightharpoonup u(t)$, $\tilde{u}_n(t) \rightharpoonup u(t)$ *in* V *for all* $t \in I$;

(iv) $\dfrac{du^{(n)}}{dt} \rightharpoonup \dfrac{du}{dt}$ *in* $L^2(I,Y)$.

Proof. Cf [73, *page* 26] ■

Chapter 2

Existence and uniqueness of the solution of an evolution problem for a quasilinear pseudo-hyperbolic equation

2.1 Statement of the problem

In the present chapter, we deal with a class of quasilinear pseudo-hyperbolic equations (T is a positive constant and Ω is a bounded open domain in $\mathbb{R}^{\mathbf{n}}$ with a Lipschitz boundary Γ):

$$\frac{\partial^2 s}{\partial t^2} + \alpha^2 \Delta^{2k} s + \beta^2 \Delta^{2k} \frac{\partial s}{\partial t} = g\left(t, x, s, \frac{\partial s}{\partial t}\right), \quad (t, x) \in \Omega \times [0, T], \tag{2.1}$$

subject to the initial conditions

$$s(0,x)=\varphi_1'(x)\,,\quad \frac{\partial}{\partial t}s(0,x)=\varphi_2'(x)\,,\qquad x\in\Omega, \tag{2.2}$$

and the boundary conditions

$$s_{|\Gamma\times(0,T)} = \psi_1,\quad \Delta^i s_{|\Gamma\times(0,T)}=\psi_i' \;\; i=0,...,k-1, \tag{2.3}$$

$$\Delta^{k+i}s_{|\Gamma\times(0,T)} = \psi_{i+2},\quad i=0,...,k-1. \tag{2.4}$$

Introducing a new unknown function $u(t,x)=s(t,x)-\psi_1(t,x)$, our problem with nonhomogeneous boundary conditions can be equivalently reduced to the problem of finding a function satisfying

$$\frac{\partial^2 u}{\partial t^2}+\alpha^2\Delta^{2k}u+\beta^2\Delta^{2k}\frac{\partial u}{\partial t}=f\left(t,x,u,\frac{\partial u}{\partial t}\right),\quad (x,t)\in\Omega\times[0,T]\,, \tag{2.5}$$

$$u(0,x)=\varphi_1(x)\,,\quad \frac{\partial}{\partial t}u(0,x)=\varphi_2(x)\,,\qquad x\in\Omega, \tag{2.6}$$

$$u_{|\Gamma\times(0,T)} = 0,\quad \Delta^i u_{|\Gamma\times(0,T)}=0 \;\; i=0,...,k-1, \tag{2.7}$$

$$\Delta^{k+i}u_{|\Gamma\times(0,T)} = 0,\quad i=0,...,k-1, \tag{2.8}$$

where

$$f(t,x,u,\frac{\partial u}{\partial t}):=g\left(t,x,u,\frac{\partial u}{\partial t}\right)-\frac{\partial^2\psi_1}{\partial t^2}-\alpha^2\Delta^{2k}\psi_1-\beta^2\Delta^{2k}\frac{\partial\psi_1}{\partial t},$$

and

$$\varphi_1(x)=\varphi_1'(x)-\psi_1(0,x)\,,\quad \varphi_2(x)=\varphi_2'(x)-\frac{\partial}{\partial t}\psi_1(0,x).$$

Hence, instead of studying directly the problem $(2.1)-(2.4)$, we concentrate our attention on problem $(2.5)-(2.8)$. Once u is known, the function s is immediately obtained through the relation $s=u+\psi_1$. Throughout the chapter, we assume that

H_1- $f(t,p,q) \in L^2(\Omega)$ for each $(t,p,q) \in I \times V \times V$, with

$$V = \left\{ v \in H\left(\Delta^k, \Omega\right);\ v \equiv 0 \text{ and } \Delta^i v \equiv 0, \quad i = 1, ..., k-1 \text{ over } \Gamma \right\}. \tag{2.9}$$

H_2- For some positive constant L, the following Lipschitz condition

$$\| f(t,p,q) - f(t',p',q') \| \leq L\left(|t-t'| + \|p-p'\| + \|q-q'\|\right),$$

is satisfied for all $t,\ t' \in I$, and all $p,\ p',\ q,\ q' \in V$.

$H_3 -$ $\varphi_1,\ \varphi_2,\ \Delta\varphi_1,\ \Delta\varphi_2 \in H\left(\Delta^{2k}, \Omega\right) \cap V$.

H_4- $\psi_i,\ \psi_i' \in L^2(\Omega), \quad i = \overline{1, 2k}$.

To close this section, we announce the main result of the chapter.

Theorem 7 *Under assumptions* $(H_1) - (H_4)$, *problem* $(2.5) - (2.8)$ *admits a unique weak solution* u *in the sense of*

$$\begin{cases} u \in AC(I,V), \\ u' \in L^2(I,V) \cap AC(I, L^2(\Omega)), \\ u'' \in L^2(I, L^2(\Omega)), \end{cases}$$

$$\begin{cases} u(0) = \varphi_1, \\ u'(0) = \varphi_2, \end{cases}$$

and

$$\begin{aligned} &\int_0^T (u''(t), v(t))\, dt + \alpha^2 \int_0^T \left(\Delta^k u(t), \Delta^k v(t)\right) dt + \beta^2 \int_0^T \left(\Delta^k u'(t), \Delta^k v(t)\right) dt \\ = &\int_0^T (f(t, u(t), u'(t)), v(t))\, dt,\ \forall v \in L^2(I,V), \end{aligned}$$

here the derivative $\frac{du}{dt}$ *is denoted by* u'.

The proof of the last result will be carried out along the following sections.

2.2 Construction of the approximate solutions

In order to solve problem (2.5)−(2.8) by the Rothe method, we divide the time interval $[0,T]$ into n subintervals $[t_{j-1},t_j]$, $j=1,...,n$, where $t_j=jh$ and $h=T/n$. Then, replacing $\frac{\partial}{\partial t}u$ and $\frac{\partial^2}{\partial t^2}u$ by the corresponding standard difference quotient, problem (2.5) − (2.8) may be approximated at each point $t=t_j$, $j=1,...,n$, by the following time discretized problem.

Find a function $u_j:\Omega\to\mathbb{R}^n$, such that

$$\frac{u_j-2u_{j-1}+u_{j-2}}{h^2}+\alpha^2\Delta^{2k}u_j+\beta^2\Delta^{2k}\frac{u_j-u_{j-1}}{h}=f_j, \tag{2.10}$$

or

$$\delta^2u_j+\alpha^2\Delta^{2k}u_j+\beta^2\Delta^{2k}\delta u_j=f_j,\ \forall j=\overline{1,n}, \tag{2.11}$$

$$u_{j|\Gamma} = 0,\quad \Delta^i u_{j|\Gamma}=0,...i=\overline{0,k-1}, \tag{2.12}$$

$$\Delta^{k+i}u_{j|\Gamma} = 0,...i=\overline{0,k-1}, \tag{2.13}$$

where

$$\delta u_j:=\frac{u_j-u_{j-1}}{h},\ \delta^2u_j:=\frac{\delta u_j-\delta u_{j-1}}{h},$$

and where

$$f_j:=f\left(t_j,x,u_{j-1},\delta u_{j-1}\right),$$

starting from

$$u_{-1}(x)=\varphi_1(x)-h\varphi_2(x),\ u_0(x)=\varphi_1(x),\ x\in\Omega. \tag{2.14}$$

From (2.10), we have

$$\frac{u_j}{h^2}+\alpha^2\Delta^{2k}u_j+\beta^2\Delta^{2k}\frac{u_j}{h}=f_j+\frac{2u_{j-1}-u_{j-2}}{h^2}+\beta^2\Delta^{2k}\frac{u_{j-1}}{h}. \tag{2.15}$$

Multiplying for all $j = 1, ..., n$, (2.11), (2.15) by $v \in V$ and integrating over Ω, we get

$$\left(\delta^2 u_j, v\right) + \alpha^2 \left(\Delta^{2k} u_j, v\right) + \beta^2 \left(\Delta^{2k} \delta u_j, v\right) = \left(f_j, v\right), \ \forall j = \overline{1.n}, \tag{2.16}$$

or

$$\begin{aligned} & \frac{1}{h^2}\left(u_j, v\right) + \alpha^2 \left(\Delta^{2k} u_j, v\right) + \beta^2 \frac{1}{h}\left(\Delta^{2k} u_j, v\right) \\ = & \left(f_j + \frac{2u_{j-1} - u_{j-2}}{h^2} + \beta^2 \Delta^{2k} \frac{u_{j-1}}{h}, v\right). \end{aligned} \tag{2.17}$$

For all $j = 1, ..., n$, and all $p = 1, ..., k$, we have

$$\begin{aligned} \left(\Delta^{2p} u_j, v\right) &= \int_\Omega \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\left(\Delta^{2p-1} u_j\right) v dx \\ &= \int_\Gamma \frac{\partial}{\partial \nu} \Delta^{2p-1} u_j v \, d\sigma - \int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i} \Delta^{2p-1} u_j \frac{\partial v}{\partial x_i} dx. \end{aligned}$$

But, due to (2.9), $v \equiv 0$ over Γ, then

$$\left(\Delta^{2p} u_j, v\right) = -\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i} \Delta^{2p-1} u_j \frac{\partial v}{\partial x_i} dx.$$

Integrating by parts the right-hand side, it follows

$$\left(\Delta^{2p} u_j, v\right) = -\int_\Gamma \sum_{i=1}^n \Delta^{2p-1} u_j \frac{\partial v}{\partial x_i} \cos\left(\nu, x_i\right) d\sigma + \int_\Omega \Delta^{2p-1} u_j \Delta v dx,$$

consequently

$$\left(\Delta^{2p} u_j, v\right) = \left(\Delta^{2p-1} u_j, \Delta v\right), \quad \forall v \in V, \ \forall p = \overline{1, k}, \tag{2.18}$$

hence, having in mind that $\Delta^i v_{|\Gamma} = 0, \ \Delta^{k+i} v_{|\Gamma} = 0, ...i = \overline{1, k-1}$

$$\left(\Delta^{2k} u_j, v\right) = \left(\Delta^k u_j, \Delta^k v\right), \quad \forall v \in V. \tag{2.19}$$

Substituting (2.19) in (2.16), (2.17) respectively, we get

$$\left(\delta^2 u_j, v\right) + \alpha^2 \left(\Delta^k u_j, \Delta^k v\right) + \beta^2 \left(\Delta^k \delta u_j, \Delta^k v\right) = \left(f_j, v\right), \quad \forall v \in V, \tag{2.20}$$

and

$$\begin{aligned}
& \frac{1}{h^2}(u_j, v) + \alpha^2(\Delta^k u_j, \Delta^k v) + \beta^2 \frac{1}{h}(\Delta^k u_j, \Delta^k v) \\
= & \left(f_j + \frac{2u_{j-1} - u_{j-2}}{h^2}, v\right) + \frac{\beta^2}{h}\left(\Delta^k u_{j-1}, \Delta^k v\right), \quad \forall v \in V. \qquad (2.21)
\end{aligned}$$

Putting

$$F_j = f_j + \frac{2u_{j-1} - u_{j-2}}{h^2} \ \left(\in L^2(\Omega)\right), \qquad (2.22)$$

and

$$a(u_j, v) = \frac{1}{h^2}(u_j, v) + \alpha^2\left(\Delta^k u_j, \Delta^k v\right) + \beta^2 \frac{1}{h}\left(\Delta^k u_j, \Delta^k v\right), \qquad (2.23)$$

identity (2.21) becomes

$$a(u_j, v) = L_j(v), \quad \forall v \in V, \ \forall j = \overline{1, p}, \qquad (2.24)$$

with

$$L_j(v) = (F_j, v) + \frac{\beta^2}{h}\left(\Delta^k u_{j-1}, \Delta^k v\right), \ \forall j = \overline{1, p}.$$

Let's prove now that the bilinear form $a(.,.)$ is *continuous* and $V-$*elliptic*.

Using (2.23), we get

$$\begin{aligned}
a(v, v) &= \frac{1}{h^2}(v, v) + \alpha^2\left(\Delta^k v, \Delta^k v\right) + \beta^2 \frac{1}{h}\left(\Delta^k v, \Delta^k v\right) \\
&= \frac{1}{h^2}\|v\|^2 + \left(\alpha^2 + \frac{1}{h}\beta^2\right)\left\|\Delta^k v\right\|^2 \\
&\geq k\|v\|_H^2, \quad \forall v \in V,
\end{aligned}$$

with

$$k = \min\left(\alpha^2 + \frac{1}{h}\beta^2, \frac{1}{h^2}\right).$$

Therefore, majorize the bilinear form $a(.,.)$ (by virtue of the Cauchy-Schwarz inequality), we get

$$\begin{aligned} a(u,v) &= \frac{1}{h^2}(u,v) + \left(\alpha^2 + \frac{1}{h}\beta^2\right)(\Delta^k u, \Delta^k v) \\ &\leq \left[\frac{1}{h^2}\|u\|\|v\| + \left(\alpha^2 + \frac{1}{h}\beta^2\right)\left\|\Delta^k u\right\|\left\|\Delta^k v\right\|\right] \\ &\leq \left[\frac{1}{h^2}\|u\|_H\|v\|_H + \left(\alpha^2 + \frac{1}{h}\beta^2\right)\|u\|_H\|v\|_H\right] \\ &\leq \left[\alpha^2 + \frac{1}{h}\beta^2 + \frac{1}{h^2}\right]\|u\|_H\|v\|_H \\ &\leq k'\|u\|_H\|v\|_H, \quad \forall u, v \in V, \end{aligned}$$

with

$$k' = \alpha^2 + \frac{1}{h}\beta^2 + \frac{1}{h^2},$$

from which it follows that the forme $a(.,.)$ is *continuous* and $V-$ *elliptic.*

On the other hand, we have

$$\begin{aligned} |L_j v| &\leq \|F_j\|\|v\| + \frac{\beta^2}{h}\left\|\Delta^k u_{j-1}\right\|\left\|\Delta^k v\right\| \\ &\leq \max(1, \frac{\beta^2}{h})\left(\|F_j\| + \left\|\Delta^k u_{j-1}\right\|\right)\|v\|_H, \qquad \forall v \in V, \end{aligned}$$

from which we deduce that the forme $L_j(\cdot)$, $j = 1, ..., n$, is *continuous* over $H(\Delta^k, \Omega)$.

Therefore, according to the Lax-Milgram theorem, for each $j = 1, ..., n$, problem $(2.11) - (2.13)$ admits a unique solution $u_j \in V$.

Denote by u_j^n, δu_j^n, $\delta^2 u_j^n$, the expressions corresponding to the divisions d_n with step lengths $h_n = \frac{T}{n}$.

2.3 A priori estimates

Proposition 8 *For each $n \in \mathbb{N}^*$ and each $j = 1, ..., n$, the solutions u_j of the time-discretized problem $(2.11) - (2.13)$ satisfy the estimates*

$$\begin{aligned} \left\| \delta^2 u_j^n \right\| &\leq \eta, \ \forall j = \overline{1, n}, && (2.25) \\ \left\| \delta u_j^n \right\|_H &\leq \frac{\eta}{|\alpha|}, \ \forall j = \overline{1, n}, && (2.26) \\ \left\| u_j^n \right\|_H &\leq \frac{\eta}{|\alpha|} T + \|\varphi_1\|_H, \ \forall j = \overline{1, n}, && (2.27) \end{aligned}$$

where

$$\eta = \sqrt{2 \left[3K_2^2 + \alpha^2 K_1^2 + TL \right] \exp \left[\gamma LT \right]},$$

with

$$\begin{aligned} K_1 &= 2TM + \|\varphi_2\| + \left(\alpha^2 + 1 \right) \left\| \Delta^k \varphi_1 \right\|, \\ K_2 &= M + \left\| \alpha^2 \Delta^{2k} \varphi_1 + \beta^2 \Delta^{2k} \varphi_2 \right\| + \left(\frac{\alpha^2}{2} + 1 \right) \left\| \Delta^k \varphi_2 \right\|, \\ M &= L \left(\|\varphi_1\| + \|\varphi_2\| \right) + \max_{0 \leq t \leq T} \|f(t, 0, 0)\|, \end{aligned}$$

and

$$\gamma = \max \left(\frac{1}{\alpha^2}, 1 \right) \left(4 + \frac{T+1}{L} \alpha^2 \right).$$

Proof. Now, for $j = 1, ..., n$, we take the difference of the relations $(2.20)_j - (2.20)_{j-1}$, tested with $v = \delta^2 u_j = (\delta u_j - \delta u_{j-1})/h$ which belongs to V, we have

$$(\delta^2 u_j - \delta^2 u_{j-1}, \delta^2 u_j) + \alpha^2 \left(\Delta^k \delta u_j, \Delta^k \left(\delta u_j - \delta u_{j-1} \right) \right) + \beta^2 h \left(\Delta^k \delta^2 u_j, \Delta^k \delta^2 u_j \right)$$

$$= (f_j - f_{j-1}, \delta^2 u_j).$$

Accordingly, due to assumption H_2 and Lemma 5, we get

$$\begin{aligned}
& \left\|\delta^2 u_j\right\|^2 + \left\|\delta^2 u_j - \delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\Delta^k \delta u_j\right\|^2 \\
& +\alpha^2 \left\|\Delta^k \left(\delta u_j - \delta u_{j-1}\right)\right\|^2 + 2h\beta^2 \left\|\Delta^k \delta^2 u_j\right\|^2 \\
\leq \; & \left\|\delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\Delta^k \delta u_{j-1}\right\|^2 \\
& +2L(|t_j - t_{j-1}| + \|u_{j-1} - u_{j-2}\| + \|\delta u_{j-1} - \delta u_{j-2}\|) \left\|\delta^2 u_j\right\|,
\end{aligned}$$

then

$$\begin{aligned}
& \left\|\delta^2 u_j\right\|^2 + \left\|\delta^2 u_j - \delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\Delta^k \delta u_j\right\|^2 \\
& +\alpha^2 \left\|\Delta^k \left(\delta u_j - \delta u_{j-1}\right)\right\|^2 + 2h\beta^2 \left\|\Delta^k \delta^2 u_j\right\|^2 \\
\leq \; & \left\|\delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\Delta^k \delta u_{j-1}\right\|^2 \\
& +2Lh(1 + \|\delta u_{j-1}\| + \left\|\delta^2 u_{j-1}\right\|) \left\|\delta^2 u_j\right\|. \qquad (2.28)
\end{aligned}$$

On the other hand, thanks to the Cauchy inequality

$$|ab| \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2, \;\; \forall a, \;\; b \in \mathbb{R}, \;\; et \; \varepsilon > 0,$$

we can write, for $\varepsilon = 1$

$$\begin{aligned}
\|\delta u_{j-1}\| \left\|\delta^2 u_j\right\| & \leq \frac{1}{2} \|\delta u_{j-1}\|^2 + \frac{1}{2} \left\|\delta^2 u_j\right\|^2, \\
\left\|\delta^2 u_{j-1}\right\| \left\|\delta^2 u_j\right\| & \leq \frac{1}{2} \left\|\delta^2 u_{j-1}\right\|^2 + \frac{1}{2} \left\|\delta^2 u_j\right\|^2,
\end{aligned}$$

and

$$\left\|\delta^2 u_j\right\| \leq \frac{1}{2} + \frac{1}{2} \left\|\delta^2 u_j\right\|^2,$$

hence

$$\begin{aligned}
\left(1+\|\delta u_{j-1}\|+\left\|\delta^2 u_{j-1}\right\|\right)\left\|\delta^2 u_j\right\| &= \left\|\delta^2 u_j\right\|+\|\delta u_{j-1}\|\left\|\delta^2 u_j\right\| \\
&\quad +\left\|\delta^2 u_{j-1}\right\|\left\|\delta^2 u_j\right\| \\
&\leq \frac{1}{2}+\frac{1}{2}\left\|\delta^2 u_j\right\|^2+\frac{1}{2}\|\delta u_{j-1}\|^2+\frac{1}{2}\left\|\delta^2 u_j\right\|^2 \\
&\quad +\frac{1}{2}\left\|\delta^2 u_{j-1}\right\|^2+\frac{1}{2}\left\|\delta^2 u_j\right\|^2 \\
&\leq \frac{1}{2}+\frac{3}{2}\left\|\delta^2 u_j\right\|^2+\frac{1}{2}\|\delta u_{j-1}\|^2 \\
&\quad +\frac{1}{2}\left\|\delta^2 u_{j-1}\right\|^2 . \qquad (2.29)
\end{aligned}$$

Substituting (2.29) in (2.28) and omitting the second, fourth and last terms in the left-hand side, this gives

$$\begin{aligned}
\left\|\delta^2 u_j\right\|^2+\alpha^2\left\|\Delta^k \delta u_j\right\|^2 &\leq \left\|\delta^2 u_{j-1}\right\|^2+\alpha^2\left\|\Delta^k \delta u_{j-1}\right\|^2 \\
&\quad +2Lh\left[\frac{1}{2}+\frac{3}{2}\left\|\delta^2 u_j\right\|^2+\frac{1}{2}\|\delta u_{j-1}\|^2+\frac{1}{2}\left\|\delta^2 u_{j-1}\right\|^2\right] \\
&\leq \left\|\delta^2 u_{j-1}\right\|^2+\alpha^2\left\|\Delta^k \delta u_{j-1}\right\|^2 \\
&\quad +Lh\left[1+3\left\|\delta^2 u_j\right\|^2+\|\delta u_{j-1}\|^2+\left\|\delta^2 u_{j-1}\right\|^2\right].
\end{aligned}$$

Observing that

$$\begin{aligned}
\|\delta u_j\|^2 &= \left\|h\delta^2 u_j+\delta u_{j-1}\right\|^2 \\
&\leq h\left(h+1\right)\left\|\delta^2 u_j\right\|^2+\left(1+h\right)\|\delta u_{j-1}\|^2 ,
\end{aligned}$$

it follows that

$$\begin{aligned}
\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_j\right\|_H^2 &\leq \left\|\delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\Delta^k \delta u_{j-1}\right\|^2 \\
&\quad + Lh\left[1 + 3\left\|\delta^2 u_j\right\|^2 + \left\|\delta u_{j-1}\right\|^2 + \left\|\delta^2 u_{j-1}\right\|^2\right] \\
&\quad + \alpha^2 h\left(h+1\right)\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_{j-1}\right\|^2 + \alpha^2 h \left\|\delta u_{j-1}\right\|^2 \\
&= \left\|\delta^2 u_{j-1}\right\|^2 + \alpha^2 \left\|\delta u_{j-1}\right\|_H^2 + Lh \\
&\quad + Lh\left(3 + \frac{\alpha^2}{L}\left(h+1\right)\right)\left\|\delta^2 u_j\right\|^2 \\
&\quad + Lh\left[\left(1 + \frac{\alpha^2}{L}\right)\left\|\delta u_{j-1}\right\|^2 + \left\|\delta^2 u_{j-1}\right\|^2\right].
\end{aligned}$$

By recurrence, we get

$$\begin{aligned}
\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_j\right\|_H^2 &\leq \left\|\delta^2 u_1\right\|^2 + \alpha^2 \left\|\delta u_1\right\|_H^2 + L\left(j-1\right)h \\
&\quad + Lh\sum_{i=2}^{j}\left(3 + \frac{\alpha^2}{L}\left(T+1\right)\right)\left\|\delta^2 u_i\right\|^2 \\
&\quad + Lh\sum_{i=1}^{j-1}\left(\left\|\delta^2 u_i\right\|^2 + \left(1 + \frac{\alpha^2}{L}\right)\left\|\delta u_i\right\|^2\right) \\
&\leq \left\|\delta^2 u_1\right\|^2 + \alpha^2 \left\|\delta u_1\right\|_H^2 + LT \\
&\quad + Lh\sum_{i=1}^{j}\left[\left(4 + \frac{\alpha^2}{L}\left(T+1\right)\right)\left\|\delta^2 u_i\right\|^2 + \left(1 + \frac{\alpha^2}{L}\right)\left\|\delta u_i\right\|^2\right] \\
&\leq \left\|\delta^2 u_1\right\|^2 + \alpha^2 \left\|\delta u_1\right\|_H^2 + LT \\
&\quad + \left(4 + \frac{\alpha^2}{L}\left(T+1\right)\right) Lh\sum_{i=1}^{j}\left[\left\|\delta^2 u_i\right\|^2 + \left\|\delta u_i\right\|^2\right],
\end{aligned}$$

hence

$$\begin{aligned}
\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_j\right\|_H^2 &\leq \left\|\delta^2 u_1\right\|^2 + \alpha^2 \left\|\delta u_1\right\|_H^2 + LT \\
&\quad + \gamma Lh\sum_{i=1}^{j}\left[\left\|\delta^2 u_i\right\|^2 + \alpha^2 \left\|\delta u_i\right\|_H^2\right],
\end{aligned}$$

with

$$\gamma = \max\left(\frac{1}{\alpha^2}, 1\right)\left(4 + \frac{T+1}{L}\alpha^2\right).$$

In virtue of Lemma 4, we can write

$$\left\|\delta^2 u_j\right\|^2+\alpha^2\left\|\delta u_j\right\|_H^2 \leq \left[\frac{\left\|\delta^2 u_1\right\|^2+\alpha^2\left\|\delta u_1\right\|_H^2+LT}{1-\gamma Lh}\right] \times \exp\left[\gamma L\,(j-1)\,h/1-\gamma Lh\right],$$

provided that $h<\frac{1}{\gamma L}$. In particular, since h is intended to tend towards zero, we can, without loss of generality, consider that $h\leq\frac{1}{2\gamma L}$. In this case we get

$$\left\|\delta^2 u_j\right\|^2+\alpha^2\left\|\delta u_j\right\|_H^2 \leq 2\left[\left\|\delta^2 u_1\right\|^2+\alpha^2\left\|\delta u_1\right\|_H^2+LT\right] \times \exp\left[2\gamma LT\right]. \tag{2.30}$$

To estimate $\left\|\delta^2 u_1\right\|^2+\alpha^2\left\|\delta u_1\right\|_H^2$, we test the relation (2.20), written for $j=1$, with $v=\delta u_1=(u_1-\varphi_1)/h$ which is an element of V and observing that $\delta u_0=\varphi_2$, $\delta^2 u_1=(\delta u_1-\varphi_2)/h$, we have

$$\frac{1}{h}(\delta u_1-\varphi_2,\delta u_1)+\alpha^2(h\Delta^k\delta u_1+\Delta^k\varphi_1,\Delta^k\delta u_1)+\beta^2(\Delta^k\delta u_1,\Delta^k\delta u_1)=(f_1,\delta u_1),$$

hence

$$\begin{aligned}&\frac{1}{2h}\left\|\delta u_1\right\|^2-\frac{1}{2h}\left\|\varphi_2\right\|^2+\frac{1}{2h}\left\|\delta u_1-\varphi_2\right\|^2+\frac{\alpha^2 h}{2}\left\|\Delta^k\delta u_1\right\|^2\\&-\frac{\alpha^2}{2h}\left\|\Delta^k\varphi_1\right\|^2+\frac{\alpha^2}{2h}\left\|h\Delta^k\delta u_1+\Delta^k\varphi_1\right\|^2+\beta^2\left\|\Delta^k\delta u_1\right\|^2\\ \leq\;& \left\|f_1\right\|\left\|\delta u_1\right\|,\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\left\|\delta u_1\right\|^2+\left\|\delta u_1-\varphi_2\right\|^2+\alpha^2h^2\left\|\Delta^k\delta u_1\right\|^2\\&+\alpha^2\left\|h\Delta^k\delta u_1+\Delta^k\varphi_1\right\|^2+2\beta^2h\left\|\Delta^k\delta u_1\right\|^2\\ \leq\;& 2h\left\|f_1\right\|\left\|\delta u_1\right\|+\left\|\varphi_2\right\|^2+\alpha^2\left\|\Delta^k\varphi_1\right\|^2,\end{aligned}$$

consequently,

$$\|\delta u_1\|^2 \leq 2h \|f_1\| \|\delta u_1\| + \|\varphi_2\|^2 + \alpha^2 \left\|\Delta^k \varphi_1\right\|^2,$$

taking into account that

$$\begin{aligned}
\|f_1\| &= \|f(t_1, \varphi_1, \varphi_2)\| \\
&\leq \|f(t_1, \varphi_1, \varphi_2) - f(t_1, 0, 0)\| + \|f(t_1, 0, 0)\| \\
&\leq L(\|\varphi_1\| + \|\varphi_2\|) + \max_{0 \leq t \leq T} \|f(t, 0, 0)\| := M < +\infty,
\end{aligned}$$

by virtue of assumption (H_2), then

$$\|\delta u_1\| \leq K_1, \tag{2.31}$$

where

$$K_1 = 2TM + \|\varphi_2\| + \left(\alpha^2 + 1\right) \left\|\Delta^k \varphi_1\right\|.$$

On the other hand, taking (2.20) written for $j = 1$ and tested with $v = \delta^2 u_1$, we obtain

$$\begin{aligned}
& (\delta^2 u_1, \delta^2 u_1) + \alpha^2 (\Delta^k u_1 - \Delta^k \varphi_1, \Delta^k \delta^2 u_1) + \beta^2 (\Delta^k \delta u_1 - \Delta^k \varphi_2, \Delta^k \delta^2 u_1) \\
= \; & (f_1, \delta^2 u_1) - \left(\alpha^2 \Delta^k \varphi_1 + \beta^2 \Delta^k \varphi_2, \Delta^k \delta^2 u_1\right),
\end{aligned}$$

therefore

$$\begin{aligned}
& (\delta^2 u_1, \delta^2 u_1) + \alpha^2 \left(\Delta^k \delta u_1, \Delta^k \delta u_1 - \Delta^k \varphi_2\right) + \beta^2 h \left(\Delta^k \delta^2 u_1, \Delta^k \delta^2 u_1\right) \\
= \; & \left(f_1, \delta^2 u_1\right) - \left(\alpha^2 \Delta^{2k} \varphi_1 + \beta^2 \Delta^{2k} \varphi_2, \delta^2 u_1\right),
\end{aligned}$$

consequently,

$$\begin{aligned}
& \left\|\delta^2 u_1\right\|^2 + \frac{\alpha^2}{2} \left\|\Delta^k \delta u_1\right\|^2 + \frac{\alpha^2}{2} \left\|\Delta^k \delta u_1 - \Delta^k \varphi_2\right\|^2 + \beta^2 h \left\|\Delta^k \delta^2 u_1\right\|^2 \\
\leq \; & \|f_1\| \left\|\delta^2 u_1\right\| + \left\|\alpha^2 \Delta^{2k} \varphi_1 + \beta^2 \Delta^{2k} \varphi_2\right\| \left\|\delta^2 u_1\right\| + \frac{\alpha^2}{2} \left\|\Delta^k \varphi_2\right\|^2.
\end{aligned}$$

Provided that $\left\|\delta^2 u_1\right\| \geq \left\|\Delta^k \varphi_2\right\|$, we can write

$$\begin{aligned}
&\left\|\delta^2 u_1\right\|^2 + \frac{\alpha^2}{2}\left\|\Delta^k \delta u_1\right\|^2 + \beta^2 h \left\|\Delta^k \delta^2 u_1\right\|^2 \\
\leq\ & M\left\|\delta^2 u_1\right\| + \left\|\alpha^2 \Delta^{2k}\varphi_1 + \beta^2 \Delta^{2k}\varphi_2\right\| \left\|\delta^2 u_1\right\| + \frac{\alpha^2}{2}\left\|\Delta^k \varphi_2\right\| \left\|\delta^2 u_1\right\|,
\end{aligned}$$

from which it follows that

$$\left\|\delta^2 u_1\right\|^2 + \frac{\alpha^2}{2}\left\|\Delta^k \delta u_1\right\|^2 + \beta^2 h \left\|\Delta^k \delta^2 u_1\right\|^2 \leq K_2 \left\|\delta^2 u_1\right\|,$$

where

$$K_2 = M + \left\|\alpha^2 \Delta^{2k}\varphi_1 + \beta^2 \Delta^{2k}\varphi_2\right\| + \left(\frac{\alpha^2}{2} + 1\right)\left\|\Delta^k \varphi_2\right\|,$$

hence

$$\left\|\delta^2 u_1\right\| \leq K_2, \tag{2.32}$$

and

$$\alpha^2 \left\|\Delta^k \delta u_1\right\|^2 \leq 2K_2^2.$$

The sum of the inequality (2.31) squared and multiplied by α^2 with the last inequality, gives

$$\alpha^2 \left\|\delta u_1\right\|_H^2 \leq \left(2K_2^2 + \alpha^2 K_1^2\right). \tag{2.33}$$

Substituting (2.32) and (2.33) in (2.30), it holds that

$$\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_j\right\|_H^2 \leq 2\left[3K_2^2 + \alpha^2 K_1^2 + TL\right] \times \exp\left[2\gamma LT\right],\ \forall j = 1, p,$$

hence

$$\left\|\delta^2 u_j\right\|^2 + \alpha^2 \left\|\delta u_j\right\|_H^2 \leq \eta^2,\ \forall j = \overline{1, n},$$

where

$$\eta = \sqrt{2\left[3K_2^2 + \alpha^2 K_1^2 + TL\right] \exp\left[\gamma LT\right]},$$

from which it follows that

$$\left\|\delta^2 u_j\right\| \leq \eta,\ \forall j = \overline{1,n}, \tag{2.34}$$

and

$$\left\|\delta u_j\right\|_H \leq \frac{\eta}{|\alpha|},\ \forall j = \overline{1,n}. \tag{2.35}$$

On the other hand, we have

$$\left\|u_j\right\|_H \leq h\left(\left\|\delta u_1\right\|_H + \left\|\delta u_2\right\|_H + ... + \left\|\delta u_j\right\|_H\right) + \left\|\varphi_1\right\|_H,$$

from where

$$\left\|u_j\right\|_H \leq \frac{\eta}{|\alpha|}T + \left\|\varphi_1\right\|_H,\ \forall j = \overline{1,n}. \tag{2.36}$$

Finally, the inequalities (2.34), (2.35) and (2.36) can be generalized for each $n \in \mathbb{N}^*$, from where we obtain the desired estimations (2.25), (2.26) et (2.27). So, the proof of the Proposition 8 is complete. ■

2.4 Convergence and existence result

Let us define, in the interval $I = [0, T]$, the abstract functions

$$u^n(t) = u^n_{j-1} + \delta u^n_j (t - t^n_{j-1}), \qquad \text{in } I^n_j, \tag{2.37}$$

$$\widetilde{u}_n(t) = \begin{cases} u^n_1, & \text{for } t = 0, \\ u^n_j, & \text{in } \tilde{I}^n_j = \left(t^n_{j-1}, t^n_j\right], \end{cases} \tag{2.38}$$

$$U_n(t) = \delta u^n_{j-1} + \delta^2 u^n_j (t - t^n_{j-1}), \qquad \text{in } I^n_j, \tag{2.39}$$

$$\widetilde{U}_n(t) = \begin{cases} \delta u^n_1, & \text{for } t = 0, \\ \delta u^n_j, & \text{in } \tilde{I}^n_j, \end{cases} \tag{2.40}$$

$$Y_n(t) = \begin{cases} \delta^2 u^n_1, & \text{for } t = 0, \\ \delta^2 u^n_j, & \text{in } \tilde{I}^n_j, \end{cases} \tag{2.41}$$

$$\widehat{u}_n(t) = \begin{cases} u^n_1, & \text{for } t = 0, \\ u^n_{j-1}, & \text{in } \tilde{I}^n_j, \end{cases} \tag{2.42}$$

$$\widehat{U}_n(t) = \begin{cases} \delta u^n_1, & \text{for } t = 0, \\ \delta u^n_{j-1}, & \text{in } \tilde{I}^n_j, \end{cases} \tag{2.43}$$

$$\widetilde{f}_n(t) = f\left(t^n_j, \widehat{u}_n(t), \widehat{U}_n(t)\right), \qquad \text{in } \tilde{I}^n_j. \tag{2.44}$$

As a consequence of Proposition 8, we have the following Corollary

Corollary 9 *There exist $C > 0$ such that the estimates*

$$\|u^n(t)\|_H \le C, \qquad \|\tilde{u}_n(t)\|_H \le C, \tag{2.45}$$

$$\left\|\widetilde{U}_n(t)\right\|_H \le C, \tag{2.46}$$

$$\|Y_n(t)\| \le C, \quad \left\|U^n - \frac{du^n}{dt}\right\|_{L^2(I.L^2(\Omega))} \le Ch_n, \tag{2.47}$$

$$\|\tilde{u}_n(t) - u^n(t)\|_H \le Ch_n, \qquad \left\|\widetilde{U}_n(t) - U^n(t)\right\| \le Ch_n, \tag{2.48}$$

$$\|u^n(t) - \widehat{u}_n(t)\|_H \le Ch_n, \qquad \left\|U^n(t) - \widehat{U}_n(t)\right\| \le Ch_n, \tag{2.49}$$

hold for all $t \in I$ *and* $n \in \mathbb{N}^*$.

Proof. Obviously, estimates (2.45) [(2.46), (2.47), respectively] are a direct consequence of estimates (2.25) [(2.26), (2.27), respectively]. On the other hand, from (2.37) and (2.38) [(2.39) and (2.40), respectively], we can write

$$\tilde{u}_n(t) - u^n(t) = \begin{cases} (t_j^n - t)\delta u_j^n, & \text{in } \widetilde{I}_j^n, \\ h_n \delta u_1^n, & \text{for } t = 0, \end{cases} \tag{2.50}$$

and

$$\widetilde{U}_n(t) - U_n(t) = \begin{cases} (t_j^n - t)\delta^2 u_j^n, & \text{in } \widetilde{I}_j^n, \\ h_n \delta^2 u_1^n, & \text{for } t = 0, \end{cases} \tag{2.51}$$

from which, for each $t \in I$, by virtue of (2.26) and (2.25) respectively;

$$\begin{aligned} \|\tilde{u}_n(t) - u^n(t)\|_H &\le h_n \max_{0 \le j \le p_n} \left\|\delta u_j^n\right\|_H \\ &\le Ch_n, \end{aligned}$$

and

$$\begin{aligned} \left\|\widetilde{U}_n(t) - U_n(t)\right\| &\le h_n \max_{0 \le j \le p_n} \left\|\delta^2 u_j^n\right\| \\ &\le Ch_n. \end{aligned}$$

Similarly, we obtain

$$u^n(t) - \widehat{u}_n(t) = \begin{cases} -h_n \delta u_1^n, & \text{for } t = 0, \\ (t - t_j^n)\delta u_j^n, & \text{in } \widetilde{I}_j^n, \end{cases} \tag{2.52}$$

and

$$U_n(t) - \widehat{U}_n(t) = \begin{cases} -h_n \delta^2 u_1^n, & \text{for } t = 0, \\ (t - t_j^n)\delta^2 u_j^n, & \text{in } \widetilde{I}_j^n, \end{cases} \tag{2.53}$$

which implies

$$\begin{aligned} \|u^n(t) - \widehat{u}_n(t)\|_H &\leq h_n \max_{0 \leq j \leq p_n} \left\|\delta u_j^n\right\|_H \\ &\leq Ch_n, \forall t \in I, \end{aligned}$$

and

$$\begin{aligned} \left\|U_n(t) - \widehat{U}_n(t)\right\| &\leq h_n \max_{0 \leq j \leq p_n} \left\|\delta^2 u_j^n\right\| \\ &\leq Ch_n, \forall t \in I, \end{aligned}$$

which achieves the proof of Corollary 9. ■

Proposition 10 *There exists a function* $u \in AC(I, V)$ *with* $u' \in L^2(I, V) \cap AC(I, L^2(\Omega))$ *and* $u'' \in L^2(I, L^2(\Omega))$, *such that*

$$u^n \longrightarrow u \qquad \text{in } C(I, V), \tag{2.54}$$

$$U^n \longrightarrow u' \qquad \text{in } C(I, L^2(\Omega)), \tag{2.55}$$

$$\widetilde{u}_n \rightharpoonup u \qquad \text{in } L^2(I, V), \tag{2.56}$$

$$\widetilde{U}_n \rightharpoonup u' \qquad \text{in } L^2(I, V), \tag{2.57}$$

$$Y_n \rightharpoonup u'' \qquad \text{in } L^2(I, L^2(\Omega)). \tag{2.58}$$

Moreover, the error estimate

$$\|u^n - u\|_{C(I,V)}^2 + \|U^n - u'\|_{C(I,L^2(\Omega))}^2 \leq Ch_n,$$

takes place for all $n \geq n_0$.

Proof. In view of identity (2.20) and due to the definitions of $\widetilde{u}_n(t)$, $\widetilde{U}_n(t)$ and $Y_n(t)$, this shows that for each n and $m \in \mathbb{N}^*$

$$\begin{aligned} & \alpha^2\left(\Delta^k \widetilde{u}_n(t), \Delta^k v(t)\right)+\beta^2\left(\Delta^k \widetilde{U}_n(t), \Delta^k v(t)\right)+\left(Y_n(t), v(t)\right) \\ = & \left(\widetilde{f}_n(t), v(t)\right), \quad \forall v \in L^2(I, V),\ a.e.\ t \in I, \end{aligned} \tag{2.59}$$

and

$$\begin{aligned} & \alpha^2\left(\Delta^k \widetilde{u}_m(t), \Delta^k v(t)\right)+\beta^2\left(\Delta^k \widetilde{U}_m(t), \Delta^k v(t)\right)+\left(Y_m(t), v(t)\right) \\ = & \left(\widetilde{f}_m(t), v(t)\right), \quad \forall v \in L^2(I, V),\ a.e.\ t \in I. \end{aligned} \tag{2.60}$$

Taking the difference of relations (2.59) and (2.60) tested with $v=\widetilde{U}_n-\widetilde{U}_m$ which belongs to V, it follows that

$$\begin{aligned} & \alpha^2\left(\Delta^k \widetilde{u}_n(t)-\Delta^k \widetilde{u}_m(t), \Delta^k \widetilde{U}_n(t)-\Delta^k \widetilde{U}_m(t)\right) \\ & +\beta^2\left(\Delta^k \widetilde{U}_n(t)-\Delta^k \widetilde{U}_m(t), \Delta^k \widetilde{U}_n(t)-\Delta^k \widetilde{U}_m(t)\right) \\ & +\left(Y_n(t)-Y_m(t), \widetilde{U}_n(t)-\widetilde{U}_m(t)\right) \\ = & \left(\widetilde{f}_n(t)-\widetilde{f}_m(t), \widetilde{U}_n(t)-\widetilde{U}_m(t)\right),\ a.e.\ t \in I. \end{aligned}$$

Ignoring the second term in the left-hand side, we have

$$\begin{aligned} & \alpha^2\left(\Delta^k \widetilde{u}_n(t)-\Delta^k \widetilde{u}_m(t), \Delta^k \widetilde{U}_n(t)-\Delta^k \widetilde{U}_m(t)\right) \\ & +\left(Y_n(t)-Y_m(t), \widetilde{U}_n(t)-\widetilde{U}_m(t)\right) \\ & \leq\left(\widetilde{f}_n(t)-\widetilde{f}_m(t), \widetilde{U}_n(t)-\widetilde{U}_m(t)\right),\ a.e.\ t \in I. \end{aligned} \tag{2.61}$$

On the other hand, observing that

$$\widetilde{u}_n-\widetilde{u}_m=\left(u^n-u^m\right)+\left(\widetilde{u}_n-u^n\right)+\left(u^m-\widetilde{u}_m\right),$$

and

$$\widetilde{U}_n - \widetilde{U}_m = (U^n - U^m) + \left(\widetilde{U}_n - U^n\right) + \left(U^m - \widetilde{U}_m\right),$$

the inequality (2.61) becomes

$$\alpha^2\left(\Delta^k u^n(t) - \Delta^k u^m(t), \Delta^k \widetilde{U}_n(t) - \Delta^k \widetilde{U}_m(t)\right) + \left(U^n(t) - U^m(t), Y_n(t) - Y_m(t)\right)$$

$$\leq -\alpha^2\left(\Delta^k \widetilde{u}_n(t) - \Delta^k u^n(t), \Delta^k \widetilde{U}_n(t) - \Delta^k \widetilde{U}_m(t)\right) - \left(\widetilde{U}_n(t) - U^n(t), Y_n(t) - Y_m(t)\right)$$

$$-\alpha^2\left(\Delta^k u^m(t) - \Delta^k \widetilde{u}_m(t), \Delta^k \widetilde{U}_n(t) - \Delta^k \widetilde{U}_m(t)\right) - \left(U^m(t) - \widetilde{U}_m(t), Y_n(t) - Y_m(t)\right)$$

$$+\left(\widetilde{f}_n(t) - \widetilde{f}_m(t), \widetilde{U}_n(t) - \widetilde{U}_m(t)\right), \ a.e.\ t \in I. \tag{2.62}$$

But, we have

$$\begin{aligned} &\left(\Delta^k u^n(t) - \Delta^k u^m(t), \Delta^k \widetilde{U}_n(t) - \Delta^k \widetilde{U}_m(t)\right) \\ = &\ \frac{1}{2}\frac{d}{dt}\left\|\Delta^k u^n(t) - \Delta^k u^m(t)\right\|^2, \ a.e.\ t \in I, \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} &\left(U^n(t) - U^m(t), Y_n(t) - Y_m(t)\right) \\ = &\ \frac{1}{2}\frac{d}{dt}\left\|U^n(t) - U^m(t)\right\|^2, \ a.e.\ t \in I. \end{aligned} \tag{2.64}$$

Accordingly, substituting (2.63) and (2.64) in (2.62), we derive, with the help of Schwarz and Cauchy inequalities

$$\frac{\alpha^2}{2}\frac{d}{dt}\left\|\Delta^k u^n(t) - \Delta^k u^m(t)\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|U^n(t) - U^m(t)\right\|^2$$

$$\begin{aligned} \leq \ &\alpha^2\left\|\Delta^k \widetilde{U}_n(t) - \Delta^k \widetilde{U}_m(t)\right\|\left[\left\|\Delta^k \widetilde{u}_n(t) - \Delta^k u^n(t)\right\| + \left\|\Delta^k u^m(t) - \Delta^k \widetilde{u}_m(t)\right\|\right] \\ &+\left\|Y_n(t) - Y_m(t)\right\|\left[\left\|\widetilde{U}_n(t) - U^n(t)\right\| + \left\|U^m(t) - \widetilde{U}_m(t)\right\|\right] \\ &+\frac{1}{2}\left\|\widetilde{U}_n(t) - \widetilde{U}_m(t)\right\|^2 + \frac{1}{2}\left\|\widetilde{f}_n(t) - \widetilde{f}_m(t)\right\|^2, \ a.e.\ t \in I, \end{aligned}$$

by virtue of Corollary 9, we have

$$\left\|\Delta^k\widetilde{U}_n(t)-\Delta^k\widetilde{U}_m(t)\right\|\le C,\ \ \left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|\le C \text{ and } \|Y_n(t)-Y_m(t)\|\le C,$$

whence

$$\begin{aligned}
&\frac{\alpha^2}{2}\frac{d}{dt}\left\|\Delta^k u^n(t)-\Delta^k u^m(t)\right\|^2+\frac{1}{2}\frac{d}{dt}\|U^n(t)-U^m(t)\|^2\\
\le\;& C\left[\left\|\Delta^k\widetilde{u}_n(t)-\Delta^k u^n(t)\right\|+\left\|\Delta^k u^m(t)-\Delta^k\widetilde{u}_m(t)\right\|\right]\\
&+C\left[\left\|\widetilde{U}_n(t)-U^n(t)\right\|+\left\|U^m(t)-\widetilde{U}_m(t)\right\|\right]\\
&+\frac{1}{2}\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|^2+\frac{1}{2}\left\|\widetilde{f}_n(t)-\widetilde{f}_m(t)\right\|^2\\
\le\;& C\left[\left\|\Delta^k\widetilde{u}_n(t)-\Delta^k u^n(t)\right\|+\left\|\Delta^k u^m(t)-\Delta^k\widetilde{u}_m(t)\right\|\right]\\
&+C\left[\left\|\widetilde{U}_n(t)-U^n(t)\right\|+\left\|U^m(t)-\widetilde{U}_m(t)\right\|\right]\\
&+\frac{3}{2}\left\|\widetilde{U}_n(t)-U^n(t)\right\|^2+\frac{3}{2}\|U^n(t)-U^m(t)\|^2\\
&+\frac{3}{2}\left\|U^m(t)-\widetilde{U}_m(t)\right\|^2+\frac{1}{2}\left\|\widetilde{f}_n(t)-\widetilde{f}_m(t)\right\|^2,\ \ a.e.\ t\in I,
\end{aligned}$$

here, the elementary inequality

$$(a+b+c)^2\le 3\left(a^2+b^2+c^2\right),$$

has been used, and on the basis of estimate (2.48), we obtain

$$\begin{aligned}
&\alpha^2\frac{d}{dt}\left\|\Delta^k u^n(t)-\Delta^k u^m(t)\right\|^2+\frac{d}{dt}\|U^n(t)-U^m(t)\|^2\\
\le\;& C(h_n+h_m)+C(h_n+h_m)^2\\
&+3\|U^n(t)-U^m(t)\|^2+\left\|\widetilde{f}_n(t)-\widetilde{f}_m(t)\right\|^2,\ \ a.e. t\in I. \qquad (2.65)
\end{aligned}$$

For every t fixed in $(0,T]$, there exist two integers p and q corresponding to the subdivision of I into n and m subintervals, respectively, such that $t\in\left[t_{p-1}^n,t_p^n\right]\cap$

$\left[t_{q-1}^m, t_q^m\right]$. Now, owing to assumption (H_2), let us majorize the last term of the inequality (2.65)

$$\begin{aligned}
\left\|\widetilde{f}_n(t)-\widetilde{f}_m(t)\right\| &= \left\|f(t_p^n,\hat{u}_n(t),\hat{U}_n(t))-f\left(t_q^m,\hat{u}_m(t),\hat{U}_m(t)\right)\right\| \\
&\leq L\left(\left|t_p^n-t_q^m\right|+\|\hat{u}_n(t)-\hat{u}_m(t)\|+\left\|\hat{U}_n(t)-\hat{U}_m(t)\right\|\right) \\
&\leq L\left((h_n+h_m)+\|\hat{u}_n(t)-\hat{u}_m(t)\|_H+\left\|\hat{U}_n(t)-\hat{U}_m(t)\right\|\right) \\
&\leq L\left((h_n+h_m)+\|\hat{u}_n(t)-u^n(t)\|_H+\|u^n(t)-u^m(t)\|_H\right. \\
&\quad +\|u^m(t)-\hat{u}_m(t)\|_H+\left\|\hat{U}_n(t)-U^n(t)\right\| \\
&\quad \left.+\|U^n(t)-U^m(t)\|+\left\|U^m(t)-\hat{U}_m(t)\right\|\right),
\end{aligned}$$

taking into account (2.49), we obtain

$$\begin{aligned}
\left\|\widetilde{f}_n(t)-\widetilde{f}_m(t)\right\| &\leq L(1+C)(h_n+h_m)+L\|u^n(t)-u^m(t)\|_H \\
&\quad +L\|U^n(t)-U^m(t)\|. \qquad (2.66)
\end{aligned}$$

Substituting (2.66) in (2.65) and integrating over $(0,t)$ with consideration to the fact that $u^n(0)=u^m(0)=\varphi_1$ and $U^n(0)=U^m(0)=\varphi_2$, this gives

$$\begin{aligned}
&\alpha^2\left\|\Delta^k u^n(t)-\Delta^k u^m(t)\right\|^2+\|U^n(t)-U^m(t)\|^2 \\
\leq\ &C(h_n+h_m)+C(h_n+h_m)^2 \\
&+C\int_0^t\left(\|u^n(s)-u^m(s)\|_H^2+\|U^n(s)-U^m(s)\|^2\right)ds,
\end{aligned}$$

from where

$$\begin{aligned}
&\left\|\Delta^k u^n(t)-\Delta^k u^m(t)\right\|^2+\|U^n(t)-U^m(t)\|^2 \\
\leq\ &C(h_n+h_m)+C(h_n+h_m)^2 \\
&+C\int_0^t\left(\|u^n(s)-u^m(s)\|_H^2+\|U^n(s)-U^m(s)\|^2\right)ds. \qquad (2.67)
\end{aligned}$$

On the other hand, for almost all $t \in I$, we have

$$\begin{aligned}
\frac{d}{dt}\left\|u^n(t)-u^m(t)\right\|^2 &= 2\left(u^n(t)-u^m(t),\widetilde{U}_n(t)-\widetilde{U}_m(t)\right)\\
&\leq 2\left\|u^n(t)-u^m(t)\right\|\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|\\
&\leq \left\|u^n(t)-u^m(t)\right\|^2+\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|^2.
\end{aligned}$$

Integrating over $(0,t)$ with consideration to the fact that $u^n(0)-u^m(0)=0$, we arrive at

$$\begin{aligned}
\left\|u^n(t)-u^m(t)\right\|^2 &\leq \int_0^t \left\|u^n(s)-u^m(s)\right\|^2 ds\\
&\quad+\int_0^t \left\|\widetilde{U}_n(s)-\widetilde{U}_m(s)\right\|^2 ds\\
&\leq \int_0^t \left\|u^n(s)-u^m(s)\right\|_H^2 ds\\
&\quad+\int_0^t \left\|\widetilde{U}_n(s)-\widetilde{U}_m(s)\right\|^2 ds.
\end{aligned} \tag{2.68}$$

We sum up the inequalities (2.67) and (2.68), we deduce that

$$\begin{aligned}
&\left\|u^n(t)-u^m(t)\right\|_H^2+\left\|U^n(t)-U^m(t)\right\|^2\\
\leq\ & C\left(h_n+h_m\right)+C\left(h_n+h_m\right)^2\\
&+C\int_0^t\left(\left\|u^n(s)-u^m(s)\right\|_H^2+\left\|U^n(s)-U^m(s)\right\|^2\right)ds\\
&+\int_0^t\left\|\widetilde{U}_n(s)-U^n(s)\right\|^2 ds+\int_0^t\left\|U^m(s)-\widetilde{U}_m(s)\right\|^2 ds\\
\leq\ & C\left(h_n+h_m\right)+C\left(h_n+h_m\right)^2\\
&+\left(\int_0^T\left\|\widetilde{U}_n(t)-U^n(t)\right\|^2 dt+\int_0^T\left\|U^m(t)-\widetilde{U}_m(t)\right\|^2 dt\right)\\
&+C\int_0^t\left(\left\|u^n(s)-u^m(s)\right\|_H^2+\left\|U^n(s)-U^m(s)\right\|^2\right)ds,
\end{aligned}$$

consequently, by virtue of corollary 9, we get

$$\begin{aligned} & \|u^n(t) - u^m(t)\|_H^2 + \|U^n(t) - U^m(t)\|^2 \\ \leq\ & C(h_n + h_m) + C(h_n + h_m)^2 \\ & + C\int_0^t \left(\|u^n(s) - u^m(s)\|_H^2 + \|U^n(s) - U^m(s)\|^2\right) ds. \end{aligned}$$

Now, let us apply the Lemma 4 to the last inequality, this gives

$$\begin{aligned} & \|u^n(t) - u^m(t)\|_H^2 + \|U^n(t) - U^m(t)\|^2 \\ \leq\ & C(h_n + h_m) + C(h_n + h_m)^2 + \\ & \left(C(h_n + h_m) + C(h_n + h_m)^2\right) e^{CT}. \end{aligned}$$

Since the right-hand side of this inequality does'nt depend on t, we pass to the supremum in the left part, it follow that

$$\begin{aligned} & \|u^n - u^m\|_{C(I,V)}^2 + \|U^n - U^m\|_{C(I,L^2(\Omega))}^2 \\ \leq\ & \left[(h_n + h_m) + (h_n + h_m)^2\right] e^{CT}, \end{aligned} \tag{2.69}$$

from which we deduce that both $\{u^n\}$ and $\{U^n\}$ are Cauchy sequences in the Banach spaces $C(I, V)$ and $C(I, L^2(\Omega))$, respectively. Accordingly, there exist two functions $u \in C(I, V)$ and $U \in C(I, L^2(\Omega))$ such that

$$\begin{cases} u^n \longrightarrow u & in\ C(I, V), \\ U^n \longrightarrow U & in\ C(I, L^2(\Omega)). \end{cases} \tag{2.70}$$

Lemma 11 *There exists a function s with the properties*

$$s \in AC(I, V)\ with\ s' \in L^2(I, V) \cap AC\left(I, L^2(\Omega)\right)\ and\ s'' \in L^2\left(I, L^2(\Omega)\right),$$

and sub-sequences

$$\{u^{n_l}\}_k \subseteq \{u^n\}_n, \{\widetilde{u}_{n_l}\}_k \subseteq \{\widetilde{u}_n\}_n, \left\{\widetilde{U}_{n_l}\right\}_k \subseteq \left\{\widetilde{U}_n\right\}_n \text{ and } \{Y_{n_l}\}_k \subseteq \{Y_n\}_n,$$

such that

$$u^{n_l} \rightharpoonup s \qquad \text{in } L^2(I,V), \tag{2.71}$$

$$\widetilde{u}_{n_l} \rightharpoonup s \qquad \text{in } L^2(I,V), \tag{2.72}$$

$$\widetilde{U}_{n_l} \rightharpoonup s' \qquad \text{in } L^2(I,V), \tag{2.73}$$

$$Y_{n_l} \rightharpoonup s'' \qquad \text{in } L^2(I,L^2(\Omega)). \tag{2.74}$$

Proof. *We integrate the estimates* (2.45) *and* (2.48a) *squared over* $(0,T)$. *Successively we obtain*

$$\begin{aligned} \|u^n\|_{L^2(I,V)} &\leq C, \\ \|\widetilde{u}_n\|_{L^2(I,V)} &\leq C, \\ \|\widetilde{u}_n - u^n\|_{L^2(I,V)} &\leq Ch_n. \end{aligned}$$

The sequences $\{u^n\}_n$ *and* $\{\widetilde{u}_n\}_n$ *are bounded in the Hilbert space* $L^2(I,V)$, *we can extract from* $\{u^n\}_n$ $[\{\widetilde{u}_n\}_n$, *respectively*$]$ *a sub-sequence* $\{u^{n_l}\}$ $[\{\widetilde{u}_{n_l}\}$, *respectively*$]$ *which converges weakly in* $L^2(I,V)$, *i.e.*

$$u^{n_l} \rightharpoonup s \qquad \text{in } L^2(I,V), \tag{2.75}$$

and

$$\widetilde{u}_{n_l} \rightharpoonup \varphi \qquad \text{in } L^2(I,V). \tag{2.76}$$

Let's prove that φ *is none other than the function* s*. For this, note that*

$$\begin{aligned}\left|(\widetilde{u}_{n_l} - s, v)_{L^2(I,V)}\right| &= \left|(\widetilde{u}_{n_l} - u^{n_l}, v)_{L^2(I,V)} + (u^{n_l} - s, v)_{L^2(I,V)}\right| \\ &\leq \|\widetilde{u}_{n_l} - u^{n_l}\|_{L^2(I,V)} \cdot \|v\|_{L^2(I,V)} + \left|(u^{n_l} - s, v)_{L^2(I,V)}\right| \\ &\leq Ch_{n_l} \cdot \|v\|_{L^2(I,V)} + \left|(u^{n_l} - s, v)_{L^2(I,V)}\right|,\end{aligned}$$

for all $v \in L^2(I,V)$*. Passing to the limit for* $n_l \to +\infty$*, owing to* (2.75)*, we obtain*

$$\left|(\widetilde{u}_{n_l} - s, v)_{L^2(I,V)}\right| \to 0,$$

i.e.

$$\widetilde{u}_{n_l} \rightharpoonup s \qquad \text{in } L^2(I,V).$$

Compared with (2.76)*, we deduce, according to the uniqueness of the limit in* $L^2(I,V)$*, that* $s = \varphi$*. Analogously, we deduce that* $\left\{\widetilde{U}_n\right\}$ *and* $\{Y_n\}$ *are bounded in* $L^2(I,V)$ *and* $L^2(I, L^2(\Omega))$*, respectively, and so, it's possible to extract a sub-sequences* $\left\{\widetilde{U}_{n_l}\right\}$ *and* $\{Y_{n_l}\}$ *such that*

$$\begin{aligned}\widetilde{U}_{n_l} &\rightharpoonup S \qquad \text{in } L^2(I,V), \\ Y_{n_l} &\rightharpoonup Y \qquad \text{in } L^2(I, L^2(\Omega)).\end{aligned} \tag{2.77}$$

By virtue of (2.37) *and* (2.40)*, we have for* $t \in \widetilde{I}_j^{n_l}$

$$\begin{aligned}\int_0^t \widetilde{U}_{n_l}(\tau)\, d\tau &= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} \delta u_i^{n_l} d\tau + \int_{t_{j-1}}^{t} \delta u_j^{n_l} d\tau \\ &= h_{n_l} \sum_{i=1}^{j-1} \delta u_i^{n_l} + (t - t_{j-1})\, \delta u_j^{n_l} \\ &= u_{j-1}^{n_l} + \delta u_j^{n_l}(t - t_{j-1}) - \varphi_1 \\ &= u^{n_l}(t) - \varphi_1,\end{aligned}$$

then

$$\int_0^t \widetilde{U}_{n_l}(\tau)\,d\tau = u^{n_l}(t) - \varphi_1,\ \forall t \in I_n. \tag{2.78}$$

■

Lemma 12 *The limit function s satisfies*

$$s(t) = \int_0^t S(\tau)\,d\tau + \varphi_1, \quad \text{in } L^2(I,V).$$

Proof. Owing to Lemma 1 and the limit relation (2.77), the integral $w(t) = \int_0^t S(\tau)\,d\tau$ exists, with the properties

$$w \in AC(I,V),\ w' = S \text{ in } L^2(I,V) \text{ and } w(0) = 0 \text{ in } V. \tag{2.79}$$

By virtue of (2.70), (2.75) and the uniqueness of the weak limit, we obtain the required result if $u^{n_l} \rightharpoonup w + \varphi_1$ in $L^2(I,V)$ i.e.

$$\lim_{n_l\to\infty} \int_0^T (u^{n_l}(t), v(t))_H\,dt = \int_0^T (w(t) + \varphi_1, v(t))_H\,dt, \quad \forall v \in L^2(I,V),$$

or

$$\lim_{n_l\to\infty} \int_0^T (u^{n_l}(t) - w(t) - \varphi_1, v(t))_H\,dt = 0.$$

Let's suppose, first, that $v(t) \equiv v \in V$, $\forall t \in I$, then by virtue of (2.78) and since the norms of functions are uniformly bounded with respect to t and n_l, owing to the dominated convergence theorem of Lebesgue we conclude

$$\begin{aligned} &\lim_{n_l\to\infty} \int_0^T (u^{n_l}(t) - (w(t) + \varphi_1), v)_H\,dt \\ =& \int_0^T \left[\lim_{n_l\to\infty} (u^{n_l}(t) - (w(t) + \varphi_1), v)_H\right] dt \\ =& \int_0^T \left[\lim_{n_l\to\infty} \int_0^t \left(\tilde{U}_{n_l}(\tau) - S(\tau), v\right)_H d\tau\right] dt, \end{aligned}$$

but $\tilde{U}^{n_l} \rightharpoonup S$ in $L^2(I,V)$, then

$$\lim_{n_l \to \infty} \int_0^T (u^{n_l}(t) - (w(t) + \varphi_1), v)_H \, dt = 0, \quad \forall t \in I.$$

Analogously, this can be extended to cases where v is a step function. Because of the step functions space is a dense subspace of $L^2(I,V)$, then, the last result is true for all $v \in L^2(I,V)$, we can thus arrive at

$$u^{n_l} \rightharpoonup w \ + \varphi_1 \qquad in \ \ L^2(I,V).$$

■

Hence, owing to (2.79), the function s satisfies

$$s \in AC(I,V), \tag{2.80}$$

$$s' = S \text{ in } L^2(I,V), \tag{2.81}$$

$$s(0) = \varphi_1 \text{ in } C(I,V). \tag{2.82}$$

Having in mind that $\widetilde{U}_{n_l} \rightharpoonup S$ in $L^2(I,V)$, in what follows $\widetilde{U}_{n_l} \rightharpoonup S$ in $L^2(I, L^2(\Omega))$. Similarly, we conclude that the integral $\int_0^t Y(\tau)\, d\tau$ exists and $\int_0^t Y(\tau)\, d\tau + \varphi_2 = S(t)$, with the properties

$$S \in AC\left(I, L^2(\Omega)\right), \ S'(t) = Y(t), \ S(0) = \varphi_2 \ \ in \ L^2(\Omega) \ \ and \ s''(t) = Y(t). \tag{2.83}$$

Owing to (2.70), (2.80), (2.81), (2.82), and (2.83), we deduce that

$$\begin{aligned} u &\in AC(I,V), \\ U &= u' \text{ in } L^2(I,V) \cap AC(I,L^2(\Omega)), \\ Y_{n_l} &\rightharpoonup u'' \text{ in } L^2(I,L^2(\Omega)), \\ u(0) &= \varphi_1, \\ u'(0) &= \varphi_2. \end{aligned}$$

Moreover, letting $m \to \infty$ in (2.69), we obtain the desired error estimate

$$\begin{aligned} & \|u^n - u\|^2_{C(I,V)} + \|U^n - u'\|^2_{C(I,L^2(\Omega))} \\ \leq & \; C\left(h_n + h_n^2\right)e^{CT} \\ \leq & \; Ch_n, \end{aligned}$$

and the proof is complete. ■

Theorem 13 *The limit function u from Proposition 10 is a weak solution to problem* $(2.5)-(2.8)$ *in the sense of:*

$$\begin{cases} u \in AC(I,V), \\ u' \in L^2(I,V) \cap AC(I,L^2(\Omega)), \\ u'' \in L^2(I,L^2(\Omega)), \end{cases}$$

$$\begin{cases} u(0) = \varphi_1, \\ u'(0) = \varphi_2, \end{cases}$$

and

$$\begin{aligned} & \int_0^T (u''(t), v(t))\,dt + \alpha^2 \int_0^T \left(\Delta^k u(t), \Delta^k v(t)\right)dt + \beta^2 \int_0^T \left(\Delta^k u'(t), \Delta^k v(t)\right)dt \\ = & \int_0^T (f(t,u(t),u'(t)), v(t))\,dt, \quad \forall v \in L^2(I,V). \end{aligned} \tag{2.84}$$

Proof. In light of the properties of the function u listed in Proposition 10, the first three conditions are already seen. On the other hand, since $u^n \to u$ in $C(I,V)$ and $U^n \to u'$ in $C(I,L^2(\Omega))$ as $n \to \infty$ and, by construction, $u^n(0) = \varphi_1$ and $U^n(0) = \varphi_2$, it follows that $u(0) = \varphi_1$ and $u\prime(0) = \varphi_2$, so the initial conditions are also fulfilled. It remains to see that the integral identity (2.84) , is obeyed by u.

In view of (2.38), (2.40), (2.41) and (2.44), the identity (2.20) becomes

$$\begin{aligned} & (Y_{n_l}(t), v(t)) + \alpha^2 \left(\Delta^k \widetilde{u}_{n_l}(t), \Delta^k v(t)\right) + \beta^2 \left(\Delta^k \widetilde{U}_{n_l}(t), \Delta^k v(t)\right) \\ = & \left(\widetilde{f}_{n_l}(t), v(t)\right), \ \forall v \in L^2(I,V), \ a.e. \ t \in I. \end{aligned} \tag{2.85}$$

Integrating (2.85) over $(0,t)$, this gives

$$\begin{aligned} & \int_0^T (Y_{n_l}(t), v(t))\, dt + \alpha^2 \int_0^T \left(\Delta^k \widetilde{u}_{n_l}(t), \Delta^k v(t)\right) dt + \beta^2 \int_0^T \left(\Delta^k \widetilde{U}_{n_l}(t), \Delta^k v(t)\right) dt \\ = & \int_0^T \left(\widetilde{f}_{n_l}(t), v(t)\right) dt, \end{aligned}$$

which may be rewritten in the form

$$\begin{aligned} & \int_0^T (Y_{n_l}(t) - u''(t), v(t))\, dt + \int_0^T (u''(t), v(t))\, dt \\ & + \alpha^2 \int_0^T \left(\Delta^k \widetilde{u}_{n_l}(t) - \Delta^k u(t), \Delta^k v(t)\right) dt + \alpha^2 \int_0^T \left(\Delta^k u(t), \Delta^k v(t)\right) dt \\ & + \beta^2 \int_0^T \left(\Delta^k \widetilde{U}_{n_l}(t) - \Delta^k u'(t), \Delta^k v(t)\right) dt + \beta^2 \int_0^T \left(\Delta^k u'(t), \Delta^k v(t)\right) dt \\ = & \int_0^T \left(\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right), v(t)\right) dt + \int_0^T \left(f\left(t, u(t), u'(t)\right), v(t)\right) dt. \end{aligned}$$

Thus, to establish relation (2.84) , we have to show that

$$\begin{aligned} & \int_0^T (Y_{n_l}(t) - u''(t), v(t))\, dt + \alpha^2 \int_0^T \left(\Delta^k \widetilde{u}_{n_l}(t) - \Delta^k u(t), \Delta^k v(t)\right) dt \\ & + \beta^2 \int_0^T \left(\Delta^k \widetilde{U}_{n_l}(t) - \Delta^k u'(t), \Delta^k v(t)\right) dt \underset{n_l \to \infty}{\longrightarrow} 0, \end{aligned} \tag{2.86}$$

and

$$\int_0^T \left(\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right), v(t)\right) dt \underset{n_l\to\infty}{\longrightarrow} 0, \tag{2.87}$$

for all $v \in L^2(I, V)$. Obviously, limit relation (2.86) is a direct consequence of (2.56), (2.57) and (2.58), while for relation (2.87), let us observe that

$$\begin{aligned}
& \left\|\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right)\right\| \\
= & \left\|f\left(t_p^{n_l}, \hat{u}_{n_l}(t), \hat{U}_{n_l}(t)\right) - f\left(t, u(t), u'(t)\right)\right\| \\
\leq & L\left(\left|t_p^{n_l} - t\right| + \|\hat{u}_{n_l}(t) - u(t)\| + \left\|\hat{U}_{n_l}(t) - u'(t)\right\|\right), \forall t \in \tilde{I}_p^{n_l},
\end{aligned}$$

and this, by virtue of condition (H_2), whence

$$\begin{aligned}
& \left\|\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right)\right\| \\
\leq & L\left(h_{n_l} + \|\hat{u}_{n_l}(t) - u(t)\|_H + \left\|\hat{U}_{n_l}(t) - u'(t)\right\|\right) \\
\leq & L\left(h_{n_l} + \|\hat{u}_{n_l}(t) - u^{n_l}(t)\|_H + \|u^{n_l}(t) - u(t)\|_H\right. \\
& \left. + \left\|\hat{U}_{n_l}(t) - U^{n_l}(t)\right\| + \|U^{n_l}(t) - u'(t)\|\right).
\end{aligned}$$

Due to estimates (2.49), we obtain

$$\begin{aligned}
& \left\|\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right)\right\| \\
\leq & L\left(Ch_{n_l} + \|u^{n_l}(t) - u(t)\|_H + \|U^{n_l}(t) - u'(t)\|\right),
\end{aligned}$$

for all $t \in I$, from where, performing a limit process $n_l \to \infty$ and taking into account (2.54) and (2.55) , we get

$$\left\|\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right)\right\| \underset{n_l\to\infty}{\longrightarrow} 0,$$

hence

$$\left\|\widetilde{f}_{n_l}(t) - f\left(t, u(t), u'(t)\right)\right\|_{L^2(I, L^2(\Omega))} \underset{n_l\to\infty}{\longrightarrow} 0,$$

from which we deduce the desired relation. ■

2.5 Uniqueness

Proposition 14 *The limit function u from Proposition 10 is the unique weak solution of the problem* (2.5) – (2.8).

Proof. Let u_1 and u_2 be two weak solutions of (2.5) – (2.8). From (2.84), for $u(t) = u_1(t) - u_2(t)$ and $v \in L^2(I, V)$, such that

$$v(t) = \begin{cases} u'(t), & \text{for } t \in [0, a] \\ 0, & \text{for } t \in]a, T] \end{cases},$$

where $a \in [0, T]$ is arbitrary, we obtain

$$\begin{aligned} & \alpha^2 \int_0^a \left(\Delta^k u(t), \Delta^k u'(t)\right) dt + \beta^2 \int_0^a \left\|\Delta^k u'(t)\right\|^2 dt + \int_0^a \left(u''(t), u'(t)\right) dt \\ \leq & \int_0^a \left\|f(t, u_1(t), u_1'(t)) - f(t, u_2(t), u_2'(t)\right\| \left\|u'(t)\right\| dt \\ \leq & \; L \int_0^a \left(\left\|u(t)\right\| + \left\|u'(t)\right\|\right) \left\|u'(t)\right\| dt. \end{aligned}$$

Hence, omitting the second term in the left-hand side of the inequality thus obtained and applying the inequality of Cauchy to the right part, we get

$$\begin{aligned} & \alpha^2 \int_0^a \left(\Delta^k u(t), \Delta^k u'(t)\right) dt + \int_0^a \left(u''(t), u'(t)\right) dt \\ \leq & \; \frac{L}{2} \int_0^a \left\|u(t)\right\|^2 dt + \frac{3L}{2} \int_0^a \left\|u'(t)\right\|^2 dt, \end{aligned}$$

from where, with consideration to the fact that $u \in AC(I, V)$,

$$\begin{aligned} & \frac{\alpha^2}{2} \left\|\Delta^k u(a)\right\|^2 - \frac{\alpha^2}{2} \left\|\Delta^k u(0)\right\|^2 + \frac{1}{2} \left\|u'(a)\right\|^2 - \frac{1}{2} \left\|u'(0)\right\|^2 \\ \leq & \; \frac{L}{2} \int_0^a \left\|u(t)\right\|^2 dt + \frac{3L}{2} \int_0^a \left\|u'(t)\right\|^2 dt, \end{aligned}$$

so, because

$$u(0) = 0 \text{ in } V \text{ and } u'(0) = 0 \text{ in } L^2(\Omega),$$

we obtain

$$\frac{\alpha^2}{2}\left\|\Delta^k u\,(a)\right\|^2 + \frac{1}{2}\left\|u'\,(a)\right\|^2 \le \frac{L}{2}\int_0^a \left\|u\,(t)\right\|^2 dt + \frac{3L}{2}\int_0^a \left\|u'(t)\right\|^2 dt, \tag{2.88}$$

here, using the elementary inequality

$$\left\|u\,(a)\right\|^2 \le \int_0^a \left(\left\|u\,(t)\right\|^2 + \left\|u'(t)\right\|^2\right) dt,$$

it follows

$$\begin{aligned} &\frac{\alpha^2}{2}\left\|\Delta^k u\,(a)\right\|^2 + \left\|u\,(a)\right\|^2 + \frac{1}{2}\left\|u'\,(a)\right\|^2 \\ \le\ & \left(1 + \frac{3L}{2}\right)\int_0^a \left(\left\|u\,(t)\right\|^2 + \left\|u'(t)\right\|^2\right) dt. \end{aligned}$$

then

$$\begin{aligned} &\min\left(\alpha^2, 1\right)\left(\left\|\Delta^k u\,(a)\right\|^2 + \left\|u(a)\right\|^2\right) + \left\|u'(a)\right\|^2 \\ \le\ & 2\left(1 + \frac{3L}{2}\right)\int_0^a \left(\left\|u\,(t)\right\|_H^2 + \left\|u'(t)\right\|^2\right) dt, \end{aligned}$$

or

$$\left\|u\,(a)\right\|_H^2 + \left\|u'(a)\right\|^2 \le \frac{2 + 3L}{\min\left(\alpha^2, 1\right)}\int_0^a \left(\left\|u\,(t)\right\|_H^2 + \left\|u'(t)\right\|^2\right) dt.$$

In light of which, due to Lemma 4, we get

$$\left\|u\,(a)\right\|_H^2 + \left\|u'(a)\right\|^2 = 0,$$

and consequently

$$u\,(a) = 0,\ \forall a \in [0, T]\,,$$

which achieves the proof. ■

Chapter 3

The weak solvability of a semilinear parabolic integrodifferential equation with nonclassical boundary conditions

3.1 Statement of the problem

The purpose of this chapter is to study the solvability of the following equation:

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)\, k'(s, v(x,s))\, ds + g(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{3.1}$$

with initial condition

$$v(x,0) = V_0(x), \quad x \in (0,1), \tag{3.2}$$

and the integral conditions

$$\int_0^1 v(x,t)dx = E(t), \quad t \in [0,T], \tag{3.3}$$

$$\int_0^1 xv(x,t)dx = G(t), \quad t \in [0,T], \tag{3.4}$$

where v is an unknown function, E, G and V_0 are a given functions supposed to be sufficiently regular, while k' and a are suitably defined functions satisfying certain conditions to be specified later and T is a positive constant.

It is convenient at the beginning to reduce problem (3.1) – (3.4) with inhomogeneous integral conditions to an equivalent one with homogeneous conditions. For this, we introduce a new unknown function u by setting

$$u(x,t) = v(x,t) - R(x,t), \quad (x,t) \in (0,1) \times [0,T],$$

where

$$R(x,t) = 6\left(2G(t) - E(t)\right)x - 2\left(3G(t) - 2E(t)\right).$$

Then, the function u is seen to be the solution of the following problem

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^t a(t-s)\,k\left(s, u(x,s)\right)ds + f(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{3.5}$$

$$u(x,0) = U_0(x), \quad x \in (0,1), \tag{3.6}$$

$$\int_0^1 u(x,t)dx = 0, \quad t \in [0,T], \tag{3.7}$$

$$\int_0^1 xu(x,t)dx = 0, \quad t \in [0,T], \tag{3.8}$$

where

$$f(x,t) = g(x,t) - \frac{\partial R(x,t)}{\partial t}, \tag{3.9}$$

$$U_0(x) = V_0(x) - R(x,0), \tag{3.10}$$

and

$$k\left(s,u\left(x,s\right)\right) = k'\left(s,u\left(x,s\right) + R(x,s)\right). \tag{3.11}$$

Hence, instead of looking for the function v, we search for the function u. The solution of problem $(3.1)-(3.4)$ will be simply given by the formula $v(x,t) = u(x,t) + R(x,t)$.

In the sequel, we make the following assumptions:

H_1- Functions $f : [0,T] \to L_2(0,1)$ and $a : [0,T] \to \mathbb{R}$ are Lipschitz continuous, i.e.

$$\exists l_1 \in \mathbb{R};\ \left\| f(t) - f(t') \right\| \leq l_1 \left| t - t' \right|, \quad \forall t \in [0,T],$$

and

$$\exists l_2 \in \mathbb{R};\ \left| a(t) - a(t') \right| \leq l_2 \left| t - t' \right|, \quad \forall t \in [0,T].$$

H_2- Mapping $k : [0,T] \times W \to L_2(0,1)$ is Lipschitz continuous in both variables, i.e.

$$\exists l_3 \in \mathbb{R};\ \left\| k(t,u) - k(t',u') \right\| \leq l_3 \left[\left| t - t' \right| + \left\| u - u' \right\| \right],$$

for all $t,\ t' \in I,\ u,\ u' \in W$, and satisfies

$$\exists l_4,\ l_5 \in \mathbb{R};\ \left\| k(t,u) \right\|_B \leq l_4 \left\| u \right\|_B + l_5,$$

for all $t \in I$ and all $u \in W$, where l_4 and l_5 are positive constants.

H_3- Function $U_0 \in H^2(0,1) \cap W,\ i.e.$

$$U_0 \in H^2(0,1);\ \int_0^1 U_0(x)\,dx = \int_0^1 xU_0(x)\,dx = 0.$$

We will be concerned with a weak solution in the following sense.

Definition 15 *A function* $u : I \to L_2(0,1)$ *is called a weak solution to problem* $(3.5)-(3.8)$ *if the following conditions are satisfied:*

(i) $\quad u \in L^\infty(I, W) \cap C(I, B_2^1(0,1))$,

(ii) $\quad u$ *is strongly differentiable a.e. in* I *and* $du/dt \in L^\infty(I, B_2^1(0,1))$,

(iii) $\quad u(0) = U_0$ *in* W,

(iv) *the identity*

$$\begin{aligned} & \left(\frac{du}{dt}(t), v\right)_B + (u(t), v) \\ = & \left(\int_0^t a(t-s)\, k(s, u(s))\, ds, v\right)_B + (f(t), v)_B, \end{aligned} \tag{3.12}$$

holds for all $v \in W$ *and a.e.* $t \in [0, T]$.

This chapter is organized as follows. In Section 2, by the Rothe discretization in time method, we construct approximate discretised solutions to problem $(3.5)-(3.8)$. Some a priori estimates for the approximations are derived in Section 3, while Section 4 is devoted to establish the existence and the uniqueness of the solutions of the problem under study.

To close this section, we announce the main result of this chapter:

Theorem 16 *Under assumptions* $(H_1)-(H_3)$, *Problem* $(3.5)-(3.8)$ *admits a unique weak solution* u, *in the sense of Definition* 15.

3.2 Construction of approximate discrete solutions

In order to solve problem $(3.5)-(3.8)$ by the Rothe method, we proceed as follows. Let n be a positive integer, we divide the time interval $I = [0, T]$ into n subintervals

$I_j^n := [t_{j-1}^n, t_j^n]$, $j = 1, ..., n$, where $t_j^n := jh_n$ and $h_n := T/n$. Then, for each $n \geq 1$, problem (3.5) – (3.8) may be approximated by the following recurrent sequence of time-discretized problems. We successively look for functions $u_j^n \in W$ such that

$$\frac{u_j^n - u_{j-1}^n}{h_n} - \frac{d^2 u_j^n}{dx^2} = h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n\right) + f_j^n, \tag{3.13}$$

$$\int_0^1 u_j^n(x)dx = 0, \tag{3.14}$$

$$\int_0^1 x u_j^n(x)dx = 0, \tag{3.15}$$

starting from

$$u_0^n = U_0, \;\; \delta u_0^n = \frac{d^2}{dx^2} U_0 + f(0), \tag{3.16}$$

for every $j = 1, ..., n$, where $u_j^n(x) := u\left(x, t_j^n\right)$, $\delta u_j^n := \left(u_j^n - u_{j-1}^n\right)/h_n$, $f_j^n(x) := f\left(x, t_j^n\right)$. For this, multiplying for all $j = 1, ..., n$, (3.1) by $\Im_x^2 v := \int_0^x \left[\int_0^\xi v(\tau)\, d\tau\right] d\xi$ and integrating over $(0, 1)$, we get

$$\begin{aligned} & \int_0^1 \delta u_j^n(x)\, \Im_x^2 v dx - \int_0^1 \frac{d^2 u_j^n}{dx^2}(x)\, \Im_x^2 v dx \\ = \;\; & h_n \int_0^1 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n\right) \Im_x^2 v dx + \int_0^1 f_j^n \Im_x^2 v dx. \end{aligned} \tag{3.17}$$

Note that, using a standard integration by parts, for any function v from the space W

$$\Im_1^2 v = \int_0^1 (1-\xi)\, v(\xi)\, d\xi = \int_0^1 v(\xi)\, d\xi - \int_0^1 \xi v(\xi)\, d\xi = 0. \tag{3.18}$$

Carrying out some integrations by parts and invoking (3.18) , we obtain for each term in (3.17) :

$$\begin{aligned}\int_0^1 \delta u_j^n(x)\,\Im_x^2 v dx &= \int_0^1 \frac{\partial}{\partial x}\Im_x\left(\delta u_j^n(x)\right)\Im_x^2 v dx\\ &= \Im_x\left(\delta u_j^n(x)\right)\Im_x^2 v\Big|_0^1 - \int_0^1 \Im_x\left(\delta u_j^n(x)\right)\Im_x v dx\\ &= -\left(\delta u_j^n, v\right)_B. \qquad (3.19)\end{aligned}$$

For the second term in left-hand side, we get

$$\begin{aligned}\int_0^1 \frac{\partial^2 u_j^n}{\partial x^2}(x)\,\Im_x^2 v dx &= \frac{\partial u_j^n}{\partial x}(x)\,\Im_x^2 v\Big|_0^1 - \int_0^1 \frac{\partial u_j^n}{\partial x}(x)\,\Im_x v dx\\ &= -\int_0^1 \frac{\partial u_j^n}{\partial x}(x)\,\Im_x v dx\\ &= -u_j(x)\,\Im_x v\big|_0^1 + \int_0^1 u_j(x)\, v dx\\ &= (u_j, v). \qquad (3.20)\end{aligned}$$

While for the first one in the right-hand side, we obtain

$$\begin{aligned}& h_n \int_0^1 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n(x)\right)\Im_x^2 v dx\\ =\;& h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\int_0^1 k\left(t_i^n, u_i^n(x)\right)\Im_x^2 v dx\\ =\;& h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\int_0^1 \frac{\partial}{\partial x}\Im_x k\left(t_i^n, u_i^n(x)\right)\Im_x^2 v dx\\ =\;& h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\left[\Im_x k\left(t_i^n, u_i^n(x)\right)\Im_x^2 v\big|_0^1 dx - \int_0^1 \Im_x k\left(t_i^n, u_i^n(x)\right)\Im_x v dx\right]\\ =\;& -h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\int_0^1 \Im_x k\left(t_i^n, u_i^n(x)\right)\Im_x v dx\\ =\;& -h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\left(k\left(t_i^n, u_i^n\right), v\right)_B, \qquad (3.21)\end{aligned}$$

and for the last one

$$\begin{aligned}\int_0^1 f_j^n(x)\,\Im_x^2 v(x)\,dx &= \int_0^1 \frac{\partial}{\partial x}\Im_x\left(f_j^n(x)\right)\Im_x^2 v dx \\ &= \Im_x\left(f_j^n(x)\right)\Im_x^2 v\Big|_0^1 - \int_0^1 \Im_x\left(f_j^n(x)\right)\Im_x v dx \\ &= -\left(f_j^n, v\right)_B . \end{aligned} \tag{3.22}$$

By virtue of (3.19), (3.20), (3.21) and (3.22), (3.17) becomes

$$\begin{aligned} &\left(\delta u_j^n, v\right)_B + \left(u_j^n, v\right) \\ &= h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\left(k\left(t_i^n, u_i^n\right), v\right)_B + \left(f_j^n, v\right)_B , \end{aligned} \tag{3.23}$$

or

$$\begin{aligned} &\left(u_j^n, v\right)_B + h_n\left(u_j^n, v\right) \\ &= h_n^2 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\left(k\left(t_i^n, u_i^n\right), v\right)_B + h_n\left(f_j^n, v\right)_B + \left(u_{j-1}^n, v\right)_B . \end{aligned}$$

Let $\eta(.,.) : W \times W \to \mathbb{R}$ and $L_j(.) : W \to \mathbb{R}$ be two functions defined by

$$\eta(u, v) = (u, v)_B + h_n(u, v), \tag{3.24}$$

$$L_j(v) = h_n^2 \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right)\left(k\left(t_i^n, u_i^n\right), v\right)_B + h_n\left(f_j^n, v\right)_B + \left(u_{j-1}^n, v\right)_B . \tag{3.25}$$

To derive the existence and uniqueness of u_j^n, we need to use the Lax-Milgram theorem. For this, let us prove that the bilinear form $\eta(.,.)$ is *continuous* and $W-$*elliptic*.

Using (3.24), we get

$$\begin{aligned}\eta(v, v) &= (v, v)_B + h_n(v, v) \\ &\geq (1 + h_n)\|v\|_B^2 \\ &\geq 2\|v\|_B^2 . \end{aligned} \tag{3.26}$$

On the other hand, we have

$$\begin{aligned} |\eta(u,v)| &= |(u,v)_B + h_n(u,v)| \\ &\leq \|u\|_B \|v\|_B + h_n \|u\| \|v\| \\ &\leq \|u\| \|v\|, \end{aligned} \tag{3.27}$$

estimates (3.26) and (3.27) implies that $\eta(.,.)$ is *continuous* and $W-elliptic$. Also, from (3.25) we have

$$\begin{aligned} |L_j(v)| &= \left| h_n^2 \sum_{i=0}^{j-1} \eta\left(t_j^n - t_i^n\right) \left(k\left(t_i^n, u_i^n\right), v\right)_B \right. \\ &\quad \left. + h_n \left(f_j^n, v\right)_B + \left(u_{j-1}^n, v\right)_B \right| \\ &\leq \left[h_n C \sum_{i=0}^{j-1} \left(C_1 \|u_i^n\| + C_2\right) + h_n \left\|f_j^n\right\| + \left\|u_{j-1}^n\right\| \right] \|v\|, \end{aligned} \tag{3.28}$$

which prove that $L_j(.)$ is *continuous* for each $j = 1, ..., n$. Since $\eta(.,.)$ is *continuous* and $W-elliptic$ and $L_j(.)$ is *continuous*, the Lax-Milgram Lemma guarantees the existence and uniqueness of $u_j^n, \quad \forall j = 1, ..., n$.

3.3 A priori estimates

Lemma 17 *There exist $C > 0$ such that, for all $n \geq 1$, the solutions u_j of the discretized problems* (3.13) – (3.16), *$j = 1, ..., n$, satisfy the estimates*

$$\left\|u_j^n\right\| \leq C, \tag{3.29}$$

$$\left\|\delta u_j^n\right\|_B \leq C. \tag{3.30}$$

Proof. Testing the difference $(3.23)_j - (3.23)_{j-1}$ with $v = \delta u_j^n \, (\in W)$, taking into account assumptions $(H_1) - (H_3)$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \left\|\delta u_j^n\right\|_B + \left\|u_j^n - u_{j-1}^n\right\|_B \\ \leq \; & \left\|\delta u_{j-1}^n\right\|_B + \frac{C_1}{3} h_n^2 \sum_{i=0}^{j-2} \|u_i^n\|_B + \frac{C_1}{3} h_n + \frac{C_1}{3} h_n \left\|u_{j-1}^n\right\|_B , \end{aligned}$$

where

$$C_1 := 3 \max\{l_2 \zeta, T l_2 \zeta + M_1 \zeta + l_1\}, \;\; M_1 := \max_{t \in I} |a(t)| \;\; \textit{and} \;\; \zeta := \max\{l_4, l_5\}.$$

Multiplying the left-hand side of the last inequality with $\left(1 - \frac{C_1}{3} h_n\right)$ $(< 1$ and positive for $n \geq n_0)$ and adding the terme

$$\frac{2}{3} C_1 h_n \left[\left\|u_j^n - u_{j-1}^n\right\|_B - \left\|\delta u_j^n\right\|_B\right] (< 0 \text{ for } n \geq n_0),$$

we get

$$\begin{aligned} & (1 - C_1 h_n) \left[\left\|\delta u_j^n\right\|_B + \left\|u_j^n\right\|_B\right] \\ \leq \; & \left[\left\|u_{j-1}^n\right\|_B + \left\|\delta u_{j-1}^n\right\|_B\right] + C_1 h_n^2 \sum_{i=0}^{j-2} \|u_i^n\|_B + C_1 h_n. \end{aligned} \tag{3.31}$$

Applying the last inequality recursively, it follows that

$$\begin{aligned} & (1 - C_1 h_n)^j \left[\left\|\delta u_j^n\right\|_B + \left\|u_j^n\right\|_B\right] \\ \leq \; & [\|u_0^n\|_B + \|\delta u_0^n\|_B + C_1 T] + T C_1 h_n \sum_{i=0}^{j-2} \|u_i^n\|_B , \end{aligned} \tag{3.32}$$

or, by virtue of Lemma 4, there exists $n_0 \in \mathbb{N}^*$ such that

$$\left\|\delta u_j^n\right\|_B + \left\|u_j^n\right\|_B \leq C_2, \quad \forall n \geq n_0,$$

where

$$C_2 \;:\; = (\exp(TC_1)+1)\left[\|\delta u_0^n\|_B + \|u_0^n\|_B + TC_1\right] \times \exp\left[(\exp(TC_1)+1)\,TC_1\right].$$

And so our proof is complete. ■

We address now the question of convergence and existence.

3.4 Convergence, existence and uniqueness

Now let us introduce the Rothe function $u^n(t) : I \to W$ obtained from the functions u_j by piecewise linear interpolation with respect to time

$$u^n(t) = u_{j-1}^n + \delta u_j^n (t - t_{j-1}^n), \quad in\ I_j^n, \tag{3.33}$$

as well the step functions $\widetilde{u}_n(t)$, $\hat{u}_n(t)$, $\widetilde{f}^n(t)$ and $\widetilde{k}(t, \widetilde{u}_n(t))$ defined as follows:

$$\widetilde{u}_n(t) = \begin{cases} u_0^n, & \text{for } t = 0, \\ u_j^n, & \text{in } \tilde{I}_j^n := \left(t_{j-1}^n, t_j^n\right], \end{cases} \qquad \hat{u}_n(t) = \begin{cases} u(0), & \text{for } t = 0, \\ u_{j-1}^n, & \text{in } \tilde{I}_j^n, \end{cases} \tag{3.34}$$

$$\widetilde{f}^n(t) = \begin{cases} f(0), & \text{for } t = 0, \\ f\left(t_j^n\right), & \text{in } \tilde{I}_j^n, \end{cases} \tag{3.35}$$

$$\widetilde{k}(t, \widetilde{u}_n(t)) = \begin{cases} 0, & \text{for } t = 0, \\ h_n \sum_{i=0}^{j-1} a\left(t_j^n - t_i^n\right) k\left(t_i^n, u_i^n\right), & \text{in } \tilde{I}_j^n = \left(t_{j-1}^n, t_j^n\right]. \end{cases} \tag{3.36}$$

Corollary 18 *There exist $C > 0$ such that the estimates*

$$\|u^n(t)\| \le C, \qquad \|\tilde{u}_n(t)\| \le C, \quad \forall t \in I, \tag{3.37}$$

$$\left\| \frac{du^n}{dt}(t) \right\|_B \leq C, \qquad for\ a.\ e.\ t \in I, \tag{3.38}$$

$$\left\| \tilde{u}_n(t) - u^n(t) \right\|_B \leq Ch_n,\ \ \left\| \hat{u}_n(t) - u^n(t) \right\|_B \leq Ch_n, \qquad \forall t \in I, \tag{3.39}$$

and

$$\left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) \right\| \leq C, \qquad \forall t \in I, \tag{3.40}$$

hold for all $n \in \mathbb{N}^*$.

Proof. For the inequalities $(3.37), (3.38)$ and (3.39) see [98, *Corollary* 4.2.], whereas for the last inequality, assumption (H_2) and estimate (3.29) guarantee the desired result. ■

Proposition 19 *The sequence* $(u^n)_n$ *converges in the norm of the space* $C(I, B_2^1(0,1))$ *to some function* $u \in C(I, B_2^1(0,1))$ *and the error estimate*

$$\|u^n - u\|_{C(I, B_2^1(0,1))} \leq C\sqrt{h_n}, \tag{3.41}$$

takes place for all $n \geq n_0$.

Proof. By virtue of (3.34), (3.35) and (3.36) the variational equation (3.23) may be rewritten in the form

$$\left(\frac{du^n}{dt}(t), v \right)_B + (\widetilde{u}_n(t), v) = \left(\widetilde{k}\left(t, \widetilde{u}_n(t)\right), v \right)_B + \left(\widetilde{f}^n(t), v \right)_B, \tag{3.42}$$

for a.e. $t \in [0, T]$. In view of (3.42), using (3.38) and (3.40) with the fact that

$$\left\| \widetilde{f}^n(t) \right\|_B \leq M_2 := \max_{t \in I} \|f(t)\|_B,$$

we obtain

$$\begin{aligned} |(\widetilde{u}_n(t), v)| &\leq \left(\left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) \right\|_B + \left\| \widetilde{f}^n(t) \right\|_B + \left\| \frac{du^n}{dt}(t) \right\|_B \right) \|v\|_B \\ &\leq C\|v\|_B,\ a.e. t \in [0, T]. \end{aligned} \tag{3.43}$$

Now, for $n,\ m$ be two positive integers, testing the difference $(3.42)^n - (3.42)^m$ with $v = u^n(t) - u^m(t)$ which is in W, with the help of the Cauchy-Schwarz inequality and taking into account that

$$2\left((du/dt)(t), u(t)\right)_B = (d/dt)\left\|u(t)\right\|_B^2,\ a.e.t \in [0,T],$$

and, by virtue of (3.43) we obtain after some rearrangements

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left\|u^n(t) - u^m(t)\right\|_B^2 + \left\|\widetilde{u}^n(t) - \widetilde{u}^m(t)\right\|^2 \\
\leq\ & C\left\|u^m(t) - \widetilde{u}_m(t)\right\|_B + C\left\|\widetilde{u}_n(t) - u^n(t)\right\|_B \\
& + \left\|\widetilde{k}(t, \widetilde{u}_n(t)) - \widetilde{k}(t, \widetilde{u}_m(t))\right\|_B \left\|u^n(t) - u^m(t)\right\|_B \\
& + \left\|\widetilde{f}^n(t) - \widetilde{f}^m(t)\right\|_B \left\|u^n(t) - u^m(t)\right\|_B,\ a.e.t \in [0,T]. \qquad (3.44)
\end{aligned}$$

To derive the required result, we need to estimate the third and the last term in the left-hand side, for this, let t be arbitrary but fixed in $(0,T]$, without loss of generality we can suppose that there exist three integers $p,\ q$ and β such that

$$t \in (t_{p-1}^n, t_p^n] \cap (t_{q-1}^m, t_q^m],\ n = \beta m,\ t_p^n = t_q^m.$$

From which, using (3.36) we can write

$$\begin{aligned}
& \left\|\widetilde{k}(t, \widetilde{u}_n(t)) - \widetilde{k}(t, \widetilde{u}_m(t))\right\|_B \\
= \ & h_m \left\|\sum_{j=0}^{p-1}\left[\sum_{i=j\beta}^{\beta(j+1)-1}\left(a\left(t_p^n - t_j^n\right)k\left(t_j^n, u_j^n\right) - a\left(t_q^m - t_i^m\right)k\left(t_i^m, u_i^m\right)\right)\right]\right\|_B.
\end{aligned}$$

Taking into account assumption (H_1) and the fact that $\left|a\left(t_p^n - t_j^n\right) - a\left(t_q^m - t_i^m\right)\right| \leq Ch_n$, thus, there exist $\varepsilon_n \in [0, Ch_n]$ such that

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
\leq\ & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} \left\|\left(Ch_n - \varepsilon_n\right) k\left(t_j^n, u_j^n\right)\right\|_B \right. \\
&\left. + \left|a\left(t_q^m - t_i^m\right)\right| \left\|k\left(t_j^n, u_j^n\right) - k\left(t_i^m, u_i^m\right)\right\|_B \right].
\end{aligned}$$

Therefore, recalling assumptions (H_1), (H_2) and having in mind that $\varepsilon_n \in [0, Ch_n]$, we estimate

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
\leq\ & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C\left(h_n + \left\|u_j^n - u_i^m\right\|_B\right) \right],
\end{aligned}$$

from where, we derive

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
\leq\ & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C\left(h_n + \left\|\widetilde{u}_n(s) - u^n(s)\right\|_B \right.\right. \\
&\left.\left. + \left\|u^n(s) - u^m(s)\right\|_B + \left\|u^m(s) - \widetilde{u}_m(s)\right\|_B\right)\right],
\end{aligned}$$

holds for all $s \in \left(t_i^m, t_{i+1}^m\right]$. We take the supremum with respect to s from 0 to t in the right-hand side, invoking the fact that $s \in \left(t_i^m, t_{i+1}^m\right] \subset \left(t_{j-1}^n, t_j^n\right]$ and estimate (3.39), we obtain

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
\leq\ & h_m \sum_{i=0}^{q-1} \left[Ch_n + C \sup_{0 \leq s \leq t} \left\|u^n(s) - u^m(s)\right\|_B \right],
\end{aligned}$$

so that

$$\left\|\widetilde{k}\left(t,\widetilde{u}_n(t)\right)-\widetilde{k}\left(t,\widetilde{u}_m(t)\right)\right\|_B \leq Ch_n + C\sup_{0\leq s\leq t}\left\|u^n\left(s\right)-u^m\left(s\right)\right\|_B. \tag{3.45}$$

Let $t\in(t^n_{p-1},t^n_p]\cap(t^m_{q-1},t^m_q]$, from assumption (H_1) it follows that

$$\begin{aligned}\left\|\widetilde{f}^n\left(t\right)-\widetilde{f}^m\left(t\right)\right\|_B &= \left\|f\left(t^n_p\right)-f\left(t^m_q\right)\right\|_B\\ &\leq l_1\left|t^n_p-t^m_q\right|\\ &\leq l_1h_n. \end{aligned}\tag{3.46}$$

Ignoring the second term in the left-hand side of (3.44) which is clearly positive and using estimates (3.37), (3.39), (3.45) and (3.46), yields

$$\begin{aligned}&\frac{d}{dt}\left\|u^n\left(t\right)-u^m\left(t\right)\right\|_B^2\\ \leq\ & C\left(h_n+h_m\right)+C\sup_{0\leq s\leq t}\left\|u^n\left(s\right)-u^m\left(s\right)\right\|_B^2,\ \ a.e.t\in[0,T].\end{aligned}$$

Integrating this inequality with respect to time from 0 to t and invoking the fact that $u^n(0)=u^m(0)=U_0$, we get

$$\left\|u^n\left(t\right)-u^m\left(t\right)\right\|_B^2\leq C\left(h_n+h_m\right)+C\int_0^t\sup_{0\leq\xi\leq t}\left\|u^n\left(\xi\right)-u^m\left(\xi\right)\right\|_B^2 d\xi,$$

whence

$$\sup_{0\leq s\leq t}\left\|u^n\left(s\right)-u^m\left(s\right)\right\|_B^2\leq C\left(h_n+h_m\right)+C\int_0^t\sup_{0\leq\xi\leq t}\left\|u^n\left(\xi\right)-u^m\left(\xi\right)\right\|_B^2 d\xi.$$

Accordingly, by Gronwall's Lemma we obtain

$$\sup_{0\leq s\leq t}\left\|u^n\left(s\right)-u^m\left(s\right)\right\|_B^2\leq C\left(h_n+h_m\right)\exp\left(ct\right),\ \ \forall t\in\left[0,T\right],$$

consequently

$$\sup_{0\leq s\leq T}\left\|u^n\left(s\right)-u^m\left(s\right)\right\|_B\leq C\sqrt{h_n+h_m}, \tag{3.47}$$

takes place for all $n,\ m \in \mathbb{N}^*$. This implies that $(u^n(t))_n$ is a Cauchy sequence in the Banach space $C(I, B_2^1(0,1))$, and hence it converges in the norm of this latter to some function $u \in C(I, B_2^1(0,1))$. Besides, passing to the limit $m \to \infty$ in (3.47), we obtain the desired error estimate, which finishes the proof. ■

Now, we present some properties of the obtained solution.

Theorem 20 *The limit function u from Proposition 19 yields the following statements*

(i) $u \in C(I, B_2^1(0,1)) \cap L^\infty(I, W)),$

(ii) *u is strongly differentiable a.e. in I and* $du/dt \in L^\infty(I, B_2^1(0,1)),$

(iii) $\widetilde{u}_n(t) \to u(t)$ *in* $B_2^1(0,1)$ *for all* $t \in I,$

(iv) $u^n(t)\,,\ \widetilde{u}_n(t) \rightharpoonup u(t)$ *in* W *for all* $t \in I,$

(v) $\frac{du^n}{dt}(t) \rightharpoonup \frac{du}{dt}(t)$ *in* $L^2(I, B_2^1(0,1)).$

Proof. On the basis of estimates (3.37) and (3.38), uniform convergence statement from Proposition 19 and the continuous embedding $W \hookrightarrow B_2^1(0,1)$, the assertions of the present theorem are direct consequences of Lemma 6. ■

Theorem 21 *Under assumptions* $(H_1) - (H_3)$, *problem* $(3.5) - (3.8)$ *admits a unique weak solution, namely the limit function u from Proposition 20, in the sense of Definition 15.*

Proof. We have to show that the limit function u satisfies all the conditions (i), (ii), (iii), (iv) of Definition 15. Obviously, in light of the properties of the function u listed in Theorem 20, the first two conditions of Definition 15 are already seen. On the other hand, since $u^n \to u$ in $C(I, W)$ as $n \to \infty$ and, by construction, $u^n(0) = U_0$, it follows that $u(0) = U_0$, so the initial condition is also fulfilled, that is, Definition 15(iii) takes place. It remains to see that the integral identity (3.12) is obeyed by u. For this, integrating (3.42) over $(0,t)$ and using the fact that $u^n(0) = U_0$, we get

$$\begin{aligned} & (u^n(t) - U_0, v)_B + \int_0^t (\widetilde{u}_n(\tau), v)\, d\tau \\ = & \int_0^t \left(\widetilde{k}(\tau, \widetilde{u}_n(\tau)), v\right)_B d\tau + \int_0^t \left(\widetilde{f}^n(\tau), v\right)_B d\tau, \end{aligned}$$

consequently, after some rearrangements

$$\begin{aligned} & (u^n(t) - U_0, v)_B + \int_0^t (\widetilde{u}_n(\tau), v)\, d\tau \\ = & \int_0^t \left(\int_0^\tau a(\tau - s)\, k(s, u(s))\, ds, v\right)_B d\tau + \int_0^t (f(\tau), v)_B\, d\tau \\ & + \int_0^t \left(\widetilde{k}(\tau, \widetilde{u}_n(\tau)) - \int_0^\tau a(\tau - s)\, k(s, u(s))\, ds, v\right)_B d\tau \\ & + \int_0^t \left(\widetilde{f}^n(\tau) - f(\tau), v\right)_B d\tau. \end{aligned} \tag{3.48}$$

Let $\hat{s}_n : I \to I$ and $\hat{s}_n : I \to I$ denotes the functions

$$\hat{s}_n(t) = \begin{cases} 0, & \text{for } t = 0 \\ t_{j-1}^n, & \text{in } \tilde{I}_j^n \end{cases}, \qquad \tilde{s}_n(t) = \begin{cases} 0, & \text{for } t = 0 \\ t_j^n, & \text{in } \tilde{I}_j^n \end{cases}. \tag{3.49}$$

To investigate the desired result, we prove some convergence statements. Using (3.34), (3.35) and (3.49) we have for all $t \in \left(t_{j-1}^n, t_j^n\right]$

$$\begin{aligned} & \widetilde{k}(t, \widetilde{u}_n(t)) - \int_0^t a(t - s)\, k(s, u(s))\, ds \\ = & \int_0^{t_j^n} \left[a\left(t_j^n - \hat{s}_n(s)\right) k(\hat{s}_n(s), \hat{u}_n(s)) - a(t - s)\, k(s, u(s))\right] ds \\ & + \int_t^{t_j^n} a(t - s)\, k(s, u(s))\, ds. \end{aligned} \tag{3.50}$$

Taking into account (3.37), (3.41) and assumptions (H_1), (H_2) it follows that

$$\left\| a\left(t_j^n - \hat{s}_n(s)\right) k(\hat{s}_n(s), \hat{u}_n(s)) - a(t - s)\, k(s, u(s)) \right\|_B \le C\sqrt{h_n}. \tag{3.51}$$

Thanks to (3.50) and (3.51) we obtain

$$\left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \int_0^t a\left(t-s\right) k\left(s, u\left(s\right)\right) ds \right\|_B \leq C\sqrt{h_n}. \tag{3.52}$$

On the other hand, in view of the assumed Lipschitz continuity of f, we have

$$\begin{aligned} \left\| \widetilde{f^n}\left(\tau\right) - f\left(\tau\right) \right\|_B &\leq \left\| f\left(\tilde{s}_n\left(\tau\right)\right) - f\left(\tau\right) \right\|_B \\ &\leq l_1 h_n. \end{aligned} \tag{3.53}$$

Now, the sequences $\{(\widetilde{u}_n(\tau), v)\}$, $\left\{\left(\widetilde{f^n}\left(\tau\right), v\right)_B\right\}$ and $\left\{\left(\widetilde{k}\left(\tau, \widetilde{u}_n(\tau)\right), v\right)_B\right\}$ are uniformly bounded with respect to both τ and n, so the Lebesgue theorem of majorized convergence is applicable to (3.48), thus, having in mind (3.39), (3.41), (3.52) and (3.53), we derive

$$\begin{aligned} &\left(u\left(t\right) - U_0, v\right)_B + \int_0^t \left(u\left(\tau\right), v\right) d\tau \\ = &\int_0^t \left(\int_0^\tau a\left(\tau - s\right) k\left(s, u\left(s\right)\right) ds, v\right)_B d\tau + \int_0^t \left(f\left(\tau\right), v\right)_B d\tau, \end{aligned} \tag{3.54}$$

takes place for all $v \in W$ and $t \in [0, T]$. Finally, differentiating (3.54) with respect to t, we get

$$\begin{aligned} &\left(\frac{d}{dt} u\left(t\right), v\right)_B + \left(u\left(t\right), v\right) \\ = &\left(\int_0^t a\left(t-s\right) k\left(s, u\left(s\right)\right) ds, v\right)_B + \left(f\left(t\right), v\right)_B, \ a.e. t \in [0, T]. \end{aligned}$$

The uniqueness may be argued in the usual manner. Indeed, exploiting an idea in $[3]$, consider u_1 and u_2 two different solutions of $(3.1) - (3.4)$ and define $w = u_1 - u_2$. Then, we have

$$\begin{aligned} &\left(\frac{d}{dt} w\left(t\right), v\right)_B + \left(w\left(t\right), v\right) \\ = &\left(\int_0^t a\left(t-s\right) \left[k\left(s, u_1\left(s\right)\right) - k\left(s, u_2\left(s\right)\right)\right] ds, v\right)_B. \end{aligned}$$

Choosing $v = w(t)$ as a test function, with the aid of Cauchy-Schwarz inequality and assumption (H_2), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t)\|_B^2 + \|w(t)\|^2 \\ \leq\ & C\int_0^t [\|k(s,u_1(s)) - k(s,u_2(s))\|_B]\,ds\,\|w(t)\|_B\,. \end{aligned} \tag{3.55}$$

Let $\xi \in [0,p]$ such that

$$\|w(\xi)\|_B = \max_{s\in[0,p]}\|w(s)\|_B\,, \tag{3.56}$$

integrating (3.55) over $(0,p)$, $0 \leq p \leq T$, using (3.56) and invoking assumption (H_2), we get

$$\int_0^p \left[\frac{1}{2}\frac{d}{dt}\|w(t)\|_B^2 + \|w(t)\|^2\right] dt \leq Cp^2\|w(\xi)\|_B^2\,,$$

consequently, with the fact that $w(0) = 0$

$$\int_0^p \left[\frac{1}{2}\frac{d}{dt}\|w(t)\|_B^2 + \|w(t)\|^2\right] dt \leq Cp\int_0^\xi \frac{d}{dt}\|w(t)\|_B^2\,dt. \tag{3.57}$$

Choosing p as constant verifying the condition

$$\exists\alpha \in \mathbb{N};\ T = \alpha p \text{ and } Cp \leq \frac{1}{2},$$

which gives, by virtue of (3.57) that

$$\int_0^p \frac{1}{2}\frac{d}{dt}\|w(t)\|_B^2\,dt + \int_0^p \|w(t)\|^2\,dt \leq \int_0^\xi \frac{1}{2}\frac{d}{dt}\|w(t)\|_B^2\,dt,$$

taking into account that $\xi \leq p$, we obtain

$$\|w(t)\| = 0, \text{ on } [0,p]\,.$$

Following the same lines as for $[0,p]$, we deduce that

$$\|w(t)\| = 0, \text{ on } [ip,(i+1)p]\,,\ i = 1,\ 2,\ 3,\ ...,$$

therefore, we derive $w(t) \equiv 0$, on $[0,T]$, then $u_1 \equiv u_2$. This achieves the proof. ■

Chapter 4

Existence and uniqueness of the solution of an evolution problem for a quasilinear hyperbolic integrodifferential equation

4.1 Statement of the problem

In this chapter, we want to study the solvability of the following integrodifferential equation

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^3 v}{\partial t \partial x^2} = \int_0^t a(t-s) k'(s,v)\, ds + g\left(t, v, \frac{\partial v}{\partial t}\right), \tag{4.1}$$

subject to the boundary integral conditions

$$\int_0^1 v(x,t)dx = E(t), \qquad \int_0^1 xv(x,t)dx = G(t), \quad t \in [0,T], \tag{4.2}$$

and starting from

$$v(x,0) = V_0(x), \qquad \frac{\partial}{\partial t}v(x,0) = W_0(x), \qquad x \in (0,1), \tag{4.3}$$

where a, k', g, V_0, W_0, E and G are sufficiently regular given functions of the indicated variables (satisfying certain conditions to be specified later) and T is a positive constant.

By the transformation

$$u(x,t) = v(x,t) - R(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{4.4}$$

where

$$R(x,t) = 6\left(2G(t) - E(t)\right)x - 2\left(3G(t) - 2E(t)\right), \tag{4.5}$$

problem (5.1) – (5.3) with inhomogeneous integral conditions (5.2) is converted to the following equivalent problem with homogeneous conditions for the new unknown function u :

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 u}{\partial t \partial x^2} = \int_0^t a(t-s)\,k(s,u)\,ds + f\left(t, u, \frac{\partial u}{\partial t}\right), \tag{4.6}$$

$$\int_0^1 u(x,t)dx = 0, \qquad \int_0^1 xu(x,t)dx = 0, \qquad t \in [0,T], \tag{4.7}$$

$$u(x,0) = \varphi(x), \qquad \frac{\partial}{\partial t}u(x,0) = \psi(x), \qquad x \in (0,1), \tag{4.8}$$

where

$$f\left(t, u, \frac{\partial u}{\partial t}\right) = g\left(t, u+R, \frac{\partial (u+R)}{\partial t}\right) - \frac{\partial^2 R(x,t)}{\partial t^2}, \tag{4.9}$$

$$\varphi(x) = V_0(x) - R(x,0), \qquad \psi(x) = W_0(x) - \frac{\partial R(x,t)}{\partial t}, \tag{4.10}$$

and

$$k\left(s, u\left(x,s\right)\right) = k'\left(s, u\left(x,s\right) + R(x,s)\right). \tag{4.11}$$

Hence, instead of looking for the function v, we seek the function u. The solution of problem $(5.1) - (5.3)$ will be given by

$$v(x,t) = u(x,t) + R(x,t). \tag{4.12}$$

This chapter is divided as follows. We begin by stating the precise assumptions of the functions involved in the posed problem and by making precise the concept of the solution. Section 2, is devoted to the construction of approximate solutions of problem $(4.6){-}(4.8)$. Then, some a priori estimates for the approximations are derived in Section 3, while the convergence of the method, uniqueness and the continuous dependence on initial data of the solution to problem under study are established in Section 4.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the usual inner product and the corresponding norm respectively in $L_2(0,1)$.

Throughout this chapter, we will make the following assumptions:

H_1- Functions $f : [0,T] \times W \times W \to L_2\left(0,1\right)$ and $a : [0,T] \to \mathbb{R}$ are Lipschitz continuous, i.e.

$$\exists l_1 \geq 0;\ \left\|f\left(t,u,v\right) - f\left(t',u',v'\right)\right\| \leq l_1\left[\left|t-t'\right| + \left\|u-u'\right\| + \left\|v-v'\right\|\right], \tag{4.13}$$

and

$$\exists l_2 \geq 0;\ \left|a\left(t\right) - a\left(t'\right)\right| \leq l_2\left|t-t'\right|, \tag{4.14}$$

for all $t, t' \in I$ and $u, v, u', v' \in W$.

H_2- Mapping $k : [0, T] \times W \to L_2(0, 1)$ is Lipschitz continuous in both variables, i.e.

$$\exists l_3 \in \mathbb{R};\ \|k(t, u) - k(t', u')\| \leq l_3 [|t - t'| + \|u - u'\|], \tag{4.15}$$

and satisfies

$$\exists l_4,\ l_5 \in \mathbb{R};\ \|k(t, u)\|_B \leq l_4 \|u\|_B + l_5, \tag{4.16}$$

for all $t,\ t' \in I,\ u,\ u' \in W$, where $l_3,\ l_4$ and l_5 are positive constants.

H_3- Functions $\varphi,\ \psi \in H^2(0, 1) \cap W,\ i.e.$

$$\varphi,\ \psi \in \left\{ v \in H^2(0, 1);\ \int_0^1 v(x)\, dx = \int_0^1 x v(x)\, dx = 0 \right\}. \tag{4.17}$$

We look for a weak solution in the following sense.

Definition 22 *A function $u : I \to L^2(0, 1)$ is called a weak solution to problem (4.6) – (4.8) if the following conditions are satisfied:*

(i) $u \in C^{0,1}(I, W)$,

(ii) *u is strongly differentiable a.e. in I with $du/dt \in L^\infty(I, W) \cap C^{0,1}(I, B_2^1(0, 1))$ and $d^2u/dt^2 \in L^\infty(I, B_2^1(0, 1))$,*

(iii) *$u(0) = \varphi$ in W and $(du/dt)(0) = \psi$ in $B_2^1(0, 1)$,*

(iv) *the identity*

$$\begin{aligned} &\left(\left(d^2u/dt^2 \right)(t), v \right)_B + ((du/dt)(t), v) + (u(t), v) \\ = &\left(\int_0^t a(t - s) k(s, u(x, s))\, ds, v \right)_B + \left(f\left(t, u(t), \frac{\partial u}{\partial t}(t) \right), v \right)_B, \end{aligned} \tag{4.18}$$

holds for all $v \in W$ and a.e. $t \in [0, T]$.

In this chapter, we will demonstrate the following main result:

Theorem 23 *Under assumptions* $(H_1)-(H_3)$, *Problem* $(4.6)-(4.8)$ *admits a unique weak solution* u, *in the sense of Definition* 23.

4.2 Construction of an approximate solution

Let n be a positive integer. To solve problem (4.6) – (4.8) by the Rothe method, we subdivide the time interval I by points $t = jh,\ j = 0, ..., n$, where $h = T/n$ is a step time. Then, we are conducted to solve successively for $j = 1, ..., n$ the following recurrent sequence of time-discretized problems:

$$\delta^2 u_j^n - \frac{\partial^2}{\partial x^2} u_j^n - \frac{\partial^2}{\partial x^2} \delta u_j^n = h_n \sum_{i=0}^{j-1} a_{j,i}^n k_i^n + f_j^n, \tag{4.19}$$

$$\int_0^1 u_j(x)dx = 0, \quad t \in [0, T], \tag{4.20}$$

$$\int_0^1 x u_j(x)dx = 0, \quad t \in [0, T], \tag{4.21}$$

starting from

$$u_0(x) = \varphi(x), \quad x \in (0, 1), \tag{4.22}$$

$$\delta u_0(x) = \psi(x), \quad x \in (0, 1), \tag{4.23}$$

where $u_j^n := u\left(t_j^n\right), \delta u_j^n := \left(u_j^n - u_{j-1}^n\right)/h_n$, $f_j^n := f\left(t_j, u_{j-1}, \delta u_{j-1}\right), a_{j,i}^n := a\left(t_j^n - t_i^n\right)$ and $k_i^n := k\left(t_i^n, u_i^n\right)$, for all $j, i = 1, ..., n$. Multiplying for all $j = 1, ..., n$, (4.19) by

$\Im_x^2 v := \int_0^x \left[\int_0^\xi v(\tau)\, d\tau\right] d\xi$ and integrating over $(0,1)$, we get

$$\begin{aligned}&\int_0^1 \delta^2 u_j^n \Im_x^2 v dx - \int_0^1 \frac{\partial^2}{\partial x^2} u_j^n \Im_x^2 v dx - \int_0^1 \frac{\partial^2}{\partial x^2} \delta u_j^n \Im_x^2 v dx \\ = \;& h_n \int_0^1 \sum_{i=0}^{j-1} a_{j,i}^n k_i^n \Im_x^2 v dx + \int_0^1 f_j^n \Im_x^2 v dx, \end{aligned} \tag{4.24}$$

carrying out some integrations by parts for each term in (4.24), with consideration to the fact that

$$\Im_1^2 v = \int_0^1 (1-\xi)\, v(\xi)\, d\xi = \int_0^1 v(\xi)\, d\xi - \int_0^1 \xi v(\xi)\, d\xi = 0,\ \forall v \in W, \tag{4.25}$$

it follows that

$$\begin{aligned}&\left(\delta^2 u_j^n, v\right)_B + \left(u_j^n, v\right) + \left(\delta u_j^n, v\right) \\ = \;& h_n \sum_{i=0}^{j-1} a_{j,i}^n \left(k_i^n, v\right)_B + \left(f_j^n, v\right)_B, \end{aligned} \tag{4.26}$$

and also

$$\begin{aligned}&\left(u_j^n, v\right)_B + h_n\left(u_j^n, v\right) + h_n^2\left(u_j^n, v\right) \\ = \;& h_n^3 \sum_{i=0}^{j-1} a_{j,i}^n \left(k_i^n, v\right)_B + h_n^2\left(f_j^n, v\right)_B \\ & + h_n\left(u_{j-1}^n, v\right) + \left(2u_{j-1}^n - u_{j-2}^n, v\right)_B. \end{aligned} \tag{4.27}$$

Let $\eta(.,.) : W \times W \to \mathbb{R}$ and $L_j(.) : W \to \mathbb{R}$ be two functions defined by

$$\eta(u,v) = \left(u_j^n, v\right)_B + h_n\left(u_j^n, v\right) + h_n^2\left(u_j^n, v\right), \tag{4.28}$$

$$\begin{aligned} L_j(v) \;=\; & h_n^3 \sum_{i=0}^{j-1} a_{j,i}^n \left(k_i^n, v\right)_B + h_n^2\left(f_j^n, v\right)_B \\ & + h_n\left(u_{j-1}^n, v\right) + \left(2u_{j-1}^n - u_{j-2}^n, v\right)_B. \end{aligned} \tag{4.29}$$

Since $\eta(.,.)$ is *continuous* and $W-$*elliptic* and $L_j(.)$ is *continuous*, then, Lax-Milgram Lemma guarantees the existence and uniqueness of u_j^n, for all $j = 1, ..., n$.

4.3 A priori estimates

Lemma 24 *There exist $C > 0$ such that, for all $n \geq 1$, the solutions u_j^n of the discretized problems* (4.19) − (4.23), $j = 1, ..., n$, *obey the estimates*

$$\left\|u_j^n\right\| \leq C, \tag{4.30}$$

$$\left\|\delta u_j^n\right\| \leq C, \tag{4.31}$$

$$\left\|\delta^2 u_j^n\right\|_B \leq C. \tag{4.32}$$

Proof. Using $v = \delta^2 u_j^n \, (\in W)$ as a test function in the difference $(4.26)_j - (4.26)_{j-1}$, we obtain

$$\begin{aligned}
&\left(\delta^2 u_j^n - \delta^2 u_{j-1}^n, \delta^2 u_j^n\right)_B + \left(\delta u_j^n, \delta u_j^n - \delta u_{j-1}^n\right) + h_n \left(\delta^2 u_j^n, \delta^2 u_j^n\right) \\
= \; & h_n \sum_{i=0}^{j-2} \left[a_{j,i}^n - a_{j-1,i}^n\right] \left(k_i^n, \delta^2 u_j^n\right)_B \\
&+ h_n a_{j,j-1}^n \left(k_{j-1}^n, \delta^2 u_j^n\right)_B + \left(f_j^n - f_{j-1}^n, \delta^2 u_j^n\right)_B ,
\end{aligned} \tag{4.33}$$

taking into account assumptions $(H_1) - (H_3)$ and the Cauchy Schwarz inequality, we get

$$\begin{aligned}
&\left\|\delta^2 u_j^n\right\|_B^2 - \left\|\delta^2 u_{j-1}^n\right\|_B^2 \\
&+ \left\|\delta u_j^n\right\|^2 - \left\|\delta u_{j-1}^n\right\|^2 + 2h_n \left\|\delta^2 u_j^n\right\|^2 \\
\leq \; & C h_n^2 \left\|\delta^2 u_j^n\right\|_B \sum_{i=0}^{j-2} \left(1 + \left\|u_i^n\right\|_B\right) \\
&+ C h_n \left(1 + \left\|u_{j-1}^n\right\|_B\right) \left\|\delta^2 u_j^n\right\|_B \\
&+ C h_n \left[1 + \left\|\delta u_{j-1}^n\right\|_B + \left\|\delta^2 u_{j-1}^n\right\|_B\right] \left\|\delta^2 u_j^n\right\|_B ,
\end{aligned} \tag{4.34}$$

hence,

$$\begin{aligned}
& \left\|\delta^2 u_j^n\right\|_B^2+\left\|\delta u_j^n\right\|^2-\left\|\delta^2 u_{j-1}^n\right\|_B^2-\left\|\delta u_{j-1}^n\right\|^2+2h_n\left\|\delta^2 u_j^n\right\|^2 \\
\leq\ & Ch_n^2\left\|\delta^2 u_j^n\right\|_B\sum_{i=0}^{j-2}\left\|u_i^n\right\|_B+Ch_n\left\|\delta^2 u_j^n\right\|_B \\
& +Ch_n\left\|u_{j-1}^n\right\|_B\left\|\delta^2 u_j^n\right\|_B \\
& +Ch_n\left\|\delta u_{j-1}^n\right\|_B^2+Ch_n\left\|\delta^2 u_j^n\right\|_B^2 \\
& +Ch_n\left\|\delta^2 u_{j-1}^n\right\|_B^2 . \qquad (4.35)
\end{aligned}$$

Noting that

$$h_n\left\|\delta u_{j-1}^n\right\|_B\geq\left\|u_{j-1}^n\right\|_B-\left\|u_{j-2}^n\right\|_B, \qquad (4.36)$$

hence

$$h_n\sum_{i=1}^{j-1}\left\|\delta u_i^n\right\|_B+\left\|\varphi\right\|_B\geq\left\|u_{j-1}^n\right\|_B . \qquad (4.37)$$

Similarly,

$$h_n\sum_{i=1}^{j-1}\left\|\delta^2 u_i^n\right\|_B+\left\|\psi\right\|_B\geq\left\|\delta u_{j-1}^n\right\|_B . \qquad (4.38)$$

Using the inequality (4.37) in (4.35), we get after some rearrangements

$$\begin{aligned}
& \left\|\delta^2 u_j^n\right\|_B^2+\left\|\delta u_j^n\right\|^2+2h_n\left\|\delta^2 u_j^n\right\|^2 \\
\leq\ & (1+Ch_n)\left[\left\|\delta u_{j-1}^n\right\|^2+\left\|\delta^2 u_{j-1}^n\right\|_B^2\right]+Ch_n\left\|\delta^2 u_j^n\right\|_B+Ch_n\left\|\delta^2 u_j^n\right\|_B^2 \\
& +Ch_n^2\left\|\delta^2 u_j^n\right\|_B\sum_{i=0}^{j-1}\left[\left\|u_i^n\right\|_B+\left\|\delta u_i^n\right\|_B\right], \qquad (4.39)
\end{aligned}$$

or, by virtue of (4.37) and (4.38)

$$\begin{aligned}
& \left\|\delta^2 u_j^n\right\|_B^2+\left\|\delta u_j^n\right\|^2+2h_n\left\|\delta^2 u_j^n\right\|^2 \\
\leq\ & (1+Ch_n)\left[\left\|\delta u_{j-1}^n\right\|^2+\left\|\delta^2 u_{j-1}^n\right\|_B^2\right]+Ch_n\left\|\delta^2 u_j^n\right\|_B+Ch_n\left\|\delta^2 u_j^n\right\|_B^2 \\
& +Ch_n^2\left\|\delta^2 u_j^n\right\|_B\sum_{i=0}^{j-1}\left[h_n\sum_{k=0}^{i}\left(\left\|\delta u_k^n\right\|_B+\left\|\delta^2 u_k^n\right\|_B\right)\right], \qquad (4.40)
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
& \left\|\delta^2 u_j^n\right\|_B^2 + \left\|\delta u_j^n\right\|^2 + 2h_n \left\|\delta^2 u_j^n\right\|^2 \\
\leq\; & (1 + Ch_n) \left[\left\|\delta u_{j-1}^n\right\|^2 + \left\|\delta^2 u_{j-1}^n\right\|_B^2\right] + Ch_n \left\|\delta^2 u_j^n\right\|_B \\
& + Ch_n^2 \sum_{i=0}^{j-1} \left(\|\delta u_i^n\|_B^2 + \left\|\delta^2 u_i^n\right\|_B^2\right) + Ch_n \left\|\delta^2 u_j^n\right\|_B^2, \qquad (4.41)
\end{aligned}$$

then, with the fact that

$$\begin{aligned}
Ch_n \left\|\delta^2 u_j^n\right\|_B &= \left(C\sqrt{h_n}\right)\left(\sqrt{h_n}\left\|\delta^2 u_j^n\right\|_B\right) \\
&\leq \frac{1}{4} C^2 h_n + h_n \left\|\delta^2 u_j^n\right\|_B^2, \qquad (4.42)
\end{aligned}$$

we obtain

$$\begin{aligned}
& (1 - Ch_n) \left[\left\|\delta^2 u_j^n\right\|_B^2 + \left\|\delta u_j^n\right\|^2\right] \\
\leq\; & (1 + Ch_n) \left[\left\|\delta^2 u_{j-1}^n\right\|_B^2 + \left\|\delta u_{j-1}^n\right\|^2\right] \\
& + Ch_n^2 \sum_{i=0}^{j-1} \left(\left\|\delta^2 u_i^n\right\|_B^2 + \|\delta u_i^n\|^2\right) + Ch_n. \qquad (4.43)
\end{aligned}$$

Hence

$$\begin{aligned}
& (1 - Ch_n) \left[\left\|\delta^2 u_j^n\right\|_B^2 + \left\|\delta u_j^n\right\|^2\right] \\
\leq\; & (1 + Ch_n) \left[\left\|\delta u_{j-1}^n\right\|^2 + \left\|\delta^2 u_{j-1}^n\right\|_B^2\right] \\
& + (\gamma + Ch_n)\, Ch_n^2 \sum_{i=0}^{j-1} \left(\|\delta u_i^n\|^2 + \left\|\delta^2 u_i^n\right\|_B^2\right) + (\gamma + Ch_n)\, Ch_n, \qquad (4.44)
\end{aligned}$$

with

$$\gamma := 2\exp(TC) \geq (1 + Ch_n)^p, \qquad \forall n \geq n_0, \qquad (4.45)$$

for all $p \leq j$. Now, let's suppose that

$$\begin{aligned}
& (1 - Ch_n)^p \left[\left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \right] \\
\leq \ & (1 + Ch_n)^p \left[\left\| \delta u_{j-p}^n \right\|^2 + \left\| \delta^2 u_{j-p}^n \right\|_B^2 \right] \\
& + (\gamma p + Ch_n) \, C h_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta u_i^n \right\|^2 + \left\| \delta^2 u_i^n \right\|_B^2 \right) \\
& + (\gamma p + Ch_n) \, Ch_n, \qquad (4.46)
\end{aligned}$$

multiplying the last inequality by $(1 - ch_n)$ and using (4.43), we get

$$\begin{aligned}
& (1 - Ch_n)^{p+1} \left[\left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \right] \\
\leq \ & (1 + Ch_n)^p \Big[(1 + Ch_n) \left[\left\| \delta u_{j-(p+1)}^n \right\|^2 + \left\| \delta^2 u_{j-(p+1)}^n \right\|_B^2 \right] \\
& + C h_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta u_i^n \right\|^2 + \left\| \delta^2 u_i^n \right\|_B^2 \right) + Ch_n \Big] \\
& + (\gamma p + Ch_n) \, C h_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta u_i^n \right\|^2 + \left\| \delta^2 u_i^n \right\|_B^2 \right) \\
& + (\gamma p + Ch_n) \, Ch_n, \qquad (4.47)
\end{aligned}$$

using (4.45) we obtain

$$\begin{aligned}
& (1 - Ch_n)^{p+1} \left[\left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \right] \\
\leq \ & (1 + Ch_n)^{p+1} \left[\left\| \delta u_{j-(p+1)}^n \right\|^2 + \left\| \delta^2 u_{j-(p+1)}^n \right\|_B^2 \right] \\
& + (\gamma (p+1) + Ch_n) \, C h_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta u_i^n \right\|^2 + \left\| \delta^2 u_i^n \right\|_B^2 \right) \\
& + (\gamma (p+1) + Ch_n) \, Ch_n, \qquad (4.48)
\end{aligned}$$

then, the following inequality is verified for all j.

$$\begin{aligned}
& (1 - Ch_n)^j \left[\left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \right] \\
\leq \; & (1 + Ch_n)^j \left[\left\| \delta u_0^n \right\|^2 + \left\| \delta^2 u_0^n \right\|_B^2 \right] \\
& + (\gamma j + Ch_n)\, Ch_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta u_i^n \right\|^2 + \left\| \delta^2 u_i^n \right\|_B^2 \right) \\
& + (\gamma j + Ch_n)\, Ch_n. \qquad (4.49)
\end{aligned}$$

Hence

$$\begin{aligned}
& \left(1 - CT\frac{1}{n}\right)^n \left[\left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \right] \\
\leq \; & \left(1 + CT\frac{1}{n}\right)^n \left[\left\| \delta^2 u_0^n \right\|_B^2 + \left\| \delta u_0^n \right\|^2 \right] \\
& + (\gamma j + Ch_n)\, Ch_n^2 \sum_{i=0}^{j-1} \left(\left\| \delta^2 u_i^n \right\|_B^2 + \left\| \delta u_i^n \right\|^2 \right) \\
& + (\gamma j + Ch_n)\, Ch_n. \qquad (4.50)
\end{aligned}$$

This shows that

$$\begin{aligned}
& \left\| \delta^2 u_j^n \right\|_B^2 + \left\| \delta u_j^n \right\|^2 \\
\leq \; & C \left[\left\| \delta^2 u_0^n \right\|_B^2 + \left\| \delta u_0^n \right\|^2 + 1 \right] \\
& + Ch_n \sum_{i=0}^{j-1} \left(\left\| \delta^2 u_i^n \right\|_B^2 + \left\| \delta u_i^n \right\|^2 \right). \qquad (4.51)
\end{aligned}$$

Applying the Gronwall's Lemma in (4.51) and taking into account the fact that

$$\begin{aligned}
h_n \sum_{i=1}^{j} \left\| \delta u_i^n \right\| \; & \geq \; h_n \left\| \delta u_j^n + \delta u_{j-1}^n + \ldots + \delta u_1^n \right\| \\
& \geq \; \left\| u_j^n \right\| - \left\| \varphi \right\|, \qquad (4.52)
\end{aligned}$$

we get the desired result. So, the proof is complete. ■

4.4 Convergence, existence and uniqueness

Let us define, in the interval $I=[0,T]$, the abstract functions

$$u^n(t) = u^n_{j-1}+\delta u^n_j(t-t^n_{j-1}), \qquad \text{in } I^n_j, \tag{4.53}$$

$$\widetilde{u}_n(t) = \begin{cases} u^n_{1,} & \text{for } t=0, \\ u^n_j, & \text{in } \tilde{I}^n_j=\left(t^n_{j-1},t^n_j\right], \end{cases} \tag{4.54}$$

$$U_n(t) = \delta u^n_{j-1}+\delta^2 u^n_j(t-t^n_{j-1}), \qquad \text{in } I^n_j, \tag{4.55}$$

$$\widetilde{U}_n(t) = \begin{cases} \delta u^n_1, & \text{for } t=0, \\ \delta u^n_j, & \text{in } \tilde{I}^n_j, \end{cases} \tag{4.56}$$

$$Y_n(t) = \begin{cases} \delta^2 u^n_1, & \text{for } t=0, \\ \delta^2 u^n_j, & \text{in } \tilde{I}^n_j, \end{cases} \tag{4.57}$$

$$\widehat{u}_n(t) = \begin{cases} u^n_1, & \text{for } t=0, \\ u^n_{j-1}, & \text{in } \tilde{I}^n_j, \end{cases} \tag{4.58}$$

$$\widehat{U}_n(t) = \begin{cases} \delta u^n_1, & \text{for } t=0, \\ \delta u^n_{j-1}, & \text{in } \tilde{I}^n_j, \end{cases} \tag{4.59}$$

$$\widetilde{f}_n(t) = f\left(t^n_j,\widehat{u}_n(t),\widehat{U}_n(t)\right), \qquad \text{in } \tilde{I}^n_j. \tag{4.60}$$

$$\widetilde{k}(t,\widetilde{u}_n(t)) = \begin{cases} 0, & \text{for } t=0, \\ h_n\sum_{i=0}^{j-1} a^n_{j,i}k^n_i, & \text{in } \tilde{I}^n_j=\left(t^n_{j-1},t^n_j\right]. \end{cases} \tag{4.61}$$

Lemma 25 *There exist $C>0$ such that the estimates*

$$\|u^n(t)\|\le C,\ \|\tilde{u}_n(t)\|\le C,\ \|U_n(t)\|\le C,\ \left\|\widetilde{U}_n(t)\right\|\le C,\ \|Y_n(t)\|_B\le C, \tag{4.62}$$

$$\left\|\frac{du^n}{dt}(t)\right\|\le C, \quad \left\|U_n(t)-\frac{du_n}{dt}(t)\right\|_B\le Ch_n, \tag{4.63}$$

$$\left\|\widetilde{k}(t,\widetilde{u}_n(t))\right\|_B\le C, \tag{4.64}$$

$$\left\|\widetilde{U}_n(t) - U_n(t)\right\|_B \leq Ch_n, \qquad \|\widetilde{u}_n(t) - u^n(t)\| \leq Ch_n, \tag{4.65}$$

$$\left\|\widehat{U}_n(t) - U_n(t)\right\|_B \leq Ch_n, \qquad \|\widehat{u}_n(t) - u^n(t)\| \leq Ch_n, \tag{4.66}$$

$$\left\|\widetilde{f}^n(t)\right\| \leq C, \tag{4.67}$$

$$\begin{aligned} \left\|\widetilde{f}^n(t) - \widetilde{f}^m(t)\right\|_B &\leq C(h_n + h_m) + C\|u_n(t) - u_m(t)\|_B \\ &\quad + C\|U_n(t) - U_m(t)\|_B, \end{aligned} \tag{4.68}$$

and

$$\left\|\widetilde{k}(t, \widetilde{u}_n(t)) - \widetilde{k}(t, \widetilde{u}_m(t))\right\|_B \leq Ch_n + C \sup_{0\leq s\leq t} \|u^n(s) - u^m(s)\|_B, \tag{4.69}$$

hold for all $t \in I$ *and* $n \geq n_0$.

Proof. Having in mind estimates $(4.30) - (4.32)$ and assumptions $(H_1) - (H_3)$, estimates $(4.62) - (4.66)$ follow immediately. Whereas, from the inequality $\left\|\widetilde{f}^n(t)\right\| \leq \left\|f\left(t_j^n, \widehat{u}_n(t), \widehat{U}_n(t)\right) - f\left(t_j^n, 0, 0\right)\right\| + \left\|f\left(t_j^n, 0, 0\right)\right\|$, it follows by means of (4.30), (4.31) and (H_1) that

$$\begin{aligned} \left\|\widetilde{f}^n(t)\right\| &\leq C\left\|u_{j-1}^n\right\| + C\left\|\delta u_{j-1}^n\right\| + \max_{t\in I}\|f(t,0,0)\| \\ &\leq C. \end{aligned} \tag{4.70}$$

For estimate (4.68), let t be arbitrary but fixed in $(0, T]$, then there exist two integers p and q corresponding to the subdivision of $(0, T]$ into n and m subintervals, respectively, such that $t \in (t_{p-1}^n, t_p^n] \cap (t_{q-1}^m, t_q^m]$. According to (4.66) and assumption (H_1)

we get

$$\left\|\widetilde{f}^n(t)-\widetilde{f}^m(t)\right\|_B = \left\|f\left(t_p^n,\widehat{u}_n(t),\widehat{U}_n(t)\right)-f\left(t_q^m,\widehat{u}_m(t),\widehat{U}_m(t)\right)\right\|_B$$
$$\leq C(h_n+h_m)+C\|u_n(t)-u_m(t)\|_B$$
$$+C\|U_n(t)-U_m(t)\|_B. \tag{4.71}$$

For the last inequality, we have

$$\left\|\widetilde{k}(t,\widetilde{u}_n(t))-\widetilde{k}(t,\widetilde{u}_m(t))\right\|_B=\left\|h_n\sum_{i=0}^{p-1}a_{p,i}^n k_i^n-h_m\sum_{i=0}^{q-1}a_{q,i}^m k_i^m\right\|_B. \tag{4.72}$$

Let l be an arbitrary positive integer such that $l=nm$, noting that

$$\left\|\widetilde{k}(t,\widetilde{u}_n(t))-\widetilde{k}(t,\widetilde{u}_m(t))\right\|_B$$
$$\leq \left\|\widetilde{k}(t,\widetilde{u}_l(t))-\widetilde{k}(t,\widetilde{u}_n(t))\right\|_B+\left\|\widetilde{k}(t,\widetilde{u}_l(t))-\widetilde{k}(t,\widetilde{u}_m(t))\right\|_B, \tag{4.73}$$

hence, to establish that

$$\left\|\widetilde{k}(t,\widetilde{u}_n(t))-\widetilde{k}(t,\widetilde{u}_m(t))\right\|_B\leq Ch_n+C\sup_{0\leq s\leq t}\|u^n(s)-u^m(s)\|_B, \tag{4.74}$$

we can suppose that there exist $\beta\in\mathbb{N}^*$ such that

$$m=\beta n. \tag{4.75}$$

On the other hand, let $t_1,\ t_2\in\left(t_{j-1}^n,t_j^n\right]$ such that $t_1\in\left(t_{\lambda-1}^m,t_\lambda^m\right],\ t_2\in\left(t_{\mu-1}^m,t_\mu^m\right]$ and $\lambda\leq\mu$, using (4.61) we write

$$\left\|\widetilde{k}(t_1,\widetilde{u}_m(t_1))-\widetilde{k}(t_2,\widetilde{u}_m(t_2))\right\|_B=\left\|h_m\sum_{i=0}^{\lambda-1}a_{\lambda,i}^m k_i^m-h_m\sum_{i=0}^{\mu-1}a_{\mu,i}^m k_i^m\right\|_B, \tag{4.76}$$

where

$$\begin{aligned}&\left\|\widetilde{k}\left(t_1,\widetilde{u}_m(t_1)\right)-\widetilde{k}\left(t_2,\widetilde{u}_m(t_2)\right)\right\|_B\\ \leq\ &\left\|h_m\sum_{i=0}^{\lambda-1}a^m_{\lambda,i}k^m_i-h_m\sum_{i=0}^{\lambda-1}a^m_{\mu,i}k^m_i\right\|_B+\left\|h_m\sum_{i=\lambda}^{\mu-1}a^m_{\mu,i}k^m_i\right\|_B\\ \leq\ &h_m\sum_{i=0}^{\lambda-1}\left|a^m_{\lambda,i}-a^m_{\mu,i}\right|\left\|k^m_i\right\|_B+h_m\sum_{i=\lambda}^{\mu-1}\left|a^m_{\mu,i}\right|\left\|k^m_i\right\|_B,\end{aligned}\tag{4.77}$$

then, by taking into account assumptions $(H_1)-(H_3)$, (4.30) and (4.75), we derive

$$\begin{aligned}&\left\|\widetilde{k}\left(t_1,\widetilde{u}_m(t_1)\right)-\widetilde{k}\left(t_2,\widetilde{u}_m(t_2)\right)\right\|_B\\ \leq\ &Ch_m\sum_{i=0}^{\lambda-1}\left|t^m_\lambda-t^m_\mu\right|+h_m\sum_{i=\lambda}^{\mu-1}C,\end{aligned}\tag{4.78}$$

and consequently, by virtue of assumption (H_2) and estimate (4.78), having in mind that $\left(t^m_{\lambda-1},t^m_\lambda\right]\cup\left(t^m_{\mu-1},t^m_\mu\right]\subset\left(t^n_{j-1},t^n_j\right]$, and $\lambda h_m\leq\mu h_m\leq T$, we get

$$\left\|\widetilde{k}\left(t_1,\widetilde{u}_m(t_1)\right)-\widetilde{k}\left(t_2,\widetilde{u}_m(t_2)\right)\right\|_B\leq Ch_n.\tag{4.79}$$

Therefore, (4.75) and (4.79) enables us to suppose that

$$\exists\beta\in\mathbb{N}^*;\ m=\beta n\text{ and }t^n_p=t^m_q.\tag{4.80}$$

From which, identity (4.72) becomes

$$\begin{aligned}&\left\|\widetilde{k}\left(t,\widetilde{u}_n(t)\right)-\widetilde{k}\left(t,\widetilde{u}_m(t)\right)\right\|_B\\ =\ &\left\|\beta h_m\sum_{j=0}^{p-1}a^n_{p,j}k^n_j-h_m\sum_{j=0}^{p-1}\left(\sum_{i=j\beta}^{\beta(j+1)-1}a^m_{q,i}k^m_i\right)\right\|_B,\end{aligned}\tag{4.81}$$

that is,

$$\left\|\widetilde{k}\left(t,\widetilde{u}_n(t)\right)-\widetilde{k}\left(t,\widetilde{u}_m(t)\right)\right\|_B=h_m\left\|\sum_{j=0}^{p-1}\left[\sum_{i=j\beta}^{\beta(j+1)-1}\left(a^n_{p,j}k^n_j-a^m_{q,i}k^m_i\right)\right]\right\|_B.\tag{4.82}$$

Taking into account assumption (H_1) and the fact that $t_p^n = t_q^m$, we infer that

$$\begin{aligned}
\left|a_{p,j}^n - a_{q,i}^m\right| &\leq C\left|t_p^n - t_j^n - t_q^m + t_i^m\right| \\
&\leq C\left|t_i^m - t_j^n\right| \\
&\leq Ch_n, \qquad (4.83)
\end{aligned}$$

we suppose that $a_{p,j}^n \geq a_{q,i}^m$, thus, there exists $\varepsilon_n \in [0, Ch_n]$ such that

$$a_{p,j}^n = a_{q,i}^m + Ch_n - \varepsilon_n. \qquad (4.84)$$

Performing the substitution $a_{p,j}^n = a_{q,i}^m + Ch_n - \varepsilon_n$ in the identity (4.82), we get

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
&\leq h_m \sum_{j=0}^{p-1}\left[\sum_{i=j\beta}^{\beta(j+1)-1}\left\|\left[a_{q,i}^m + Ch_n - \varepsilon_n\right]k_j^n - a_{q,i}^m k_i^m\right\|_B\right], \qquad (4.85)
\end{aligned}$$

whence

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
&\leq h_m \sum_{j=0}^{p-1}\left[\sum_{i=j\beta}^{\beta(j+1)-1}\left\|\left(Ch_n - \varepsilon_n\right)k_j^n\right\|_B + \left|a_{q,i}^m\right|\left\|k_j^n - k_i^m\right\|_B\right]. \qquad (4.86)
\end{aligned}$$

Therefore, recalling assumptions (H_1), (H_2), using estimate (4.86) and having in mind that $\varepsilon_n \in [0, Ch_n]$, we estimate

$$\begin{aligned}
&\left\|\widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right)\right\|_B \\
&\leq h_m \sum_{j=0}^{p-1}\left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C\left(h_n + \left\|u_j^n - u_i^m\right\|_B\right)\right], \qquad (4.87)
\end{aligned}$$

this consists the fact that $a^n_{p,j} \geq a^m_{q,i}$. If not, we follows the same lines as above, from where, we derive

$$\begin{aligned}
& \left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right) \right\|_B \\
\leq \; & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C \left(h_n + \left\| u^n_j - u^n(s) \right\|_B \right.\right. \\
& \left.\left. + \left\| u^n(s) - u^m(s) \right\|_B + \left\| u^m(s) - u^m_i \right\|_B \right) \right], \\
\leq \; & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C \left(h_n + \left\| \widetilde{u}_n(s) - u^n(s) \right\|_B \right.\right. \\
& \left.\left. + \left\| u^n(s) - u^m(s) \right\|_B + \left\| u^m(s) - \widetilde{u}_m(s) \right\|_B \right) \right], \qquad (4.88)
\end{aligned}$$

holds for all $s \in \left(t^m_i, t^m_{i+1}\right]$. Hence, we take the supremum with respect to s from 0 to t in the right-hand side, invoking the fact that $s \in \left(t^m_i, t^m_{i+1}\right] \subset \left(t^n_{j-1}, t^n_j\right]$ and estimate (4.65), we obtain

$$\begin{aligned}
& \left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right) \right\|_B \\
\leq \; & h_m \sum_{j=0}^{p-1} \left[\sum_{i=j\beta}^{\beta(j+1)-1} Ch_n + C \sup_{0 \leq s \leq t} \left\| u^n(s) - u^m(s) \right\|_B \right], \qquad (4.89)
\end{aligned}$$

which implies that

$$\begin{aligned}
& \left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right) \right\|_B \\
\leq \; & h_m \sum_{i=0}^{q-1} \left[Ch_n + C \sup_{0 \leq s \leq t} \left\| u^n(s) - u^m(s) \right\|_B \right], \qquad (4.90)
\end{aligned}$$

and finally,

$$\left\| \widetilde{k}\left(t, \widetilde{u}_n(t)\right) - \widetilde{k}\left(t, \widetilde{u}_m(t)\right) \right\|_B \leq Ch_n + C \sup_{0 \leq s \leq t} \left\| u^n(s) - u^m(s) \right\|_B, \qquad (4.91)$$

hence the proof is complete. ■

Proposition 26 *There exists a function u such that*

(i) $u \in C^{0,1}(I,W)$, $du/dt \in L^\infty(I,W) \cap C^{0,1}(I,B_2^1(0,1))$ *and* $d^2u/dt^2 \in L^\infty(I,B_2^1(0,1))$,

(ii) $u^n \to u$ *in* $C(I,W)$,

(iii) $\widetilde{u}_n(t) \rightharpoonup u(t)$, *in* W *for all* $t \in I$,

(iv) $U_n \to du/dt$ *in* $C(I,B(0,1))$,

(v) $\widetilde{U}_n(t) \rightharpoonup du/dt$ *in* W *for all* $t \in I$,

(vi) $du^n/dt \rightharpoonup du/dt$ *in* $L^2(I,W)$,

(vii) $(d/dt)\,U_n \rightharpoonup d^2u/dt^2$ *in* $L^2(I,B(0,1))$.

Moreover, the error estimates

$$\|u^n - u\|_{C(I,W)} \le C\sqrt{h_n}, \tag{4.92}$$

$$\|U^n - du/dt\|_{C(I,B_2^1(0,1))} \le C\sqrt{h_n}, \tag{4.93}$$

take splace for all $n \ge n_0$.

Proof. By virtue of $(4.54)-(4.61)$, the variational equation (4.26) may be rewritten in the form

$$\begin{aligned} &(Y_n(t), v)_B + (\widetilde{u}_n(t), v) + \left(\widetilde{U}_n(t), v\right) \\ = \; &\left(\widetilde{k}(t, \widetilde{u}_n(t)), v\right)_B + \left(\widetilde{f}^n(t), v\right)_B. \end{aligned} \tag{4.94}$$

Testing the difference $(4.94)_n - (4.94)_m$ with $v = \widetilde{U}_n(t) - \widetilde{U}_m(t)$, taking into account the fact that

$$\begin{aligned} &\left(u^n(t) - u^m(t), \widetilde{U}_n(t) - \widetilde{U}_m(t)\right) \\ = \; &\frac{1}{2}\frac{d}{dt}\|u^n(t) - u^m(t)\|^2, \; a.e. \; t \in I, \end{aligned} \tag{4.95}$$

and

$$\begin{aligned}&\left(U_n(t)-U_m(t),Y_n(t)-Y_m(t)\right)\\ =\;&\frac{1}{2}\frac{d}{dt}\left\|U_n(t)-U_m(t)\right\|^2,\ a.e.\ t\in I,\end{aligned}\tag{4.96}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|U^n(t)-U^m(t)\right\|_B^2+\frac{1}{2}\frac{d}{dt}\left\|u^n(t)-u^m(t)\right\|^2\\ \leq\;&\left\|u^m(t)-\widetilde{u}_m(t)\right\|\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|+\left\|\widetilde{u}_n(t)-u^n(t)\right\|\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|+\\ &\left\|Y_n(t)-Y_m(t)\right\|_B\left\|\widetilde{U}_n(t)-U^n(t)\right\|_B\\ &+\left\|Y_n(t)-Y_m(t)\right\|_B\left\|U^m(t)-\widetilde{U}_m(t)\right\|_B\\ &+\left\|\widetilde{k}(t,\widetilde{u}_n(t))-\widetilde{k}(t,\widetilde{u}_m(t))\right\|_B\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|_B\\ &+\left\|\widetilde{f}^n(t)-\widetilde{f}^m(t)\right\|_B\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|_B.\end{aligned}\tag{4.97}$$

Or, by virtue of Lemma 4, we can write

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|U_n(t)-U_m(t)\right\|_B^2+\frac{1}{2}\frac{d}{dt}\left\|u^n(t)-u^m(t)\right\|^2\\ \leq\;&C\left(h_m+h_n\right)+C\sup_{0\leq s\leq t}\left\|u^n(s)-u^m(s)\right\|\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|_B\\ &+C\left\|U_n(s)-U_m(s)\right\|_B\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|_B.\end{aligned}\tag{4.98}$$

Noting that

$$\begin{aligned}\left\|\widetilde{U}_n(t)-\widetilde{U}_m(t)\right\|_B&\leq\left\|\widetilde{U}_n(t)-U_n(t)\right\|_B+\left\|U_n(t)-U_m(t)\right\|_B+\left\|U_m(t)-\widetilde{U}_m(t)\right\|_B\\ &\leq C\left(h_m+h_n\right)+\left\|U_n(t)-U_m(t)\right\|_B,\end{aligned}\tag{4.99}$$

using (4.99) in (4.98) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|U^{n}(t)-U^{m}(t)\right\|_{B}^{2}+\frac{1}{2}\frac{d}{dt}\left\|u^{n}(t)-u^{m}(t)\right\|^{2}\\
\leq\ & C\left(h_{m}+h_{n}\right)+C\sup_{0\leq s\leq t}\left\|u^{n}(s)-u^{m}(s)\right\|^{2}\\
&+C\left\|U_{n}(t)-U_{m}(t)\right\|_{B}\left\|\widetilde{U}_{n}(t)-\widetilde{U}_{m}(t)\right\|_{B},
\end{aligned}\tag{4.100}$$

recalling (4.99) in this last, it follows that

$$\begin{aligned}
&\frac{d}{dt}\left\|U_{n}(t)-U_{m}(t)\right\|_{B}^{2}+\frac{d}{dt}\left\|u^{n}(t)-u^{m}(t)\right\|^{2}\\
\leq\ & C\left(h_{m}+h_{n}\right)+C\left[\left\|U_{n}(t)-U_{m}(t)\right\|_{B}^{2}+\sup_{0\leq s\leq t}\left\|u^{n}(s)-u^{m}(s)\right\|^{2}\right].
\end{aligned}\tag{4.101}$$

Integrating this inequality with respect to time from 0 to t and invoking the fact that $u^n(0)=u^m(0)$ and $U_n(0)=U_m(0)$, we get

$$\begin{aligned}
&\left\|U_{n}(t)-U_{m}(t)\right\|_{B}^{2}+\left\|u^{n}(t)-u^{m}(t)\right\|^{2}\\
&\leq C\left(h_{m}+h_{n}\right)+C\int_{0}^{t}\left[\left\|U_{n}(s)-U_{m}(s)\right\|_{B}^{2}+\sup_{0\leq\tau\leq s}\left\|u^{n}(\tau)-u^{m}(\tau)\right\|^{2}\right]ds,
\end{aligned}\tag{4.102}$$

from which we deduce that

$$\begin{aligned}
&\left\|U_{n}(t)-U_{m}(t)\right\|_{B}^{2}+\sup_{0\leq s\leq t}\left\|u^{n}(s)-u^{m}(s)\right\|^{2}\\
&\leq C\left(h_{m}+h_{n}\right)+C\int_{0}^{t}\left[\left\|U_{n}(s)-U_{m}(s)\right\|_{B}^{2}+\sup_{0\leq\tau\leq s}\left\|u^{n}(\tau)-u^{m}(\tau)\right\|^{2}\right]ds.
\end{aligned}\tag{4.103}$$

By virtue of Gronwall's Lemma and the above inequality, we obtain

$$\sup_{0\leq s\leq T}\left\|U_{n}(s)-U_{m}(s)\right\|_{B}^{2}+\sup_{0\leq s\leq T}\left\|u^{n}(s)-u^{m}(s)\right\|^{2}\leq C\left(h_{m}+h_{n}\right),\tag{4.104}$$

takes place for all $n,\ m\in\mathbb{N}^*$. This implies that $\left(U_n(t)\right)_n,\ \left(u^n(t)\right)_n$ are a Cauchy sequences in the Banach spaces $C(I,B_2^1(0,1))$ and $C(I,W)$, respectively. Hence, there exist two functions $u\in C(I,W)$ and $U\in C(I,B_2^1(0,1))$ such that

$$\left(u^{n}\right)_{n}\rightarrow u\qquad\text{in } C(I,W),\tag{4.105}$$

$$(U_n)_n \to U \qquad \text{in } C(I, B_2^1(0,1)). \tag{4.106}$$

Now, on the basis of the estimates (4.30)$-$(4.32) and the convergence results (4.105), (4.106), Lemma 6 enables us to state the following assertions

(i) $\quad u \in C^{0,1}(I, W)$,

(ii) $\quad u$ is strongly differentiable a.e.in I and $du/dt \in L^\infty(I, W)$,

(iii) $\quad \widetilde{u}_n(t) \rightharpoonup u(t)$, for all $t \in I$,

(iv) $\quad du^n/dt \rightharpoonup du/dt$ in $L^2(I, W)$,

as well as

(1) $\quad U \in C^{0,1}(I, B(0,1))$,

(2) $\quad U$ is strongly differentiable a.e.in I and $dU/dt \in L^\infty(I, B(0,1))$,

(3) $\quad \widetilde{U}_n(t) \rightharpoonup U$ in W for all $t \in I$,

(4) $\quad (d/dt)\, U_n \rightharpoonup dU/dt$ in $L^2(I, B(0,1))$.

On the other hand, by virtue of (4.63), (4.106) and the convergence property (iv) stated above, we get

$$\begin{aligned}(U(t) - du(t)/dt, v)_B &= \lim_n (U(t) - U_n(t), v)_B \\ &\quad + \lim_n \left(U_n(t) - \frac{d}{dt} u^n(t), v \right)_B + \lim_n \left(\frac{d}{dt} u^n(t) - du(t)/dt, v \right)_B \\ &\longrightarrow {}_{n \longrightarrow \infty} 0,\end{aligned} \tag{4.107}$$

from which, we deduce that $U = du/dt$ and consequently $dU/dt = d^2u/dt^2$. Finally, letting $m \to \infty$ in (4.104), we obtain the desired error estimate. So, the proof is complete. ■

Theorem 27 *Under assumptions* $(H_1) - (H_3)$, *problem* $(4.6) - (4.8)$ *admits a unique* *weak solution, namely the limit function u from Proposition 26, in the sense of Definition* 22.

Proof. Note that in light of what precedes, the limit function u satisfies all the conditions (i), (ii), (iii) and (iv) of Definition 22. It remains to see that u obeys the integral identity (4.18). For this, integrating (4.94) over $(0,t)$, we get

$$\begin{aligned} & (U_n(t)-\psi,v)_B+(u^n(t)-\varphi,v)+\int_0^t(\widetilde{u}_n(\tau),v)\,d\tau \\ = & \int_0^t\left(\widetilde{k}\left(\tau,\widetilde{u}_n(\tau)\right),v\right)_B d\tau+\int_0^t\left(\widetilde{f}^n(\tau),v\right)_B d\tau, \end{aligned} \tag{4.108}$$

consequently, after some rearrangement

$$\begin{aligned} & (U_n(t)-\psi,v)_B+(u^n(t)-\varphi,v)+\int_0^t(\widetilde{u}_n(\tau),v)\,d\tau \\ = & \int_0^t\left(\int_0^\tau a(\tau-s)\,k(s,u(x,s))\,ds,v\right)_B d\tau+\int_0^t\left(f\left(\tau,u(\tau),\frac{\partial u}{\partial t}(\tau)\right),v\right)_B d\tau \\ & +\int_0^t\left(\widetilde{k}\left(\tau,\widetilde{u}_n(\tau)\right)-\int_0^\tau a(\tau-s)\,k(s,u(x,s))\,ds,v\right)_B d\tau \\ & +\int_0^t\left(\widetilde{f}^n(\tau)-f\left(\tau,u(\tau),\frac{\partial u}{\partial t}(\tau)\right),v\right)_B d\tau. \end{aligned} \tag{4.109}$$

Using (4.58) and (4.61) we have for all $t\in\left(t_{j-1}^n,t_j^n\right]$

$$\begin{aligned} & \widetilde{k}\left(t,\widetilde{u}_n(t)\right)-\int_0^t a(t-s)\,k(s,u(.,s))\,ds \\ = & h_n\sum_{i=0}^{j-1}a_{j,i}^n k_i^n-\int_0^t a(t-s)\,k(s,u(.,s))\,ds \\ = & \int_0^{t_j^n}a\left(t_j^n-\hat{s}_n(s)\right)k\left(\hat{s}_n(s),\hat{u}_n(s)\right)ds-\int_0^t a(t-s)\,k(s,u(.,s))\,ds \\ = & \int_0^{t_j^n}a\left(t_j^n-\hat{s}_n(s)\right)k\left(\hat{s}_n(s),\hat{u}_n(s)\right)ds-\int_0^{t_j^n}a(t-s)\,k(s,u(.,s))\,ds \\ & +\int_t^{t_j^n}a(t-s)\,k(s,u(.,s))\,ds \\ = & \int_0^{t_j^n}\left[a\left(t_j^n-\hat{s}_n(s)\right)k\left(\hat{s}_n(s),\hat{u}_n(s)\right)-a(t-s)\,k(s,u(.,s))\right]ds \\ & +\int_t^{t_j^n}a(t-s)\,k(s,u(.,s))\,ds, \end{aligned} \tag{4.110}$$

where $\hat{s}_n : I \to I$ denotes the function

$$\hat{s}_n(t) = \begin{cases} 0, & \text{for } t = 0, \\ t_{j-1}^n, & \text{in } \tilde{I}_j^n. \end{cases} \tag{4.111}$$

Thus, estimating the term

$$\left| a\left(t_j^n - \hat{s}_n(s)\right) - a(t-s) \right|, \tag{4.112}$$

owing to assumption (H_1), taking into account that $t \in I_j^n$ we obtain

$$\begin{aligned} \left| a\left(t_j^n - \hat{s}_n(s)\right) - a(t-s) \right| &\leq C\left| t_j^n - \hat{s}_n(s) - t + s \right| \\ &\leq C\left(\left| t_j^n - t \right| + \left| s - \hat{s}_n(s) \right| \right) \\ &\leq Ch_n, \end{aligned} \tag{4.113}$$

which clearly follows that

$$\exists \varepsilon_n \geq 0; \ \left| a\left(t_j^n - \hat{s}_n(s)\right) - a(t-s) \right| + \varepsilon_n = Ch_n. \tag{4.114}$$

For the term

$$k\left(\hat{s}_n(s), \hat{u}_n(s)\right) - k\left(s, u(., s)\right), \tag{4.115}$$

we have, using assumption (H_2)

$$\begin{aligned} &\left\| k\left(\hat{s}_n(s), \hat{u}_n(s)\right) - k\left(s, u(., s)\right) \right\|_B \\ \leq\ & C\left[\left| \hat{s}_n(s) - s \right| + \left\| \hat{u}_n(s) - u(., s) \right\|_B \right], \end{aligned} \tag{4.116}$$

or, by (4.66), (4.92), (4.111) and the fact that

$$\left\| \hat{u}_n(s) - u(., s) \right\|_B \leq \left\| \hat{u}_n(s) - u^n(s) \right\|_B + \left\| u^n(s) - u(., s) \right\|_B, \tag{4.117}$$

we get

$$\left\| k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) - k\left(s, u\left(., s\right)\right)\right\|_B \leq C\sqrt{h_n}. \tag{4.118}$$

Taking into account estimate (4.84) and assumptions (H_1), (H_2) we estimate

$$\begin{aligned}
& \left\| a\left(t_j^n - \hat{s}_n\left(s\right)\right) k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) - a\left(t-s\right) k\left(s, u\left(., s\right)\right)\right\|_B \\
\leq\ & \left\| a\left(t_j^n - \hat{s}_n\left(s\right)\right) k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) \right. \\
& \left. - \left(a\left(t_j^n - \hat{s}_n\left(s\right)\right) + Ch_n - \varepsilon_n\right) k\left(s, u\left(., s\right)\right)\right\|_B \\
\leq\ & \left\| \left(Ch_n - \varepsilon_n\right) k\left(s, u\left(., s\right)\right)\right\|_B \\
& + \left\| a\left(t_j^n - \hat{s}_n\left(s\right)\right) k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) - a\left(t_j^n - \hat{s}_n\left(s\right)\right) k\left(s, u\left(., s\right)\right)\right\|_B \\
\leq\ & C\left(Ch_n - \varepsilon_n\right)\left(1 + \left\| u\left(., s\right)\right\|_B\right) \\
& + \left| a\left(t_j^n - \hat{s}_n\left(s\right)\right)\right| \left\| k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) - k\left(s, u\left(., s\right)\right)\right\|_B,
\end{aligned} \tag{4.119}$$

hence, by virtue of (4.30) and (4.118), it follows that

$$\begin{aligned}
& \left\| a\left(t_j^n - \hat{s}_n\left(s\right)\right) k\left(\hat{s}_n\left(s\right), \hat{u}_n\left(s\right)\right) - a\left(t-s\right) k\left(s, u\left(., s\right)\right)\right\|_B \\
\leq\ & C\left(Ch_n\right)\left(1 + C\right) + C\sqrt{h_n} \leq C\sqrt{h_n} \\
\longrightarrow\ & {}_{n\longrightarrow\infty} 0,
\end{aligned} \tag{4.120}$$

this consists the fact that $a\left(t-s\right) \geq a\left(t_j^n - \hat{s}_n\left(s\right)\right)$. If not, we follows the same lines as above. On the other hand, in view of the assumed Lipschitz continuity of f, we have

$$\begin{aligned}
\left\| \widetilde{f}^n\left(\tau\right) - f\left(\tau, u\left(\tau\right), \frac{\partial u}{\partial t}\left(\tau\right)\right)\right\|_B & \leq \left\| f\left(\tilde{s}_n\left(\tau\right), \widehat{u}_n(\tau), \widehat{U}_n(\tau)\right) - f\left(\tau, u(\tau), u'(\tau)\right)\right\|_B \\
& \leq C\left[h_n + \left\| \widehat{u}_n(\tau) - u(\tau)\right\|_B + \left\| \widehat{U}_n(\tau) - u'(\tau)\right\|_B\right] \\
& \longrightarrow {}_{n\longrightarrow\infty} 0.
\end{aligned} \tag{4.121}$$

Now, the sequences $\{(\widetilde{u}_n(\tau),v)\}$, $\left\{\left(\widetilde{f}^n(\tau),v\right)_B\right\}$ and $\left\{\left(\widetilde{k}(\tau,\widetilde{u}_n(\tau)),v\right)_B\right\}$ are uniformly bounded with respect to both τ and n, so the Lebesgue theorem of majorized convergence is applicable to (4.109), thus, having in mind (4.110), (4.120), and (4.121), we derive

$$\begin{aligned}&(U(t)-\psi,v)_B+(u(t)-\varphi,v)+\int_0^t(u(\tau),v)\,d\tau\\ =\ &\int_0^t\left(\int_0^\tau a(\tau-s)\,k(s,u(x,s))\,ds,v\right)_B d\tau+\int_0^t\left(f\left(\tau,u(\tau),\frac{\partial u}{\partial t}(\tau)\right),v\right)_B \end{aligned} \tag{4.122}$$

takes place for all $v\in W$ and $t\in[0,T]$. Finally, differentiating (4.122) with respect to t, we get

$$\begin{aligned}&(u''(t),v)_B+(u'(t),v)+(u(t),v)\\ =\ &\left(\int_0^t a(t-s)\,k(s,u(x,s))\,ds,v\right)_B+\left(f\left(t,u(t),\frac{\partial u}{\partial t}(t)\right),v\right)_B,\end{aligned} \tag{4.123}$$

which achieves the proof of the existance. The uniqueness may be argued as follow. Let r be another solution for $(4.6)-(4.8)$ and $w=u-r$. Then, with $v=w'(t)$ as test function in the difference $(4.123)_u-(4.123)_r$ and taking into account assumptions (H_1), (H_2) and the fact that

$$((d/dt)\,w(t),w(t))_B=(1/2)\,(d/dt)\,\|w(t)\|_B^2, \tag{4.124}$$

we obtain

$$\begin{aligned}&\frac{1}{2}(d/dt)\,\|w'(t)\|_B^2+\|w'(t)\|^2+\frac{1}{2}(d/dt)\,\|w(t)\|_B^2\\ \le\ &\left\|\int_0^t a(t-s)\,k(s,u(x,s))\,ds-\int_0^t a(t-s)\,k(s,r(x,s))\,ds\right\|_B\|w'(t)\|_B\\ &+\left\|f\left(t,u(t),\frac{\partial u}{\partial t}(t)\right)-f\left(t,r(t),\frac{\partial r}{\partial t}(t)\right)\right\|_B\|w'(t)\|_B,\ \forall t\in[0,T]\end{aligned}$$

from which, we deduce that

$$\begin{aligned}
&\frac{1}{2}(d/dt)\|w'(t)\|_B^2+\|w'(t)\|^2+\frac{1}{2}(d/dt)\|w(t)\|_B^2\\
\leq\ & C\|w'(t)\|_B\sup_{t\in[0,T]}|a(t)|\int_0^t\|w(s)\|_B\,ds\\
&+C(\|w(t)\|_B+\|w'(t)\|_B)\|w'(t)\|_B,\ \forall t\in[0,T]
\end{aligned}$$

or, by virtue of the property (i) from Proposition 26

$$\begin{aligned}
&\frac{1}{2}(d/dt)\|w'(t)\|_B^2+\frac{1}{2}(d/dt)\|w(t)\|_B^2\\
\leq\ & C\sup_{s\in[p_1,t]}\|w(s)\|_B\|w'(t)\|_B+C\|w(t)\|_B^2+C\|w'(t)\|_B^2,
\end{aligned}$$

for all $t\in[p_1,p_2]\subset[0,T]$,where

$$\begin{aligned}
p\ \ &:\ =p_2-p_1,\ w'(t)=0,\ \forall t\in[0,p_1]\\
\textit{and } w'(t)\ &\neq\ 0,\ \forall t\in\,]p_1,p_2[\,,
\end{aligned}\tag{4.125}$$

from which, with (4.125) , it follows

$$\begin{aligned}
&(d/dt)\|w'(t)\|_B^2+(d/dt)\|w(t)\|_B^2\\
\leq\ & C\sup_{s\in[p_1,t]}\|w(s)\|_B^2+C\|w'(t)\|_B^2.
\end{aligned}$$

this for all $t\in[p_1,p_2]$. Integrating the above inequality on $(p_1,t)\subset[p_1,p_2]$, we get

$$\begin{aligned}
&\|w'(t)\|_B^2+\|w(t)\|_B^2\\
\leq\ & C\int_{p_1}^t\left[\|w'(s)\|_B^2+\sup_{\xi\in[p_1,s]}\|w(\xi)\|_B^2\right]ds,
\end{aligned}$$

or

$$\begin{aligned}
&\|w'(t)\|_B^2+\sup_{s\in[p_1,t]}\|w(s)\|_B^2\\
\leq\ & C\int_{p_1}^t\left[\|w'(s)\|_B^2+\sup_{\xi\in[p_1,s]}\|w(\xi)\|_B^2\right]ds.
\end{aligned}$$

Applying Gronwall's inequality, we get

$$\|w'(t)\|_B^2 + \sup_{s\in[p_1,t]} \|w(t)\|_B^2 \leq 0, \ \forall t \in [p_1, p_2].$$

Contradiction with (4.125). This achieves the proof. ■

Finally, we introduce the result of continuous dependence of the solution upon the data.

Theorem 28 *Let* u^* *be the weak solution of problem* (4.6) – (4.8) *corresponding to* $(\varphi^*, \psi^*, a^*, k^*, f^*)$ *instead of* (φ, ψ, a, k, f), *then the inequality*

$$\begin{aligned}
& \|u(t) - u^*(t)\|^2 \\
\leq\ & \|\psi - \psi^*\|_B^2 + \|\varphi - \varphi^*\|^2 \\
& + \int_0^t \left[\int_0^\tau \|a(\tau - s) k(s, u(s)) ds - a^*(\tau - s) k^*(s, u^*(s))\|_B ds \right]^2 d\tau \\
& + \int_0^t \left\| f\left(\tau, u(\tau), \frac{d}{d\tau} u(\tau)\right) - f^*\left(\tau, u^*(\tau), \frac{d}{d\tau} u^*(\tau)\right) \right\|_B^2 d\tau, \qquad (4.126)
\end{aligned}$$

takes place for all $t \in I$.

Proof. Subtracting the identity (4.123) for u and u^* with $w(t) = u(t) - u^*(t)$ as a test fuction in the resulting relation, we get by integration over $(0, t)$

$$\begin{aligned}
& \frac{1}{2}\|w'(t)\|_B^2 - \frac{1}{2}\|w'(0)\|_B^2 + \int_0^t \|w'(t)\|^2 + \frac{1}{2}\|w(t)\|^2 - \frac{1}{2}\|w(0)\|^2 \\
\leq\ & \int_0^t \int_0^\tau \|a(\tau - s) k(s, u(s)) ds - a^*(\tau - s) k^*(s, u^*(s))\|_B ds \|w'(\tau)\|_B d\tau \\
& + \int_0^t \left\| f\left(\tau, u(\tau), \frac{d}{dt} u(\tau)\right) - f^*\left(\tau, u^*(\tau), \frac{d}{dt} u^*(\tau)\right) \right\|_B \|w'(\tau)\|_B d\tau,
\end{aligned}$$

hence

$$
\begin{aligned}
& \|w'(t)\|_B^2 - \|w'(0)\|_B^2 + 2\int_0^t \|w'(t)\|^2 + \|w(t)\|^2 - \|w(0)\|^2 \\
\leq \; & \int_0^t \left[\int_0^\tau \|a(\tau - s)\, k(s, u(s))\, ds - a^*(\tau - s)\, k^*(s, u^*(s))\|_B \, ds \right]^2 d\tau \\
& + \int_0^t \left\| f\left(\tau, u(\tau), \frac{d}{d\tau} u(\tau)\right) - f^*\left(\tau, u^*(\tau), \frac{d}{d\tau} u^*(\tau)\right) \right\|_B^2 d\tau \\
& + 2\int_0^t \|w'(\tau)\|_B^2 \, d\tau, \qquad (4.127)
\end{aligned}
$$

from which we deduce the desired result and so the continuous dependence of the solution of (4.6) − (4.8) upon data. So the proof is complete. ■

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ABSTRACT

ملخص: في هذه الرسالة، نثبت وجود ووحدانية الحل لبعض المسائل التطورية، الأولى مسألة من أجل معادلة نصف زائديه شبه خطية مع شروط ابتدائية وحدية غير متجانسة' المسألة الثانية تخص معادلة تكافئية نصف خطية تكاملية-تفاضلية مع شروط حدية غير كلاسيكية و المسألة الأخيرة تخص معادلة زائديهشبه خطية تكاملية-تفاضلية مع شروط حدية غير كلاسيكية. الطريقة المستعملة هي طريقة التقطيع النصفي' المسماة أيضا طريقة روت لتقطيع الزمن.

RESUME:Dans cette thèse on démontre l'existence et l'unicité de quelques problèmes d'évolution. On commence par un problème avec une équation pseudo-hyperbolique quasi-linéaire avec des conditions initiales et aux limites non-homogènes.Le second chapitre est consacré à l'étude d'un problème relatif à une équation semi-linéaire parabolique intégro-différentielle avec des conditions aux limites non classiques et le dernier chapitre est pour l'étude d'un problème avec une équation quasi-linéaire pseudo-hyperbolique intégro-différentielle avec des conditions aux limites non classiques. La méthodeutilisée est celle de discrétisation de Rothe.

ABSTRACT: The thesis is concerned with the study of the existence and the uniqueness of evolution problems. We began with a problem for a quasi-linear pseudo-hyperbolic equation with nonhomogeneous boundary and initial conditions. The second is for a semi-linear parabolic integro-differential equation with non-classical boundary conditions, and the last one is for a quasi-linear hyperbolic integro-differential equation with non-classical boundary conditions. We use the Rothe-time discretization method.