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Stochastic Homogenization of Non Convex Integral Functionals and Ergodic Theory

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Introduction générale.

Le présent travail porte sur l'étude de l'épi-convergence presque sûre de fonctionnelles intégrales aléatoires et a pour origine la modélisation de problèmes provenant pour la plupart de la mécanique ou de l'éléctrostatique, le milieu étudié présentant une étérogénéité microscopique répartie aléatoirement. Ainsi, nous ferons, concernant ces fonctionnelles, une hypothèse probabiliste de périodicité en loi, généralesant la périodicité classique. Pour l'étude de la limite presque sûre au sens de l'épi-convergence, nous utilisons la Théorie ergodique de processus sous-additifs.

Cette thèse se compose de cinq chapitre. Dans le premier chapitre, on généralise, au cas probabiliste, les travaux de S.Müller concernant l'homogénéisation de fonctionnelles intégrales non convexes. Pour cela nous ferons sur les intégrandes associées, les mêmes hypothèses de croissance et de continuité, l'hypothèse de périodicité étant remplacée par une hypothèse de périodicité en loi et d'ergodicité. Un théorème ergodique sou-additif dû à M.A.Ackoglu &U.Krengel permet de surmonter les défficultés liées à l'aléatoire.

Dan le second chapitre, on retrouve par une autre technique les résultats pécèdents. L'intérêt de la méthode ainsi mise en oeuvre réside dans sa flexibilité. Elle a été notament utilisée avec succés pour la résolution de problèmes de structures différentes.On démontre dans un premier temps un résultat de convergence faible presque sûre de mesures aléatoires à l'aide d'un théorème ergodique additif. On obtient ainsi, la convergence faible presque sûre des fonctions test construites à partir des problèmes d'optimisations localisés sur des celules kY, $k \in \mathbb{N}^*$, $Y=]0, 1[^d$, généralisant ainsi la convergence faible des fonctions test classiques obtenues par prolongement périodique d'une solution d'un problème local.

Le chapitre trois est consacré à l'homogénéisation d'un problème de Neumann à "trous" aléatoires et d'un problème à "fissures" aléatoires. Pour cela, dans le but de caractériser le domaine de l'épi-limite, nous faisons une hypothèse d'ordre géométrique sur la répartition aléatoire des trous ou des fissures. L'espace de probabilité de base est alors un espace de Bernoulli ce qui permet d'utiliser les techniques de prolongement et se ramener plus au moins au cadre coercif.

Le chapitre quatre est consacré à l'étude de la dualité. Classiquement on y définit le Lagrangien L_n du problème (P_n) et le Lagrangien L^{hom} du problème homogénéisé (P_{hom}) à partir des fonctions de perturbation ψ_n et ψ_{hom} définies sur l'espace $W_0^{1,p}(\Omega)\times V(\Omega)$ où $V(\Omega)$ est le sous espace des tenseurs symétriques de $(L^{p'}(\Omega))^{m\times m}$. On démontre que ψ_n épi-converge presque sûrement vers ψ_{hom} pour la topologie produit des topologies faible de $W_0^{1,p}(\Omega)$ et forte de $V(\Omega)$. Utilisant alors un résultat de H.Attouch, D.Azé & R.J.B.Wets, on obtient, lorsque L^{hom} a un unique point selle (σ, u), la convergence faible presque sûre de tout point selle de L_n vers (σ, u) et on retrouve, dans le cas probabiliste la relation $\sigma\in\partial f^{hom}(e(u))$.

Dans le dernier chapitre nous donnons quelques résultats partiels. Dans un premier paragraphe, on aborde l'analyse numérique. Utilisant le théorème ergodique du premier chapitre, nous introduisons, pour tout ω d'une partie de probabilité 1 de l'espace de probabilité de base

$$f^{hom}(a)=\lim_n \mathcal{M}_Y^N(F_n(\omega), a)$$

où

$$\mathcal{M}_Y^N(F_n(\omega), a):=\mathrm{Inf}\left\{F_n(\omega)(u+l_a, Y);\ u\in W_0^{1,2,N}(Y)\right\}$$

avec $l_a(x)=ax$ et nous démontrons que l'intégrande f^{hom} de la fonctionnelle homogénéisée vérifie

$$f^{hom}(a)=\mathrm{Inf}_N\, f^{hom,N}(a)=\lim_n \lim_N \mathcal{M}_Y^N(F_n(\omega), a).$$

Dans le paragraphe suivant, nous donnons un résultat partiel d'épi-convrgence dans le cas où nos fonctionnelles sont définies dans l'espace non réflexif $W^{1,1}(\Omega)$. La fonctionnelle épi-limite conjecturée est de domaine $BV(\Omega)$ ce qui introduit quelques defficultés technique non encore surmontées définitivement pour prouver l'inégalité presque sûre

$$F^{hom}\leq\tau\text{-épi-lim inf}\,F_n(\omega)$$

où τ désigne la topologie forte de $L^1(\Omega)$.

Ce travail a fait l'objet de deux publications en collaboration avec G.Michaille: une premiere publication intitulée " Homogénéisation stochastique de certains problèmes non coercifs " parue dans Séminaire d'Analyse Convexe de Montpellier vol.20, 1990, exp.11, une seconde intitulée " Stochastic homogenization of non convex integral functionals" parue dans Mathematical Modelling and Numerical Analysis, 329 - 356 vol. 28, n° 3, 1994. Enfin le chapitre deux écrit églement en collaboration et compléter par d'autre exemples fera peut-etre l'objet d'une publcation fiture.

Chapter I

Stochastic Homogenization of Non Convex Integral Functionals

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Stochastic Homogenization of Non convex Integral Functionals.

1.1 Introduction.

In this chapter, we propose a method for stochastic homogenization of a process $(F_n)_{n\in\mathbb{N}}$ with a state space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$ where $\mathcal{B}(\mathcal{F})$ is a σ-field on the class $\mathcal{F}$ of integral functionals G of the type

$$G(u, A)=\int_A g(x, \nabla u(x))dx$$

in a sense explained later, A being a bounded regular domain in $\mathbb{R}^d$, $u: A \to \mathbb{R}^m$ a vector valued function, $g: \mathbb{R}^d \times M^{m\times d} \to \mathbb{R}$ an equi-coercive and equi-bounded function, measurable with respect to its first variable and continuous with respect to the matrix variable of $M^{m\times d}$, but not necessary convex.

Given a probability space $(\Sigma, \mathcal{T}, P)$ and a measurable map $F:(\Sigma, \mathcal{T}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F}))$ with

$$F(\omega)(u, A)=\int_A f(\omega)(x, \nabla u(x))dx.$$

If the law of F possesses some ergodic and periodic properties, the process $(F_n)_{n\in\mathbb{N}}$ defined by

$$F_n(\omega)(u, A)=\int_A f(\omega)\left(\frac{x}{\varepsilon_n}, \nabla u(x)\right)dx,$$

epi-converges almost surely when ε_n tends to 0 towards a constant F^{hom} in $\mathcal{F}$ whose integrand f^{hom} is quasi-convex (and so convex in the scalar case m=1). More precisely, there exists a subset Σ' of Σ with $P(\Sigma')=1$, such that, for every ω in Σ', every bounded regular domain A

$$F^{hom}(u, A)=\tau\text{-epi}\lim_{n\to+\infty} F_n(\omega)(u, A)$$

in $W^{1,p}(A, \mathbb{R}^m)$, equipped with its weak topology or strong topology of $L^p(A, \mathbb{R}^m)$ denoted τ, where

$$F^{hom}(u, A)=\int_A f^{hom}(\nabla u(x))dx;$$

and for every a in $M^{m\times d}$

$$f^{hom}(a)=\operatorname*{Inf}_{n\in\mathbb{N}^*}\frac{1}{meas(nY)}\int_\Sigma \operatorname{Inf}\{\int_{nY} f(\omega)(x,\nabla u(x)+a)dx,\ u\in W_0^{1,p}(nY,\ \mathbb{R}^m)\}dP(\omega),$$

Y denoting the unit cube $]0, 1[^d$.

Under few hypothesis on Φ from $W^{1,p}(A, \mathbb{R}^m)$ into $\mathbb{R}$ and on a subspace V of $W^{1,p}(A, \mathbb{R}^m)$, variational properties of epi-convergence lead to almost sure convergence of Inf $\{F_n(\omega)(u, A)+\Phi(u);\ u\in V\}$ towards $\min\{F^{hom}(u, A)+\Phi(u);\ u\in V\}$, this last statement justifying the epi-convergence process.

These results generalize ones obtained by G.Dal Maso & L.Modica [6], G.Facchinetti & A.Gavioli [11], K.Sab [14] in stochastic convex case and S.Müller [13] in periodic non convex case. We give a new proof by using a direct method, where sequences of functions, to obtain the lower bound in definition of epi-convergence, are construct thanks to an ergodic theorem which was first used in the calculus of variation by G.Dal Maso & L.Modica [7] in the convex case by means of compactness method.

This non convex approach finds its motivation in non linear elasticity where f(ω) is the stored energy density of a composite material with random inclusion. Hevertheless, let us point out that our method requires an equiboundedness property on f(ω) and that the class $\mathcal{F}$ is not a correct model in non linear elasticity. Homogenization of functionals from a class $\mathcal{F}$ constructed with polyconvex functions g which takes its values in $\mathbb{R}^{*+}\cup\{+\infty\}$ seems to be open.

Let us clarify the plan of this chapter. In part 1.2, we give definition and main properties of epi-convergence and Ergodicity . In part 1.3, we give some notations and preliminary results about $A\mapsto\mathcal{M}_A(., a)$. In part 1.4 we define the process $\{F_n;\ F^{hom}\ n\in\mathbb{N}\}$. The main results are proposition 1.6 and Corollary 1.7 where we use the Ackoglu & Krengel ergodic theorem to define F^{hom}. In part 1.5, we prove our main theorem 1.8 by means of two lemmas [1.9, 1.11] (lower bound (i) and upper bound (ii) in epi limit process), and give in corollary 1.12, the almost sure convergence of corresponding optimization problems. In part 1.6, we give a standard example of non homogenous random function f(ω) which is a model of stored energy density for material with random spherical inclusions distributed with a given proportion in an independence way in $\mathbb{R}^3$. The corresponding integral functional F(ω) is then periodic in law, ergodic and theorem 1.8 can be applied.

1.2 Epi-convergence and Ergodicity.

Epi-convergence is a convergence notion for sequences of functions specially designed in order to study convergence of solutions and values of corresponding minimization problems: it is a "variational convergence".

Let us denote (X,τ) a space X with a topology τ and consider a sequence $(F_n)_{n\in\mathbb{N}}$ of functions from X into $\bar{\mathbb{R}}$, the extended reals. For simplicity we give definitions in the case τ metrizable, for further details about epi-convergence or Γ-convergence in a general setting we refer to H.Attouch [2], Dal Maso & Modica [5], De Giorgi [8], and De Giorgi & Dal Maso [9].

Definition 1.1. The sequence of functions $(F_n)_{n\in\mathbb{N}}$ from X into $\bar{\mathbb{R}}$ is said to be τ-epi-convergent to $F: X\to\bar{\mathbb{R}}$ at the point $u\in X$ if the two following sentences hold

(i) For every converging sequence $(u_n)_{n\in\mathbb{N}}$, $u_n\in X$ $u_n\to u$ in (X,τ)

$$F(u)\le\liminf_{n\to+\infty} F_n(u_n).$$

(ii) There exists a sequence, $(u_n)_{n\in\mathbb{N}}$, $u_n\in X$ $u_n\to u$ converging in (X,τ) such that

$$F(u)\ge\limsup_{n\to+\infty} F_n(u_n).$$

We then write

$$F(u)=\tau\text{-epi}\lim F_n(u).$$

When this property hold for every $u\in X$, the sequence $(F_n)_{n\in\mathbb{N}}$ is said to be τ-epi-convergent to F and we write $F=\tau\text{-epi}\lim F_n$.

Functions defined by:

$$\tau\text{-epi}\liminf F_n(u):=\min\{\liminf_{n\to+\infty} F_n(u_n);\ u=\tau\text{-}\lim_{n\to+\infty} u_n\};$$

$$\tau\text{-epi}\limsup F_n(u):=\min\{\limsup_{n\to+\infty} F_n(u_n);\ u=\tau\text{-}\lim_{n\to+\infty} u_n\},$$

are the lower and upper epi-limits of the sequence $(F_n)_{n\in\mathbb{N}}$. It is straightforward to check that: $F=\tau\text{-epi}\lim_{n\to+\infty} F_n$ if and only if $\tau\text{-epi}\limsup_{n\to+\infty} F_n\le F\le\tau\text{-epi}\liminf_{n\to+\infty} F_n$ in X.

Theorem 1.2 (variational properties of epi-convergence). Let $(F_n)_{n\in\mathbb{N}}$ a sequence of functions from (X,τ) into $\bar{\mathbb{R}}$ which is τ-epi-convergent, $F=\tau\text{-epi}\lim_{n\to+\infty} F_n$.

(i) Let us assume there exists a "minimizing sequence" $(u_n)_{n\in\mathbb{N}}$ i.e

$$F_n(u_n)\le\operatorname*{Inf}_{u\in X} F_n(u)+\varepsilon_n \text{ with } \varepsilon_n\to 0$$

which is τ-relatively compact. Then

$$\operatorname*{Inf}_{u\in X} F_n(u)\to\operatorname*{Min}_{u\in X} F(u) \text{ as } n\to+\infty,$$

and every τ-cluster point of the sequence $(u_n)_{n\in\mathbb{N}}$ does minimize F on X.

(ii) For every τ-continuous function G: $X \to \mathbb{R}$;

$$F+G=\tau\text{-}\operatorname{epi}\lim_{n\to+\infty}(F_n+G).$$

Let us give now few definitions and results about Ergodicity. Let $(\Sigma, \mathcal{T}, P)$ be any probability space and $(\tau_z)_{z\in\mathbb{Z}^d}$ a group of P-preserving transformations on $(\Sigma, \mathcal{T})$, that is to say

(i) τ_z is $\mathcal{T}$-measurable;

(ii) $P\circ\tau_z(E)=P(E)$ for every E in $\mathcal{T}$ and z in $\mathbb{Z}^d$, where $\tau_z(E)=z+E$;

(iii) $\tau_z\circ\tau_t=\tau_{z+t}$, $\tau_{-z}=\tau_z^{-1}$, for every z and t in $\mathbb{Z}^d$.

In addition, if for every set E in $\mathcal{T}$ satisfying for every z in $\mathbb{Z}^d$, $\tau_z(E)=E$, we have $P(E)\in\{0;1\}$, $(\tau_z)_{z\in\mathbb{Z}^d}$ is said *Ergodic*.. A sufficient condition to ensure ergodicity of $(\tau_z)_{z\in\mathbb{Z}^d}$ is the following *mixing property*: for every E and F in $\mathcal{T}$

$$\lim_{|z|\to+\infty} P(\tau_z E\cap F)=P(E)\,P(F).$$

$\mathcal{J}$ denotes the set of intervals $[x, y[$ in $\mathbb{R}^d$ where x and y belong to $\mathbb{Z}^d$ and consider a set function $\mathcal{Y}$ from $\mathcal{J}$ into $L^1(\Sigma, \mathcal{T}, P)$ verifing the three conditions:

(i) $\mathcal{Y}$ is superadditifve, that is, for every A in $\mathcal{J}$ such that there exists a finite family $(A_i)_{i\in I}$ of disjoint sets in $\mathcal{J}$ whose union A belongs to $\mathcal{J}$, then

$$\mathcal{Y}_A(.)\geq\sum_{i\in I}\mathcal{Y}_{A_i}(.),$$

(ii) $\mathcal{Y}$ is covariant, that is, for every A in $\mathcal{J}$ and every z in $\mathbb{Z}^d$,

$$\mathcal{Y}_{A+z}=\mathcal{Y}_A\circ\tau_z,$$

(iii) $\sup\{\frac{1}{\text{meas}(A)}\int_\Sigma \mathcal{Y}_A\, dP,\ A\in\mathcal{J},\ \text{meas}(A)\neq\emptyset.\}<+\infty.$

Following M.A.Ackoglu &U.Krengel [1], $\mathcal{Y}$ is called a discrete superadditive process. If $-\mathcal{Y}$ is superadditive, $\mathcal{Y}$ is said subadditive. The following useful almost sure convergence result holds (see M.A.Ackoglu &U.Krengel [1] Theorem 2.4, Lemma 3.4 and Remark p.59):

Theorem 1.3. When n tends to $+\infty$, $\frac{1}{n^d}\mathcal{Y}_{[0,n[^d}$ converges almost surely. Moreover, if $(\tau_z)_{z\in\mathbb{Z}^d}$ is Ergodic , we have almost surely :

$$\lim_{n\to+\infty}\frac{1}{n^d}\mathcal{Y}_{[0,n[^d}(\omega)=\sup_{n\in\mathbb{N}^*}\frac{1}{n^d}E(\mathcal{Y}_{[0,n[^d}(.))$$

where E(.) denotes the probability average operator.

1.3 Notations and preliminary results

For m, $d\in\mathbb{N}^*$, $M^{m\times d}$ denotes the space of real m×d matrices $a=(a_{i,j})_{\substack{i=1,\dots,m\\ j=1,\dots,d}}$ equipped with the Euclidean Hilbert-Schmitt product $a{:}b=\text{trace}(a\,{}^t b)$: In that follows, we shall denote indifferently the norms in $\mathbb{R}^m$ and $M^{m\times d}$

$\mathcal{O}$ will denote the set of all open bounded subset in $\mathbb{R}^d$ with Lipschitz boundary. For $1<p<+\infty$ and $A\in\mathcal{O}$, we consider the two classical Banach spaces

$$L^p(A, \mathbb{R}^m):=\{u: A\to\mathbb{R}^m;\ u=(u_i)_i,\ u_i\in L^p(A),\ i=1,\dots,m\},$$
$$W^{1,p}(A, \mathbb{R}^m):=\{u\in L^p(A, \mathbb{R}^m); \frac{\partial u_i}{\partial x_j}\in L^p(A);\ i=1,\dots,m,\ j=1,\dots,d\},$$

respectively equipped with the two following norms

$$|u|_{0,A}:=\left(\int_A |u(x)|^p dx\right)^{1/p};$$
$$|u|_{1,A}:=\left(\int_A |u(x)|^p dx+\int_A |\nabla u(x)|^p dx\right)^{1/p},$$

where ∇u denotes the matrix valued distributions $\left(\frac{\partial u_i}{\partial x_j}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,d}}$.

$W_0^{1,p}(A, \mathbb{R}^m)$ is the subspace of functions u in $W^{1,p}(A, \mathbb{R}^m)$ with null trace on the boundary ∂A of A and $W^{1,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ is the space of vector valued functions u, measurable in $\mathbb{R}^d$ satisfying the following condition: every x in $\mathbb{R}^d$ possesses a neighborhood A such that the restriction of u to A belongs to $W^{1,p}(A, \mathbb{R}^m)$.

α, β, being two given positive constants, we define the subset $\mathcal{F}$ of the product space $\mathbb{R}^{W^{1,p}_{loc}(\mathbb{R}^d,\mathbb{R}^m)\times\mathcal{O}}$ as follows:

G belongs to $\mathcal{F}$ iff there exists a function g: $\mathbb{R}^d\times M^{m\times d}\to\mathbb{R}$ measurable with respect to its first variable, and a positive constant L such that, for every a, b in $M^{m\times d}$ and x a.e

(1.1) $\alpha|a|^p\le g(x,a)\le\beta(1+|a|^p)$;

(1.2) $|g(x,a)-g(x,b)|\le L(1+|a|^{p-1}+|b|^{p-1})\,|a-b|$,

with, for every $A\in\mathcal{O}$ and $u\in W^{1,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$

$$G(u, A)=\int_A g(x, \nabla u(x))dx.$$

For every $z\in\mathbb{Z}^d$, every $r\in\mathbb{R}^{*+}$, we define on $\mathcal{F}$ the two operators τ_z and ρ_r by

$$\tau_z G(u, A):=G(\tau_z u, z+A)=\int_A g(x+z, \nabla u(x))dx; \tag{1.3}$$
$$\rho_r G(u, A):=r^d G(\rho_r u, \tfrac{1}{r}A)=\int_A g(\tfrac{x}{r}, \nabla u(x))dx$$

with

$$\tau_z u(x)=u(x-z) \text{ and } \rho_r u(x)=\frac{1}{r}u(rx).$$

For every $a\in M^{m\times d}$, $A\in\mathcal{O}$ and $G\in\mathcal{F}$ we set

$$\mathcal{M}_A(G, a):=\mathrm{Inf}\{G(u+l_a, A);\ u\in W_0^{1,p}(A, \mathbb{R}^m)\}.$$

where l_a denotes the linear vector valued function whose gradient is a. We shall use in the sequel the following elementary properties

Proposition 1.4.

(i) $\dfrac{\mathcal{M}_A(\rho_r G, a)}{\text{meas}(A)} = \dfrac{\mathcal{M}_{1/r\,A}(G, a)}{\text{meas}(1/r\ A)}$, $\mathcal{M}_A(\tau_z G, a) = \mathcal{M}_{A+z}(G, a)$;

(ii) There exists a positive constant L' depending only on L, α, β and p such that, for every a, b in $M^{m\times d}$

$$\left|\frac{\mathcal{M}_A(G, a)}{\text{meas}(A)} - \frac{\mathcal{M}_A(G, b)}{\text{meas}(A)}\right| \leq L'(1+|a|^{p-1}+|b|^{p-1})\ |a-b|.$$

Proof. It is straightforward to check (i). We only prove (ii). For every $a \in M^{m\times d}$, let us set

$$m(a) = \frac{\mathcal{M}_A(G, a)}{\text{meas}(A)}$$

Let $\eta > 0$ and $u_\eta \in W_0^{1,p}(A, \mathbb{R}^m)$ such that

$$m(b) > \frac{1}{\text{meas}(A)}(G(u_\eta + l_b, A) - \eta\ ).$$

We have

$$m(a) - m(b) < \frac{1}{\text{meas}(A)}(G(u_\eta + l_a, A) - G(u_\eta + l_b, A) + \eta)$$

$$\leq \frac{1}{\text{meas}(A)} \int_A |g(x, \nabla u_\eta(x) + a) - g(x\ , \nabla u_\eta(x) + b)|\ dx + \frac{\eta}{\text{meas}(A)}$$

Using (1.2) and Hölder's inequality, with p' denotes the conjugate exponent of p, we obtain

$$m(a) - m(b) < \frac{L|a-b|}{\text{meas}(A)} \int_A (1 + |a + \nabla u_\eta(x)|^{p-1} + |b + \nabla u_\eta(x)|^{p-1}) dx + \frac{\eta}{\text{meas}(A)}$$

$$\leq L|a-b| \left(\frac{1}{\text{meas}(A)} \int_A (1 + |a + \nabla u_\eta(x)|^{p-1} + |b + \nabla u_\eta(x)|^{p-1})^{p'} dx\right)^{1/p'} + \frac{\eta}{\text{meas}(A)}.$$

Therefore

(1.4) $$m(a) - m(b) \leq CL|a-b| \left(\frac{1}{\text{meas}(A)} \int_A (1 + |a|^p + |b|^p + |b + \nabla u_\eta(x)|^p) dx\right)^{1/p'} + \frac{\eta}{\text{meas}(A)}$$

where C is a constant that depends only on p. On the other hand, by (1.1),

$$\frac{1}{\text{meas}(A)} \int_A |b + \nabla u_\eta(x)|^p dx \leq \frac{1}{\alpha\ \text{meas}(A)} G(u_\eta + l_b, A)$$

$$\leq \frac{1}{\alpha}\left(m(b) + \frac{\eta}{\text{meas}(A)}\right)$$

$$\leq \frac{\beta}{\alpha}(1 + |b|^p) + \frac{\eta}{\alpha\ \text{mea}(A)}$$

From (1.4) and after making $\eta \to 0$

$$m(a) - m(b) \leq L'\ |a-b|\ (1 + |a|^{p-1} + |b|^{p-1})$$

where L' depends only on p, α and β. We conclude the proof by intrechanging the roles of a and b.∎

1.4 Difinition of the process $\{F_n ; F^{hom}, n\in \mathbb{N}\}$.

Let $\mathcal{B}(\mathcal{F})$ the trace on $\mathcal{F}$ of the product σ-field of $\mathbb{R}^{W^{1,P}_{loc}(\mathbb{R}^d,\mathbb{R}^m)\times\mathcal{O}}$, that is the smallest σ-field on $\mathcal{F}$ such that all the evaluation maps

$$G\mapsto G(u, A),\ u\in W^{1,P}_{loc}(\mathbb{R}^d, \mathbb{R}^m),\ A\in\mathcal{O}$$

are $(\mathcal{B}(\mathcal{F}),\ \mathcal{B}(\mathbb{R}))$ measurable, $\mathcal{B}(\mathbb{R})$ denoting the Borelian σ-field of $\mathbb{R}$, as a direct consequence of the definition of $\mathcal{B}(\mathcal{F})$, we have

Proposition 1.5. For every $z\in\mathbb{Z}$ and $r\in\mathbb{R}^{*+}$, τ_z and ρ_r are measurable from $(\mathcal{F},\ \mathcal{B}(\mathcal{F}))$ into itself.

We define now the process $\{F_n ; n\in\mathbb{N}\}$. $(\Sigma, \mathcal{T}, P)$ is a given probability space and F a given measurable map

$$F: (\Sigma, \mathcal{T})\to(\mathcal{F},\ \mathcal{B}(\mathcal{F}))$$
$$\omega\mapsto F(\omega)$$

where

$$F(\omega)(u, A)=\int_A f(\omega)(x,\ \nabla u(x))dx.$$

We assume that $(\tau_z)_{z\in\mathbb{Z}^d}$ defined in (1.3) is a group of μ-preserving transformations on the probability space $(\mathcal{F},\ \mathcal{B}(\mathcal{F}), \mu)$, where μ is the probability image $P\circ F^{-1}$ of P in $\mathcal{F}$ (or the law of F)

We summarize these properties upon F by saying that F is *a random integral functional, periodic in law and ergodic.*

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{*+}$ which tends to 0, we define the process $\{F_n ; n\in\mathbb{N}\}$ by

$$F_n: (\Sigma, \mathcal{T})\to(\mathcal{F},\ \mathcal{B}(\mathcal{F}))$$
$$\omega\mapsto F_n(\omega)$$

where

$$F_n(\omega)(u, A):=\rho_{\varepsilon_n} F(\omega)(u, A)=\int_A f(\omega)(\frac{x}{\varepsilon_n}, \nabla u(x))dx.$$

Note that the measurability of F_n comes from proposition 1.5. In part 1.5, we shall study in the sense of epi convergence, the asymptotic behaviour of $\{F_n ; n\in\mathbb{N}\}$. The main tool to define the expected epi limite is the theorem 1.3 applied to the map $A\mapsto -\mathcal{M}_A(., a)$ when A belongs to the set of all open intervals $]x, y[$ in $\mathbb{Z}^d$ (or equivalently to the set $\mathcal{J}$). Let us give some properties of this map.

Proposition 1.6. For every a in $M^{m\times d}$, the map

$$\mathcal{J}\to L^1(\mathcal{F},\ \mathcal{B}(\mathcal{F}), \mu)$$

$$A \mapsto \mathcal{M}_A(., a)$$

is a discrete subadditive ergodic process in $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$. Moreover $\mathcal{M}_A(., a)$ satisfies the following upper growth condition on $L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$

$$|\mathcal{M}_A(., a)|_{L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)} \leq \beta(1+|a|^p) \text{ meas}(A).$$

Proof. Let us prove the $(\mathcal{B}(\mathcal{F}), \mathcal{B}(\mathbb{R}))$ measurability of $A \mapsto \mathcal{M}_A(G, a)$. From the separability of $W_0^{1,p}(A, \mathbb{R}^m)$, and the continuity of the map $u \mapsto G(u+l_a, A)$ there exists a dense countable subset $\{u_k; k \in \mathbb{N}\}$ of $W_0^{1,p}(A, \mathbb{R}^m)$ such that

$$\mathcal{M}_A(G, a) = \operatorname*{Inf}_{k \in \mathbb{N}} \{G(u_k+l_a, A)\}$$

where from the definition of the σ-field $\mathcal{B}(\mathcal{F})$, the map $G \mapsto G(u_k+l_a, A)$ are $(\mathcal{B}(\mathcal{F}), \mathcal{B}(\mathbb{R}))$ measurable. The upper growth condition is a direct consequence of (1.1). So $\mathcal{M}_A(., a) \in L^1(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mu)$. For the subadditivity, we consider a finite familly $(A_i)_{i \in I}$ of disjoint sets of $\mathcal{J}$ with $A_i \subset A$ and $\text{meas}(A \setminus \bigcup_{i \in I} A_i) = 0$. Let $\eta > 0$, $G \in \mathcal{F}$, $i \in I$ and $u_\eta^i \in W_0^{1,p}(A_i, \mathbb{R}^m)$ which satisfies, $G(u_\eta^i+l_a, A_i) \leq \mathcal{M}_{A_i}(G, a) + \frac{\eta}{\text{Card}(I)}$ and define u_η in $W_0^{1,p}(A, \mathbb{R}^m)$ by setting $u_\eta = u_\eta^i$ on A_i. We have

$$\begin{aligned} \mathcal{M}_A(G, a) &\leq G(u_\eta+l_a, A) \\ &= \sum_{i \in I} G(u_\eta^i+l_a, A_i) \\ &\leq \sum_{i \in I} \mathcal{M}_{A_i}(G, a) + \eta, \end{aligned}$$

which conclude the proof after making η tends to 0 and noticing that covariance property is given by proposition 1.4 (i). ∎

We are now in position to define the integal function F^{hom} in $\mathcal{F}$ which will be the expected epi limite.

Corollary 1.7. There exist Σ' in Σ with $P(\Sigma') = 1$ and a function $f^{hom}: M^{m\times d} \to \mathbb{R}$ such that, for all ω in Σ', all cube Q in $\mathbb{R}^d$ and all a in $M^{m\times d}$

$$\begin{aligned} f^{hom}(a) := \lim_{\substack{t \to +\infty \\ t \in \mathbb{R}}} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{\text{meas}(t\,Q)} \\ = \operatorname*{Inf}_{n \in \mathbb{N}^*} \left(\int_\Sigma \frac{\mathcal{M}_{nY}(F(\omega), a)}{\text{meas}(nY)} dP(\omega) \right\}. \end{aligned}$$

Moreover f^{hom} satisfies (1.1) and (1.2) with L' defined in proposition 1.4 (ii).

Proof. By step.*First step* We assume that a belongs to the subset $M'^{m\times d}$ of $M^{m\times d}$ with rational entries. Combining proposition 1.6, theorem 1.3, with the probability space $(\mathcal{F},$

$\mathcal{B}(\mathcal{F})$, μ), for every $a \in M'^{m\times d}$ we obtain existence of $E_a \in \mathcal{B}(\mathcal{F})$, $\mu(E_a)= 1$ and $f^{hom}(a) \in \mathbb{R}$ such that, for all $G \in E_a$

$$f^{hom}(a) := \lim_{n\to+\infty} \frac{\mathcal{M}_{nY}(G, a)}{meas(nY)}$$
$$= \operatorname*{Inf}_{n\in\mathbb{N}^*} \left\{ \int_{\mathcal{F}} \frac{\mathcal{M}_{nY}(H, a)}{meas(nY)}\, d\mu(H) \right\}.$$

Recall that Y denoting the unit cube $]0, 1[^d$.
Setting $\Sigma' = F^{-1}\left(\bigcap_{a\in M'^{m\times d}} E_a \right)$, we obtain, from above

$$f^{hom}(a) := \lim_{n\to+\infty} \frac{\mathcal{M}_{nY}(G, a)}{meas(nY)} \tag{1.5}$$
$$= \operatorname*{Inf}_{n\in\mathbb{N}^*} \left\{ \int_{\Sigma} \frac{\mathcal{M}_{nY}(F(\omega), a)}{meas(nY)}\, dp(\omega) \right\}$$

for every a in $M'^{m\times d}$ and every ω in Σ'.

Let Q be any cube in $\mathbb{R}^d$ wih side η and, for every t in $\mathbb{R}^{+*}$ set $k^- = [t\eta]-1$, $k^+ = [t\eta]+1$, and consider $Q^- = k^-(Y+z)$, $Q^+ = k^+(Y+z')$ the two cubes such that $z, z' \in \mathbb{Z}^d$, $Q^- \subset tQ \subset Q^+$. Thanks to the inequality

$$\mathcal{M}_A(F(\omega), a) \le \mathcal{M}_B(F(\omega), a) + \beta(1+|a|^p)\, meas(A\setminus B)$$

whenever $B\subset A$ in $\mathcal{O}$ and noticing that meas(tQ) is equivalent to meas(k^+Y) and meas(k^-Y) whenever t tends to $+\infty$, we get from (1.5) and the covariance property,

$$f^{hom}(a) = \lim_{t\to+\infty} \frac{\mathcal{M}_{k^+Y}(F(\tau_z\omega), a)}{meas(tQ)}$$
$$\le \liminf_{t\to+\infty} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{meas(tQ)}$$
$$\le \limsup_{t\to+\infty} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{meas(tQ)}$$
$$\le \lim_{t\to+\infty} \frac{\mathcal{M}_{k^-Y}(F(\tau_z\omega), a)}{meas(tQ)}$$
$$= f^{hom}(a)$$

for every a in $M^{m\times d}$ and ω in Σ', which conclude this step.

Second step . We extend the result of previous step to every a in $M^{m\times d}$. In that follows, ω will be a fixed element of Σ'. Using proposition 1.4 and above step, it is clear that f^{hom} satisfies the locally Lipschitz condition (1.2) with the new constant L' for every a and b in $M^{m\times d}$. So, by a classical argument , one can extend f^{hom} to $M^{m\times d}$ by setting, for every r in $M^{m\times d}$., $f^{hom}(r) = \lim_{n\to+\infty} f^{hom}(a_n)$ where a_n is any sequence in $M'^{m\times d}$ converging towards r. It is straightforward to check that this extention verifies the same condition (1.2). On the other hand, from

$$|f^{hom}(a) - \frac{\mathcal{M}_{tQ}(F(\omega), r)}{meas(tQ)}| \leq |f^{hom}(r) - f^{hom}(a_n)| +$$

$$+ |f^{hom}(a_n) - \frac{\mathcal{M}_{tQ}(F(\omega), a_n)}{meas(tQ)}| + |\frac{\mathcal{M}_{tQ}(F(\omega), a_n)}{meas(tQ)} - \frac{\mathcal{M}_{tQ}(F(\omega), r)}{meas(tQ)}|$$

we conclude this step by using proposition 1.4 (ii) and letting t tends to $+\infty$ and a_n tends to r.

It remains to prove that f^{hom} satisfies the growth condition (1.1). The upper bound is just a consequence of the proposition 1.6. In the other hand from (1.1) and the convexity of $r \to |r|^p$ we get

$$\frac{\mathcal{M}_{tQ}(F(\omega), a)}{meas(tQ)} \geq \alpha \operatorname{Inf}\left\{\frac{1}{meas(tQ)}\int_{tQ} |a+\nabla u(x)|^p\, dx,\ u \in W_0^{1,p}(tQ, \mathbb{R}^d)\right\}$$

$$\geq \alpha |a|^p,$$

which gives the lower bound after going to the limit in t.∎

We now define F^{hom} in $\mathcal{F}$ by

$$F^{hom}(u, A) := \int_A f^{hom}(\nabla u(x))dx,$$

1.5 Almost sure epi-convergence of the process $\{F_n\ ;\ F^{hom},\ n \in \mathbb{N}\}$.

Our main result is the following almost sure epi convergence theorem.

Theorem 1.8. Let Σ' be the subset of Σ with $P(\Sigma')=1$ defined in the corollary 1.7. For all ω in Σ' and A in $\mathcal{O}$, we have

$$F^{hom}(u, A) = \tau\text{-epi} \lim_{n \to +\infty} F_n(\omega)(u, A) \text{ in } W^{1,p}(A, \mathbb{R}^m),$$

in $W^{1,p}(A, \mathbb{R}^m)$ equipped with its weak topology τ or the strong topology of $L^p(A, \mathbb{R}^m)$.

We shall give the proof with τ denoting the strong topology of $L^p(A, \mathbb{R}^m)$. From (1.1) and the compact imbedding from $W^{1,p}(A, \mathbb{R}^m)$ into $L^p(A, \mathbb{R}^m)$, we conclude in the other case.
The proof of lheorem 1.8 will be established by means of two lemmas: the upper bound in definition of epi-convergence is proved in lemma 1.9, the lower bound in lemma 1.11, lemma 1.10 being just a simple technical lemma. In all what follows, ω denotes a fixed element of Σ'.

Lemma 1.9. For every $A \in \mathcal{O}$ and every u in $W^{1,p}(A, \mathbb{R}^m)$

$$F^{hom}(u, A) \leq \tau\text{-epi} \liminf F_n(\omega)(u, A),$$

that is to say, for every sequence u_n, τ-converging towards u,

$$F^{hom}(u, A) \leq \liminf_{n \to +\infty} F_n(\omega)(u_n, A).$$

Proof .*First stage.* Assume that A is un open cube Q in $\mathbb{R}^d$ and $u=l_a$, $a\in M^{m\times d}$. It is convenient and involves no loss of generality, to assume that u_n-l_a belongs to $W_0^{1,p}(Q, \mathbb{R}^m)$ (see for instance S.Müller [13] or G.Dal Maso & L.Modica [5], [6]). From definition of F^{hom} corollary 1.7 and proposition 1.4 (i), we get

$$F^{hom}(u, Q)=meas(Q)f^{hom}(a),$$

$$F^{hom}(u, Q)=meas(Q)\lim_{n\to+\infty}\frac{\mathfrak{M}_{1/\varepsilon_n Q}(F(\omega), a)}{meas(1/\varepsilon_n Q)}$$

$$=meas(Q)\lim_{n\to+\infty}\frac{\mathfrak{M}_Q(F_n(\omega), a)}{meas(Q)}$$

$$\le\liminf_{n\to+\infty} F_n(\omega)(u_n, Q)$$

which ends the first stap.

Second step. Assume that $A\in\mathcal{O}$ and $u=l_a$.

For $\eta > 0$, there exists a finite family $(Q_i)_{i\in I}$ of disjoint open cubes include in A such that $meas(A\setminus\bigcup_{i\in I}Q_i)\le\eta$. Since f^{hom} satisfies (1.2), we get

$$F^{hom}(u, A)\le\sum_{i\in I}F^{hom}(u, Q_i)+\beta\eta(1+|a|^p).$$

Using previous step, superadditivity and non decreasing properties of the set function $B\mapsto\tau$-epi lim inf $F_n(\omega)(., B)$ from $\mathcal{O}$ into $\overline{\mathbb{R}}^{*+}$ (cf H. Attouch [2, p.156-157]), we obtain

$$F^{hom}(u, A)\le\sum_{i\in I}\tau\text{-epi}\liminf F_n(\omega)(u, Q_i)+\eta\beta(1+|a|^p)$$

$$\le\tau\text{-epi}\liminf F_n(\omega)(u, A)+\eta\beta(1+|a|^p)$$

and we conclude by letting η tends to 0.

Third stage. $A\in\mathcal{O}$ and $u\in W^{1,p}(A, \mathbb{R}^m)$. We use the previous step and the density of the set of piecewise affine continuous functions in $W^{1,p}(A, \mathbb{R}^m)$ (cf I.Ekeland & R.Temam [10]). Let u, u_n in $W^{1,p}(A, \mathbb{R}^m)$ such that $u=\tau\text{-}\lim_{n\to+\infty}u_n$. For $\eta>0$, there exist a finite partition $(A_i)_{i\in I}$ of A, $A_i\in\mathcal{O}$ and u_η in $W^{1,p}(A, \mathbb{R}^m)$ such that $|u-u_\eta|_{1,A}\le\eta$ and its restriction u^i_η to A_i is affine.

Set $v_{n,\eta}=u_\eta+u_n-u$ and denote by $v^i_{n,\eta}$ its restriction to A_i. By using the second step, we get for every $i\in I$

$$\begin{cases} u^i_\eta=\tau\text{-}\lim_{n\to+\infty}v^i_{n,\eta} \text{ in } L^p(A_i, \mathbb{R}^m), \\ F^{hom}(u^i_\eta, A_i)\le\liminf_{n\to+\infty}F_n(\omega)(v^i_{n,\eta}, A_i). \end{cases}$$

After summation over i, with superadditivity of lim inf, we obten

$$F^{hom}(u_\eta, A)\le\liminf_{n\to+\infty}F_n(\omega)(v_{n,\eta}, A). \tag{1.6}$$

On the other hand, by (1.2)

$$F_n(\omega)(v_{n,\eta}, A)\leq F_n(\omega)(u_n, A)+L\int_A (1+|\nabla u_n(x)|^{p-1}+|\nabla v_{n,\eta}(x)|^{p-1})\,|\nabla u_n(x)-\nabla v_{n,\eta}(x)|\,dx,$$

and after using Hölder inequality, we get to a further subsequence with respect to n

$$\begin{aligned}(1.7)\qquad F_n(\omega)(v_{n,\eta}, A)&\leq F_n(\omega)(u_n, A)+C|u_\eta-u|_{1,A}\\ &\leq F_n(\omega)(u_n, A)+C\eta,\end{aligned}$$

where C will denote any constant that does not depends on η and n. Note that we have assumed that $\liminf_{n\to+\infty} F_n(\omega)(u_n, A)<+\infty$ and so, thinks to (1.1), up to a further subsequence with respect to n, u_n and $v_{n,\eta}$ bounded in $W^{1,p}(A, \mathbb{R}^m)$. On the other hand, by continuity property of F^{hom},

$$\begin{aligned}(1.8)\qquad F^{hom}(u_\eta, A)&\geq F^{hom}(u, A)-L'\int_A (1+|\nabla u(x)|^{p-1}+|\nabla u_\eta(x)|^{p-1})(|\nabla u(x)-\nabla u_\eta(x)|)dx\\ &\geq F^{hom}(u, A)-C\eta.\end{aligned}$$

From (1.6), (1.7) and (1.8), after letting η tends to 0, we get

$$F^{hom}(u, A)\leq\liminf_{n\to+\infty} F_n(\omega)(u_n, A)$$

which ends the proof of Lemma 1.9.∎

Before proving the lower bound in the definition of epi-convergence, we shall need the following estimaton for η-approximating minimizer of $\mathcal{M}_Q(F_n(\omega), a)$.

Lemma 1.10. Let $\eta>0$, Q be an open cube in $\mathbb{R}^d$ with side η of the lattice in $\mathbb{R}^d$ spanned by $]0, \eta[$, and $v_{n,\eta}(\omega)$ in $W_0^{1,p}(Q, \mathbb{R}^m)$ such that

$$F_n(\omega)(v_{n,\eta}(\omega)+l_a, Q)\leq\mathcal{M}_Q(F_n(\omega), a)+\eta.$$

Then

$$|v_{n,\eta}(\omega)|_{0,Q}^p\leq C\eta^p(\mathrm{meas}(Q)+\eta)$$

where the constant C depends only on α, β and a.

Proof of Lemma 1.10. In that follows, C will denote different constants which depend only on α, β and a. By (1.1), omitting the variable ω we get

$$\begin{aligned}(1.9)\qquad |\nabla v_{n,\eta}+a|_{0,Q}^p &\leq \frac{1}{\alpha}F_n(\omega)(v_{n,\eta}+l_a, Q)\\ &\leq\frac{1}{\alpha}\mathcal{M}_Q(F_n(\omega), a)+\frac{\eta}{\alpha}\\ &\leq\frac{\beta}{\alpha}(1+|a|^p)\mathrm{meas}(Q)+\frac{\eta}{\alpha}\\ &\leq C\mathrm{meas}(Q)+\frac{\eta}{\alpha}\end{aligned}$$

On the other hand

$$|v_{n,\eta}|_{0,Q}^p\leq C\eta^p|\nabla v_{n,\eta}|_{0,Q}^p$$

where C is the Poincaré's constant in $W_0^{1,p}(Y, \mathbb{R}^m)$. Recalling (1.9) we obtain

$$|v_{n,\eta}|_{0,Q}^{p} \leq C\eta^p(\text{meas}(Q)+\eta),$$

which closes the proof of lemma 1.10.∎

Lemma 1.11. For every A in $\mathcal{O}$ and u in $W^{1,p}(A, \mathbb{R}^m)$, there exists a sequence $(u_n(\omega))_{n\in\mathbb{N}}$ in $W^{1,p}(A, \mathbb{R}^m)$ such that

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n(\omega), \\ F^{hom}(u, A) \geq \limsup\limits_{n\to+\infty} F_n(\omega)(u_n(\omega), A). \end{cases}$$

Proof. By step.*First step.* Assume that $u = l_a$, $a\in M^{m\times d}$. Let $\eta>0$ and $(Q_i)_{i\in I}$, $(Q_i)_{i\in J}$ tow finite family of open disjoint cubes with side η of the lattice in $\mathbb{R}^d$ spanned by $]0, \eta[$ such that $\bigcup\limits_{i\in I} Q_i \subset A \subset \bigcup\limits_{i\in I\cup J} Q_i$, $\text{meas}(\bigcup\limits_{i\in J} Q_i)=\delta(\eta)$, with $\lim\limits_{\eta\to+\infty}\delta(\eta)=0$ (note that I and J depend on η).

Using definition of F^{hom}, corollary 1.7 and proposition 1.4, we get

$$\begin{aligned} F^{hom}(u, A) &\geq F^{hom}(u, \bigcup_{i\in I} Q_i) \\ &= \sum_{i\in I} \text{meas}(Q_i) f^{hom}(a) \\ &= \lim_{n\to+\infty} \sum_{i\in I} \mathcal{M}_{Q_i}(F_n(\omega), a). \end{aligned} \tag{1.10}$$

The suitable sequence of functions $(u_n(\omega))_{n\in\mathbb{N}}$ will be deduced from the approximite minimizers of $\mathcal{M}_{Q_i}(F_n(\omega), a)$. Precisely let $v^i_{n,\eta}(\omega)$ in $W_0^{1,p}(Q_i, \mathbb{R}^m)$ such that

$$F_n(\omega)(v^i_{n,\eta}(\omega)+l_a, Q_i) \leq \mathcal{M}_{Q_i}(F_n(\omega), a) + \frac{\eta}{\text{Card}(I\cup J)},$$

and define $v_{n,\eta}$, $u_{n,\eta}$ in $W^{1,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m)$ by $v_{n,\eta}=v^i_{n,\eta}$ in Q_i, and $u_{n,\eta}=v_{n,\eta}+l_a$. Recalling (1.10) we get

$$\begin{aligned} F^{hom}(u, A) &\geq \limsup_{n\to+\infty} F_n(\omega)(u_{n,\eta}, \bigcup_{i\in I} Q_i) - \eta \\ &\geq \limsup_{n\to+\infty} F_n(\omega)(u_{n,\eta}, A) - \beta(1+|a|^p)\delta(\eta) - 2\eta. \end{aligned}$$

Therefore

$$F^{hom}(u, A) \geq \limsup_{\eta\to 0}\limsup_{n\to+\infty} F_n(\omega)(u_{n,\eta}, A). \tag{1.11}$$

On the other hand

$$|u_{n,\eta} - l_a|^p_{0,A} = |v_{n,\eta}|^p_{0,A}$$

and thanks to the lemma 1.10

$$|u_{n,\eta} - l_a|^p_{0,A} \leq C \sum_{i\in I\cup J} \eta^p(\text{meas}(Q_i) + \frac{\eta}{\text{card}(I\cup J)}) \tag{1.12}$$

$$\leq C\eta^p(\text{meas}(B)+\eta)$$

where C is a constant that depends only on p, α, β, a and B is any bounded set that containing A. From (1.11), (1.12) and using a diagonalization argument (see H. Attouch [2, cor 1.16]), there exists a map $n \mapsto \eta(n)$ such that $\eta(n) \to 0$ when $n \to +\infty$ with

$$\begin{cases} u = \tau\text{-}\lim_{n\to+\infty} u_{n,\eta(n)}, \\ F^{hom}(u, A) \geq \limsup_{n\to+\infty} F_n(\omega)(u_{n,\eta(n)}, A). \end{cases}$$

It suffices to set $u_n := u_{n,\eta(n)}$

Second step. We assume that u is any element in $W^{1,p}(A, \mathbb{R}^m)$. By continuity of $F^{hom}(., A)$ in $W^{1,p}(A, \mathbb{R}^m)$, it suffices to assume u to be piecewise affine and continuous function, and we conclude by using first step and again a diagonalization argument. More precisely, there exists a finite partition $(A_i)_{i\in I}$ of A, $A_i \in \mathcal{O}$, such that $u = l_{a_i} + b_i$ in A_i, where $a_i \in M^{m\times d}$ and $b_i \in \mathbb{R}^m$. Using previous step, there exists v_n^i in $W^{1,p}(A_i, \mathbb{R}^m)$, possibly depending on ω, such that

$$\begin{cases} u = \tau\text{-}\lim_{n\to+\infty} v_n^i \text{ in } L^p(A_i, \mathbb{R}^m), \\ F^{hom}(u, A_i) \geq \limsup_{n\to+\infty} F_n(\omega)(v_n^i, A_i). \end{cases}$$

By an argument proved in G.Dal Maso & L.Modica [5],[6] or with some different technics in K.Messaoudi & G.Michaille [12], we can construct, by modifying v_n^i, another sequence u_n^i in $W^{1,p}(A_i, \mathbb{R}^m)$, such that

$$\begin{cases} u = \tau\text{-}\lim_{n\to+\infty} u_n^i \text{ in } L^p(A_i, \mathbb{R}^m),\ u_n^i = u \text{ on } \partial A_i, \\ F^{hom}(u, A_i) \geq \limsup_{n\to+\infty} F_n(\omega)(u_n^i, A_i). \end{cases}$$

The sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = u_n^i$ on A_i satisfies, after summing over i

$$\begin{cases} u = \tau\text{-}\lim_{n\to+\infty} u_n, \\ F^{hom}(u, A) \geq \limsup_{n\to+\infty} F_n(\omega)(u_n, A). \end{cases}$$

When u belongs to $W^{1,p}(A, \mathbb{R}^m)$, we conclude like in the last step in the proof of lemma 1.9, by density and diagonalization argument .(see also S.Müller [13]). ∎

We give now the following consequence of theorem 1.8.

Corollary 1.12. Let Ω be a given element in $\mathcal{O}$, Γ_0 a subset of boundary $\partial\Omega$ of Ω with strictly positive surface measure and V the subset

$$\{u\in W^{1,p}(\Omega, \mathbb{R}^m),\ u=u_0 \text{ on } \Gamma_0\}$$

where $u_0\in W^{1,p}(\Omega, \mathbb{R}^m)$, V being equipped with the weak topology of $W^{1,p}(\Omega, \mathbb{R}^m)$. If F is a random integral functional, periodic in law and ergodic, Φ a continuous map from V into $\mathbb{R}$, then $F^{hom}(u, \Omega)$ is lower semi continuous for the weak topology of $W^{1,p}(\Omega, \mathbb{R}^m)$, f^{hom} is quasiconvex, and

$$\inf\{F_n(\omega)(u, \Omega)+\Phi(u)\ ;\ u\in V\},$$

converges almost surely towards

$$\min\{F^{hom}(u, \Omega)+\Phi(u);\ u\in V\}.$$

Proof. Let ω be a fixed element in Σ'. Since every τ-epi limit is τ-lower semi continuous (see H.Attouch [2]), it follows, from theorem 1.8, that $F^{hom}(u, \Omega)$ is lower semi continuous for the weak topology of $W^{1,p}(\Omega, \mathbb{R}^m)$ and that f^{hom} is quasiconvex (see J.M.Ball & F.Murat [3]).
For the last statement, it remains to prove that

$$F^{hom}(\omega)(., \Omega)+\Phi=\tau\text{-epi}\lim\,(\,F_n(\omega)(., \Omega)+\Phi\,) \text{ in V}.$$

But Φ being τ-continuous perturbation of the sequence $(F_n(\omega))_{n\in\mathbb{N}}$ and so (see theorem 1.2 (ii)), it suffices to prove that

$$F^{hom}(\omega)(., \Omega)=\tau\text{-epi}\lim\ F_n(\omega)(., \Omega) \text{ in V}$$

and thus, that, for every u in V there exists a sequence $(u_n(\omega))_{n\in\mathbb{N}}$ in V such that

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n(\omega)\,, \\ F^{hom}(u, A)\geq\limsup\limits_{n\to+\infty} F_n(\omega)(u_n(\omega), A). \end{cases}$$

For this, it suffices to modify, in a neighbourhoud of $\partial\Omega$, the sequence of functions $u_n(\omega)$ obtained in lemma 1.11, in such a way to preserve above condition, with, in addition, $u=u_n(\omega)$ in $\partial\Omega$ (see again G.Dal Maso & L.Modica [5],[6]).∎

1.6 A model of random integral functional.

We would like to give in this section, un example of non homogeneous random function $f(\omega)$ which will be a model of stored energy density for material with inclusions distributed at random and for which, the corresponding integral functional is *a random integral functional, periodic in law and ergodic.*

Let us denote by $\mathcal{G}$ the set a functions g defined as in part 1.3, equipped with the trace σ-field $\mathcal{B}(\mathcal{G})$ of the product σ-field of $\mathbb{R}^{\mathbb{R}^d \times M^{m\times d}}$ and define the groupe of transformations $(\tau_z)_{z\in\mathbb{Z}}$ on $\mathcal{G}$, by

$$\tau_z\, g(x, a)=g(x+z, a).$$

Consider a map f from $\Sigma\times\mathbb{R}^d\times M^{m\times d}$ into $\mathbb{R}$, which is $(\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(M^{m\times d}), \mathcal{B}(\mathbb{R}))$ measurable function and such that, for every ω in Σ, $f(\omega,.,.)$ belongs to $\mathcal{G}$. It is clear that for every z in $\mathbb{Z}^d$ the maps $\tau_z f$ from Σ into $\mathcal{G}$ are $(\mathcal{T}, \mathcal{B}(\mathcal{G}))$ measurable. We say that:

f is periodic in law if, for every z in $\mathbb{Z}^d$ $P\circ f^{-1}=P\circ(\tau_z f)^{-1}$;

f is ergodic if, for every E in $\mathcal{B}(\mathcal{G})$ such that for every z in $\mathbb{Z}^d$ $\tau_z E=E$, we have $P\circ f^{-1}(E)\in\{0,1\}$.

With some slight modification of the proof of G.Dal Maso & L.Modica [7] one can easily show that corresponding random integral functional $\omega\longmapsto F(\omega)$ from Σ into $\mathcal{F}$ defined by:

$$F(\omega)(u, A)=\int_A f(\omega)(x, \nabla u(x))dx;\ u\in W^{1,p}(A, \mathbb{R}^m) \text{ and } A\in\mathcal{O},$$

is *periodic in law and ergodic* in the sense of part 1.3. (Note that no convexity assumption is required to obtain this last result in the proof of [7]). So, we have, by definition of σ-field $\mathcal{B}(\mathcal{G})$, the two following sufficient conditions to obtain the periodicity in law and ergodicity of F (see also G.Dal Maso & L.Modica [7]).

Proposition 1.13.

(i) If, for all finite family $(x_i, a_i)_{i\in I}$ of $\mathbb{R}^d\times M^{mxd}$, the random vectors $(f(., x_i, a_i))_{i\in I}$ and $(f(., x_i+z, a_i))_{i\in I}$ have the same law for every z in $\mathbb{Z}^d$, then F is periodic in law.

(ii) If, for all finite family $(x_i, a_i, r_i)_{i\in I}$, $(y_j, b_j, s_j)_{j\in J}$ in $\mathbb{R}^d\times M^{mxd}\times\mathbb{R}$

$$\lim_{\substack{|z|\to+\infty\\ z\in\mathbb{Z}^d}} P([f(., x_i, a_i)>r_i]\cap[f(., z+y_j, b_j)>s_j])$$
$$=P([f(., x_i, a_i)>r_i])\ P([f(., y_j, b_j)>s_j]).$$

Then F is ergodic.

We now give our example. Let g, h: $M^{m\times d}\to\mathbb{R}$ be two homogeneous stored energy density which satisfy (1.1) and (1.2), and consider a ponctual Poisson process $\omega\longmapsto\mathcal{N}(\omega, .)$ from $(\Sigma, \mathcal{T}, P)$ into $\mathbb{N}^{\mathcal{B}(\mathbb{R}^d)}$ of parameter $\mu>0$, which satisfies (see for instance N.Bouleau [4]):

(i) For every bounded borel set A in $\mathbb{R}^d$,

$$\mathcal{N}(\omega,A)=\sum_{y\in D(\omega)}\delta_y(A)$$

where $\delta_y(A)$ denotes the Dirac measure with support $\{y\}$ and $D(\omega)$ is a given countable subset of $\mathbb{R}^d$ without cluster point.

(ii) For every finite family $(A_i)_{i\in I}$ of bounded Borel set in $\mathbb{R}^d$, two by two disjoint $(\mathcal{N}(.,A_i))_{i\in I}$ are independent *random variables.*

(iii) For every bounded Borel set A in $\mathbb{R}^d$, every k in $\mathbb{N}^*$

$$P\{(\mathcal{N}(.,A)=k)\}=\mu^k\ (\text{meas}(A))^k\ \frac{e^{-\mu\text{meas}(A)}}{k!}$$

(Note that $\mathcal{N}(\omega,A)=\text{Card}(A\cap D(\omega))$, $\int_\Sigma \mathcal{N}(\omega,A)dP(\omega)=\mu\ \text{meas}(A)$).

For a given r>0, we define the random non homogenous stored energy density by

$$f(\omega,x,a):=g(a)+(h(a)-g(a))\min(1,\mathcal{N}(\omega,B(x,r))$$

that is

$$f(\omega,x,a)=\begin{cases} h(a) \text{ if } x\in \bigcup_{y\in D(\omega)} B(y,r), \\ g(a) \text{ if not.}\end{cases}$$

f is then a model for a stored energy density of a composit material in $\mathbb{R}^d$, $B((y,r))_{y\in D(\omega)}$ being the rescaled random inclusions with a probability expectation μ meas(A) in every bounded Borel set A. On can see that f satisfies the hypothesis of proposition 1.13, and so defines a *random integral functional, periodic in low and ergodic.*

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Chapter II.

Ergodic Theory: Some Tools for The Calculus of Variations. Application to Stochastic Homogenization of Non convex Integral Functionals.

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Ergodic Theory: Some Tools for The Calculus of Variations. Application to Stochastic Homogenization of Non convex Integral Functionals.

2.1 Introduction.

Let Ω be an open regular set in $\mathbb{R}^d$, μ a given random Borel measure from a probability space $(\Sigma, \mathcal{T}, P)$ into the set $M(\mathbb{R}^d)$ of non negative regular Borel measures on $\mathbb{R}^d$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers which tends to 0. Define the sequence $(\mu_n)_{n\in\mathbb{N}}$ of maps from Σ into the set $M(\Omega)$ of non negative bounded regular Borel measures in Ω by

$$\mu_n(\omega)(A) := \varepsilon_n^d \, \mu(\omega) \left(\frac{1}{\varepsilon_n} A\right)$$

for every Borel set A in Ω and ω in Σ.

Under appropriate integrability assumptions on μ, we study the almost sure weak convergence of $(\mu_n(.))_{n\in\mathbb{N}}$ in $M(\Omega)$. We shall assume that μ satisfies the so called covariance property in ergodic Theory

$$\mu(\tau_z\omega)(A) = \mu(\omega)(z+A)$$

for every bounded Borel set A in $\mathbb{R}^d$, where $(\tau_z)_{z\in S}$ is a group of P-preserving transformations of $(\Sigma, \mathcal{T}, P)$ and S any subgroup $k\mathbb{Z}^d$ of $(\mathbb{Z}^d, +)$.

More precisely, using an adaptation of an additive ergodic theorem due to Nguyen Xuan Xanh & H.Zessin [7] proved in part 2.2 (see theorem 2.1), we shall establish in part 2.3 the following convergence theorem:

Theorem 2.4.

(i) If almost surely $(\mu_n(\omega))_{n\in\mathbb{N}}$ is tight, then almost surely $\mu_n(\omega)$ converges for the narrow topology towards $\theta(\omega)dx$ where

$$\theta(\omega) := \frac{1}{k^d} \mathbb{E}^{\mathcal{F}_k} \mu(.) ([0, k[^d),$$

$\mathbb{E}^{\mathcal{F}_k}$ denoting the conditional expectation operator with respect to the σ-field

$$\mathcal{F}_k = \{E \in \mathcal{T};\ \tau_z(E) = E \ \forall z \in k\mathbb{Z}^d\}.$$

(ii) If $(\tau_z)_{z\in S}$ is ergodic, that is to say, if $\mathcal{F}_k$ contains only sets of $\mathcal{T}$ with probability 0 or 1, then, almost surely $\mu_n(\omega)$ converges for the narrow topology towards θdx where

$$\theta := \frac{1}{k^d} E\,\mu(.)\,([0,k[^d).$$

(For the relevant definitions and notations see Section 2.2).

In part 2.4, we give a stronger result in the particular case where $\mu(\omega)=u(\omega,\ .)dx$, $u(\omega,\ .)$ belonging to $L^p_{loc}(\mathbb{R}^d,\ \mathbb{R}^m)$, $1\leq p\leq+\infty$. Under suitable integrability assumptions, we prove the following result:

Theorem 2.6. Setting $u_n(\omega, x) := u(\omega, \frac{x}{\varepsilon_n})$, we have

(i) in the case $1<p\leq+\infty$, almost surely $u_n(\omega, x)$ converges towards

$$E^{\mathcal{F}_k} \fint_{]0,k[^d} u(., x)dx$$

in $L^p(\Omega,\ \mathbb{R}^m)$ weak if $p\neq+\infty$ (in $L^\infty(\Omega,\ \mathbb{R}^m)$ weak* if $p=+\infty$).

(ii) in the case $p=1$, when $(\tau_z)_{z\in S}$ is ergodic, almost surely $u_n(\omega,\ .)$ converges towards

$$E \fint_{]0,k[^d} u(., x)dx \quad \text{in } L^1(\Omega,\ \mathbb{R}^m) \text{ weak.}$$

This last theorem generalizes, in the probability case, the well known weak convergence result about the sequence constructed from a periodic function u in $L^p_{loc}(\mathbb{R}^d,\ \mathbb{R}^m)$ by:

$$u_n(x) := u(\frac{x}{\varepsilon_n}).$$

Note that the covariance property on the measure $u(\omega,\ .)dx$ is equivalent to $u(\tau_z\omega,\ x)= u(\omega,\ x+z)$ and so, in the non probability case, to the periodicity of u. For a proof of this classical result, see for instance F.Murat & J.Ball [6] or B.Dacorogna [3].

In part 2.5, we give an application of theorem 2.4, 2.6 in homogenization of non convex random integral functionals defined in $W^{1,p}(\Omega,\ \mathbb{R}^m)$ by

$$F_n(\omega)(u, \Omega) = \int_\Omega f(\omega)(\frac{x}{\varepsilon_n}, \nabla u(x))dx\,.$$

Using theorem 2.6, we construct a sequence $(u_n(\omega,\ .))_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega,\ \mathbb{R}^m)$ such that $f(\omega)(x/\varepsilon_n,\ \nabla u_n(\omega,\ .)+a)dx$ converges almost surely for the narrow topology, towards $f^{hom}(a)dx$.

The functional defined by

$$F^{hom}(u)=\int_{\Omega} f^{hom}(\nabla u(x))dx,$$

will be almost surely the expected epi limit of $F_n(\omega)(u, \Omega)$ for the strong topology of $L^p(\Omega, \mathbb{R}^m)$ in $W^{1,p}(\Omega, \mathbb{R}^m)$.

2.2 Notations and additive ergodic theorem.

Let $(\Sigma, \mathcal{T}, P)$ be a probability space. For $k\in \mathbb{N}^*$, we consider the subgroup $k\mathbb{Z}^d$ of $(\mathbb{Z}^d, +)$ and a group $(\tau_z)_{z\in k\mathbb{Z}^d}$ of P-preserving transformations on $(\Sigma, \mathcal{T})$. let us recall that τ_z satisfies for every z, t in $k\mathbb{Z}^d$:

(i) τ_z is $\mathcal{T}$-measurable;

(ii) $P\circ\tau_z(E)=P(E)$ for every E in $\mathcal{T}$;

(iii) $\tau_z\circ\tau_t=\tau_{z+t}$, $\tau_{-z}=\tau_z^{-1}$.

$\mathcal{F}_k$ denotes the invariant sub σ-field of $\mathcal{T}$ for $(\tau_z)_{z\in k\mathbb{Z}^d}$, that is:

$$\mathcal{F}_k=\{E\in \mathcal{T};\ \tau_z(E)=E \quad \forall z\in k\mathbb{Z}^d\}.$$

If X is any topological space, $\mathcal{B}(X)$ will denote the Borel σ-field of X.

For every function f from Σ into $\mathbb{R}^m$ which belongs to $\mathcal{L}^1(\Sigma, \mathcal{T}, P)^m$, $E^{\mathcal{F}_k} f$ will denote the vectoriel valued conditional expectation of f with respect to $\mathcal{F}_k$:

$$E^{\mathcal{F}_k} f=(E^{\mathcal{F}_k} f_i)_{i=1,\dots,m}.$$

$(\tau_z)_{z\in k\mathbb{Z}^d}$ is said ergodic if $\mathcal{F}_k$ is reduce to the sub σ-field of sets with probability 0 or 1. In this case, by a classical probability argument , we have almost surely

$$E^{\mathcal{F}_k} f=Ef,$$

where E f is the vector valued expectation of f:

$$Ef=(Ef_i)_{i=1,\dots,m}.$$

$\mathcal{B}_b(\mathbb{R}^d)$ will denote the family of bounded Borel sets with positive Lebesgue measure. We adopt the notation meas for denoting Lebesgue measure on $\mathbb{R}^d$. A sequence $(A_n)_{n\in \mathbb{N}}$ of convex sets in $\mathcal{B}_b(\mathbb{R}^d)$ is called regular in the sense of Nguyen Xuan Xanh & H.Zessin [7] if there exist an increasing sequence of intervals $(I_n)_{n\in \mathbb{N}}$ in $\mathcal{B}_b(\mathbb{R}^d)$ and a finite constant C such that, for all n in $\mathbb{N}$

$$\text{meas}(I_n)\leq C\ \text{meas}(A_n)$$
$$A_n\subset I_n.$$

For any set A of $\mathcal{B}_b(\mathbb{R}^d)$, $\varrho(A)$ denotes the following supremum

$$\varrho(A):=\sup\{r\geq 0,\ \exists \bar{B}(x, r)\subset A\},$$

where $\bar{B}(x, r)=\{y\in\mathbb{R}^d;\ \|x-y\|\leq r\}$ is the closed ball in $\mathbb{R}^d$.

Theorem below is a straightforward consequence of Nguyen Xuan Xanh & H.Zessin theorem for additive processes (see Nguyen & Zessin [7] or U.Krengel [5, p. 209-211]).

Theorem 2.1. Let $\mathcal{A}$ be a map from $\mathcal{B}_b(\mathbb{R}^d)$ into $\mathcal{L}^1(\Sigma, \mathcal{T}, P)$ such that:

Additivity: for disjoint A_1, A_2 in $\mathcal{B}_b(\mathbb{R}^d)$, $\mathcal{A}_{A_1\cup A_2}=\mathcal{A}_{A_1}+\mathcal{A}_{A_2}$;

Covariance: for every z in $k\mathbb{Z}^d$ and A in $\mathcal{B}_b(\mathbb{R}^d)$, $\mathcal{A}_A\circ\tau_z=\mathcal{A}_{A+z}$;

Domination: there exists a>0 in $\mathcal{L}^1(\Sigma, \mathcal{T}, P)$ such that $|\mathcal{A}_A|\leq a$ for all convex set $A\in\mathcal{B}_b(\mathbb{R}^d)$ with $A\subset[0, k[^d$.

Then, for any regular sequence $(A_n)_{n\in\mathbb{N}}$ of convex sets such that $\lim_{n\to+\infty}\varrho(A_n)=+\infty$

$$\lim_{n\to+\infty}\frac{\mathcal{A}_{A_n}}{\text{meas}(A_n)}=\frac{1}{k^d}E^{\mathcal{F}_k}\mathcal{A}_{[0,k[^d}, \text{ almost surely.}$$

Proof. For proving above result, following the proof of Nguyen Xuan Xanh & H.Zessin [7, p.143], we use the fondamental results of theorem 3.7 and corollary 3.10 p.138 in above paper (see also U.Krengel [5, p.209-210]) which are summarized in the following lemma:

Lemma 2.2. Let a, b be two random variables with a, b in $\mathcal{L}^1(\Sigma, \mathcal{T}, P)$, $a\geq 0$, $(A_n)_{n\in\mathbb{N}}$ be a regular sequence of convex sets in $\mathbb{R}^d$ with $\lim_{n\to+\infty}\varrho(A_n)=+\infty$ and $(\tau_z)_{z\in\mathbb{Z}^d}$ a group of P-preserving transformations whose invariant σ-field is $\mathcal{F}$ (k=1) then almost surely

(i) $$\lim_{n\to+\infty}\frac{1}{\text{meas}(A_n)}\sum_{z\in\mathbb{Z}^d\cap A_n} b\circ\tau_z=E^{\mathcal{F}}b,$$

(ii) $$\lim_{n\to+\infty}\frac{1}{\text{meas}(A_n)}\sum_{z\in\mathbb{Z}^d\cap\tilde{A}_n\setminus \underset{\sim}{A}_n} a\circ\tau_z=0,$$

where:

$$\tilde{A}_n=\bigcup_{z\in X_n}(z+[0, k[^d);\quad \underset{\sim}{A}_n=\bigcup_{z\in Y_n}(z+[0, k[^d),$$

$$X_n=\{z\in k\mathbb{Z}^d\ ;\ z+[0, k[^d\subset A_n\};\quad Y_n=\{z\in k\mathbb{Z}^d\ ;\ z+[0, k[^d\cap A_n\neq\emptyset\};\ k\in\mathbb{N}^*.$$

Let us now prove Theorem 2.1. For every z in $k\mathbb{Z}^d$, we set $z=kt$, $t\in\mathbb{Z}^d$ and we define the P-preserving group of transformation $(\tau'_t)_{t\in\mathbb{Z}^d}$ by $\tau'_t=\tau_{kt}$ whose invariant σ-field is $\mathcal{F}_k$. Applying (i) of above lemma to $b=\mathcal{A}_{[0,k[^d}$ and to the regular sequence of convex sets $(\frac{1}{k}A_n)_{n\in\mathbb{N}}$, we get almost surely,

$$\lim_{n\to+\infty}\frac{k^d}{\mathrm{meas}(A_n)}\sum_{t\in\mathbb{Z}^d\cap 1/k\,A_n}\mathcal{A}_{[0,k[^d}\circ\tau'_t=E^{\mathcal{F}_k}\mathcal{A}_{[0,k[^d}$$

that is:

$$\lim_{n\to+\infty}\frac{1}{\mathrm{meas}(A_n)}\sum_{z\in k\mathbb{Z}^d\cap A_n}\mathcal{A}_{[0,k[^d}\circ\tau_z=\frac{1}{k^d}E^{\mathcal{F}_k}\mathcal{A}_{[0,k[^d}$$

On the other hand, by additivity, covariance of $\mathcal{A}$ with respect to $(\tau_z)_{z\in k\mathbb{Z}^d}$ and domination, we get:

$$\Big|\mathcal{A}_{A_n}(\omega)-\sum_{z\in k\mathbb{Z}^d\cap A_n}\mathcal{A}_{[0,k[^d}\circ\tau_z(\omega)\Big|\le\sum_{z\in k\mathbb{Z}^d\cap\tilde{A}_n\setminus\underset{\sim}{A}_n}a\circ\tau_z(\omega)$$

which gives our result thanks to (ii) of above lemma.∎

2.3 Almost sure weak convergence of sequence of random Borel measures.

$M(\mathbb{R}^d)$ denotes the set of non negative regular Borel measures on $\mathbb{R}^d$ equipped with $\mathcal{M}$, the trace on $M(\mathbb{R}^d)$ of the product σ-field of $\mathbb{R}^{\mathcal{B}(\mathbb{R}^d)}$.

Definition 2.3. Every map from Σ into $M(\mathbb{R}^d)$ which is $(\mathcal{T},\mathcal{M})$ measurable will be called a random Borel measure.

Note that above measurability is equivalent to the measurability of maps $\omega\mapsto\mu(\omega)(A)$ from Σ into $\mathbb{R}$, for every A in $\mathcal{B}(\mathbb{R}^d)$.

In that follows, we consider a given random Borel measure μ which satisfies almost surely, the two conditions

(2.1) $\omega\mapsto\mu(\omega)(A)$ belongs to $\mathcal{L}^1(\Sigma,\mathcal{T},P)$ for every bounded set A in $\mathcal{B}(\mathbb{R}^d)$,

(2.2) $\mu(\omega)(A+z)=\mu(\tau_z\omega)(A)$ for every bounded set A in $\mathcal{B}(\mathbb{R}^d)$.

Ω being a bounded convex open set in $\mathbb{R}^d$, M(Ω) the set of non negative bounded regular Borel measure on Ω and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers which tends to 0^+, we define, for every ω in Σ, every set A in $\mathcal{B}(\Omega)$ the sequence $(\mu_n(\omega))_{n\in\mathbb{N}}$ in M(Ω) by

$$\mu_n(\omega)(A):=\varepsilon_n^d\,\mu(\omega)(\frac{1}{\varepsilon_n}A).$$

We shall study the asymptotic behaviour of the process $(\mu_n)_{n\in\mathbb{N}}$ for the narrow topology, and more precisely, the almost sure convergence of $(\mu_n(\omega))_{n\in\mathbb{N}}$ for the $\sigma(C_b'(\Omega),\ C_b(\Omega))$ topology, $C_b(\Omega)$ denoting the Banach space of continuous bounded functions from Ω into $\mathbb{R}$. For this let us recall that a subset H of M (Ω) is said to be tight if, for every $\eta>0$, there exists a compact subset K of Ω such that

$$\nu(K^c)<\eta \quad \forall\ \nu\in H,$$

and that the Prokhorov compacity theorem asserts that every tight and bounded subset of $M(\Omega)$ is relatively sequentialy compact for the narrow topology. We are now in position to prove

Theorm 2.4.

(i) If almost surely $(\mu_n(\omega))_{n\in\mathbb{N}}$ is tight, then almost surely $\mu_n(\omega)$ converges for the narrow topology towards $\theta(\omega)dx$ where

$$\theta(\omega):=\frac{1}{k^d}E^{\mathcal{F}_k}\mu(.)([0,k[^d)$$

(ii) If $(\tau_z)_{z\in k\mathbb{Z}^d}$ is ergodic, then almost surely $\mu_n(\omega)$ converges for the narrow topology towards θdx, where

$$\theta:=\frac{1}{k^d}E\mu(.)([0,k[^d).$$

Proof. *Proof of (i).*

First step. We prove that there exists a set Σ' in $\mathcal{T}$ with $P(\Sigma')=1$ such that, for every ω in Σ' the sequence $(\mu_n(\omega))_{n\in\mathbb{N}}$ is bounded. Thanks to Prokhorov's theorem we shall deduce, for every fixed $\omega\in\Sigma'$, the existence of a subsequence $(\mu_{\sigma(n)}(\omega))_{n\in\mathbb{N}}$ which converges for the narrow topology. (Note that $\sigma(n)$ is eventually depending on ω).

For this, for every bounded Borel set A, and every ω in Σ, let us define

$$\mathcal{A}_A(\omega)=\mu(\omega)(A).$$

$\mathcal{A}$ is an additive process from $\mathcal{B}_b(\mathbb{R}^d)$ into $\mathcal{L}^1(\Sigma,\mathcal{T},P)$ which satisfies all conditions of theorm 2.1. Applying this theorem for the regular family $(\frac{1}{\varepsilon_n}\Omega)_{n\in\mathbb{N}}$, we obtain existence of Σ' in $\mathcal{T}$ with $P(\Sigma')=1$ such that, for every ω in Σ'

$$\lim_{n\to+\infty}\frac{\mathcal{A}_{1/\varepsilon_n\Omega}(\omega)}{\mathrm{meas}(1/\varepsilon_n\Omega)}=\frac{1}{k^d}E^{\mathcal{F}_k}(\mu(.)([0,k[^d)),$$

and so

$$\lim_{n\to+\infty}\mu_n(\omega)(\Omega)=\theta(\omega)\,\mathrm{meas}(\Omega).$$

For every fixed ω in Σ', the sequence $(\mu_n(\omega))_{n\in\mathbb{N}}$ is then bounded in $C_b'(\Omega)$ and tight (by hypothesis).Thanks to Prokhorov's theorem there exists a subsequence $(\mu_{\sigma(n)}(\omega))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty}\mu_{\sigma(n)}(\omega)=\nu(\omega)$$

for the narrow topology. The problem is now to identify $\nu(\omega)$.

Second step. Raisoning like above step, for every Borel set A on the form $Q\cap\Omega$, $(Q\cap\Omega)^\eta$ or $(Q\cap\Omega)_\eta$ where

$$(Q\cap\Omega)^\eta:=\{x\in\mathbb{R}^m;\ d(x,(Q\cap\Omega)^c)\geq\eta\};$$
$$(Q\cap\Omega)_\eta:=\{x\in\mathbb{R}^m;\ d(x,\ Q\cap\Omega)<\eta\},$$

Q varying in the family of open intervals in $\mathbb{R}^d$ with vertices in $\mathbb{Q}^d$ and η varying in $\mathbb{Q}^+$, we get existence of $\Sigma''=\bigcap_A\Sigma_A$ with $P(\Sigma'')=1$ such that, for every ω in Σ'' and every A in above family:

$$\lim_{n\to+\infty}\mu_n(\omega)(A)=\theta(\omega)\text{meas}(A).$$

Third step. Let us fixe ω in $\Sigma'\cap\Sigma''$ and prove

$$\nu(\omega)(\partial(\Omega\cap Q))=0.$$

We have

$$\nu(\omega)(\overline{\Omega\cap Q})\leq\nu(\omega)\left((\Omega\cap Q)^{1/m}\right). \tag{2.3}$$

But, by properties of weak convergence of measures and the two above steps,

$$\nu(\omega)((\Omega\cap Q)^{1/m})\leq\liminf_{n\to+\infty}\mu_{\sigma(n)}(\omega)((\Omega\cap Q)^{1/m})$$
$$=\theta(\omega)\ \text{meas}((\Omega\cap Q)^{1/m}),$$

so that, with (2.3)

$$\nu(\omega)(\overline{\Omega\cap Q})\leq\theta(\omega)\lim_{m\to+\infty}\text{meas}((\Omega\cap Q)^{1/m})$$
$$\leq\theta(\omega)\ \text{meas}(\overline{\Omega\cap Q}).$$

On the other hand

$$\nu(\omega)(\Omega\cap Q)\geq\nu(\omega)((\Omega\cap Q)_{1/m}),$$

and, again by properties of weak convergence of measures and the two above step,

$$\nu(\omega)((\Omega\cap Q)_{1/m})\geq\limsup_{n\to+\infty}\mu_{\sigma(n)}(\omega)((\Omega\cap Q)_{1/m})$$
$$=\theta(\omega)\ \text{meas}((\Omega\cap Q)_{1/m}),$$

so that,

$$\nu(\omega)(\Omega\cap Q)\geq\theta(\omega)\ \text{meas}(\overline{\Omega\cap Q}).$$

We finally get, thanks the above inequalities,

$$\nu(\omega)(\partial(\Omega\cap Q))=0.$$

Last step. Applying first step, and a classical result about narrow convergence, we get, almost surely

$$\lim_{n\to+\infty}\int_\Omega fd\mu_{\sigma(n)}(\omega)=\int_\Omega fd\nu(\omega)$$

for every bounded function f from Ω into $\mathbb{R}$ which is $\mu_{\sigma(n)}(\omega)$-measurable for every $n\in\mathbb{N}$ and such that the set of its discontinuous points has a $\nu(\omega)$ null measure.

In particular taking $f=\chi_A$ where A is a set on the form $\Omega\cap Q$, Q being any open interval with vertices in $\mathbb{Q}^d$, we get thanks to third step

$$\lim_{n\to+\infty}\mu_{\sigma(n)}(\omega)(A)=\nu(\omega)(A),$$

and so, with the second step, $\nu(\omega)(A)=\theta(\omega)\,\text{meas}(A)$. Since the Borel σ-field $\mathcal{B}(\Omega)$ is generated by the family of such A, we obtain $\nu(\omega)=\theta(\omega)dx$. With classical properties of narrow convergence, for every ω in $\Sigma'\cap\Sigma''$, all the sequence $(\mu_n(\omega))_{n\in\mathbb{N}}$ converges towards $\theta(\omega)dx$, which closes the proof of (i).

Proof of (ii). It remains to establish that under the *Ergodicity* assumption, almost surely, the sequence $(\mu_n(\omega))_{n\in\mathbb{N}}$ is tight.

Let $\eta\in\mathbb{Q}^+$ and K a compact convex subset of Ω such that (one can suppose $\theta\neq0$)

$$\text{meas}(K^c)<\frac{\eta}{\theta}$$

Raisoning like in the first step for the proof of (i), we get existence of Σ_η in $\mathcal{T}$ with $P(\Sigma_\eta)=1$ such that, for every ω in $\Sigma'=\bigcap_{\eta\in\mathbb{Q}+}\Sigma_\eta$

$$\begin{aligned}\lim_{n\to+\infty}\mu_n(\omega)(K^c)&=\lim_{n\to+\infty}\mu_n(\omega)(\Omega)-\lim_{n\to+\infty}\mu_n(\omega)(K)\\&=\text{meas}(K^c)\theta.\end{aligned}$$

So, for $n>N(\eta)$

$$\mu_n(\omega)(K^c)<\eta.$$

On the other hand, measures $\mu_n(\omega)$, $n=1,\dots,N(\eta)$ being *regular*, there exists a compact $K_n(\omega)$ in Ω such that

$$\mu_n(\omega)(K_n^c)<\eta\quad\text{for } n=1,\dots,N(\eta).$$

Setting $K'(\omega)=K\cup(\bigcup_{n=1,\dots,N(\eta)}K_n(\omega))$, we obtain for every ω in Σ' and every $n\in\mathbb{N}$

$$\mu_n(\omega)(K'^c(\omega))<\eta$$

which closes proof of (ii).∎

2.4 Almost sure weak convergence of a sequence of random functions.

In this section we consider any function u from $\Sigma\times\mathbb{R}^d$ into $\mathbb{R}^m$ satisfying the following properties for $1\le p\le+\infty$.

Case $p\neq 1$.

(2.4) $u(\omega,.)$ belongs to $L^p_{loc}(\mathbb{R}^d,\mathbb{R}^m)$ almost surely and for every A in $\mathcal{B}_b(\mathbb{R}^d)$, the map $\omega\mapsto\int_A u(\omega,x)dx$ is $(\mathcal{T},\ \mathcal{B}(\mathbb{R}^m))$ measurable,

(2.5) For every A in $\mathcal{B}_b(\mathbb{R}^d)$, the map $\omega\mapsto\int_A |u(\omega,x)|^p dx$ belongs to $\mathcal{L}^1(\Sigma,\mathcal{T},P)$,

(2.6) For every z in $k\mathbb{Z}^d$, $u(\omega, x+z)=u(\tau_z\omega, x)$ a. s., a. e.

Case $p=1$. In this case, in addition to (2.6), we make stronger hypothesis than (2.4), (2.5) that is

(2.4.5) The map $(\omega, x)\mapsto u(\omega,x)$ from $\Sigma\times A$ into $\mathbb{R}^m$ belongs to $\mathcal{L}^1(\Sigma\times A, dP\otimes dx)^m$ for every A in $\mathcal{B}_b(\mathbb{R}^d)$.

Remark 2.5. In many applications arising from the calculus of variations, $u(\omega,\ .)$ is a minimizer or an η-minimizer of $\operatorname{Inf}_X F(\omega,v)$, where F is a normal integrand from $\Sigma\times X$ into $\overline{\mathbb{R}}$, that is $\omega\rightrightarrows \text{epi}\,F(\omega,.)$ is a closed set multifuntion, X being a subspace of $L^p(\Omega,\mathbb{R}^m)$. This property on F provides measurability of $\omega\mapsto\operatorname{Inf}_X F(\omega,v)$ and, thanks to a measurability selection theorem, we get the existence of a map $\omega\mapsto u(\omega,.)$ which is $(\mathcal{T},\ \mathcal{B}(X))$ measurable. Consequently, $\omega\mapsto\int_A |u(\omega,x)|^p dx$ is $(\mathcal{T},\ \mathcal{B}(\mathbb{R}))$ measurable and, with the following diagram where φ is linear continuous (use Hölder inequality)

$$\begin{tikzcd}
\Sigma \arrow[rr, "u"] \arrow[dddr, "\varphi o u"'] & & X \arrow[dddl, "\varphi"] \\
\omega \arrow[rr] \arrow[dr] & & u(\omega, .) \arrow[dl] \\
& \int_A u(\omega,x)\, dx & \\
& \mathbb{R}^m &
\end{tikzcd}$$

we shall obtain the $(\mathcal{T}, \mathcal{B}(\mathbb{R}^m))$ measurability of $\omega \mapsto \int_A u(\omega, x)dx$.

Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{*+}$ that tends to 0^+ and define the sequence $(u_n)_{n\in\mathbb{N}}$ from $\Sigma\times\mathbb{R}^d$ into $\mathbb{R}^m$ by $u_n(\omega, x) := u(\omega, \frac{x}{\varepsilon_n})$, we have:

Theorem 2.6. With assumptions (2.4), (2.5), (2.6) if $p\neq 1$ and (2.4.5), (2.6) if $p=1$, for every bounded convex open set Ω in $\mathbb{R}^d$, we have:

(i) Case $1<p\leq+\infty$. Almost surely the sequence $(u_n(\omega, .))_{n\in\mathbb{N}}$ converges towards

$$E^{\mathcal{F}_k} \fint_{]0,k[^d} u(., x)dx$$

in $L^p(\Omega, \mathbb{R}^m)$ weak if $p\neq+\infty$, in $L^\infty(\Omega, \mathbb{R}^m)$ weak-$*$ if not.

(ii) Case $p=1$. If $(\tau_z)_{z\in k\mathbb{Z}^d}$ is ergodic, almost surely the sequence $(u_n(\omega, .))_{n\in\mathbb{N}}$ converges towards

$$E \fint_{]0,k[^d} u(., x)dx,$$

in $L^1(\Omega, \mathbb{R}^m)$ weak.

The proof of the theorem 2.6, will be a direct consequence of the following lemma

Lemma 2.7. Under hypothesis (2.4), (2.5), (2.6) if $p\neq 1$ and (2.4.5), (2.6) if $p=1$, for every bounded convex open set Ω in $\mathbb{R}^d$, there exists Σ' in $\mathcal{T}$ with $P(\Sigma')=1$ such that, for every $\omega\in\Sigma'$:

(i) The sequence $(u_n(\omega, .))_{n\in\mathbb{N}}$ is equibounded in $L^p(\Omega, \mathbb{R}^m)$, more precisely

$$\lim_{n\to+\infty} |u_n(\omega,.)|^p_{L^p(\Omega,\mathbb{R}^m)}=\text{meas}(\Omega)\, E^{\mathcal{F}_k} \fint_{]0,k[^d} |u(.,x)|^p dx,$$

actually, this assersion holds under (2.5) and (2.6) if $p\neq 1$;

(ii) For every interval Q with vertices in $\mathbb{Q}^d$,

$$\int_{Q\cap\Omega} u_n(\omega, x)dx$$

converges towards

$$\text{meas}(Q\cap\Omega)\, E^{\mathcal{F}_k} \fint_{]0,k[^d} u(.,x)dx;$$

(iii) In the specific case $p=1$, if $(\tau_z)_{z\in k\mathbb{Z}^d}$ is ergodic, the sequence $(u_n(\omega,.))_{n\in\mathbb{N}}$ is equiintegrable, that is: for every $\varepsilon>0$, there exists $\eta>0$ (possibly depending on fixed ω) such that, for every Borel subset A of Ω with meas(A)$<\eta$,

$$\sup_n \int_A |u_n(\omega, x)|dx<\varepsilon.$$

Remark 2.8. In the case $p=1$ (i) and (iii) in above lemma imply uniform integrability of the sequence $(u_n(\omega,.))_{n\in\mathbb{N}}$, that is equivalent by Dunford-Pettis theorem, to the relative compacity of the sequence $(u_n(\omega,.))_{n\in\mathbb{N}}$ for the weak topology of $L^1(\Omega,\mathbb{R}^m)$. Assertion (ii) allows us to identify the weak limit.

Assuming for the moment lemma 2.7, let us prove theorem 2.6. For this, ω is a fixed element in Σ' and the technique reproduced bellow is then classical (see Dacorogna [3, p.19-20] for instance)

Proof of Theorem 2.6. *Proof of (i)* ($1<p\leq+\infty$). Even if it means doing the sequence $(u_n(\omega,.))_{n\in\mathbb{N}}$ substitute for

$$u_n(\omega,.) - E^{\mathcal{F}_k} \fint_{]0,k[^d} u(.,x)dx,$$

thanks to (ii) of lemma 2.7, it suffices, under hypothesis

$$\lim_{n\to+\infty}\int_{Q\cap\Omega} u_n(\omega, x)dx=0$$

for every interval Q with vertices in $\mathbb{Q}^d$, to prove that for every v in $L^{p'}(\Omega, \mathbb{R}^m)$

$$\lim_{n\to+\infty}\int_{\Omega} u_n(\omega,x)v(x)dx=0$$

where

$$p'=\begin{cases}\frac{p}{p-1} \text{ if } p\neq+\infty,\\ 1 \text{ if } p=+\infty.\end{cases}$$

Let v in $L^{p'}(\Omega, \mathbb{R}^m)$, $\varepsilon>0$ and $\sum_{i\in I}\alpha_i\chi_{Q_i\cap\Omega}$ a step function, Q_i being any interval with vertices in $\mathbb{Q}^d$, such that

$$|v-\sum_{i\in I}\alpha_i\chi_{Q_i\cap\Omega}|_{L^{p'}(\Omega, \mathbb{R}^m)}<\varepsilon.$$

We get

$$\left|\int_{\Omega}u_n(\omega, x).v(x)\ dx\right|\leq|u_n(\omega, .)|_{L^p(\Omega, \mathbb{R}^m)}|v-\sum_{i\in I}\alpha_i\chi_{Q_i\cap\Omega}|_{L^{p'}(\Omega, \mathbb{R}^m)}+$$

$$+\left|\sum_{i\in I}\alpha_i\int_{Q_i\cap\Omega}u_n(\omega, x)dx\right|$$

$$\leq C\varepsilon+\sum_{i\in I}|\alpha_i|\left|\int_{Q_i\cap\Omega}u_n(\omega, x)dx\right|,$$

C denoting any constant that does not depends on n. After making n tends to $+\infty$ and using (ii) of lemma 1.7, we get

$$\lim_{n\to+\infty}\left|\int_{\Omega}u_n(\omega, x)v(x)dx\right|\leq C\ \varepsilon.$$

Proof of (ii) (p=1). Thanks to equiintegrability (iii) of lemma 2.7, for $\varepsilon>0$, there exists $\delta>0$ such that

$$\sup_n\int_{\Omega_\delta}|u_n(\omega, x)|dx<\varepsilon, \tag{2.7}$$

where

$$\Omega_\delta = \bigcap_n \{x \in \Omega,\ |u_n(\omega, x)| > \delta\}.$$

On the other hand, let $v \in L^\infty(\Omega, \mathbb{R}^m)$ and $\sum_{i\in I} \alpha_i \chi_{Q_i \cap \Omega}$ a step function like in proof of (i) such that

$$(2.8)\qquad \begin{cases} \left| v - \sum_{i\in I} \alpha_i \chi_{Q_i\cap\Omega} \right|_{L^1(\Omega,\, \mathbb{R}^m)} < \dfrac{\varepsilon}{\delta}, \\ |\alpha_i| \le C. \end{cases}$$

where C is a positive constant that depends only on $|v|_{L^\infty(\Omega,\, \mathbb{R}^m)}$. We get, like in proof of (i):

$$\left| \int_\Omega u_n(\omega, x) v(x) dx \right| \le \int_\Omega |u_n(\omega, x)| \left| v(x) - \sum_{i\in I} \alpha_i \chi_{Q_i}(x) \right| dx +$$

$$+ C \sum_{i\in I} \left| \int_{Q_i \cap \Omega} u_n(\omega, x) dx \right|$$

$$\le \int_{\Omega_\delta} |u_n(\omega,x)| \left(|v(x)| + \sum_{i\in I} |\alpha_i| \right) dx +$$

$$+ \delta \int_{\Omega \setminus \Omega_\delta} \left| v - \sum_{i\in I} \alpha_i \chi_{Q_i\cap\Omega} \right| dx +$$

$$+ C \sum_{i\in I} \left| \int_{Q_i \cap \Omega} u_n(\omega, x) dx \right|,$$

so by (2.7) and (2.8)

$$\left| \int_\Omega u_n(\omega, x) v(x) dx \right| \le C\varepsilon + \varepsilon + C \sum_{i\in I} \left| \int_{Q_i \cap \Omega} u_n(\omega, x) dx \right|.$$

We conclude thanks to (ii) of lemma 2.7 after making n tends to $+\infty$. ∎

It remains to prove Lemma 2.7.

Proof of (i). It suffices to applied the first step of the proof of theorem 2.4 where $\mu(\omega):=|u(\omega,.)|^p\,dx$. (Note that (2.5) or (2.4.5) if p=1 and (2.6) imply respectively (2.1) and (2.2)).

Proof of (ii). We apply second step of theorem 2.4 to each measure $\mu_i(\omega):=u^i(\omega,.)dx$ $i=1,\ldots,m$, where $u^i(\omega,.)$ are components of $u(\omega,.)$. (Note that if $p\neq 1$ (2.4) and (2.5) imply (2.1) thanks to Hölder inequality).

Proof of (iii) . Raisoning on each component, one can assume $u(\omega,.)$ be a scalar valued function. Let $\delta\in\mathbb{Q}^+$ a truncation parameter destined to tend to $+\infty$ and set

$$u_\delta(\omega,x)=\begin{cases}\min(\delta,\ u(\omega,x)) & \text{if } u(\omega,x)\geq 0\\ -\min(\delta,\ -u(\omega,x)) & \text{if } u(\omega,x)<0,\end{cases}$$

$$v_\delta(\omega,x)=u(\omega,x)-u_\delta(\omega,x).$$

It is straightforward to check that v_δ satisfies (2.4.5) and (2.6). Applying (i) proved above to v_δ, we get existence of Σ_δ in $\mathcal{T}$ with $P(\Sigma_\delta)=1$ such that, for ω in $\Sigma':=\bigcap_{\delta\in\mathbb{Q}^+}\Sigma_\delta$

$$\lim_{n\to+\infty}\int_\Omega |v_{\delta,n}(\omega,x)|\,dx=\text{meas}(\Omega)\,\mathbb{E}⨍_{]0,k[^d}|v_\delta(.,x)|\,dx. \tag{2.9}$$

For $\varepsilon>0$, let δ be large enough so that

$$\mathbb{E}⨍_{]0,k[^d}|v_\delta(.,x)|\,dx<\frac{\varepsilon}{2}.$$

Indeed by (2.4.5), the map $(\omega,x)\mapsto u(\omega,x)$ belongs to $\mathcal{L}^1(\Sigma\times]0,k[^d,\ dP\otimes dx)$ and

$$\int_\Sigma ⨍_{]0,k[^d}|v_\delta(\omega,x)|\,dP(\omega)\,dx=\frac{2}{k^d}\int_{\{(\omega,x),\ |u(\omega,x)|>\delta\}}|u(\omega,x)|\,dP(\omega)dx.$$

So, recalling (2.9), for a fixed ω in Σ', we get existence of $N(\varepsilon)\in\mathbb{N}$ such that

$$\sup_{n>N(\varepsilon)}\int_\Omega |v_{\delta,n}(\omega,x)|\,dx<\frac{\varepsilon}{2}. \tag{2.10}$$

Let A be a measurable subset of Ω. For $n>N(\varepsilon)$ and by (2.10) we have

$$\int_A |u_n(\omega, x)|dx \le \int_A |v_{\delta,n}(\omega, x)|dx + \int_A |u_{\delta,n}(\omega, x)|dx$$

$$<\frac{\varepsilon}{2}+\delta \text{ meas}(A),$$

and, for meas(A)$<\frac{\varepsilon}{2\delta}=\eta$ we obtain

$$\sup_{n>N(\varepsilon)} \int_A |u_n(\omega, x)|dx<\varepsilon. \blacksquare$$

2.5 Application to stochastic homogenization of non convex integral functionals

Let $M^{m\times d}$ be the space of real m×d matrices, α, β and L three given positive constants and $(\tau_z)_{z\in \mathbb{Z}^d}$ an ergodic group of P-preserving transformation (note that it is easy to show that $(\tau_z)_{z\in k\mathbb{Z}^d}$ is ergodic for every $k\in \mathbb{N}$) in a probability space $(\Sigma, \mathcal{T}, P)$, $\mathcal{T}$ being P-complete. For every ω in Σ, every A in $\mathcal{O}$, the family of bounded open subsets in $\mathbb{R}^d$, every u in $W^{1,p}(A, \mathbb{R}^m)$ and every p; $1<p<+\infty$, set

$$F(\omega)(u, A):=\int_A f(\omega)(x, \nabla u(x))dx$$

where, $f(\omega)$ is a real function defined in $\mathbb{R}^d \times M^{m\times d}$, measurable with respect to its first variable and satisfying for every a, b in $M^{m\times d}$, almost surely and almost everywhere

(2.11) $\alpha|a|^p \le f(\omega)(x, a) \le \beta(1+|a|^p)$;

(2.12) $|f(\omega)(x, a)-f(\omega)(x, b)| \le L(1+|a|^{p-1}+|b|^{p-1})|a-b|$;

(2.13) $f(\tau_z\omega)(x, a)=f(\omega)(\tau_z x, a)$ for every z in $\mathbb{Z}^d$.

Let us remark that, thanks to continuity property (2.12) on $f(\omega)$, the map $u \mapsto F(\omega)(u, A)$ from $W^{1,p}(A, \mathbb{R}^m)$ into $\mathbb{R}$ is continuous for the strong topology of $W^{1,p}(A, \mathbb{R}^m)$.

We assume moreover that the map $\omega \mapsto F(\omega)$ from Σ into $\mathbb{R}^{W^{1,p}_{loc}(\mathbb{R}^d, \mathbb{R}^m)\times \mathcal{O}}$ is measurable, the second space being equipped with the standart product σ-field, so that all maps $\omega \mapsto F(\omega)(u, A)$, A in $\mathcal{O}$, u in $W^{1,p}(A, \mathbb{R}^m)$ are random variables. A sufficient condition for a such measurability is the $(\mathcal{T}\otimes \mathcal{B}(\mathbb{R}^d)\otimes \mathcal{B}(M^{m\times d}), \mathcal{B}(\mathbb{R}))$ measurability of the map $(\omega, x, a) \mapsto f(\omega)(x, a)$ from $\Sigma\times\mathbb{R}^d\times M^{m\times d}$ into $\mathbb{R}$.

Ω being a given bounded convex open set in $\mathbb{R}^d$ with piecewise $C^{0,1}$ boundary, and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}^{*+}$ which tends to 0, consider the sequence $(F_n(\omega)(.,\ \Omega))_{n\in\mathbb{N}}$ defined by

$$F_n(\omega)(u,\Omega)=\int_\Omega f(\omega)(\frac{x}{\varepsilon_n}, \nabla u(x))dx \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^m).$$

Our purpose is then to establish the following theorem which is an essential step in the proof of epi-convergence of the sequence $(F_n(.,\ \Omega))_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega, \mathbb{R}^m)$ equipped with the strong topology of $L^p(\Omega, \mathbb{R}^m)$.

Theorem 2.9. There exists $v_n(\omega)$ in $W^{1,p}(\Omega, \mathbb{R}^m)$ such that, almost surely

$$\begin{cases} \lim\limits_{n\to+\infty} v_n(\omega)=l_a & \text{in } L^p(\Omega, \mathbb{R}^m) \text{ strong,} \\ \lim\limits_{n\to+\infty} F_n(\omega)(v_n(\omega))=\operatorname*{Inf}\limits_{k\in\mathbb{N}^*} E\frac{1}{k^d}\mathcal{M}_{kY}(F(.), a), \end{cases}$$

where l_a is the linear function from $\mathbb{R}^d$ into $\mathbb{R}^m$ defined by $l_a(x)=a.x$, $a\in M^{m\times d}$ and

$$\mathcal{M}_{kY}(F(\omega), a):=\operatorname{Inf}\left\{F(\omega)(u+l_a, kY)\ ;\ u\in W_0^{1,p}(kY, \mathbb{R}^m)\right\},$$

Y being the unit cube $]0, 1[^d$.

For proving above theorem, we shall need some measurability properties about the closed set valued multifunction Γ: $\Sigma \rightrightarrows W_0^{1,p}(kY, \mathbb{R}^m)$ defined by:

$$\Gamma(\omega):=\{u\in W_0^{1,p}(kY, \mathbb{R}^m)\ ;\ F(\omega)(u+l_a, kY)\le \mathcal{M}_{kY}(F(\omega), a)+\eta\},$$

where η belongs to $\mathbb{Q}^{*+}$.

Proposition 2.10. The closed set valued multifunction Γ is measurable and consequently, possesses a $(\mathcal{T},\ \mathcal{B}(W_0^{1,p}(kY,\ \mathbb{R}^m)))$ measurable selection $\omega\mapsto u_{k,\eta}(\omega)$ from Σ into $W_0^{1,p}(kY, \mathbb{R}^m)$.

Proof. Continuity of the map $u\mapsto F(\omega)(u, kY)$ and measurability hypothesis on F imply that $(\omega, u)\mapsto F(\omega)(u + l_a, kY)$ is a Caratheodory function from $\Sigma\times W^{1,p}(kY, \mathbb{R}^m)$ into $\mathbb{R}$. $W_0^{1,p}(kY)$

being separable, by a classical argument the map $\omega \mapsto \mathcal{M}_{kY}(F(\omega), a)$ from Σ into $\mathbb{R}$ is a random variable.

On the other hand, the Caratheodory function $(\omega, u) \mapsto F(\omega)(u+l_a, kY)$ from $\Sigma \times W_0^{1,p}(kY, \mathbb{R}^m)$ into $\mathbb{R}$ is actually a normal integrand: it is a direct consequence of the separability of $W_0^{1,p}(kY, \mathbb{R}^m)$. The level set multifunction constructed from this normal integrand and the random variable $\omega \mapsto \mathcal{M}_{kY}(F(\omega), a)$ is then measurable and possesses a measurable selection (see C.Castaing&M.Valadier [2], C.Hess [4], J.P.Aubin & H.Frankowska [1]).∎

Applying proposition 2.10, there exists a $(\mathcal{T}, \mathcal{B}(W_0^{1,p}(kY, \mathbb{R}^m)))$ measurable selection $\omega \mapsto u_{k,\eta}(\omega)$ from Σ into $W_0^{1,p}(kY, \mathbb{R}^m)$ such that

$$(2.14) \qquad \mathcal{M}_{kY}(F(\omega), a) \leq F(\omega)(u_{k,\eta}(\omega)+l_a, kY) \leq \mathcal{M}_{kY}(F(\omega), a)+\eta.$$

Let us extend $u_{k,\eta}(\omega)$ on $W_{loc}^{1,p}(\mathbb{R}^d, \mathbb{R}^m)$ in the following way:

$$\tilde{u}_{k,\eta}(\omega, x) := u_{k,\eta}(\tau_z \omega, x-z) \text{ if } x \text{ belong to } z+kY.$$

It is clear that

$$(2.15) \qquad \tilde{u}_{k,\eta}(\omega, x+z) = \tilde{u}_{k,\eta}(\tau_z \omega, x) \text{ for every } z \text{ in } k\mathbb{Z}^d.$$

On the other hand, noticing that on A in $\mathcal{B}_b(\mathbb{R}^d)$

$$\tilde{u}_{k,\eta}(\omega, .) = \sum_{z \in I(A)} \chi_{z+kY \cap A}\, u_{k,\eta}(\tau_z \omega, .-z),$$

where $I(A) = \{z \in k\mathbb{Z}^d \ ; \ z+kY \cap A \neq \emptyset\}$ and χ_E is the caracteristic function of any set E, we obtain by proposition 2.10, the measurability of $\tilde{u}_{k,\eta}$ from Σ into $W^{1,p}(A, \mathbb{R}^m)$.

Finally, we define a map g from $\Sigma \times \mathbb{R}^d$ into $\mathbb{R}$ by:

$$g(\omega, x) := f(\omega)(x, a+\nabla \tilde{u}_{k,\eta}(\omega, x)),$$

and a test function $v_{k,\eta,n}(\omega, x)$ by

$$v_{k,\eta,n}(\omega, x) := ax + \varepsilon_n \tilde{u}_{k,\eta}(\omega, \frac{x}{\varepsilon_n}).$$

We have

Proposition 2.11. There exists $\Sigma_{k,\eta}$ in Σ with $P(\Sigma_{k,\eta})=1$ such that, the two following assertions hold

(i) $\lim_{n\to+\infty} v_{k,\eta,n}(\omega)=l_a$ strongly in $L^p(A, \mathbb{R}^m)$,

(ii) $\lim_{n\to+\infty} F_n(\omega)(v_{k,\eta,n}, A)=\lim_{n\to+\infty}\int_A g_n(\omega, x)dx$

$$=\text{meas}(A)\, E \fint_{kY} f(\omega)(x, a+\nabla\tilde{u}_{k,\eta}(\omega, x))\, dx.$$

Proof. *Proof of (i).* It suffices to apply lemma 2.7 (i) to the function $\tilde{u}_{k,\eta}(\omega, .)$. We must prove that the map $\omega\mapsto\int_A |\tilde{u}_{k,\eta}(\omega, .)|^p dx$ belongs to $\mathcal{L}^1(\Sigma, \mathcal{T}, P)$ for every A in $\mathcal{B}_b(\mathbb{R}^d)$ (see (2.5)), (2.6) being satisfied thanks to (2.15). Measurability is a direct consequence of measurability of $\omega\mapsto\tilde{u}_{k,\eta}(\omega, .)$ from Σ into $W^{1,p}(A, \mathbb{R}^m)$. Moreover

$$(2.16)\quad \int_\Sigma\int_A |\tilde{u}_{k,\eta}(\omega, x)|^p dx\, dP(\omega)\le\int_\Sigma\int_{\bigcup\limits_{z\in I(A)} z+kY} |\tilde{u}_{k,\eta}(\omega, x)|^p dx dP(\omega)$$

$$=\int_\Sigma \sum_{z\in I(A)}\int_{kY} |\tilde{u}_{k,\eta}(\omega, x+z)|^p dx dP(\omega)$$

$$=\sum_{z\in I(A)}\int_{kY}\int_\Sigma |\tilde{u}_{k,\eta}(\tau_z\omega, x)|^p dP(\omega)dx$$

$$=\text{card } I(A)\int_{kY}\int_\Sigma |u_{k,\eta}(\omega, x)|^p dP(\omega)dx,$$

where we have used the P-preserving property of $(\tau_z)_{z\in k\mathbb{Z}^d}$.

Using Poincaré inequality, growth condition (2.11) imposed on $f(\omega)$ and definition of $u_{k,\eta}(\omega)$ (see (2.14)), we get

$$\int_{kY} |u_{k,\eta}(\omega, x)|^p dx\le C\frac{\beta}{\alpha}(1+|a|^p)\text{meas}(kY)+\eta,$$

where C is the Poincaré constant.

So that (2.16) implies

$$\int_\Sigma \int_A |\tilde{u}_{k,\eta}(\omega, x)|^p dx dP(\omega) < +\infty.$$

proof of (ii). We apply theorem 2.4 (ii) of part 2.3 to the random Borel measure μ on Σ defined by:

$$\mu(\omega) = g(\omega, .)dx.$$

We must prove (2.1) and (2.2) . (2.2) is a direct consequence of (2.13), (2.15) and definition of g. The ($\mathcal{T}$, $\mathcal{B}(\mathbb{R})$) measurability of

$$\omega \mapsto \int_A g(\omega, x)dx$$

from Σ into $\mathbb{R}$ comes from the ($\mathcal{T}$, $\mathcal{B}(W^{1,p}(A, \mathbb{R}^m))$) measurability of the map $\omega \mapsto u_{k,\eta}(\omega, .)+l_a$ from Σ into $W^{1,p}A, \mathbb{R}^m)$, the fact that $(\omega, u) \mapsto F(\omega)(u, A)$ is a Caratheodory function (see J.P.Aubin & H.Frankowska [1, lemma 8.23, p.311] for instance) and

$$\int_A g(\omega, x)dx = F(\omega)(\tilde{u}_{k,\eta}(\omega, .)+l_a, A).$$

Finally, with growth condition (2.11), after raisoning like above proof of (i),

$$\int_\Sigma \int_A g(\omega, x)dxdP(\omega) \leq \beta \int_\Sigma \int_A (1+|a+\nabla\tilde{u}_{k,\eta}(\omega, x)|^p)dxdP(\omega) < +\infty. \blacksquare$$

We are in position to prove theorem 2.9.

Proof of Theorem 2.9. Proposition 2.11 above implies, for every ω in $\Sigma' = \bigcap_{k,\eta} \Sigma_{k,\eta}$

$$\lim_{k\to+\infty} \lim_{\eta\to 0} \lim_{n\to+\infty} F_n(\omega)(v_{k,n,\eta}(\omega), \Omega) = \lim_{k\to+\infty} \lim_{\eta\to 0} E \fint_{kY} f(\omega)(x, a+\nabla u_{k,\eta}(\omega, .))\, dx \text{ meas}(\Omega)$$

$$= \lim_{k\to+\infty} E \frac{\mathcal{M}_{kY}(F(\omega, a))}{\text{meas}(kY)} \text{ meas}(\Omega)$$

$$= \operatorname{Inf}_{k\in\mathbb{N}^*} E \frac{1}{k^d} \mathcal{M}_{kY}(F(.), a) \text{ meas}(\Omega),$$

where we have used the Lebesgue dominated theorem for convergence with respect to η and the convergence of

$$E\,\frac{\mathfrak{M}_{kY}(F(\omega),\ a)}{\mathrm{meas}(kY)}$$

towards

$$\inf_{k\in\mathbb{N}^*} E\,\frac{1}{k^d}\mathfrak{M}_{kY}\ (F(.),\ a),$$

(see U.Krengel [5, lemma 2.2 p.202] for instance for this last result). We end the proof by a diagonalization argument. ∎

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Chapter III

Non coercive Random Integral Functionals and Epi-convergence.

Chapter III

Non coercive Random Integral Functionals and Epi-convergence.

3.1 Introduction.

The purpose of the present chapter is the study of homogenization of elastic material with many small "holes" or "fissures" distributed in on random way.

To overcome the lack of coerciveness we shall take some hypothesis in relation with geometry of "holes" or "fissures" distribution in such a way that an extension technic can be applied see D.Cioranescu & J.Saint Jean Paulin [4] in the case of "holes", H. Attouh &F.Murat [3] and J.J.Telega & T.Lewinski [5] in the case of "fissures".

For each both problems, the probabilised space $(\Sigma, \mathcal{T}, P)$ being given, we build a class of random integral functionals which has the required properties.

Finally we apply method developped in chapter I to obtain in each case the epi limit expression and shall give a result of almost sure convergence for corresponding optimization problems. These results generalize ones obtained by H.Attouch [2] and D.Cioranescu & J.Saint Jean Paulin [4] for "holes" and H.Attouch & F.Murat [3] for "fissures" in the periodic case.

3.2 A random integral functionals related to the problem of "holes" of Neumann type.

Let $Y=]0, 1[^d$ be a unit open cube and Γ a finite set of compact T of Y whose the boundary is C^1 piecewise. For every $z\in \mathbb{Z}^d$, T_z denotes a compact of z+Y such that $-z+T_z\in\Gamma$. We shall consider, the family $(T_z)_{z\in\mathbb{Z}^d}$ as an element ω of $\Sigma:=\Gamma^{\mathbb{Z}^d}$ or as a part $\bigcup_{z\in\mathbb{Z}^d} T_z$ of $\mathbb{R}^d$.

Let $(\Sigma, \mathcal{T}, P)$ be a probability space which is the product of Bernoulli's space $(\Gamma, \mathcal{P}, \Pi)$, where $\mathcal{P}$ denotes the set of all parts of Γ, and Π is a probability measure on Γ constructed from the presence probability of element in Γ, that is $(\Sigma, \mathcal{T}, P):=(\Sigma, \mathcal{P}, \Pi)^{\mathbb{Z}^d}$.

For technical reasons, we consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of the type $\varepsilon_n=t^{-n}$, $t\in\mathbb{N}^*$ and $\mathcal{O}$ the family of sets which are a finite union of cubes $\varepsilon_q I$, where q and I varying respectively in $\mathbb{N}$ and $\mathcal{J}$ (recall that $\mathcal{J}$ is the set of all open paving in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$).

Remaks 3.1.For every $A\in\mathcal{O}$, there exists $n_A\in\mathbb{N}^*$ such that for every $n>n_A$, $\partial A\cap\varepsilon_n\omega=\emptyset$.

We define the group $(\tau_z)_{z\in\mathbb{Z}^d}$ of (ergodic, P-preserving) transformations on Σ as follows:

$$\tau_z\omega=\omega+z.$$

It is obvious that for every $A\in\mathcal{O}$ and every $\omega\in\Sigma$ the caracteristic function $\chi_{A\setminus\omega}$ of $A\setminus\omega$ defined by:

$$a(\omega,.):=\chi_{A\setminus\omega}(.),$$

satisfies for every $z\in\mathbb{Z}^d$ and every $x\in A$

$$a(\tau_z\omega, x)=a(\omega, x-z).$$

Let ω be a fixed element in Σ, for every $A\in\mathcal{O}$ and $u\in L^p(A, \mathbb{R}^m)$, we define the integral functional $F(\omega)$ from $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)\times\mathcal{O}$ into $\mathbb{R}^+\cup\{+\infty\}$ by:

$$F(\omega)(u, A)=\begin{cases}\displaystyle\int_{A\setminus\omega} f(\nabla u(x))dx \text{ if } u_{A\setminus\omega}\in W^{1,p}(A\setminus\omega, \mathbb{R}^m)\\ +\infty \text{ if not,}\end{cases}$$

where $u_{A\setminus\omega}$ denotes the restriction of u to $A\setminus\omega$ and f is a given function from $M^{m\times d}$ into $\mathbb{R}$ which satisfies:

α, β and L being three given positive constants, for every a, b in $M^{m\times d}$

(3.1) $\alpha|a|^p\le f(a)\le\beta(1+|a|^p)$ a.e;

(3.2) $|f(a)-f(b)|\le L(1+|a|^{p-1}+|b|^{p-1})|a-b|$ a.e.

We define the sequence $(F_n(\omega))_{n\in\mathbb{N}}$ by:

$$F_n(\omega):=\rho_{\varepsilon_n}F(\omega)=F(\varepsilon_n\omega) \text{ that is}$$

$$F_n(\omega)(u, A)=\begin{cases}\displaystyle\int_{A\setminus\varepsilon_n\omega} f(\nabla u(x))dx \text{ if } u_{A\setminus\varepsilon_n\omega}\in W^{1,p}(A\setminus\varepsilon_n\omega, \mathbb{R}^m),\\ +\infty \quad \text{if not.}\end{cases} \tag{3.3}$$

Our goal is the study by epi-convergence the asymptotic behaviour of the above sequence when ε_n tends to zero.

$$F^{hom}(u, A)=\begin{cases}\displaystyle\int_A f^{hom}(\nabla u(x))dx & \text{if } u\in W^{1,p}(A, \mathbb{R}^m),\\ +\infty & \text{if not.}\end{cases}$$

We shall show that for every $A\in\mathcal{O}$ and $u\in L^p(A, \mathbb{R}^m)$,

$$F^{hom}(., A)=\tau\text{-epi}\lim F_n(\omega)(., A) \text{ almost surely in } L^p(A, \mathbb{R}^m)$$

where τ denotes the strong topology of $L^p(A, \mathbb{R}^m)$.

The two following propositions are essential to show that domain of τ-epi lim $F_n(\omega)$ is $W^{1,p}(A, \mathbb{R}^m)$.

Proposition 3.4. For every $A\in\mathcal{O}$ and every sufficiently large $n\in\mathbb{N}$, there exists a linear continuous operator $\mathcal{P}_n$ from $W^{1,p}(A\setminus\varepsilon_n\omega, \mathbb{R}^m)$ into $W^{1,p}(A, \mathbb{R}^m)$ (which eventualy depends on ω) satisfying, for every $u\in W^{1,p}(A\setminus\varepsilon_n\omega, \mathbb{R}^m)$,

(i) $\mathcal{P}_n u=u$ on $A\setminus\varepsilon_n\omega$;

(ii) $|\mathcal{P}_n u|_{0,A}\leq C|u|_{0,A\setminus\varepsilon_n\omega}$;

(ii) $|\nabla\mathcal{P}_n u|_{0,A}\leq C|\nabla u|_{0,A\setminus\varepsilon_n\omega}$,

where C is a constant depending only on ω.

Proof. We have (cf remark (3.1)) $\partial A\cap\varepsilon_n\omega=\emptyset$ for n large enough so that we can, thanks to the fact that Γ is finished, use the D.Cioranescu & J.Saint.Paulin [4] results.∎

Proposition 3.5. For every $A\in\mathcal{O}$, there exists Σ'' in Σ with $P(\Sigma'')=1$ such that for every $\omega\in\Sigma''$, the sequence $(a_n(\omega))_{n\in\mathbb{N}}$ defined by:

$$a_n(\omega, .):=a(\omega, \frac{\cdot}{\varepsilon_n});$$

converges almost surely towards meas(A)θ, with $\theta=\int_\Sigma \text{meas}(Y\setminus\omega)dP(\omega)$ for the weak-* topology of $L^\infty(A)$.

Proof. It suffices to apply theorem 2.6, chapter II.∎

3.2.2 The main result.

Let $\omega\in\Sigma'''=\Sigma''\cap\Sigma'$ where Σ' and Σ'' are given by theorem 3.2 and proposition 3.5. Since the family $\mathcal{O}$ of sets A is countable, by a classical diagonalization argument, there exists a subsequence of $(F_n(\omega))_{n\in\mathbb{N}}$ still denoted $(F_n(\omega))_{n\in\mathbb{N}}$ such that τ-epi lim $F_n(\omega)(., A)$ exists in $L^p(A, \mathbb{R}^m)$ for all A in $\mathcal{O}$. In that follows, we consider this subsequence and we shall show that, for every A in $\mathcal{O}$

$$F^{hom}(., A)=\tau\text{-epi}\lim F_n(\omega)(., A) \quad\text{in } L^p(A, \mathbb{R}^m)$$

Before proving this result, let us show that almost surely τ-epi lim $F_n(\omega)(., A)$ "lives" in $W^{1,p}(A, \mathbb{R}^m)$:

Proposition 3.6. If we set $G:=\tau$-epi lim $F_n(\omega)(., A)$,then almost surely $\mathrm{dom}G=W^{1,p}(A, \mathbb{R}^m)$.

Proof. Let $u\in \mathrm{dom}\, G=\{u\in L^p(A, \mathbb{R}^m), G(u)<+\infty\}$, then by epi-convergence there exists a sequence $(u_n)_{n\in\mathbb{N}}$ (we omit the dependence on ω); $u_n\in L^p(A, \mathbb{R}^m)$ such that almost surely

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n, \\ \lim\limits_{n\to+\infty} F_n(\omega)(u_n)<+\infty \end{cases}$$

Therefore from (3.1) the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(A\setminus \varepsilon_n\omega, \mathbb{R}^m)$ independently on n. Using proposition 3.4, the sequence $(\mathcal{P}_n u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(A, \mathbb{R}^m)$ independently on n and there exist a subsequence of $(\mathcal{P}_n u_n)_{n\in\mathbb{N}}$ still denoted $(\mathcal{P}_n u_n)_{n\in\mathbb{N}}$ and u^* in $W^{1,p}(A)$ (possibly depending on ω) such that

$$\begin{cases} u^*=\lim\limits_{n\to+\infty} \mathcal{P}_n u_n \text{ for the weak topology of } W^{1,p}(A, \mathbb{R}^m), \\ u^*=\lim\limits_{n\to+\infty} \mathcal{P}_n u_n \text{ for the strong topology of } L^p(A, \mathbb{R}^m) \end{cases}$$

Making n tends to $+\infty$ in the equality,

$$\mathcal{P}_n u_n\, a(\omega, \frac{\cdot}{\varepsilon_n})=u_n\, a(\omega, \frac{\cdot}{\varepsilon_n}),$$

and using proposition 3.5, we obtain $u=u^*\in W^{1,p}(A, \mathbb{R}^m)$ almost surely. Other inclusion is obvious.∎

Theorem 3.7. For all ω in Σ''' and for all A in $\mathcal{O}$, we have

$$F^{hom}(u, A)=\tau\text{-epi}\lim F_n(\omega)(u, A) \quad \text{in } L^p(A, \mathbb{R}^m).$$

Proof. It remains to check the assertions (i) and (ii) in definition of epi-convergence (see definition 1.1 chapter I).

The proof of (i) is like of the proof of lemma 1.9 chapter I, where all cube used are such that a cube of remark 3.3 (i). While to check (ii), it suffices to note that if A=Q is a such open cube of the lattices in $\mathbb{R}^d$ spanned by $]0, \varepsilon_q[^d$ and $u=l_a$, $a\in\mathbb{R}^d$,

$$F^{hom}(u, Q)=\mathrm{meas}(Q)\, f^{hom}(a)=\lim_{n\to+\infty} \mathcal{M}_Q(F_n(\omega), a),$$

and (we omit the dependence on ω) for $v_{n,\varepsilon_q}\in W_0^{1,p}(Q)$ such that

$$F_n(\omega)(v_{n,\varepsilon_q}+l_a, Q)\le\mathcal{M}_Q(F_n(\omega), a)+\varepsilon_q,$$

we have

$$F^{hom}(u, Q)\ge \limsup_{n\to+\infty} F_n(\omega)(v_{n,\varepsilon_q}+l_a, Q)-\varepsilon_q$$

$$\geq \limsup_{n\to+\infty} F_n(\omega)(\mathcal{P}_n v_{n,\varepsilon_q}+l_a, Q)-\varepsilon_q$$

It is obvious that $\mathcal{P}_n v_{n,\varepsilon_q}\in W_0^{1,p}(Q, \mathbb{R}^m)$. Let us define u_{n,ε_q} by

$$u_{n,\varepsilon_q}=u+\mathcal{P}_n v_{n,\varepsilon_q}.$$

A diagonalization argument leads to

$$\begin{cases} u=\tau\text{-}\lim_{n\to+\infty} u_{n,\varepsilon_{q(n)}}, \\ F^{hom}(u, Q)\geq \limsup_{n\to+\infty} F_n(\omega)(u_{n,\varepsilon_{q(n)}}+l_a, Q) \end{cases}$$

Indeed from the Poincare's inequality and proposition 3.4 (iii), there exists a constant C(Y) depending only on $Y=]0, 1[^d$ such that:

$$|\mathcal{P}_n v_{n,\varepsilon_q}|^p_{0,Q}\leq C(Y)\varepsilon_q^p|\nabla \mathcal{P}_n v_{n,\varepsilon_q}|^p_{0,Q}$$

$$\leq C\varepsilon_q^p|\nabla \mathcal{P}_n v_{n,\varepsilon_q}|^p_{0,Q\setminus \varepsilon_n\omega}$$

$$\leq C\varepsilon_q^p(|\nabla v_{n,\varepsilon_q}+a|^p_{0,Q\setminus \varepsilon_n\omega} + |a|^p \text{ meas}(Q))$$

$$\leq C\varepsilon_q^p(\frac{1}{\alpha} F_n(\omega)(v_{n,\varepsilon_q}+l_a, Q)+|a|^p \text{ meas}(Q))$$

$$\leq C\varepsilon_q^p(\frac{\beta}{\alpha}(1+|a|^p)+|a|^p) \text{ meas}(Q)$$

$$=C\varepsilon_q^p \text{ meas}(Q),$$

where C denotes different constants independent on n and ε_q. If u is any function in $W^{1,p}(A, \mathbb{R}^m)$, we conclude like in chapter I.∎

Remark 3.8. The proof of (ii) is also a straightforward application of the theorem 2.9, chapter II where

$$g(\omega, x):=a(\omega, x)f(a+\nabla\tilde{u}_{k,\eta}(\omega, x)).$$

We shall show that almost sure epi-convergence of the sequence $(F_n(\omega))_{n\in\mathbb{N}}$ implies almost sure convergence of corresponding optimization problems. More precisely

Theorem 3.9. Let Ω be a fixed element in $\mathcal{O}$ and g a given element in $L^p(\Omega, \mathbb{R}^m)$. Then

$$\text{Inf}\{F_n(\omega)(u)+\int_{\Omega\setminus\varepsilon_n\omega} gu\, dx;\ u\in W_0^{1,p}(\Omega, \mathbb{R}^m)\}$$

converges almost surely towards

$$\text{Inf}\{F^{hom}(u)+v\int_{\Omega} gu\, dx;\ u\in W_0^{1,p}(\Omega, \mathbb{R}^m)\},$$

where $v=\text{meas}(\Omega)\theta$; θ is the matimatical expectation of meas$(Y\setminus\omega)$.

Proof. It suffices to apply proposition 3.5 and theorem 3.7 to show that the sequence $(H_n(\omega))_{n\in\mathbb{N}}$ defined on $W_0^{1,p}(\Omega, \mathbb{R}^m)$ by

$$H_n(\omega)(u)=F_n(\omega)(u)+\int_{\Omega\setminus\varepsilon_n\omega} gu\,dx,$$

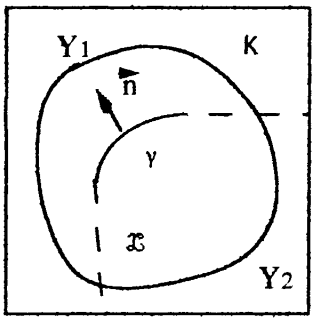
epi-converges almost surely towards

$$H^{hom}(u)=F^{hom}(u)+v\int_{\Omega} gu\,dx \quad \text{in } W_0^{1,p}(\Omega, \mathbb{R}^m),$$

and use variational properties of epi-convergence theorem 1.2 chapter I.∎

3.3 A random integral functionals related to the problem of "fissures".

Let $Y=]0, 1[^d$ be a unit open cube and Γ a finite set of parts γ of $\mathbb{R}^d$ satisfying the following properties:
There is a compact set K with regular boundary such that $\gamma\subset K\subset Y$, there exist two open sets Y_1 and Y_2 in $\mathbb{R}^d$ with common boudary $\mathfrak{S}$, which is a manifold of dimension d-1 and of class C^1 in $\mathbb{R}^d$ such that $Y=Y_1\cup Y_2\cup\mathfrak{S}$ and $\gamma\subset\mathfrak{S}$ (see figure)



$\gamma\subset\mathfrak{S}$

$Y=Y_1\cup Y_2\cup K$

A vector $\vec{n}$ which is normal to $\mathfrak{S}$ being chosen, for all u in

$$W^{1,p}(Y_1, \mathbb{R}^m)\cap W^{1,p}(Y_2, \mathbb{R}^m)=W^{1,p}(Y\setminus\mathfrak{S}, \mathbb{R}^m),$$

we define two traces on γ and therfore, a jump denoted by $[u]$.

For all z in $\mathbb{Z}^d$, γ_z denotes a subset of z+Y such that $-z+\gamma_z$ belongs to Γ. We shall consider, the family $(\gamma_z)_{z\in\mathbb{Z}^d}$ as an element ω of $\Sigma:=\Gamma^{\mathbb{Z}^d}$ or as a part $\bigcup_{z\in\mathbb{Z}^d}\gamma_z$ of $\mathbb{R}^d$. Let $(\Sigma, \mathcal{T}, P)$ be a probability space which is the product of Bernoulli's space $(\Gamma, \mathcal{P}, \Pi)$, where $\mathcal{P}$ denotes the set of all parts of Γ, and Π is a probability measure on Γ constructed from the presence probability of element in Γ that is $(\Sigma, \mathcal{T}, P):=(\Sigma, \mathcal{P}, \Pi)^{\mathbb{Z}^d}$.

We consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, the family $\mathcal{O}$, $\mathcal{J}$ and the group $(\tau_z)_{z\in\mathbb{Z}^d}$ as in part 3.2.

Let ω be a fixed element in Σ, for every $A\in\mathcal{O}$ and $u\in L^p(A, \mathbb{R}^m)$, we define $F(\omega)$ from $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)\times\mathcal{O}$ into $\mathbb{R}^+\cup\{+\infty\}$ by

$$F(\omega)(u, A)=\begin{cases}\displaystyle\int_{A\setminus\omega} f(\nabla u(x))dx & \text{if } u_{A\setminus\omega}\in C(A\setminus\omega)\\ +\infty & \text{if not}\end{cases}$$

where

$$C(A\setminus\omega):=\{u\in W^{1,p}(A\setminus\omega, \mathbb{R}^m);\ [u]\geq 0 \ \text{ in } \omega\cap A\},$$

and f is a given function from $M^{m\times d}$ into $\mathbb{R}$ which satisfies (3.1) and (3.2). We define $F_n(\omega)$ from $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)\times\mathcal{O}$ into $\mathbb{R}^+\cup\{+\infty\}$by:

$$(3.4)\qquad F_n(\omega)(u, A)=\begin{cases}\displaystyle\int_{A\setminus\varepsilon_n\omega} f(\nabla u(x))dx & \text{if } u_{A\setminus\varepsilon_n\omega}\in C(A\setminus\varepsilon_n\omega),\\ +\infty & \text{if not.}\end{cases}$$

Our goal is to study by an epi-convergence method the asymptotic behaviour of above sequence when ε_n goes to zero.

3.3.1 Definition of the integral functional F^{hom}.

For every A in $\mathcal{J}$, every ω in Σ and every a in $\mathbb{R}^d$, we define

$$\mathcal{M}_A(F(\omega), a):=\mathrm{Inf}\{F(\omega)(u+l_a, A);\ u\in C(A\setminus\omega),\ u=0 \text{ on } \partial A\}.$$

A straightforward adaptation of proposition 1.4, 1.5 and 1.6 in chapter I, shows that the set function $Q\mapsto\mathcal{M}_Q(F(\omega), a)$ from $\mathcal{J}$ into $\mathbb{R}^+$ is an ergodic discret sub-additive process.The measurability of $\omega\mapsto\mathcal{M}_A(F(\omega), a)$ comes from equality

$$\mathcal{M}_A(F(\omega), a):=\mathrm{Inf}\{F(\omega)(u+l_a,\ A),\ u\in V\}$$

where $V=\{u\in W^{1,p}(A\setminus \bigcup_{z\in\mathbb{Z}^d} K+z, \mathbb{R}^m)\ ;\quad u=0 \text{ on } \partial A\}$, and the measurability of the multifunction $\omega\rightrightarrows\mathrm{epi}\, F(\omega)(u+l_a, A)$ from Σ into $V\times\overline{\mathbb{R}}$.

Applying the M.A.Ackoglu & U.Krengel [1] subadditive ergodic theorem, we obtain

Theorem 3.10. There exist a subset Σ' in $\mathcal{T}$ with $P(\Sigma')=1$ and a function f^{hom} from $M^{m\times d}$ into $\mathbb{R}$ such that for every cube Q in $\mathcal{J}$

$$f^{hom}(a):=\lim_{n\to+\infty}\frac{\mathcal{M}_{1/\varepsilon_n Q}(F(\omega), a)}{\mathrm{meas}(\frac{1}{\varepsilon_n}Q)}$$

$$=\mathop{\mathrm{Inf}}_{n\in\mathbb{N}^*}\left\{\int_\Sigma \frac{\mathcal{M}_{nY}(F(\omega), a)}{\mathrm{meas}(nY)}\,dP(\omega)\right\}.$$

Moreover f^{hom} satisfies (3.1) and (3.2) with an other constant L' obtained as in proposition 1.4 (ii) of chapter I.

Proof. It is a simple adaptation of the proof of corollary 1.7 chapter I.∎

We now define, for every $A\in\mathcal{O}$ and $u\in L^p(A, \mathbb{R}^m)$, the integral functional F^{hom} from $L^p_{loc}(\mathbb{R}^d, \mathbb{R}^m)\times\mathcal{O}$ into $\mathbb{R}^+\cup\{+\infty\}$ by:

$$F^{hom}(u, A)=\begin{cases}\displaystyle\int_A f^{hom}(\nabla u(x))dx & \text{if } u\in W^{1,p}(A, \mathbb{R}^m),\\ +\infty & \text{if not.}\end{cases}$$

We shall show that for every $A\in\mathcal{O}$ and $u\in L^p(A, \mathbb{R}^m)$,

$$F^{hom}(., A)=\tau\text{-epi}\lim F_n(\omega)(., A) \text{ almost surely in } L^p(A, \mathbb{R}^m) \text{ strong.}$$

From remark 3.1 and the fact that Γ is finished we prove cf H.Attouch & F.Murat [3] the following main tool.

Proposition 3.11. For every $A\in\mathcal{O}$ and every sufficiently large $n\in\mathbb{N}$, there exists a linear continuous operator $\mathcal{Q}_n$ from $W^{1,p}(A\setminus\varepsilon_n\omega, \mathbb{R}^m)$ into $W^{1,p}(A, \mathbb{R}^m)$ (which eventualy depends on ω) satisfying, for every $u\in W^{1,p}(A\setminus\varepsilon_n\omega, \mathbb{R}^m)$,

(i) $\mathcal{Q}_n u=u$ in a neighbourhoud of ∂A,

(ii) $|\mathcal{Q}_n u|_{0,A}\le C|u|_{0,A\setminus\varepsilon_n\omega}$,

(iii) $|\nabla\mathcal{Q}_n u|_{0,A}\le C|\nabla u|_{0,A\setminus\varepsilon_n\omega}$,

(iv) $|\nabla\mathcal{Q}_n u-u|_{0,A}\le C\varepsilon_n|\nabla u|_{0,A\setminus\varepsilon_n\omega}$,

where C is a constant depending only on compact K.

3.3.2 The main result.

Fix ω in Σ' where Σ' is given by theorem 3.10. Since the family $\mathcal{O}$ of sets A is countable, by a classical diagonalization argument, there exists a subsequence of $(F_n(\omega))_{n\in\mathbb{N}}$ still denoted $(F_n(\omega))_{n\in\mathbb{N}}$ such that τ-epi lim $F_n(\omega)(., A)$ exists in $L^p(A, \mathbb{R}^m)$ for all A in $\mathcal{O}$. In that follows, we consider this subsequence and we shall show that, for every A in $\mathcal{O}$

$$F^{hom}(., A)=\tau\text{-epi}\lim F_n(\omega)(., A) \quad\text{in } L^p(A, \mathbb{R}^m).$$

Before proving this result, let us show that almost surely τ-epi lim $F_n(\omega)(., A)$ "lives" in $W^{1,p}(A, \mathbb{R}^m)$.

Proposition 3.12. If we set $G:=\tau$-epi lim $F_n(\omega)(., A)$, then almost surely dom$G=W^{1,p}(A, \mathbb{R}^m)$.

Proof. Let $u\in$ dom G be, then by epi-convergence there exists a sequence $(u_n)_{n\in\mathbb{N}}$ (we omit the dependence on ω), $u_n\in L^p(A, \mathbb{R}^m)$ such that almost surely

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n, \\ \lim\limits_{n\to+\infty} F_n(\omega)(u_n)<+\infty. \end{cases}$$

Therefore from (3.1) the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(A\setminus \varepsilon_n\omega, \mathbb{R}^m)$ independently on n. From (ii) and (iii) in proposition 3.11, the sequence $(\mathcal{Q}_n u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(A, \mathbb{R}^m)$ independently on n. So there exist a subsequence of $(\mathcal{Q}_n u_n)_{n\in\mathbb{N}}$ still denoted $(\mathcal{Q}_n u_n)_{n\in\mathbb{N}}$ and u^* in $W^{1,p}(A)$ (possibly depending on ω) such that

$$\begin{cases} u^*=\lim\limits_{n\to+\infty} \mathcal{Q}_n u_n \text{ for the weak topology of } W^{1,p}(A, \mathbb{R}^m), \\ u^*=\lim\limits_{n\to+\infty} \mathcal{Q}_n u_n \text{ for the strong topology of } L^p(A, \mathbb{R}^m). \end{cases}$$

After making n tends to $+\infty$ in the equality,

$$u_n=\mathcal{Q}_n u_n+u_n-\mathcal{Q}_n u_n$$

and using proposition 3.11 (iv), we obtain $u=u^*\in W^{1,p}(A, \mathbb{R}^m)$ almost surely. Other inclusion is obvious.∎

Theorem 3.13. For all ω in Σ' and all A in $\mathcal{O}$, we have

$$F^{hom}(u, A)=\tau\text{-epi}\lim F_n(\omega)(u, A) \quad \text{in } L^p(A, \mathbb{R}^m).$$

Proof. It is a straightforword adaptation of the proof of theorem 3.7 part 3.2.2.∎

3.4 References.

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Chapter IV

Stochastic Homogenization and Duality in The Convex case.

Chapter IV

Stochastic Homogenization and Duality in The Convex case.

4.1 Preliminaries.

In this chapter, we study the asymptotic behaviour of the classical perturbed optimization problem when $f(\omega, x, .)$ is convex, leading to the limit of its dual formulation. We get in this way, the structural equation $\sigma\in\partial f^{hom}(e(u))$ where $e(u)=\frac{\nabla u+{}^t\nabla u}{2}$, which links the limits u and σ of solutions of *primal and dual* problems corresponding to $F_n(\omega)$. We adopt again an epi-convergence process on the sequence of perturbed functionals, which provides almost sure weak convergence of the saddle points sequence towards the saddle point of Lagrangian of the homogenized problem.

The situation and notations are the same as in chapter I, but here d=m and more specifically, we study the asymptotic behaviour of the dual formulation of the problem

$$(\mathcal{P}_n)\quad \operatorname{Inf}\{F_n(\omega)(u, \Omega)+\Phi(u);\ \ u\in V\}$$

and asymptotic behaviour of corresponding saddle points in linearized elasticity, which is introduced by D.Azé [3] in the periodic case. We assume that

$$F(\omega)(u, A)=\int_A f(\omega)(x, e(u)(x))dx,$$

where $e(u)=\frac{\nabla u+{}^t\nabla u}{2}$, and where $f(\omega)$ is measurable on x, convex with respect to the matrix variable and satisfies almost surely the following condition, for every a in the subspace $M_s^{d\times d}$ of symetric elements of $M^{d\times d}$

$$\alpha|a|^p\leq f(\omega)(x, a)\leq\beta(1+|a|^p) \tag{4.1}$$

α and β are two given positive real numbers, with $0<\alpha\leq\beta$. It is easy to see that (1.2) of the section 1.3 in the chapter I is automatically satisfied. Indeed, every σ that belongs to the subdifferential $\partial f(\omega)(x, a)$ satisfies $|\sigma|\leq C(1+|a|^{p-1})$ where C is a constant dependind only on β (see H.Attouch [1], p.52 for p=2 or B. Dacorogna [5] in a more general setting) and with this bound, the convexity inequality leads to (1.2).

V will be the space $W_0^{1,p}(\Omega, \mathbb{R}^d)$ and Φ, the functional defined by

$$\Phi(u)=\int_\Omega \varphi(x)u(x)dx,$$

where φ is any function that belongs to $L^{p'}(\Omega, \mathbb{R}^d)$, and p' denotes the conjugate exponent of p. Thanks to Korn inequality (cf G.Duvaut & J.Lions [6] for instance),

$$\left(\int_A |u(x)|^p\, dx + \int_A |e(u)(x)|^p\, dx\right)^{1/p}$$

defines an equivalent norm in $W^{1,p}(A, \mathbb{R}^d)$ still denoted $|u|_{1,A}$.

With these new hypothesis, one could obtain similar results of chapter I for function of the form

$$F(\omega)(u, A) = \int_A f(\omega)(x, e(u)(x))dx$$

and infimum become minimum.

A classical way to perturb our optimization problem, is to define for every A in $\mathcal{O}$ the following bivariate functional $\Psi_n(\omega)(., A)$ from $W^{1,p}(A, \mathbb{R}^d)\times\Sigma(A)$ into $\mathbb{R}$

$$\Psi_n(\omega)((u,\sigma), A) := \int_A f(\omega)\left(\frac{x}{\varepsilon_n}, e(u)(x)+\sigma(x)\right)dx + \int_\Omega \varphi(x)u(x)dx,$$

where

$$\Sigma(A) := \left\{\sigma: A \to M_s^{d\times d};\ \ \sigma=(\sigma_{i,j}),\ \ \sigma_{i,j}=\sigma_{j,i},\ \ \sigma_{i,j}\in L^{p'}(A),\ \ i, j=1,\dots,d\right\}.$$

The primal $(\mathcal{P}_n)$ and dual $(\mathcal{P}_n)^*$ problems for a fixed element Ω in $\mathcal{O}$, take the form:

$$(\mathcal{P}_n) \qquad \min\left\{\Psi_n(\omega)((u, 0), \Omega);\ \ u\in W_0^{1,p}(\Omega, \mathbb{R}^d)\right\};$$

$$(\mathcal{P}_n)^* \qquad \sup\left\{-\Psi_n^*(\omega)((0, \sigma), \Omega);\ \ \sigma\in\Sigma(\Omega)\right\}$$

$$= \min\left\{\int_\Omega f^*(\omega)\left(\frac{x}{\varepsilon_n}, \sigma(x)\right)dx;\ \operatorname{div}\sigma=\varphi,\ \ \ \sigma\in\Sigma(\Omega)\right\},$$

where $\Psi_n^*(\omega)(., \Omega)$ and $f^*(\omega)$ denote the *Fenchel conjugates* of $\Psi_n(\omega)(., \Omega)$ and $f(\omega)$.

Similarly the following perturbation of the homogenized limit problem difined in part 1.5 of the chapter I

$$\Psi^{hom}((u,\sigma), A) := \int_A f^{hom}(e(u)(x)+\sigma(x))dx + \int_A \varphi(x)\, u(x)dx$$

leads to the primal $(\mathcal{P}^{hom})$ and dual $(\mathcal{P}^{hom})^*$ problems

$$(\mathcal{P}^{hom}) \qquad \min\left\{\Psi^{hom}((u, 0), \Omega);\ \ u\in W_0^{1,p}(\Omega, \mathbb{R}^d)\right\};$$

$$(\mathcal{P}^{hom})^* \qquad \sup\left\{-\,(\Psi^{hom})^*((0, \sigma), \Omega);\ \ \sigma\in\Sigma(\Omega)\right\}$$

$$= \min\left\{\int_\Omega (f^{hom})^*\ (\sigma(x))dx;\ \ \operatorname{div}\sigma=\varphi;\ \ \sigma\in\Sigma(\Omega)\right\}.$$

$u_n(\omega)$ and $\sigma_n(\omega)$ being respectively a solution of $(\mathcal{P}_n)$ and $(\mathcal{P}_n)^*$, $(u_n(\omega), \sigma_n(\omega))$ is a saddle point of the associated *Lagrangian* defined from $W^{1,p}(\Omega, \mathbb{R}^d)\times\Sigma(\Omega)$ into $\mathbb{R}$ by

$$L_n(\omega)((u,\sigma)) = -\,\Psi_n^{*/\sigma}(\omega)((u, \sigma), \Omega)$$

$$=\int_\Omega \sigma(x):e(u)(x)dx - \int_\Omega \varphi(x)u(x)dx - \int_\Omega f^*(\omega)(\frac{x}{\varepsilon_n}, \sigma(x))dx$$

where $\Psi_n^{*/\sigma}(\omega)(., \Omega)$ denotes the Fenchel conjugate of $\Psi n(\omega)(., \Omega)$ with respect to its second variable.

Finally, if u and σ are respectively solutions of $(\mathcal{P}^{hom})$ and $(\mathcal{P}^{hom})^*$, (u, σ) is a saddle point of the associated Lagrangian

$$L^{hom}(u, \sigma)=\int_\Omega \sigma(x):e(u)(x)dx-\int_\Omega \varphi(x)\ u(x)dx-\int_\Omega (f^{hom})^*)(\sigma(x)dx.$$

For further details about above notions, we refer to I.Ekeland & R.Temam [7].
Let ω be a fixed element in Σ, we have the following result.

Proposition 4.1. Every saddle point $(u_n(\omega), \sigma_n(\omega))$ of the Lagrangian $L_n(\omega)$, is bounded in $W_0^{1,p}(\Omega, \mathbb{R}^d)\times\Sigma(\Omega)$. Therefore, there exists $((u(\omega), \sigma(\omega))$ in $W_0^{1,p}(\Omega, \mathbb{R}^d)\times\Sigma(\Omega)$ such that, to a further subsequence, $(u_n(\omega), \sigma_n(\omega))$ tends towards $((u(\omega), \sigma(\omega))$ in $W_0^{1,p}(\Omega, \mathbb{R}^d)\times\Sigma(\Omega)$, equiped with the product of the weak topology of $W_0^{1,p}(\Omega, \mathbb{R}^d)$ and $L^{p'}(A,M_s^{d\times d})$

Proof. It is easy to show, thanks to the growth condition (4.1) that $u_n(\omega)$ is bounded in $W_0^{1,p}(\Omega, \mathbb{R}^d)$. On the other hand, again by (4.1) and the convexity assumption, one can prove that every element σ that belongs to $\partial f(\omega)(\frac{x}{\varepsilon_n}, e(u_n(\omega)))$ satisfies $|\sigma|\leq C(1+|e(u_n(\omega))|^{p-1})$ which, with the property $\sigma_n(\omega)\in\partial f(\omega)(\frac{x}{\varepsilon_n}, e(u_n(\omega)))$ leads to the conclution.∎

In the classical periodic homogenization the structural equation which links u to σ is given by $\sigma\in\partial f^{hom}(e(u(.)))$, this last equation being obtained by using energy method introduced by L.Tartar [9] and partialy written in F.Murat [8]. We can't adopt this approach in the stochastic case because of the presence of set in $\mathcal{T}$ with null probability that depends on every sequence considered. This is the reason for which we adopt again an epi-convergence process.

We show that almost surely, every cluser point $(u(\omega), \sigma(\omega))$ of a saddle point $(u_n(\omega), \sigma_n(\omega))$ is a saddle point (u, σ) of the Lagrangian L^{hom} and so does not depends on ω and satisfies: $\sigma\in\partial f^{hom}(e(u))$, u and σ are respectively solution of $(\mathcal{P}^{hom})$ and $(\mathcal{P}^{hom})^*$.

The main tool is the following proposition, direct consequence of H.Attouch, D.Aze & R.Wets [2], theorems 2.4 and 3.2.

Proposition 4.2. If ω is a fixed element of Σ such that

$$\Psi^{hom}(.,\Omega)=\tau\times s\text{-epi}\lim \Psi_n(\omega)(.,\Omega),$$

where $\tau\times s$ denotes the topology product of the weak topology of $W_0^{1,p}(\Omega,\mathbb{R}^d)$ and the strong topology of $\Sigma(\Omega)$. Then every cluster point $((u(\omega),\sigma(\omega))$ of proposition 4.1 is a saddle point of L^{hom}.

4.3 **The main result.**

We are now in position to prove the main result of this chapter. Let Σ' be the subset of the probability one defined in corollary 1.7, part 1.4 chapter I. We have

Theorem 4.3. For every ω in Σ',

$$\Psi^{hom}(.,\Omega)=\tau\times s\text{-epi}\lim \Psi_n(\omega)(.,\Omega).$$

Moreover every cluster point $(u(\omega),\sigma(\omega))$, in the sense of the proposition 4.1, of the sequence of saddle point $(u_n(\omega),\sigma_n(\omega))$ of $L_n(\omega)$, is a saddle point of L^{hom}, and so deos not depends on ω. σ is then a solution of the dual problem $(\mathcal{P}^{hom})^*$ where

$$(f^{hom})^*(a)=\operatorname*{Sup}_{n\in\mathbb{N}^*}\frac{1}{n^d}\times\Big(\text{epi}\int_\Sigma \min\Big\{\int_{nY} f(\omega)^*(x,\sigma+.)dx,\quad \sigma\in K(nY)\Big)\Big\}dP(\omega)\Big)(a),$$

where $\text{epi}\int_\Sigma$ denotes the continuous infimal convolution defined by:

$$\Big[\text{epi}\int_\Sigma g(\omega)(.)dP(\omega)\Big](a):=\inf\Big\{\int_\Sigma g(\omega)(a(\omega))dP(\omega);\quad \int_\Sigma a(\omega)dP(\omega)=a\Big\},$$

and where

$$K(nY):=\Big\{\sigma\in\Sigma(nY);\quad \int_{nY}\sigma(y)dy=0,\quad \operatorname{div}\sigma=0\Big\}.$$

Proof. Above expression of $(f^{hom})^*$ is a straightforward consequence of the definition of the Fenchel conjugate, permutation of two sup, property of the continuous infimal convolution which is, in our case the Fenchel conjugate of

$$\int_\Sigma \min\Big\{\int_{nY} f(\omega)(x,e(u)(x)+.)dx,\ u\in W_0^{1,p}(nY,\mathbb{R}^d)\Big\}dP(\omega),$$

and finally, classical expression of Fenchel conjugate of

$$\min\Big\{\int_{nY} f(\omega)(x,e(u)(x)+.)dx,\quad u\in W_0^{1,p}(nY)\Big\},$$

which is

$$\min\left\{\int_{nY} f(\omega)^*(x, \sigma+.\,)dx, \quad \sigma\in K(nY)\right\}.$$

We refer to H.Attouch [1] for this last result and to C.Castaing &M.Valadier [4] for more about continuous infimal convolution.

It remains to prove that

$$\Psi^{hom}(., \Omega)=\tau\times s\text{-epi}\lim \Psi_n(\omega)(., \Omega).$$

Noticing that for every A in $\mathcal{O}$, $u\mapsto\int_A \varphi(x)u(x)dx$ is τ-continuous perturbation so we can neglect the presence of this term in the expression of $\Psi_n(\omega)(., \Omega)$ and $\Psi^{hom}(., \Omega)$ (see theorem 1.2 (ii)).

On the other hand, with this convention we get, when σ is constant

$$\Psi_n(\omega)((u,\sigma), A)=F_n(\omega)(u+l_\sigma, A),$$
$$\Psi^{hom}((u,\sigma), A)=F^{hom}(u+l_\sigma, A).$$

These remarks lead to the following steps:

first step. We prove $\Psi^{hom}(., \Omega)=\tau\times s\text{-epi}\lim \Psi_n(\omega)(., \Omega)$ in $W_0^{1,p}(\Omega, \mathbb{R}^d)\times\mathfrak{E}(\Omega)$ where $\mathfrak{E}(\Omega)$ is the subspace of piecewise constant functions of $\Sigma(\Omega)$.

(i)Upper bound. Let $u=\tau\text{-}\lim_{n\to+\infty} u_n$ and $\sigma=s\text{-}\lim_{n\to+\infty}\sigma_n$ with $(u_n, \sigma_n)\in W_0^{1,p}(\Omega, \mathbb{R}^d)\times\mathfrak{E}(\Omega)$.

We have $\sigma=\sum_{i\in I} a_i\chi_{\Omega_i}$ where $(\Omega_i)_{i\in I}$ is a finite partition of Ω and $u+l_{a_i}=\tau\text{-}\lim_{n\to+\infty} u_n+l_{a_i}$.

So, by theorem 1.8 of chapter I,

$$F^{hom}(u+l_{a_i}, \Omega_i)\le\liminf_{n\to+\infty} F_n(\omega)(u_n+l_{a_i}, \Omega_i),$$

that is

$$\Phi^{hom}((u,\sigma), \Omega_i)\le\liminf_{n\to+\infty}\Phi_n(\omega)((u_n, \sigma), \Omega_i). \tag{4.2}$$

But, by convexity

$$\Phi_n(\omega)((u_n, \sigma_n),\Omega_i)\ge\Phi_n(\omega)((u_n, \sigma), \Omega_i)+\int_{\Omega i} q(\omega)(x, e(u_n)(x)+\sigma(x)):(\sigma_n-\sigma)(x)\, dx \tag{4.3}$$

where $x\mapsto q(\omega)(x, e(u_n)(x)+\sigma(x))$ is an integrable selection of the closed set multivalied function $x\mapsto\partial(\omega)(x, e(u_n)(x)+\sigma(x))$ (for more about integral of set valued maps and existence of integrable selections, we refer to C.Castaing &M.Valadier [4]).

so (4.2) and (4.3), after summing over i, lead to

$$\Phi^{hom}((u, \sigma), \Omega)\le\liminf_{n\to+\infty}\Phi_n(\omega)((u_n, \sigma_n), \Omega)$$

where we have use Hölder's inequality and the estimation

$$|q(\omega)(x, e(u_n)(x)+\sigma(x))|\le C(1+|e(u_n)(x)+\sigma(x)|^{p-1})$$

in the last term of (4.3).

(ii) Lower bound. Let $(u, \sigma)\in W_0^{1,p}(\Omega, \mathbb{R}^d)\times\mathfrak{T}(\Omega)$. By theorem 1.8 of the chapter 1, there exists $v_n^i(\omega)$ in $W^{1,p}(\Omega_i, \mathbb{R}^d)$ such that $u+l_{a_i}=\tau\text{-}\lim_{n\to+\infty} v_n^i(\omega)$ and $v_n^i(\omega)=l_{a_i}$ on $\partial\Omega_i$. Setting $u_n(\omega):=v_n^i(\omega)-l_{a_i}$ in every Ω_i,

$$F^{hom}(u+l_{a_i}, \Omega_i)\geq\limsup_{n\to+\infty} F_n(\omega)(u_n(\omega)+l_{a_i}, \Omega_i).$$

and, after summing over i

$$\begin{cases} u=\tau\text{-}\lim_{n\to+\infty} u_n(\omega) \text{ and } \sigma=\sigma_n, \\ \Phi^{hom}((u,\sigma),\Omega)\geq\limsup_{n\to+\infty} \Phi_n(\omega)((u_n(\omega), \sigma_n), \Omega) \end{cases}$$

Second step. We end the proof by using the s-density of $\mathfrak{T}(\Omega)$ in $\Sigma(\Omega)$, a continuity and a diagonalization argument like in the proof of theorem 1.8 in chapter I.∎

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Chapter V

Open Problems and Partial Results.

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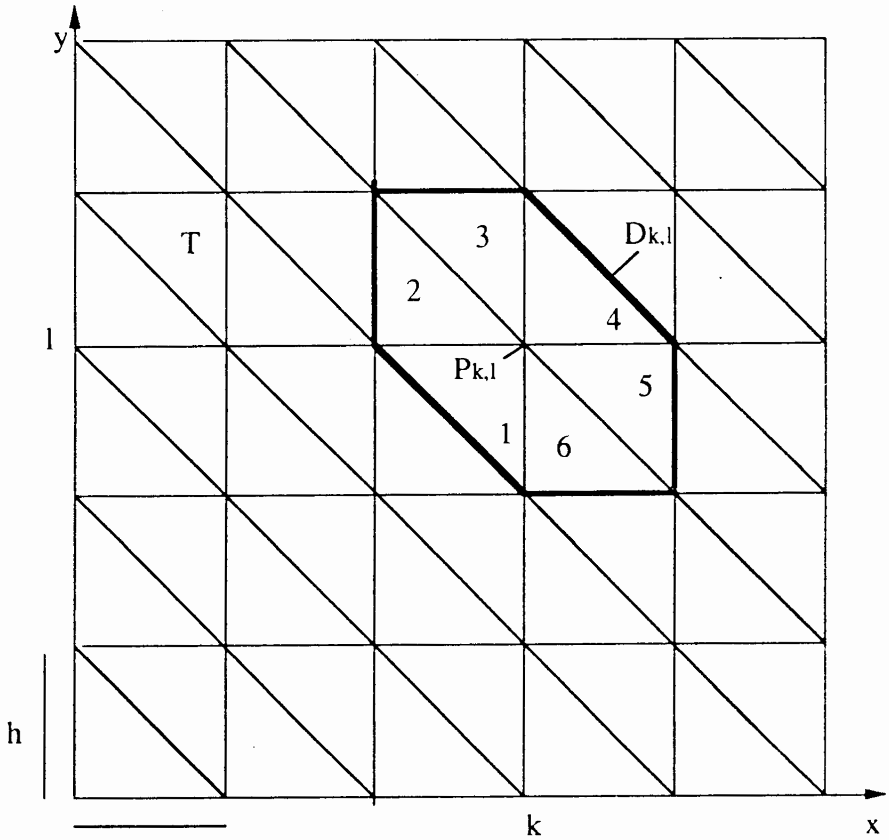
5.1 Numerical approach.

The situation and notations are the same as in chapter I with d=2, m=1, p=2 and $\varepsilon_n=\frac{1}{n}$ $n\in\mathbb{N}^*$. Let Q in $\mathcal{J}$, we subdivise Q on N small cubes of size $h=\frac{1}{N+1}$; $N\in\mathbb{N}$ (the number of interior "points " or "nodes" of Q is N^2), h is the step of the subdivision of Q. (hl, hk) denotes the coordinates of interior points $P_{k,l}$, $(k, l)\in N^{*2}$. T any elementary triangle which decompose Q and $D_{k,l}$ the reunion of six triangles $D^i_{k,l}$ i=1,...,6 of common vertices $P_{k,l}$ (see figure) .



We define by a classical way the family of functions $(\varphi_{k,l})_{k,l\in N}$ by

$$\varphi_{k,l}(P_{i,j})=\begin{cases}1 \text{ if } (i, j)=(k, l),\\ 0 \text{ if not,}\end{cases}$$

$\varphi_{k,l}$ is continuous in Q and coincides with an affine function on each triangle T.

For every A in $\mathcal{J}$, $W^{2,2}(A)$ will denote the set of all functions $u\in L^2(A)$, whose distributional derivatives up to the order 2 belong to $L^2(A)$ and

$$|u|_{2,A}=\left(\int_A \sum_{|\alpha|\le 2} (\partial^\alpha u)^2 dx\right)^{1/2}$$

the norm in $W^{2,2}(A)$, where for every φ in $C^\infty(A)$, $<\partial^\alpha u, \varphi>=(-1)^{|\alpha|}<u, \partial^\alpha\varphi>$; $\partial^\alpha\varphi=\dfrac{\partial^{|\alpha|}\varphi}{\partial^{\alpha_1}x_1\,\partial^{\alpha_2}x_2}$, $\alpha=(\alpha_1, \alpha_2)\in\mathbb{N}^2$, $|\alpha|=\alpha_1+\alpha_2$.

Consider the Sobolev subspace $W_0^{1,2,N}(Q)$ of $W_0^{1,2}(Q)$ generated by $\varphi_{k,l}$,

$$W_0^{1,2,N}(Q)=\mathrm{Vect}\{\varphi_{k,l};\ (k, l)\in N^{*2}\}.$$

The following proposition is classical (see P.A.Raviart & J.M.Thomas [6] for instance).

Proposition 5.1. For every u in $W_0^{1,2}(Q)\cap W^{2,2}(Q)$, there exists u_N in $W_0^{1,2,N}(Q)$ such that

$$|u-u_N|_{1,Q}\le\frac{C}{1+N}|u|_{2,Q},$$

where C is a constant independent on N.

Let us consider the following optimization problems

$$\mathcal{M}_Q(F_n(\omega), a):=\mathrm{Inf}\{F_n(\omega)(u+l_a, Q);\ \ u\in W_0^{1,2}(Q)\}$$

and

$$\mathcal{M}_Q^N(F_n(\omega), a):=\mathrm{Inf}\{F_n(\omega)(u+l_a, Q);\ \ u\in W_0^{1,2,N}(Q)\}.$$

Recall that (cf chapter I) from the M.A.Ackoglu & U.Krengel subadditive ergodic theorem almost surely

$$f^{hom}(a)=\lim_{n\to+\infty}\mathcal{M}_Y(F_n(\omega), a)=\mathrm{Inf}_n\int_\Sigma \mathcal{M}_Y(F_n(\omega), a)\, dP(\omega). \tag{5.1}$$

Using again M.A.Ackoglu & U.Krengel [1], we get almost surely

$$\lim_{n\to+\infty}\mathcal{M}_Y^N(F_n(\omega), a)=\mathrm{Inf}_n\int_\Sigma \mathcal{M}_Y^N(F_n(\omega), a)\, dP(\omega).$$

Set

$$f^{hom,N}(a)=\lim_{n\to+\infty}\mathcal{M}_Y^N(F_n(\omega), a) \text{ for every a in } M^{m\times d}. \tag{5.2}$$

Lemma 5.2. $\lim_{N\to+\infty}\mathcal{M}_Y^N(F_n(\omega), a)=\mathcal{M}_Y(F_n(\omega), a)=\mathrm{Inf}_N\,\mathcal{M}_Y^N(F_n(\omega), a)$.

Proof. Since $W_0^{1,2,N}(Y) \subset W_0^{1,2}(Y)$, almost surely

$$\mathcal{M}_Y(F_n(\omega), a) \le \mathcal{M}_Y^N(F_n(\omega), a),$$

therefore

$$\mathcal{M}_Y(F_n(\omega), a) \le \liminf_{N\to+\infty} \mathcal{M}_Y^N(F_n(\omega), a).$$

On the other hand for ε in $\mathbb{R}^{*+}$ there exists $u_{n,\varepsilon}$ in $W_0^{1,2}(Q) \cap W^{2,2}(Q)$, witch possibly depending on ω such that

$$F_n(\omega)(u_{n,\varepsilon}+l_a, Y) \le \mathcal{M}_Y(F_n(\omega), a)+\varepsilon,$$

and u_N in $W_0^{1,2,N}(Y)$ satisfying (see proposition 5.1)

$$|u_{n,\varepsilon}-u_N|_{1,Y} \le \frac{C}{1+N} |u_{n,\varepsilon}|_{2,Y}. \tag{5.3}$$

Noticing that

$$F_n(\omega)(u_{n,\varepsilon}+l_a, Y) = F_n(\omega)(u_{n,\varepsilon}+l_a, Y) - F_n(\omega)(u_N+l_a, Y) + F_n(\omega)(u_N+l_a, Y)$$
$$\le \mathcal{M}_Y(F_n(\omega), a)+\varepsilon,$$

we get

$$\mathcal{M}_Y^N(F_n(\omega), a) - |F_n(\omega)(u_{n,\varepsilon}+l_a, Y) - F_n(\omega)(u_N+l_a, Y)| \le$$
$$\le \mathcal{M}_Y(F_n(\omega), a)+\varepsilon,$$

and from (1.2) in chapter I

$$\mathcal{M}_Y^N(F_n(\omega), a) - C(1+|\nabla u_N+a|_{0,Y}^2+|\nabla u_{n,\varepsilon}+a|_{0,Y}^2)^{1/2} |\nabla u_{n,\varepsilon}-\nabla u_N|_{0,Y} \le$$
$$\le \mathcal{M}_Y(F_n(\omega), a)+\varepsilon,$$

or equivalently

$$\mathcal{M}_Y^N(F_n(\omega), a) \le \mathcal{M}_Y(F_n(\omega), a)+\varepsilon+R_{\varepsilon,n}^N,$$

where

$$R_{\varepsilon,n}^N = C(1+|\nabla u_N+a|_{0,Y}^2+|\nabla u_{n,\varepsilon}+a|_{0,Y}^2)^{1/2} |\nabla u_{n,\varepsilon}-\nabla u_N|_{0,Y}.$$

Let us show that

$$\lim_{N\to+\infty} R_{\varepsilon,n}^N = 0.$$

For this we use the following estimations (direct consequence of proposition 5.1, and (1.1) chapter I)

$$|\nabla u_{n,\varepsilon}-\nabla u_N|_{0,Y} \le |u_{n,\varepsilon}-u_N|_{1,Y} \le C(\varepsilon, n)\frac{1}{1+N}; \tag{5.4}$$

$$|\nabla u_N+a|_{0,Y} \le |\nabla u_N|_{0,Y}+|a| \le C(\varepsilon, n, a); \tag{5.5}$$

$$\doteq \operatorname*{Inf}_N f^{hom.N}(a).\blacksquare$$

5.2 Random integral functionals in non reflexive case.

Let $(\Sigma, \mathcal{T}, P)$ be a probability space, $\mathcal{O}$ as in chapter I, α, β and γ being three given positive constants; $0<\alpha\leq\beta$. For a fixed element $\omega\in\Sigma$, we define the class $\mathcal{F}$ of functionals $(u, A)\mapsto F(\omega)(u, A)$ from $L^1_{loc}(\mathbb{R}^d)\times\mathcal{O}$ into $\mathbb{R}^{*+}\cup\{+\infty\}$ by

$$F(\omega)(u, A) = \begin{cases} \int_A f(\omega)(x, \nabla u)\, dx \text{ if } u\in W^{1,1}(A) \\ +\infty \text{ if not,} \end{cases}$$

the function $(x, a)\mapsto f(\omega)(x, a)$ from $\mathbb{R}^d\times\mathbb{R}^d$ into $\mathbb{R}$ is measurable on x; convex on a and satisfies a following linear growth condition with respect to a

(5.7) $\qquad \alpha|a|-\gamma \leq f(\omega)(x, a)\leq\beta(1+|a|)$ a.e., for every a in $\mathbb{R}^d$.

We define the sequence $(F_n(\omega))_{n\in\mathbb{N}}$ by

$$F_n(\omega)(u, A)=\begin{cases} \int_A f(\omega)(\frac{x}{\varepsilon_n}, \nabla u)\, dx \text{ if } u\in W^{1,1}(A) \\ +\infty \text{ if not.} \end{cases}$$

Let $\mathcal{B}(\mathcal{F})$ be the trace σ-field on $\mathcal{F}$ of the product σ-field of $\mathbb{R}^{W^{1,1}_{loc}(\mathbb{R}^d)\times\mathcal{O}}$. We shall interest to stochastic homogenization in $L^1(\Omega)$ strong, of the process$(F_n)_{n\in\mathbb{N}}$with a state space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. This type of problem provides its motivation in plastcity theory. Using an R.Temam [8] approximation result stated in G.Bouchitté [3] theorem 2.11, we get an almost sure partial epi-convergence result.

For every A in $\mathcal{O}$, every ω in Σ and every a in $\mathbb{R}^d$, we set

$$\mathcal{M}_A(F(\omega), a):=\mathrm{Min}\{F(\omega)(u+l_a, A);\ u\in W^{1,1}_0(A)\}.$$

Assume that the map $\omega\mapsto F(\omega)$ from Σ into $\mathcal{F}$ is periodic in law and ergodic. As usual, from M.A.Ackoglu & U.Krengel [1] subadditive ergodic theorem, there exist $\Sigma'\subset\Sigma$ with $P(\Sigma')=1$ and $a\mapsto f^{hom}(a)$ from $\mathbb{R}^d$ into $\mathbb{R}$ such that, for every cube Q in $\mathbb{R}^d$ and every ω in Σ'

$$f^{hom}(a):= \lim_{\substack{t\to+\infty \\ t\in\mathbb{R}}} \frac{\mathcal{M}_{tQ}(F(\omega), a)}{\mathrm{meas}(t\,Q)}$$

$$= \operatorname*{Inf}_{n\in\mathbb{N}^*} \{\int_\Sigma \frac{\mathcal{M}_{nY}(F(\omega), a)}{\mathrm{meas}(nY)}\, dP(\omega)\}.$$

Moreover f^{hom} is obviously convex and satisfies (5.7).

For a fixed element Ω in $\mathcal{O}$, let $M_b(\Omega, \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$ valued Radon measures μ with bounded total variation on Ω i.e., the total variation norm

$$\int_\Omega |\mu| = \sup\{<\mu, \varphi>, \quad \varphi \in C_0^\infty(\Omega) \quad |\varphi(x)| \le 1\}$$

is bounded. $<\mu, \varphi> := \int_\Omega \varphi\mu = \int_\Omega \varphi d\mu$ denotes the integral of φ with respect to measure μ. With the above norm, $M_b(\Omega, \mathbb{R}^d)$ is a natural dual of the space $C_0(\Omega, \mathbb{R}^d)$ of continuous functions on Ω with null trace on $\partial\Omega$ equipped with the uniform-convergence norm. $L^1(\Omega, \mathbb{R}^d)$ is a closed subset of $M_b(\Omega, \mathbb{R}^d)$.

Definition 5.4. For every convex function g verifying (5.4), we associate the functional G from $M_b(\Omega, \mathbb{R}^d)$ into $\mathbb{R}$ denoted $\int_\Omega g(\mu)$ and defined by

$$G(\mu) := \sup\{\int_\Omega v\mu - \int_\Omega g^*(v(x))dx; \quad v \in C_0(\Omega, \mathbb{R}^d)\},$$

g^* is the Fenchel conjugate of g.

The space $BV(\Omega)$ of the functions of bounded variation is defined as the space of all functions $u \in L^1_{loc}(\Omega)$ whose distributional gradient, ∇u belongs to $M_b(\Omega, \mathbb{R}^d)$.

Equipped with the norm

$$|u|_{BV(\Omega)} = \int_\Omega |u| dx + \int_\Omega |\nabla u|,$$

$BV(\Omega)$ is a separable non reflexive Banach space, whose $W^{1,1}(\Omega)$ is a closed subset. It is included in $L^p(\Omega)$, with continuous injection if $1 \le p \le \frac{d}{d-1}$. This injection is compact if in addition $1 \le p < \frac{d}{d-1}$. For the general properties of $BV(\Omega)$ we refer to E.Giusti [4], V.G.Maz'ya [5], L.M.Simon [7] and A.I.Vol'pert & S.I.Hudjaev [9]. Let τ be the strong topology of $L^1(\Omega)$.

Proposition 5.5. If τ-epi lim $F_n(\omega)$ exists almost surely, then its domain is $BV(\Omega)$.

Proof. let u be an element in domain of τ-epi lim $F_n(\omega)$, by epi-convergence, there exists a sequence $u_n \in L^1(\Omega)$ such that

$$\begin{cases} u = \tau\text{-}\lim_{n\to+\infty} u_n, \\ \lim_{n\to+\infty} F_n(\omega)(u_n) < +\infty. \end{cases}$$

From (5.7), ∇u_n is bounded in $L^1(\Omega, \mathbb{R}^d)$, therefore (to a near subsequence) converges towards ∇u in $M_b(\Omega, \mathbb{R}^d)$ weak, hence $u \in BV(\Omega)$. For the other inclusion it suffices to use (5.7) and to recall (cf R.Temam [8, p.126] in the general framework of the space $BD(\Omega)$) that, the set $C^\infty(\overline{\Omega}, \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued functions of class C^∞ in $\overline{\Omega}$ is dense in $BV(\Omega)$ in the following sense: for every $u \in BV(\Omega)$, there exists $u_n \in C^\infty(\overline{\Omega})$ such that

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n \text{ in } L^1(\Omega), \\ \nabla u_n \mapsto \nabla u \text{ in } M_b(\Omega, \mathbb{R}^d) \text{ weak (i.e.} \sigma(M_b, C_0)), \\ \int_\Omega |\nabla u_n|\, dx \mapsto \int_\Omega |\nabla u|. \blacksquare \end{cases}$$

We now define the integral functional F^{hom} by setting

$$F^{hom}(u, \Omega)=\begin{cases} \int_\Omega f^{hom}(\nabla u)\, dx \text{ if } u \in BV(\Omega), \\ +\infty \text{ if } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases} \tag{5.8}$$

A plausible conjecture is $F^{hom}=\tau$-epi lim $F_n(\omega)$ almost surely in $L^1(\Omega)$. We now state the partial result.

Theorem 5.6. For every $u \in L^1(\Omega)$, there exists a sequence $u_n \in L^1(\Omega)$ satisfying

$$\begin{cases} u=\tau\text{-}\lim\limits_{n\to+\infty} u_n, \\ F^{hom} \geq \tau\text{-epi} \lim\sup F_n(\omega) \text{ almost surely.} \end{cases}$$

Proof. We invoke our method detailed in chapter I, to get for a fixed ω in Σ'

$$F^{hom} \geq \tau\text{-epi} \lim\sup F_n(\omega) \text{ almost surely in } W^{1,1}(\Omega).$$

Setting

$$\tilde{F}(u, \Omega)=\begin{cases} F^{hom}(u, \Omega) \text{ if } u \in W^{1,1}(\Omega), \\ +\infty \text{ if } u \in L^1(\Omega) \setminus W^{1,1}(\Omega). \end{cases}$$

Then, we obtain almost surely

$$\tilde{F}(u, \Omega) \geq \tau\text{-epi} \lim\sup F_n(\omega)(u, \Omega) \text{ for every } u \text{ in } L^1(\Omega).$$

Going to the lower semicontinuous regularization in above inequality, we get almost surely

$$F^{hom}(u, \Omega) \geq \tau\text{-epi} \lim\sup F_n(\omega)(u, \Omega) \text{ for every } u \text{ in } L^1(\Omega),$$

where we have used cf H.Attouch [2] the lower semicontinuity of τ-epi lim sup $F_n(\omega)$ and the fact that cf R.Temam[8], $F^{hom}(u, \Omega)$ defined in (5.8) is the lower semicontinuous regularization of $\tilde{F}(u, \Omega)$.$\blacksquare$

5.3 Non equi-bounded random integral functionals in non linear elasticity.

It is known that f(ω) given in chapter I (part 1.2) is never convex. On the other hand, it is known that f(ω) is an explicit function $\tilde{f}$ of matrice a, comatrice com(a) and determinant det(a) with

$$\lim_{\det(a)\to 0^+} \tilde{f}(\omega)(a,\mathrm{com}(a),\det(a))=+\infty.$$

In this way, it would be interesting to improve our results by taking a class $\mathfrak{F}$ that contains Integral functionals with such integrand. Monotone troncature process on f(ω), to obt equibounded functions $f_m(\omega)$ (see for example H.Attouch [2]) seems unifortunately to fail in a so general setting because we cannot control the link between the two parameters n and m at the limit . A last improvement would be to introduce in the set V of optimization problem, the constraint $\det\nabla u>0$ which garantees that u is an orientation preserving deformation. It seems to be more difficult to deal with this last condition.

5.4 References.

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Résumé. L'objectif des travaux de cette thèse est l'étude par le concept d'épiconvergence du comportement asympthotique des suites de fonctionnelles integrales aléatoires non nécessairement convexes et non nécessairement coercives. On présente une méthode directe utilisant le théorème ergodique des procéssus additifs, retrouvant ainssi et précisant un résultat de S. Mûller obtenu dans le cas périodique. Dans le cas convexe les variables primales et duales aléatoires sont étudiées Entre autres, un résultat de convergence faible presque sûre d'une suite de mesures de Borel aléatoires a été établi et a été utilisé pour résoudre les problèmes à "trous" et à "fissures" aléatoires.

Mots-Clés. Homogéniésation, Epiconvergence, Théorie Ergdique Des Processus Additifs et Sous Additifs, Dualité.

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