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Control and Synchronization in Fractional-Order Systems

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Contribution in research

During these four years of research in order to obtain my Doctorate in science, my efforts culminated in the following scientific works :

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Abstract

In this graduation thesis, we have considered different classes of fractional order discrete time systems. The proposed systems are defined using Caputo h-difference operator, in addition, we have studied the existence of chaos and its control and synchronization. As a conclusive remark, we would point out that the main contributions and innovations of this thesis can be summarized as follows, novel fractional-order discrete time systems based on the Caputo h-difference operator, chaos synchronization using linear control laws, and chaos stabilization based on very simple controllers.

Keywords: Chaos, Control, Synchronization, Discrete fractional calculus, Bifurcation, Caputo h-difference operator.

Résumé

Dans cette thèse, nous avons considéré différentes classes de systèmes à temps discret d'ordre fractionnaire. Les systèmes proposés sont définis à l'aide de l'opérateur de différence *h*-Caputo. De plus, nous avons étudié l'existence du chaos et son contrôle et synchronisation. En guise de conclusion, signalons que les principaux apports et innovations de cette thèse peuvent être résumés comme suit, de nouveaux systèmes à temps discret d'ordre fractionnaire basés sur l'opérateur de différence *h*-Caputo, la synchronisation du chaos utilisant des lois de commande linéaires, et la stabilisation du chaos basée sur des contrôleurs très simples.

Mots clés : Chaos, Contrôle, Synchronisation, Calcul fractionnaire discret, Bifurcation, opérateur de différence *h*-Caputo .

ملخص

في هذه الأطروحة، درسنا فئات مختلفة من الأنظمة المتقطعة ذات الأس الكسري. يتم تحديد الأنظمة المقترحة باستخدام عامل الفرق المتقطع ح ـ كابيتو. إن الهدف الرئيسي من هذه الأطروحة هو تلخيص خصائص هذه الأنظمة المتقطعة ذات الأس الكسري، حيث قمنا بدراسة وجود الفوضى إستقرارها ومزامنتها. في الختام، نشير إلى أن المساهمات والابتكارات الرئيسية لهذه الأطروحة يمكن تلخيصها على النحو التالي أنظمة زمنية متقطعة ذات أس كسري تعتمد على عامل الفرق المتقطع ح ـ كابيتو، مزامنة الفوضى باستخدام قوانين تحكم خطية و التخلص من الفوضى على أساس أدوات تحكم بسيطة للغاية.

الكلمات المفتاحية : الفوضى ، التحكم ، المزامنة ، الحساب الكسري المتقطع ، التشعبات ، عامل الفرق المتقطع ح ـ كابيتو

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General Introduction

Nonlinear dynamics, chaos control, and chaos synchronization represent important research topics. In particular, referring to synchronization and control, new advances have been recently reported, for both integer-order systems and fractional-order systems [1, 2]. In particular, referring to continuous-time systems described by fractional derivative, some interesting techniques involving adaptive synchronization have been recently illustrated in [1, 2]. However, there is a remarkable difference in fractional calculus regarding continuous-time and discrete-time systems. Namely, while fractional derivatives made their first appearance in a letter that Gottfried Wilhelm Leibniz wrote to Guillaume de l'Hopital in 1695, discrete fractional calculus has been introduced by Diaz and Olser only in 1974 [3]. Indeed, the authors of [3] presented the first definition of a discrete fractional operator, obtained by discretizing a continuous-time fractional operator. Over the years, several types of difference operators have been introduced in the field of discrete fractional calculus [4, 5, 6]. In particular, a number of fractional h-difference operators, which represent generalizations of the fractional difference operators, have been investigated in [5].

Based on fractional difference equations, in recent years some chaotic discrete-time systems have been studied [7, 8, 10]. These systems are fractional-order maps, which show complex unpredictable behaviors due to the nonlinearities included in their difference equations [5]. With the introduction of fractional chaotic maps, attention has been also focused on the issues related to the synchronization and control of these systems [11]. For example, in [9] the fractional logistic map and its chaotic behaviors have been illustrated, whereas in [10] the presence of chaos in fractional sine and standard maps has been discussed. In [7], discrete chaos in the fractional Hénon map is reported, where as in [8] the chaotic dynamics of the fractional delayed logistic map are analyzed in detail. In [11], three different discrete-time systems, namely, the fractional Lozi map, the fractional Lorenz map, and the fractional flow map, have been studied, along with the

control laws for stabilizing and synchronizing these three maps. In [12], the fractional generalized hyper chaotic Hénon map has been introduced, whereas in [13], the dynamics of the Ikeda map have been investigated via phase plots and bifurcation diagrams. In [14], three fractional chaotic maps, namely, the Stefanski map, the Rossler map, and the Wang map have been studied, along with the synchronization properties of these systems. In [15], dynamics and control of the fractional version of the discrete double-scroll hyperchaotic map are investigated in detail. In [16], bifurcations, entropy, and control of a quadratic fractional map without equilibrium points are analyzed, whereas in [17] the dynamics of fractional maps with fixed points located on closed curves are studied.
A challenging topic in discrete fractional calculus is to study dynamics, synchronization, and control of very complex systems, such as the chaotic three-dimensional (3D) maps [6]. Namely, by computing the approximate entropy, it can be shown that 3D maps highlight a higher degree of complexity with respect to one-dimensional (1D) or two-dimensional (2D) fractional maps [18, 19]. Since the increased complexity can enhance the applicability of 3D maps in pseudo-random number generators and image encryption techniques [20], it is important to analyze their dynamics as well as conceive improved synchronization and control schemes for these maps. In this regard, some interesting results have been recently published [21, 22, 23]. In [21], synchronization and control schemes for a new 3D generalized Hénon map have been proposed, whereas in [23] control and synchronization properties of a 3D fractional map without equilibria have been analyzed in detail. In [22], the fractional form of the Grassi-Miller map has been introduced using the ν-Caputo delta difference. In particular, phase portraits and bifurcation diagrams have been illustrated in [22], with the aim of deriving the fractional-order range for which the system is chaotic. In addition, two nonlinear control laws have been proposed in [22], one for stabilizing the system dynamics and the other for synchronizing a master-slave pair of maps. Although the methods developed in [21, 23] are interesting, a drawback is represented by the fact that very complex control laws have been exploited for controlling and synchronizing the corresponding 3D fractional maps. For example, in [21], synchronization and control in the 3D generalized Hénon maps have been achieved using nonlinear control laws. Moreover, in [23], the 3D fractional maps with hidden attractors have been synchronized and controlled via nonlinear control laws that include several nonlinear terms. We would observe that it might be difficult to implement very complex control laws in practical applications of fractional maps. This drawback also regards the Grassi-Miller map in [22], since its

introduction via the Caputo delta difference has led to complex nonlinear control laws to achieve synchronization and control of its chaotic dynamics.
Inspired by the mentioned above considerations, this thesis provides a further contribution to the topic of dynamics, control, and synchronization of fractional 3D maps by presenting a novel version of the Grassi-Miller map, along with improved schemes for controlling and synchronizing its dynamics [24].
Over the last few years, several fractional-order difference models have been discretized based on efficient tools introduced by the DFC. The so-called fractional-order chaotic discrete systems (FoCDSs) are the most significant among those models [25].
In general, when dealing with FoCDSs, two main aspects should be explored, i.e., control and synchronization of their chaotic modes [26]. Controlling these systems consists in proposing a suitable adaptive controller for their chaotic modes, so that their states are forced to be asymptotically stable, or are stabilized at zero [27, 28, 29]. Control issues are, for instance, of great importance in several industrial processes, like in robotics where chaotic motions of a rigid body need to be controlled [30, 31]. On the other hand, synchronization, which has been considered a key concept in chaos theory over the last three decades, targets to compel the states of a slave system to tend towards the exact trajectories that are determined by a master system, assuming that both systems start from different initial points in phase space [25]. Different synchronization and control techniques have been suggested and implemented on some FoCDSs [25, 27, 32, 33]. One could observe that all the aforementioned works that have discussed both issues of synchronization and control, have employed some linearization methods or some nonlinear laws to implement their strategies [34]. As far as we know, the topic of controlling and synchronizing FoCDSs based on h-DDOs remains, to date, a new and almost unexplored field.
Based on these considerations, this thesis makes a contribution to the topic of fractional-order chaotic discrete systems by presenting novel versions of two and three dimensional Lorenz and Wang fractional chaotic maps, respectively, as well as by providing efficient improvements in the schemes for controlling and synchronizing their dynamics. This objective is achieved by introducing novel theorems that exploit Lyapunov-based approaches [35].
The thesis is divided into six chapters: **The first chapter** presents some elements on the discrete fractional calculus, basic concepts, as well as the three types of discrete fractional operators. **The second chapter** gives the stability of fractional discrete systems

that will aid the reader in understanding the analysis to come. **Chapter three** is meant to provide a simple and heuristic illustration of some basic features of chaos theory in nonlinear discrete dynamical systems with fractional order.

The fourth chapter is an introduction aimed to give an overview about control theory, and chaos synchronization as well as the different types of synchronization. **Chapter five**, provides a contribution to the topic by presenting a novel version of the fractional Grassi-Miller map, along with improved schemes for controlling and synchronizing its dynamics by exploiting the Caputo h-difference operator. In **the sixth chapter**, we presents a novel version of the chaotic fractional Grassi-Miller map, based on the Caputo h-difference operator.

Chapter 1

Discrete fractional calculus

1.1 Introduction

In this chapter, we introduce the basic discrete fractional calculus that will be useful for our later studies. Beginning with integer order discrete calculus, then we introduce the fractional sum and difference of Rimann Liouville and Caputo operators.

1.2 Basic concepts

1.2.1 Integer order difference operator

Definition 1.1. *[36] Assume $f : \mathbb{N} \to \mathbb{R}$, then we define the* ***forward difference*** *operator Δ by:*

$$\Delta f(k) := f(k+1) - f(k), \quad \forall k \in \mathbb{N}. \tag{1.1}$$

Also, the operators Δ^N, $N = 1, 2, 3, \cdots$ is defined recursively by $\Delta^N f(k) = \Delta(\Delta^{N-1} f(k))$ for $k \in \mathbb{N}$. Finally, Δ^0 denotes the identity operator, i.e , $\Delta^0 f(k) = f(k)$.

Theorem 1.1. *[36] Assume $f, g : \mathbb{N} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, then for $k \in \mathbb{N}$ we have:*

(i) $\Delta \alpha = 0$.

(ii) $\Delta \alpha f(k) = \alpha \Delta f(k)$.

(iii) $\Delta [f + g](k) = \Delta f(k) + \Delta g(k)$.

(iv) $\Delta\alpha^{k+\beta} = (\alpha-1)\alpha^{k+\beta}$.

(v) $\Delta[fg](k) = f(k+1)\Delta g(k) + \Delta f(k) g(k)$.

(vi) $\Delta\left(\frac{f}{g}\right)(k) = \frac{g(k)\Delta f(k) - f(k)\Delta g(k)}{g(k)g(k+1)}$,

*where in **(vi)** we assume* $g(k) \neq 0, \forall k \in \mathbb{N}$.

Proposition 1.1. *[36] Assume* $f : \mathbb{N} \to \mathbb{R}$*, then for* $k_0 \in \mathbb{N}$ *we have:*

$$\Delta \sum_{j=k_0}^{k-1} f(j) = f(k), \quad \forall k \in \mathbb{N}, \tag{1.2}$$

and

$$\sum_{j=k_0}^{k-1} \Delta f(j) = f(k) - f(k_0), \quad \forall k \in \mathbb{N}. \tag{1.3}$$

Remark 1.1. *If* $k \leq k_0$*, we have the common convention:*

$$\sum_{j=k_0}^{k-1} f(j) := 0. \tag{1.4}$$

Theorem 1.2. ***(Summation by parts)****[36] Given two functions* $u, v : \mathbb{N} \to \mathbb{R}$ *and* $a, b \in \mathbb{N}$, $a < b$*, we have the summation by parts formulas:*

$$\sum_{j=a}^{b-1} u(j)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j+1)\Delta u(j), \tag{1.5}$$

and

$$\sum_{j=a}^{b-1} u(j+1)\Delta v(j) = u(j)v(j)|_a^b - \sum_{i=a}^{b-1} v(j)\Delta u(j), \tag{1.6}$$

where $u(j)v(j)|_a^b = u(b)v(b) - u(a)v(a)$.

Proof. Let $u, v : \mathbb{N} \to \mathbb{R}$ be given, and $a, b \in \mathbb{N}$, $a < b$, then by Theorem 1.1 (propertie **(*v*)**)

$$\sum_{j=a}^{b-1} \Delta(uv)(j) = \sum_{j=a}^{b-1} \left(v(j+1)\Delta u(j) + \Delta v(j) u(j)\right),$$

after simplification we find

$$u(b)v(b) - u(a)v(a) = \sum_{j=a}^{b-1} v(j+1)\Delta u(j) + \sum_{j=a}^{b-1} u(j)\Delta v(j),$$

this imply

$$\sum_{j=a}^{b-1} u(j)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j+1)\Delta u(j),$$

and

$$\sum_{j=a}^{b-1} u(j+1)\Delta v(j) = u(j)v(j)|_a^b - \sum_{j=a}^{b-1} v(j)\Delta u(j).$$

□

1.2.2 Gamma function

Definition 1.2. *[36] The gamma function is defined by:*

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.7}$$

for those complex numbers z for which the real part of z is positive (it can be shown that the above improper integral converges for all such z).

We have

$$\begin{aligned} \Gamma(1) &= \int_0^\infty e^{-t} dt \\ &= 1, \end{aligned}$$

and by integration by parts for $x > 0 : \Gamma(x+1) = x\Gamma(x)$, thats give for any positive integer k

$$\Gamma(k) = (k-1)!.$$

We also have the important estimation:

$$\lim_{k\to\infty} \frac{\Gamma(k+\gamma)}{\Gamma(k)k^\gamma} = 1, \quad k \in \mathbb{N} \quad \text{and} \quad \gamma \in \mathbb{C}.$$

1.2.3 Falling function

Definition 1.3. *[36] For $r \in \mathbb{R}$, we define the falling function, $t^{(r)}$, read t to the r falling, by:*

$$t^{(r)} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)} = t(t-1)(t-2)\cdots(t-r+1), \tag{1.8}$$

for those values of t and r such that the right-hand side of this last equation make sense. We then extend this definition by making the common convention that $t^{(r)} := 0$ when $t-r+1$ is a nonpositive integer.

Remark 1.2. *When $r = 0$, we set $t^{(0)} := 1$.*

Theorem 1.3. ***(Power Rules)**[36] For $r, a, t \in \mathbb{R}$, we have the following power rules:*

$$\Delta(t+a)^{(r)} = r(t+a)^{(r-1)}, \tag{1.9}$$

and

$$\Delta(a-t)^{(r)} = -r(a-t-1)^{(r-1)}, \tag{1.10}$$

hold, whenever the expressions in these two formulas are well defined.

1.2.4 Binomial coefficient

Definition 1.4. *[36] The **(generalized)** binomial coefficient $\binom{t}{r}$ is defined by:*

$$\binom{t}{r} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)\Gamma(r+1)} = \frac{t^{(r)}}{\Gamma(r+1)}, \tag{1.11}$$

for those values of t and r so that the right-hand side is well defined.

Remark 1.3. *For $0 < t \leq 1$ and $r \in \mathbb{N}$,*

$$\begin{aligned}\binom{t+r-1}{r} = \frac{(t+r-1)^{(r)}}{\Gamma(r+1)} &= \frac{(t+r-1)(t+r-2)...(t)}{r!}\\ &= (-1)^r \frac{(-t)(-t-1)...(-t-r+1)}{r!}\\ &= (-1)^r \binom{-t}{r}.\end{aligned}$$

Theorem 1.4. ***(Pascal rule)**[36] For $r, t \in \mathbb{R}$, we heve:*

$$\binom{t-1}{r} + \binom{t-1}{r-1} = \binom{t}{r}. \tag{1.12}$$

Proof. Consider

$$\begin{aligned}\binom{t-1}{r} + \binom{t-1}{r-1} &= \frac{\Gamma(t)}{\Gamma(t-r)\Gamma(r+1)} + \frac{\Gamma(t)}{\Gamma(t-r+1)\Gamma(r)}\\ &= \Gamma(t)\left(\frac{t-r}{\Gamma(t-r+1)\Gamma(r+1)} + \frac{r}{\Gamma(t-r+1)\Gamma(r+1)}\right)\\ &= \Gamma(t)\frac{t}{\Gamma(t-r+1)\Gamma(r+1)}\\ &= \frac{\Gamma(t+1)}{\Gamma(t-r+1)\Gamma(r+1)}\\ &= \binom{t}{r}.\end{aligned}$$

□

Theorem 1.5. *(Repeated summation rule)*[36] *Let $f:\mathbb{N}\to\mathbb{R}$ be given and $a\in\mathbb{N}$ then:*

$$\sum_{\tau_1=a}^{t-1}\sum_{\tau_2=a}^{\tau_1-1}\cdots\sum_{\tau_p=a}^{\tau_{p-1}-1} f(\tau_p)=\sum_{j=a}^{t-1}\frac{(t-j-1)^{(p-1)}}{(p-1)!}f(j), \tag{1.13}$$

for $t=p+a,p+a+1,\cdots$.

Proof. We will prove this by induction on p for $p\geq 1$. The case $p=1$ is trivially true. Assume (1.13) holds for some $p\geq 1$. It remains to show that (1.13) then holds when p is replaced by $p+1$. To this end, let

$$y(t):=\sum_{\tau_1=a}^{t-1}\sum_{\tau_2=a}^{\tau_1-1}\cdots\sum_{\tau_{p+1}=a}^{\tau_p-1} f(\tau_{p+1}).$$

Let $g(\tau_p)=\sum\limits_{\tau_{p+1}=a}^{\tau_p-1} f(\tau_{p+1})$, then it follows from the induction assumption that

$$y(t)=\sum_{j=a}^{t-1}\frac{(t-j-1)^{(p-1)}}{(p-1)!}g(j),$$

using summation by parts formula

$$y(t)=-\frac{(t-j)^{(p)}}{(p)!}\sum_{i=a}^{j-1}f(i)\Bigg|_a^t+\sum_{j=a}^{t-1}\frac{(t-j-1)^{(p)}}{(p)!}f(j)=\sum_{j=a}^{t-1}\frac{(t-j-1)^{(p)}}{(p)!}f(j).$$

□

Motivated by (1.13), we define the p-th integer sum operator Δ_a^{-p} for positive integers p, by:

$$\Delta_a^{-p}f(t):=\sum_{j=a}^{t-1}\frac{(t-j-1)^{(p-1)}}{(p-1)!}f(j).$$

But, sence

$$\frac{(t-j-1)^{(p-1)}}{(p-1)!}=0,\quad j=t-1,t-2,\cdots,t-p+1,$$

we obtaine

$$\Delta_a^{-p}f(t)=\sum_{j=a}^{t-p}\frac{(t-j-1)^{(p-1)}}{(p-1)!}f(j). \tag{1.14}$$

Remark 1.4. *We have for* $t = p + a, p + a + 1, \cdots$

$$\Delta_a^{-p} \Delta^p f(t) = f(t) - \sum_{j=0}^{p-1} \frac{(t-a)^{(j)}}{j!} \Delta^j f(a),$$

and

$$\Delta^p \Delta_a^{-p} f(t) = f(t).$$

1.2.5 The Z-transform

Definition 1.5. *[37] The Z-transform of a sequence* $(x(k))_{k\in\mathbb{Z}}$*, which is identically zero for negative integers k (i.e.,* $x(k) = 0$ *for* $k = -1, -2, \cdots$*), is defined by:*

$$\tilde{x}(z) = Z(x(k)) := \sum_{k=0}^{\infty} x(k) z^{-k}, \tag{1.15}$$

where z is a complex number.

The set of numbers z in the complex plane for which series (1.15) converges is called the region of convergence of $\tilde{x}(z)$. The most commonly used method to find the region of convergence of the series (1.15) is the ratio test. Suppose that:

$$\lim_{k\to\infty} \left| \frac{x(k+1)}{x(k)} \right| = R.$$

The series (1.15) converges in the region $|z| > R$ and diverges for $|z| < R$.

1.3 Fractional difference operators

In this section, we define **fractional sum** and **Riemann Liouville** and **Caputo** difference operators, give some of their properties and the relation between them. Let us denote $(h\mathbb{N})_a$ the set of functions defined by $\mathbb{N}_a := \{a, a+1, a+2, ...\}$, where $a \in \mathbb{R}$.

1.3.1 Fractional sum operator

First we begine with the fractonal sum, we use the (**Repeated Summation Rule**) and (**Gamma function**) properties, and replaced the natural number p bay a real positive number α we get the definition of fractional order sum.

Definition 1.6. *[38] Let $\alpha > 0$. Then, the $\alpha - th$ fractional sum of $f : \mathbb{N}_a \to \mathbb{R}$ is defined by:*

$$\Delta_a^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s), \tag{1.16}$$

for any $t \in \mathbb{N}_{a+\alpha}$.

Remark 1.5. *Note that $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.*

Lemma 1.1. *[36] Assume $\mu \geq 0$ and $\alpha > 0$. Then:*

$$\Delta_{a+\mu}^{-\alpha} (t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{(\mu+\alpha)}, \tag{1.17}$$

for any $t \in \mathbb{N}_{a+\mu+\alpha}$.

Proof. Let

$$g_1(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-a)^{(\mu+\alpha)}$$

and

$$g_2(t) = \Delta_{a+\mu}^{-\alpha} (t-a)^{(\mu)} = \frac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s-a)^{(\mu)},$$

for $t \in \mathbb{N}_{a+\mu+\alpha}$. To get advanced in this proof we will show that both of these functions satisfy the initial value problem

$$(t-a-(\mu+\alpha)+1)\Delta g(t) = (\mu+\alpha) g(t), \tag{1.18}$$

$$g(a+\mu+\alpha) = \Gamma(\mu+1). \tag{1.19}$$

Since

$$\begin{aligned} g_1(a+\mu+\alpha) &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (\mu+\alpha)^{(\mu+\alpha)} \\ &= \Gamma(\mu+1), \end{aligned}$$

and

$$\begin{aligned} g_2(a+\mu+\alpha) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+\mu}^{a+\mu} (a+\mu+\alpha-s-1)^{(\alpha-1)} (s-a)^{(\mu)} \\ &= \frac{1}{\Gamma(\alpha)} (\alpha-1)^{(\alpha-1)} \mu^{(\mu)} \\ &= \Gamma(\mu+1), \end{aligned}$$

we have that $g_i(t), i = 1,2$ both satisfy the initial condition . We next show that $g_1(t)$ satisfies the difference equation (1.18). Note that

$$\Delta g_1(t) = (\mu+\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha-1)}.$$

Multiplying both sides by $(t-a-(\mu+\alpha)+1)$ we obtain

$$\begin{aligned}(t-a-(\mu+\alpha)+1)\Delta g_1(t) &= (\mu+\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}[t-a-(\mu+\alpha-1)](t-a)^{(\mu+\alpha-1)}\\ &= (\mu+\alpha)\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha)}\\ &= (\mu+\alpha)g_1(t),\end{aligned}$$

for $t \in \mathbb{N}_{a+\mu+\alpha}$. That is, $g_1(t)$ is a solution of (1.18).
It remains to show that $g_2(t)$ satisfies (1.18). We have that

$$\begin{aligned}g_2(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}\\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}[(t-s-1)-(\alpha-2)](t-s-1)^{(\alpha-2)}(s-a)^{(\mu)}\\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}[(t-a-(\mu+\alpha)+1)-(s-a-\mu)](t-s-1)^{(\alpha-2)}(s-a)^{(\mu)}\\ &= \frac{t-a-(\mu+\alpha)+1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu)}\\ &-\frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a-\mu)(s-a)^{(\mu)}\\ &= h(t)-k(t),\end{aligned}$$

where

$$h(t) = \frac{t-a-(\mu+\alpha)+1}{\Gamma(\alpha)}\sum_{r=a+\mu}^{t-\alpha}(t-r-1)^{(\alpha-2)}(r-a)^{(\mu)}$$

and

$$\begin{aligned}k(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a-\mu)(s-a)^{(\mu)}\\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu+1)}.\end{aligned}$$

So we get

$$\begin{aligned}
\Delta g_2(t) &= \frac{1}{\Gamma(\alpha)}\Delta\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}\\
&= \frac{1}{\Gamma(\alpha)}\left[\sum_{s=a+\mu}^{t+1-\alpha}(t-s)^{(\alpha-1)}(s-a)^{(\mu)} - \sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}\right]\\
&= \frac{1}{\Gamma(\alpha)}\left[\sum_{s=a+\mu}^{t-\alpha}\Delta_t(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)} + (\alpha-1)^{(\alpha-1)}(t+1-\alpha-a)^{(\mu)}\right]\\
&= \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} + \frac{1}{\Gamma(\alpha)}(\alpha-1)^{(\alpha-1)}(t+1-\alpha-a)^{(\mu)}\\
&= \frac{\alpha-1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu)} + (t+1-\alpha-a)^{(\mu)}.
\end{aligned}$$

It follows that

$$(t-a-(\mu+\alpha)+1)\Delta g_2(t) = (\alpha-1)h(t) + (t+1-\alpha-a)^{(\mu+1)}.$$

Also, by summation by parts we get

$$\begin{aligned}
k(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-2)}(s-a)^{(\mu+1)}\\
&= \frac{-(t-s)^{(\alpha-1)}(s-a)^{(\mu+1)}}{(\alpha-1)\Gamma(\alpha)}\Bigg|_{s=a+\mu}^{s=t+1-\alpha}\\
&+ \frac{\mu+1}{(\alpha-1)\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}\\
&= -\frac{(t+1-\alpha-a)^{(\mu+1)}}{(\alpha-1)} + \frac{\mu+1}{(\alpha-1)\Gamma(\alpha)}\sum_{s=a+\mu}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s-a)^{(\mu)}.
\end{aligned}$$

It follows that

$$(t+1-\alpha-a)^{(\mu+1)} = -(\alpha-1)k(t) + (\mu+1)g_2(t).$$

Finally, we get

$$\begin{aligned}
(t-a-(\mu+\alpha)+1)\Delta g_2(t) &= (\alpha-1)h(t) + (t+1-\alpha-a)^{(\mu+1)}\\
&= (\alpha-1)h(t) - (\alpha-1)k(t) + (\mu+1)g_2(t)\\
&= (\mu+\alpha)g_2(t).
\end{aligned}$$

This completes the proof. □

Theorem 1.6. *[38] Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, and let $\mu, \nu > 0$. Then:*

$$\Delta_{a+\mu}^{-\nu}[\Delta_a^{-\mu} f(t)] = \Delta_a^{-(\mu+\nu)} f(t) = \Delta_{a+\nu}^{-\mu}[\Delta_a^{-\nu} f(t)], \tag{1.20}$$

for any $t \in \mathbb{N}_{a+\mu+\nu}$.

Proof. By definition of fractional sum, we have

$$\begin{aligned}
\Delta_{a+\nu}^{-\mu}[\Delta_a^{-\nu} f(t)] &= \frac{1}{\Gamma(\nu)} \Delta_{a+\nu}^{-\mu} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a+\nu}^{t-\mu} (t-r-1)^{(\mu-1)} \sum_{s=a}^{r-\nu} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a+\nu}^{t-\mu} \sum_{s=a}^{r-\nu} (t-r-1)^{(\mu-1)} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a}^{t-(\mu+\nu)} \sum_{r=s+\nu}^{t-\mu} (t-r-1)^{(\mu-1)} (r-s-1)^{(\nu-1)} f(s) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a}^{t-(\mu+\nu)} \left(\sum_{x=\nu-1}^{t-s-1-\mu} (t-x-s-2)^{(\mu-1)} x^{(\nu-1)} \right) f(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-(\mu+\nu)} \left(\Delta_{\nu-1}^{-\mu} (\tau)^{(\nu-1)} \Big|_{\tau=t-s-1} \right) f(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} (t-s-1)^{(\nu+\mu-1)} f(s) \quad \text{(by using Lemma1.1)} \\
&= \Delta_a^{-(\mu+\nu)} f(t).
\end{aligned}$$

□

Theorem 1.7. *[38] Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, for any $\alpha > 0$ and any positive integer p, we have:*

$$\Delta_a^{-\alpha} \Delta^p f(t) = \Delta^p \Delta_a^{-\alpha} f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)} \Delta^r f(a), \tag{1.21}$$

for any $t \in \mathbb{N}_{a+\alpha}$.

Proof. We will prove this by induction on p for $p \geq 1$.

- The case where $p = 1$. We have by definition of fractional sum:

$$
\begin{aligned}
\Delta_a^{-\alpha}\Delta f(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}(\Delta f)(s) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}(f(s+1)-f(s)) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s+1) - \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t+1-\alpha}(t-s)^{(\alpha-1)}f(s) - \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s) \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t+1-\alpha}(t+1-s-1)^{(\alpha-1)}f(s) - \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a) \\
&= \Delta\left(\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s)\right) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a) \\
&= \Delta\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a).
\end{aligned}
$$

- Assume (1.21) holds for some $p \geq 1$. It remains to show that (1.21) then holds when p is replaced by $p+1$. We have

$$
\begin{aligned}
\Delta_a^{-\alpha}\Delta^{p+1}f(t) &= \Delta_a^{-\alpha}\Delta^{p}(\Delta f)(t) \\
&= \Delta^p\Delta_a^{-\alpha}\Delta f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}(\Delta^r\Delta f)(a) \quad \text{(from the induction assumption)} \\
&= \Delta^p\Delta_a^{-\alpha}\Delta f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta^{r+1}f(a) \\
&= \Delta^p\left(\Delta_a^{-\alpha}\Delta f(t)\right) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a) \\
&= \Delta^p\left(\Delta\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)}f(a)\right) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a) \\
&= \Delta^{p+1}\Delta_a^{-\alpha}f(t) - \frac{(t-a)^{(\alpha-1-p)}}{\Gamma(\alpha-p)}f(a) - \sum_{r=1}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a) \\
&= \Delta^{p+1}\Delta_a^{-\alpha}f(t) - \sum_{r=0}^{p}\frac{(t-a)^{(\alpha-p+r-1)}}{\Gamma(\alpha+r-p)}\Delta^r f(a).
\end{aligned}
$$

□

1.3.2 Rimann Liouville operator

Definition 1.7. *[38] Let $\alpha > 0$. Then, the $\alpha-$order **Rimann Liouville** fractional difference of a function f defined on $\mathbb{N}_a$ are defined by:*

$$\Delta_a^{\alpha} f(t) := \Delta^p \Delta_a^{-(p-\alpha)} f(t) = \frac{1}{\Gamma(p-\alpha)} \Delta^p \sum_{s=a}^{t-(p-\alpha)} (t-s-1)^{(p-\alpha-1)} f(s), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.22}$$

where $p = [\alpha] + 1$.

Remark 1.6. *If $\alpha \longrightarrow p \in \mathbb{N}$, then*

$$\begin{aligned} \lim_{\alpha \to p} \Delta_a^{\alpha} f(t) &= \Delta^p \Delta^{(0)} f(t) \\ &= \Delta^p f(t), \end{aligned}$$

and

$$\begin{aligned} \lim_{\alpha \to p-1} \Delta_a^{\alpha} f(t) &= \Delta^p \Delta^{-1} f(t) \\ &= \Delta^{p-1} f(t). \end{aligned}$$

We can say that the Δ_a^{α} operator is an interpolation of Δ^p operators. Also, it is clear that Δ_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(p-\alpha)}$.
Note that:

$$\begin{aligned} \Delta_a^{\alpha} 1 &= \Delta^p \Delta_a^{-(p-\alpha)} 1 \\ &= \frac{1}{\Gamma(p-\alpha)} \Delta^p \sum_{s=a}^{t-(p-\alpha)} (t-s-1)^{(p-\alpha-1)} \\ &= \frac{1}{\Gamma(1-\alpha)} (t-a)^{(-\alpha)} \\ &\neq 0, \end{aligned}$$

generally.

Theorem 1.8. *[38] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$\Delta_{a+\alpha}^{\alpha} \Delta_a^{-\alpha} f(t) = f(t), \quad \forall t \in \mathbb{N}_{a+p}, \tag{1.23}$$

where $p = [\alpha] + 1$.

Proof. Simple by using Definition 1.7 and Theorem1.6, for any $t \in \mathbb{N}_{a+p}$:

$$\begin{aligned}\Delta_{a+\alpha}^{\alpha}\Delta_a^{-\alpha}f(t) &= \Delta^p\Delta_{a+\alpha}^{-(p-\alpha)}\Delta^{-\alpha}f(t)\\ &= \Delta^p\Delta^{-p}f(t)\\ &= f(t).\end{aligned}$$

□

Remark 1.7. *We see that $\Delta_{a+\alpha}^{\alpha}$ have a right inverse $(\Delta_a^{-\alpha})$, or $\Delta_{a+\alpha}^{-\alpha}$ have a left inverse (Δ_a^{α}).*

Theorem 1.9. *[38] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) = f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta_a^{r-(p-\alpha)}f(a), \quad \forall t \in \mathbb{N}_{a+p}, \tag{1.24}$$

where $p = [\alpha]+1$.

Proof. By Definition1.7, for any t from $\mathbb{N}_{a+p}$, we have

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) = \Delta_{a+p-\alpha}^{-\alpha}\Delta^p\Delta_a^{-(p-\alpha)}f(t),$$

using Theorem1.7

$$\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) = \Delta^p\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{-(p-\alpha)}f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta^r\Delta_a^{-(p-\alpha)}f(a),$$

by using Theorem1.6

$$\begin{aligned}\Delta_{a+p-\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) &= \Delta^p\Delta_a^{-p}f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta_a^{r-(p-\alpha)}f(a)\\ &= f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(\alpha-p+r)}}{\Gamma(\alpha+r-p+1)}\Delta_a^{r-(p-\alpha)}f(a).\end{aligned}$$

□

Remark 1.8. *We see that*

$$\Delta_{a+\alpha}^{-\alpha}\Delta_a^{\alpha}f(t) \neq f(t)$$

generally.

1.3.3 Caputo operator

Definition 1.8. *[38] Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then, the $\alpha-$order* **Caputo** *fractional difference of a function f defined on $\mathbb{N}_a$, are defined by:*

$$ {}^C\Delta_a^\alpha f(t) := \Delta_a^{-(p-\alpha)} \Delta^p f(t) = \frac{1}{\Gamma(p-\alpha)} \sum_{s=a}^{t-(p-\alpha)} (t-s-1)^{(p-\alpha-1)} \Delta^p f(s), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.25} $$

where $p = [\alpha] + 1$.

Remark 1.9. *When $\alpha \to p-1 \in \mathbb{N}$, we get*

$$ \begin{aligned} \lim_{\alpha \to p-1} {}^C\Delta_a^\alpha f(t) &= \lim_{\alpha \to p-1} \Delta_a^{-(p-\alpha)} \Delta^p f(t) \\ &= \Delta_a^{-1} \Delta^p f(t) \\ &= \Delta^{p-1} f(t) - \Delta^{p-1} f(a). \end{aligned} $$

So ${}^C\Delta_a^\alpha$ is not an interpolation of Δ^p. However, it is used in modeling because it is consistent with the classical initial and boundary conditions. To not be confused we defined ${}^C\Delta_a^p$ by ${}^C\Delta_a^p f(t) := \Delta^p f(t)$, where $p \in \mathbb{N}$. Also, it is clear that ${}^C\Delta_a^\alpha$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(p-\alpha)}$.

Remark 1.10. *The fractional Caputo difference of a constant $c \in \mathbb{R}$ is*

$$ {}^C\Delta_a^\alpha c = \Delta_a^{-(p-\alpha)} \Delta^p c = 0. $$

Theorem 1.10. *[38] Assume $\alpha > 0$ and f is defined on $\mathbb{N}_a$. Then:*

$$ \Delta_{a+(p-\alpha)}^{-\alpha} \, {}^C\Delta_a^\alpha \, f(t) = f(t) - \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a), \quad \forall t \in \mathbb{N}_a, \tag{1.26} $$

where $p = [\alpha] + 1$.
In particular, if $0 < \alpha \leq 1$ then

$$ \Delta_{a+(1-\alpha)}^{-\alpha} \, {}^C\Delta_a^\alpha \, f(t) = f(t) - f(a), \quad \forall t \in \mathbb{N}_a. \tag{1.27} $$

Proof. By Definition1.8, for any t from $\mathbb{N}_a$, we have:

$$ \Delta_{a+(p-\alpha)}^{-\alpha} \, {}^C\Delta_a^\alpha \, f(t) = \Delta_{a+(p-\alpha)}^{-\alpha} \Delta_a^{-(p-\alpha)} \Delta^p f(t), $$

using Theorem1.8

$$\Delta_{a+(p-\alpha)}^{-\alpha}\,{}^{C}\Delta_a^{\alpha} f(t) = \Delta_a^{-p}\Delta^p f(t)$$
$$= f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(r)}}{r!}\Delta^r f(a).$$

□

Theorem 1.11. *[38] Let $\alpha > 0$, and f is defined on $\mathbb{N}_a$. Then:*

$$^{C}\Delta_a^{\alpha} f(t) = \Delta_a^{\alpha} f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)}\Delta^r f(a), \quad \forall t \in \mathbb{N}_{a+p-\alpha}, \tag{1.28}$$

where $p = [\alpha]+1$.
In particular, when $0 < \alpha < 1$, we have

$$^{C}\Delta_a^{\alpha}\; f(t) = \Delta_a^{\alpha} f(t) - \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)} f(a), \quad \forall t \in \mathbb{N}_{a-\alpha+1}. \tag{1.29}$$

Proof. By Definition1.8

$$^{C}\Delta_a^{\alpha} f(t) = \Delta_a^{-(p-\alpha)}\Delta^p f(t),$$

by using Theorem1.7

$$^{C}\Delta_a^{\alpha} f(t) = \Delta^p \Delta_a^{-(p-\alpha)} f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)}\Delta^r f(a)$$
$$= \Delta_a^{\alpha} f(t) - \sum_{r=0}^{p-1}\frac{(t-a)^{(r-\alpha)}}{\Gamma(r-\alpha+1)}\Delta^r f(a).$$

□

Theorem 1.12. ***(Discrete Taylor's Formula)****[38] Let f be defined on $\mathbb{N}_a$. Then, for all $t \in \mathbb{N}_a$ and $p \in \mathbb{N}$ withe $p \geq 1$:*

$$f(t) = \sum_{r=0}^{p-1}\frac{(t-a)^{(r)}}{r!}\Delta^r f(a) + \frac{1}{(p-1)!}\sum_{r=a}^{t-p}(t-r-1)^{(p-1)}\Delta^p f(r). \tag{1.30}$$

Proof. The proof is by induction. For $p = 1$. (1.30) is the same as

$$f(t) = f(a) + \sum_{r=a}^{t-1}\Delta f(r) = f(a) + f(t) - f(a) = f(t).$$

Assume (1.30) holds for some $p \geq 1$. It remains to show that (1.30) then holds when p is replaced by $p+1$. We have

$$\begin{aligned} f(t) &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{(p-1)!} \sum_{r=a}^{t-p} (t-r-1)^{(p-1)} \Delta^p f(r) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) - \frac{1}{p!} (t-r)^{(p)} \Delta^p f(r) \,|_{r=a}^{r=t-p+1} + \frac{1}{p!} \sum_{r=a}^{t-p} (t-r-1)^{(p)} \Delta^{p+1} f(r) \\ &= \sum_{r=0}^{p} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{p!} \sum_{r=a}^{t-(p+1)} (t-r-1)^{(p)} \Delta^{p+1} f(r). \end{aligned}$$

□

Theorem 1.13. *[40] Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and f is defined on $\mathbb{N}_a$. Then:*

$$f(t) = \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{\Gamma(\alpha)} \sum_{r=a+p-\alpha}^{t-\alpha} (t-r-1)^{(\alpha-1)} \ {}^{C}\Delta_a^{\alpha} \ f(r), \quad \forall t \in \mathbb{N}_{a+p}, \tag{1.31}$$

where $p = [\alpha] + 1$.

Proof. Notice that by (Discrete Taylor's Formula)

$$\begin{aligned} f(t) &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{(p-1)!} \sum_{r=a}^{t-p} (t-r-1)^{(p-1)} \Delta^p f(r) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_a^{-p} \Delta^p f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_a^{-(p-\alpha)-\alpha} \Delta^p f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_{a+p-\alpha}^{-\alpha} \Delta_a^{-(p-\alpha)} \Delta^p f(t) \quad \text{(by using Theorem1.6)} \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \Delta_{a+p-\alpha}^{-\alpha} \ {}^{C}\Delta_a^{\alpha} \ f(t) \\ &= \sum_{r=0}^{p-1} \frac{(t-a)^{(r)}}{r!} \Delta^r f(a) + \frac{1}{\Gamma(\alpha)} \sum_{r=a+p-\alpha}^{t-\alpha} (t-r-1)^{(\alpha-1)} \ {}^{C}\Delta_a^{\alpha} \ f(r), \quad \forall t \in \mathbb{N}_{a+p}. \end{aligned}$$

□

1.4 Conclusion

In this chapter, we have presented some elements on the discrete fractional calculus, basic concepts, as well as the three types of discrete fractional operators.

Chapter 2

Stability of fractional discrete systems

2.1 Introduction

In this chapter, we will present some stability theorems of fractional order difference systems. First, we will give a background on stability of integer order difference systems. Then we will move to the study of stability in fractional order case.

2.2 General form

The systems we will be concerned in studying their stability are written in the general form as follow:

$$\begin{cases} {}^C\Delta_{t_0}^{\alpha} x(t) = f(t+\alpha-1, x(t+\alpha-1)), \quad t \in \mathbb{N}_{a+1-\alpha}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}_1, t_0 \in \mathbb{N}_a, x(t) \in \mathbb{R}^n$, $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$.
A point $x_e \in \mathbb{R}^n$ is a fixed (equilibrium) point of (2.1) if and only if

$$f(t, x_e) = 0, \quad \forall t \in \mathbb{N}_a.$$

We will be concerned with the stability of the equilibrium point $x_e = 0$ (all cases can be transferred to be 0 the equilibrium point) of the discrete time systems (2.1).
Let $\|.\|$ be a norm on $\mathbb{R}^n$.

Definition 2.1. *[41] The trivial solution $x(t) = 0$ (or the equilibrium point $x = 0$) of (2.1) is said to be:*

(i) *stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{N}_a$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for any solution $x(t) = x(t, t_0, x_0)$, with $\|x_0\| < \delta$ one has $\|x(t)\| < \epsilon$, for all $t \in \mathbb{N}_{t_0} \subseteq \mathbb{N}_a$,*

(ii) *uniformly stable if it is stable and δ depends solely on ϵ,*

(iii) *asymptotically stable if it is stable and for all $t_0 \in \mathbb{N}_a$ there exists $\delta = \delta(t_0) > 0$ if $\|x_0\| < \delta$ implies that $\lim_{t\to\infty} x(t, t_0, x_0) = 0$,*

(iv) *uniformly asymptotically stable if it is uniformly stable and, for each $\epsilon > 0$, there exists $T = T(\epsilon) \in \mathbb{N}$ and $\delta_0 > 0$ such that $\|x_0\| < \delta_0$ implies $\|x(t)\| < \epsilon$ for all $t \in \mathbb{N}_{t_0+T}$ and for all $t_0 \in \mathbb{N}_a$,*

(v) *globally asymptotically stable if it is asymptotically stable for all $x_0 \in \mathbb{R}^n$,*

(vi) *globally uniformly asymptotically stable if it is uniformly asymptotically stable for all $x_0 \in \mathbb{R}^n$.*

2.3 Background on stability of integer order difference systems

2.3.1 Stability of linear systems

Consider the integer order difference system:

$$\begin{cases} \Delta x(k) = Ax(k), & k \in \mathbb{N}, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.2}$$

where $x(k) = (x_1(k), x_2(k), \cdots, x_n(k))^T \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix.
It has an equilibrium point at the origin ($x = 0$). The solution of the linear system (2.2) starting from x_0 has the form

$$x(k) = (A + I_n)^k x_0, \quad \forall k \in \mathbb{N} \tag{2.3}$$

where I_n is the identity matrix. We have the following result on the stability of linear system (2.2).

Theorem 2.1. *[37] If all the eigenvalues λ_j of A satisfies $\left|\lambda_j + 1\right| < 1$, $1 \leq j \leq n$, then the trivial solution of (2.2) is globally asymptotically stable on $\mathbb{N}$.*

Furthermore, if there is an eigenvalue λ of A withe $|\lambda+1|>1$, then the trivial solution of (2.2) is unstable on $\mathbb{N}$.

Example 2.1. *Consider the following linear system:*

$$\Delta x(k) = \begin{pmatrix} 0 & -5 \\ \frac{1}{4} & -2 \end{pmatrix} x(k), \quad k \in \mathbb{N} \tag{2.4}$$

The characteristic equation for $A = \begin{pmatrix} 0 & -5 \\ \frac{1}{4} & -2 \end{pmatrix}$ is $\lambda^2 + 2\lambda + \frac{5}{4} = 0$ and hence the eigenvalues of A are $\lambda_1 = -1 + \frac{i}{2}$ and $\lambda_2 = -1 - \frac{i}{2}$. Since

$$|\lambda_1 + 1| = |\lambda_2 + 1| = \frac{1}{2} < 1,$$

then, by Theorem2.1 the trivial solution of (2.4) is globally asymptotically stable on $\mathbb{N}$.

Example 2.2. *Consider the following linear system:*

$$\Delta x(k) = \begin{pmatrix} 1 & 6 \\ 0 & -2 \end{pmatrix} x(k), \quad k \in \mathbb{N} \tag{2.5}$$

The characteristic equation for $A = \begin{pmatrix} 1 & 6 \\ 0 & -2 \end{pmatrix}$ is $\lambda^2 + 3\lambda + 2 = 0$ and hence the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -2$. Since

$$|\lambda_2| = 2 > 1,$$

then, by Theorem2.1 the trivial solution of (2.5) is unstable on $\mathbb{N}$.

Remark 2.1. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A. Assume $\left|\lambda_j + 1\right| \leq 1$, $1 \leq j \leq n$.*
If whenever $\left|\lambda_j + 1\right| = 1$, then λ_j is a simple eigenvalue of A. Then the trivial solution of (2.2) is stable on $\mathbb{N}$.
If there is a non simple eigenvalue λ of A satisfying $|\lambda + 1| = 1$, then we can't conclude.

Example 2.3. *Consider the following system:*

$$\Delta x(k) = \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{pmatrix} x(k), \quad k \in \mathbb{N} \tag{2.6}$$

where θ is a real number. For each θ the eigenvalues of the coefficient matrix in (2.6) are $\lambda_{1,2} = e^{\pm i\theta}_{-} - 1$. Since $|\lambda_1 + 1| = |\lambda_2 + 1| = 1$ and both eigenvalues are simple, we have by Remark2.1 that the trivial solution of (2.6) is stable on $\mathbb{N}$.

2.3.2 Stability of non-linear systems

Consider the following non-linear system:

$$\begin{cases} \Delta x(k) = f(x(k)), & k \in \mathbb{N}, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.7).

2.3.2.1 Linearization method

Theorem 2.2. *[37] Let J be the Jacobian matrix of f at 0.*
If all the eigenvalues $\lambda_j, 1 \leq j \leq n$, of J satisfies $\left|\lambda_j + 1\right| < 1$, then the trivial solution of (2.7) is asymptotically stable on $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ of A withe $|\lambda + 1| > 1$, then the trivial solution of (2.7) is unstable on $\mathbb{N}$.

Example 2.4. *Consider the following non-linear system:*

$$\begin{cases} \Delta x_1(k) = \frac{2x_2(k)}{(1+x_1^2(k))} - x_1(k), \\ \Delta x_2(k) = \frac{x_1(k)}{(1+x_2^2(k))} - x_2(k). \end{cases} \tag{2.8}$$

Let $f = (f_1, f_2)^T$, where $f_1 = \frac{2x_2(k)}{(1+x_1^2(k))} - x_1(k)$ and $f_2 = \frac{x_1(k)}{(1+x_2^2(k))} - x_2(k)$, then the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

the characteristic equation for $J = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - 2 = 0$ and hence the eigenvalues of J are $\lambda_{1,2} = \pm\sqrt{2}$. Since

$$|\lambda_1 + 1| = 1 + \sqrt{2} > 1, \tag{2.9}$$

then, by Theorem2.1 the trivial solution of (2.8) is unstable on $\mathbb{N}$.

2.3.2.2 Lyapunov direct method

Theorem 2.3. *[37] If there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$, (called **Lyapunov function**) which is continuous and such that:*

$$\begin{aligned} &V(0) = 0 \text{ and } V(x(k)) > 0, \ \forall x(k) \neq 0, \\ &\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq 0, \ \forall k \in \mathbb{N}. \end{aligned} \tag{2.10}$$

Then the trivial solution of (2.7) is stable. Moreover if

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0, \ \forall k \in \mathbb{N}. \tag{2.11}$$

Then, the trivial solution of (2.7) is asymptotically stable.

Remark 2.2. *This theorem, helps to study stability not need to know the forme of solution, its dispointing fact is that there is no specific way to generate the Lyapunov functions.*

Example 2.5. *Consider the following non-linear system:*

$$\Delta x(k) = \frac{x(k)}{2 + x^2(k)} - x(k). \tag{2.12}$$

It has an equilibrium point at the origin. Our first choice of a Liapunov function will be $V(x) = x^2$, this is clearly continuous and positive definite on $\mathbb{R}$,

$$\Delta V(x(k)) = \left(\frac{x(k)}{2 + x^2(k)}\right)^2 - x^2(k) < 0.$$

Then, by Theorem2.3 the trivial solution of (2.12) is asymptotically stable.

2.4 Stability of fractional order difference systems

2.4.1 Stability of linear systems

Now, we investigate the stability of the equilibrium point $x = 0$ of the α−th order linear system of difference equations:

$$\begin{cases} {}^C\Delta_a^\alpha x(t) = Ax(t + \alpha - 1), & t \in \mathbb{N}_{a+1-\alpha}, \\ x(a) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.13}$$

where $0 < \alpha \leq 1$, $a \in \mathbb{R}$, and A is an $n \times n$ constant matrix.

Theorem 2.4. *[42] Suppose $0 < \alpha < 1$. Then the trivial solution of (2.13) is asymptotically stable if and only if the isolated zeroes, off the nonnegative real axis, of*

$$\det\left(I_n - z^{-1}(1-z^{-1})^{-\alpha}A\right), \tag{2.14}$$

lie inside the unit disk.

Remark 2.3. *If $\alpha \to 1^-$, then the condition of Theorem2.4 simplifies to the isolated zeroes, off the nonnegative real axis, of*

$$\det\left(I_n - \frac{1}{z-1}A\right) = \frac{1}{z-1}\det\left((z-1)I_n - A\right),$$

lie inside the unit disk, this means all the eigenvalues $\lambda = (z-1)$ of A satisfy $|\lambda+1| = |z| < 1$, it is the same result of Theorem2.1.

Proof. By Theorem1.13, the solution of (2.13) is given by

$$\begin{aligned} x(t) &= x(a) + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\,{}^{C}\Delta_a^{\alpha}x(s) \\ &= x(a) + \frac{1}{\Gamma(\alpha)}A\sum_{s=a+1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}x(s+\alpha-1), \quad \forall t \in \mathbb{N}_{a+1}. \end{aligned} \tag{2.15}$$

Thus, for $t = a+k+1, k = 0,1,2,\cdots$, (2.15) simplifies to

$$x_k = x_{-1} + \sum_{s=0}^{k} B(k-s)x_{s-1}. \tag{2.16}$$

Here, the kernel $B(k)$ and x_k, is given by

$$B(k) = \frac{1}{(k)!}A\prod_{j=1}^{k}(\alpha+j-1), \text{ and } x_k = x(a+k+1),\ k = -1,0,1,2,\cdots. \tag{2.17}$$

Notice that (2.16) is a nonhomogeneous Volterra difference equation of the convolution type. Moreover, the aforementioned equation will be used heavily in the sequel.
We focus our attention on the scalar case with $A = \lambda$. Without loss of generality, we shall take $x_{-1} = 1$.
If $\lambda > 0$. Since $x_k > 1$ for $k \geq 1$,

$$x_k > 1 + \lambda\sum_{s=0}^{k}\frac{1}{(k-s)!}\left(\prod_{j=1}^{k-s}(\alpha+j-1)\right) = 1 + \lambda\sum_{s=0}^{k}\frac{1}{s!}\left(\prod_{j=1}^{s}(\alpha+j-1)\right).$$

But

$$\frac{\prod_{j=1}^{s}(\alpha+j-1)/s!}{1/(\alpha+s)} = \frac{\prod_{j=0}^{s}(\alpha+j)}{s!} = \alpha\frac{\prod_{j=1}^{s}(\alpha+j)}{s!} > \alpha,$$

this implies

$$x_k > 1 + \lambda\sum_{s=0}^{k}\frac{\alpha}{(\alpha+s)},$$

by the limit comparison test the solution $\{x_k\}$, of (2.13) diverges to infinity.
If $\lambda = 0$, then $x_k = x_{-1}$ for $k = 0, 1, 2, \cdots$.
Hence, from now on, we assume that $\lambda < 0$, let $\tilde{x}(z) = Z(\{x_k\})$ be the unilateral Z-transform of the sequence $\{x_k\}$. Then

$$\tilde{x}(z) = (1-z^{-1})^{-1} + \lambda(1-z^{-1})^{-\alpha}(1+z^{-1}\tilde{x}(z)), \quad |z| > R \geq 1.$$

Solving for $\tilde{x}$ yields

$$\tilde{x}(z) = \frac{(1-z^{-1})^{-1} + \lambda(1-z^{-1})^{-\alpha}}{1-\lambda z^{-1}(1-z^{-1})^{-\alpha}} = \frac{\frac{z}{z-1} + \lambda(\frac{z}{z-1})^{\alpha}}{1-\lambda\frac{1}{z}(\frac{z}{z-1})^{\alpha}}.$$

That says

$$x_k = \frac{1}{2\pi i}\int_C z^{k-1}\tilde{x}(z)dz,$$

where C is any positively-oriented simple-closed contour in the analyticity region of $\tilde{x}$ that encircles all singular points of $\tilde{x}(z)$.
With that in mind, we consider the contour $C\rho$, depicted in Figure2.1 below. The inner circles are of radius ρ which we take it small enough so that all isolated singularities of $\tilde{x}$ are inside the $C\rho$.

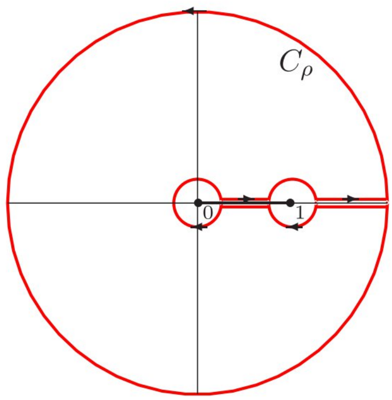


Figure 2.1: The contour $C\rho$.

Since the line integrals of $z^{k-1}\tilde{x}(z)$ over the inner circles in Figure2.1 tend to zero as ρ tends to zero, we have

$$\int_C z^{k-1}\tilde{x}(z)dz = \lim_{\rho\to 0}\int_{C_\rho} z^{k-1}\tilde{x}(z)dz.$$

But, by the Residue Theorem,where the z_i's are the isolated singularities $\tilde{x}(z)$. Indeed, the singularities of $\tilde{x}(z)$ are simple poles given by the zeros of

$$Q(z) = 1 - \lambda\frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha}.$$

To see this, suppose z_i is a zero of $Q(z)$. Then

$$\lambda\left(\frac{z_i}{z_i-1}\right)^{\alpha} = z_i,$$

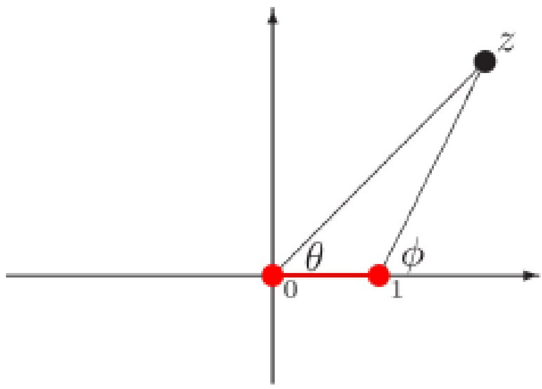
and so

$$Q'(z_i) = \frac{1-\alpha-z_i}{z_i(1-z_i)} \neq 0,$$

because $Q(1-\alpha)\neq 0$ if $0<\alpha<1$. Furthermore, with $P(z) = z/(z-1)+\lambda(z/(z-a))^{\alpha}$,

$$P(z_i) = \frac{z_i}{z_i-1} + z_i \neq 0.$$

The "function" $(z/(z-1))^{\alpha} = z^{\alpha}/(z-1)^{\alpha}$ is multi-valued. As such, we introduce the branch cut shown in Figure2.2



$$|z|>0,\ -\pi\le\theta<\pi \text{ and } |z-1|>0,\ -\pi\le\phi<\pi$$

Figure 2.2: Branch cut of $(z/(z-1))^{\alpha}$.

Given the branch depicted in Figure2.2, let

$$z = |z|e^{i\theta} \quad \text{and} \quad z-1 = |z-1|e^{i\phi},$$

where

$$|z|, |z-1| > 0 \quad \text{and} \quad -\pi \le \theta, \phi < \pi.$$

That says,

$$Q(z) = 0 \Leftrightarrow \lambda \frac{|z|^{\alpha-1}}{|z-1|^{\alpha}} e^{-i[(1-\alpha)\theta+\alpha\phi]} = 1$$
$$\Leftrightarrow (1-\alpha)\theta + \alpha\phi = -\pi \quad \text{and} \quad \lambda \frac{|z|^{\alpha-1}}{|z-1|^{\alpha}} = -1.$$

Furthermore, since $-\pi \le \theta$, $\phi < \pi$, then $\theta = \phi = -\pi$ and, consequently, $|z-1| = |z|+1$.
In view of the above argument, there are finitely many poles inside C_ρ and the residue of each is finite. Hence the result.
Following the same lines of reasoning employed in the scalar case and applying the Z-transform to (2.16), one obtains

$$\tilde{x}(z) = (1-z^{-1})^{-1}x_{-1} + A(1-z^{-1})^{-\alpha}(x_{-1} + z^{-1}\tilde{x}).$$

Re-arranging the terms and solving for $\tilde{x}(z)$, one gets

$$\tilde{x}(z) = \left(I_n - z^{-1}(1-z^{-1})A\right)^{-1}\left((1-z^{-1})^{-1} + (1-z^{-1})^{-\alpha}A\right)x_{-1},$$

where I_n is the identity matrix of order n.
Thus, like the above, we formulate the basic result. □

An immediate consequence of Theorem2.4 is the following corollary.

Corollary 2.1. *[42] If $\alpha = 1/2$ and A is a triangular matrix with diagonal elements $\lambda_i, i = 1, \cdots, n$, then the zero solution of (2.13) is asymptotically stable if and only if*

$$-\sqrt{2} < \lambda_i < 0, \quad \forall i = 1, \cdots, n. \tag{2.18}$$

Example 2.6. *Consider the following 1/2-order system of difference equations:*

$$\left({}^{C}\Delta^{1/2}_{a-1/2}\, x\right)(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x(t-1/2), \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \text{and}\ \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.19}$$

subject to the initial condition

$$x(a-1/2)=\begin{pmatrix}1\\1\end{pmatrix}. \tag{2.20}$$

Recall that Corollary2.1 asserts that the zero solution of (2.19)(2.20) *is asymptotically stable if* $-\sqrt{2}<\lambda_1,\lambda_2<0$. *This, for different values of* $\lambda_i, i=1,2$, *is confirmed in Figure2.3 below. The figure depicts the 2-norm of the solution* $\{x_k\}$ *for the first 100 iterates.*

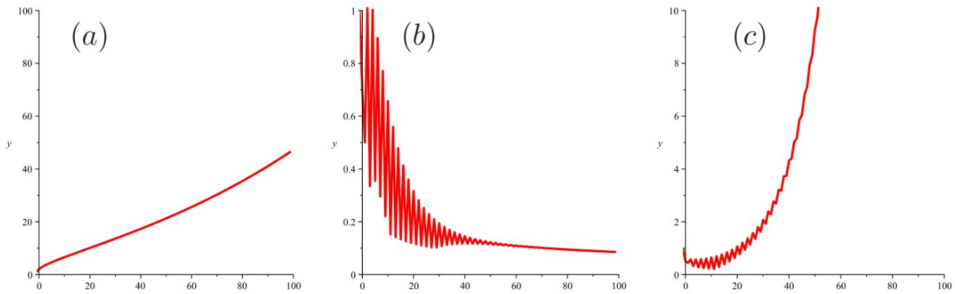


Figure 2.3: The first 100 iterates of 2-norm of the solution $\{x_k\}$ of Example2.6, for (a) $\lambda_1=0.1,\lambda_2=0.01(b)\lambda_1=-1.3,\lambda_2=-1$ and (c) $\lambda_1=-1.5,\lambda_2=-0.7$

These results gave a condition for stability of , but they remain difficult to implement. Below we present a practical result, we introduce the set:

$$S^{\alpha}=\left\{z\in\mathbb{C}:|z|<\left(2\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\text{ and }|\arg z|>\frac{\alpha\pi}{2}\right\}. \tag{2.21}$$

Theorem 2.5. *[43] Let* $\alpha\in(0,1)$ *and* A *is an* $n\times n$ *constant matrix. If* $\lambda\in S^{\alpha}$ *for all the eigenvalues* λ *of* A, *then the trivial solution of*(2.13) *is asymptotically stable. In this case, the solutions of*(2.13) *decay towards zero algebraically (and not exponentially), more precisely*

$$\|x(t)\|=O(t^{-\alpha})\quad \text{as } t\to\infty,$$

for any solution x *of* (2.13). *Furthermore, if* $\lambda\in\mathbb{C}\setminus cl(S^{\alpha})$ *for an eigenvalue* λ *of* A, *the zero solution of*(2.13) *is not stable.*

Proof. Without loss of generality, we shall take $a=0$, and use the variable change $t=k+1-\alpha$, system (2.13) become

$$({}^{C}\Delta_0^{\alpha}x)(k+1-\alpha)=Ax(k),\quad k=0,1,\cdots. \tag{2.22}$$

We have by definition

$$\begin{aligned}({}^C\Delta_0^\alpha x)(k+1-\alpha) &= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}(k-\alpha-s)^{(-\alpha)}\Delta x(s)\\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{\Gamma(k-\alpha-s+1)}{\Gamma(k-s+1)}\Delta x(s)\\ &= \sum_{s=0}^{k}\binom{k-\alpha-s}{k-s}\Delta x(s)\\ &= \sum_{s=1}^{k}\left[\binom{k-\alpha-s+1}{k-s+1}-\binom{k-\alpha-s}{k-s}\right]x(s)-\binom{k-\alpha}{k}x(0)+\binom{-\alpha}{0}x(k+1).\end{aligned}$$

Using Pascal rule yields:

$$\begin{aligned}({}^C\Delta_0^\alpha x)(k+1-\alpha) &= \sum_{s=1}^{k}\binom{k-\alpha-s}{k-s+1}x(s)-\binom{k-\alpha}{k}x(0)+x(k+1)\\ &= \sum_{s=0}^{k}\binom{k-\alpha-s}{k-s+1}x(s)-\left[\binom{k-\alpha}{k}+\binom{k-\alpha}{k+1}\right]x(0)+x(k+1)\\ &= \sum_{s=0}^{k}(-1)^{k-s+1}\binom{\alpha}{k-s+1}x(s)-(-1)^{k+1}\binom{\alpha-1}{k+1}x(0)+x(k+1),\end{aligned}$$

so a sequence $x(k)$ is a solution of(2.13) if and only if it is a solution of

$$x(k+1) = Ax(k)+\sum_{s=0}^{k}B(k-s)x(s)+g(k), \quad k=0,1,\cdots, \tag{2.23}$$

where

$$B(k) = (-1)^k\binom{\alpha}{k+1}I_n \quad \text{and} \quad g(k) = (-1)^{k+1}\binom{\alpha-1}{k+1}x(0). \tag{2.24}$$

Comparing both the Volterra difference equations, the system (2.16) seems to be formally simpler and more suitable to analyze. However, the system (2.23) has an important advantage compared to (2.16), namely the property that the elements of its convolution kernel B given by (2.24) belong to the space $\ell^1(\mathbb{N}^{n\times n})$ of absolutely summable sequences, while the elements of kernel B given by (2.17) not. This will enable us to apply some relevant results of the theory of Volterra difference equations Theorem, where the assumption $B \in \ell^1(\mathbb{N}^{n\times n})$ is crucial.

(1) We consider (2.13) in its equivalent form (2.23) and first analyze its homogeneous part

$$x(k+1) = Ax(k) + \sum_{s=0}^{k} B(k-s)x(s), \quad k = 0, 1, \cdots. \tag{2.25}$$

To apply Theorem1.6, we perform some necessary calculations related to the Z-transform of the kernel $b(k) = (-1)^k \binom{\alpha}{k+1}$. Using expansion into the binomial series (with the radius of convergence $R = 1$) we have

$$\tilde{b}(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k+1} z^{-k} = z - z\left(1 - \frac{1}{z}\right)^{\alpha}. \tag{2.26}$$

Then, based on Theorem1.6, the homogeneous system (2.25) is asymptotically stable if all the zeros of the characteristic equation

$$\det\left(z\left(1 - \frac{1}{z}\right)^{\alpha} I_n - A\right) = 0 \tag{2.27}$$

are located inside the unit circle. Obviously, z is a zero of this equation if and only if $z(1-z^{-1})^{\alpha}$ is an eigenvalue of A.
Now we consider the curve

$$\Gamma^{\alpha} = \left\{ z\left(1 - \frac{1}{z}\right)^{\alpha} : |z| = 1 \right\},$$

defining the stability boundary for (2.13) and describe its structure. Let $z = e^{i\varphi}$ for $0 \leq \varphi < 2\pi$ and let $1 - z^{-1} = re^{i\omega}$ for $r = r(\varphi) \geq 0$ and $\omega = \omega(\varphi), 0 \leq \omega < 2\pi$. Then

$$1 - e^{-i\varphi} = re^{i\omega},$$

which, after equating real and imaginary parts, turns into

$$1 - \cos\varphi = r\cos\omega, \quad \sin\varphi = r\sin\omega.$$

If $\varphi = 0$, then $r = 0$. If $\varphi \neq 0$, then

$$\tan\omega = \frac{\sin\varphi}{1 - \cos\varphi}.$$

Since

$$\frac{\sin\varphi}{1-\cos\varphi} = \frac{2\sin(\varphi/2)\cos(\varphi/2)}{2\sin^2(\varphi/2)} = \cot\frac{\varphi}{2} = \tan(\frac{\pi}{2} - \frac{\varphi}{2}),$$

we can write $\omega = \pi/2 - \varphi/2$. Further

$$r = 2\sin\frac{\varphi}{2}.$$

From here we get

$$\Gamma^{\alpha} = \left\{ \left(2\sin\frac{\varphi}{2}\right)^{\alpha} \exp(i\frac{2\varphi + (\pi - \varphi)\alpha}{2}) : 0 \le \varphi < 2\pi \right\}.$$

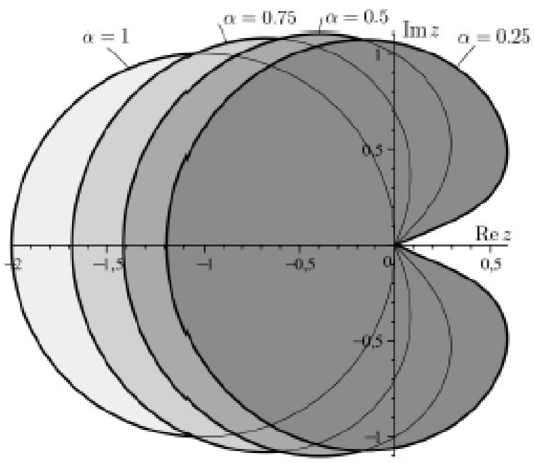


Figure 2.4: Asymptotic stability sets S^{α} for several values of α

If we set $\theta = -\omega = \varphi/2 - \pi/2$, then

$$\Gamma^{\alpha} = \left\{ -(2\cos\theta)^{\alpha} \exp(i(2-\alpha)\theta) : -\frac{\pi}{2} \le \theta < \frac{\pi}{2} \right\}.$$

Using the polar form, $|z| = (2\cos\theta)^{\alpha}$, where

$$\theta = \begin{cases} \frac{\arg z - \pi}{2-\alpha} & \text{if} \quad \arg z > 0, \\ \frac{\arg z + \pi}{2-\alpha} & \text{if} \quad \arg z < 0. \end{cases}$$

This is equivalent to

$$\Gamma^{\alpha} = \left\{ z \in \mathbb{C} : |z| = \left(2\cos\frac{|\arg z| - \pi}{2-\alpha}\right)^{\alpha} \text{ and } |\arg z| \ge \frac{\alpha\pi}{2} \right\}. \tag{2.28}$$

(2) As a next step, we show that $w_{\alpha}(z) = z(1-z^{-1})^{\alpha}$ maps the unit circle $D = \{z \in \mathbb{C} : |z| < 1\}$ onto S^{α} . First we consider the upper part of D in the form $D_u = \{z \in \mathbb{C} : |z| < 1 \text{ and } Imz > 0\}$.

Since w_α is holomorphic and nonconstant on D_u, it maps D_u (by the open mapping Theorem) to an open set. It means that for any $z_0 \in D_u$ there exists a neighborhood of $w_\alpha(z_0)$ contained in $w_\alpha(D_u)$. In other words, a point of D_u can not be mapped to the boundary of $w_\alpha(D_u)$. Further, the function w_α is continuously extendable on the closure $cl(D_u)$ (the singularity at the point $z = 0$ is removable as $\lim_{z\to 0} w_\alpha(z) = 0$) which is a compact set in $\mathbb{C}$ and thus $w_\alpha(D_u)$ is bounded. The same conclusion holds if we take the lower part of D in the form $D_\ell = \{z \in \mathbb{C} : |z| < 1 \text{ and } Imz < 0\}$. Finally, the interval $(-1, 0)$ of the real axis (which is a part of the boundary ∂D_u as well as ∂D_ℓ) is mapped by w_α onto the interval $(-2^\alpha, 0)$ of the real axis, while the continuous extensions of w_α on $cl(D_u)$ and $cl(D_\ell)$ map the interval $[0, 1)$ onto abscissas lying in $w_\alpha(cl(D_u))$ and $w_\alpha(cl(D_\ell))$, respectively. It means that the interval $[-2^\alpha, 0]$ is a common part of the boundaries $\partial w_\alpha(D_\ell)$ and $\partial w_\alpha(D_u)$. In view of $w_\alpha(\partial D) = \Gamma^\alpha$ (see the previous step), the above arguments imply that $w_\alpha(D) = S^\alpha$.
(3) Let all the eigenvalues λ of A belong to S^α and let x be the solution of (2.23). Then by the variation of constants formula, we obtain

$$x(k) = R(k)x(0) + \sum_{s=0}^{k-1} R(s)g(k-s-1),$$

where $R(k)$ is the resolvent matrix of (2.25). So

$$\begin{aligned} x(k) &= R(k)x(0) + \sum_{s=0}^{k-1} R(s)g(k-s-1) \\ &= R(k)x(0) + \sum_{s=0}^{k-1} R(s)(-1)^{k-s}\binom{\alpha-1}{k-s}x(0) \\ &= \sum_{s=0}^{k} R(s)(-1)^{k-s}\binom{\alpha-1}{k-s}x(0) = \sum_{s=0}^{k} R(k-s)(-1)^{s}\binom{\alpha-1}{s}x(0). \end{aligned}$$

Using the asymptotic equivalence

$$(-1)^k\binom{\alpha-1}{k} \sim Ck^{-\alpha} \quad \text{as } k \to \infty,$$

(with the constant C depending on α only) and taking the norms, we have

$$\begin{aligned}\|x(k)\| &= \left\|\sum_{s=0}^{k} R(k-s)(-1)^s \binom{\alpha-1}{s} x(0)\right\| \\ &\leq \left\|\sum_{s=0}^{k} R(k-s)(-1)^s \binom{\alpha-1}{s}\right\| \|x(0)\| \\ &\leq C_1 \sum_{s=0}^{k} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \\ &= C_1 \left(\sum_{s=0}^{\lfloor k/2 \rfloor} \frac{1}{(s+1)^\alpha} \|R(k-s)\| + \sum_{s=\lfloor k/2 \rfloor+1}^{k} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \right),\end{aligned}$$

where $C_1 > 0$ is a suitable real constant and the symbol $\lfloor . \rfloor$ stands for the floor function. Since each component of R belongs to $\ell^1(\mathbb{N}_0)$, we have $\|R(k)\| = O(k^{-1})$ as $k \to \infty$ and there exist $C_2, C_3 > 0$ such that

$$\sum_{s=0}^{\lfloor k/2 \rfloor} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \leq \frac{C_2}{k+1} \sum_{s=0}^{\lfloor k/2 \rfloor} \frac{1}{(s+1)^\alpha} \leq \frac{C_3}{(k+1)^\alpha},$$

where we have used the inequality $\sum_{s=1}^{k} (s+1)^{-\alpha} \leq \int_0^k (x+1)^{-\alpha} dx$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor k/2 \rfloor+1}^{k} \frac{1}{(s+1)^\alpha} \|R(k-s)\| \leq \frac{C_4}{(k+1)^\alpha} \sum_{s=\lfloor k/2 \rfloor+1}^{k} \|R(k-s)\| \leq \frac{C_5}{(k+1)^\alpha},$$

for suitable $C_4, C_5 > 0$. In summary, we have $\|x(k)\| \leq C_5 (k+1)^{-\alpha}$, hence $\|x(k)\| = O(k^{-\alpha})$ as $k \to \infty$.

(4) It remains to show that if $\lambda \in \mathbb{C} \backslash cl(S^\alpha)$ for an eigenvalue λ of A, then the zero solution of (2.13) is not stable (equivalently, if there is a zero of $(\det\left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right) = 0)$ with $|z| > 1$, then (2.13) is not stable).

If $|z| > 1$, then $\tilde{g}(z) = (1 - z^{-1})^{\alpha-1} x(0)$, and the Z-transform of the solution x of (2.13) takes the form

$$\tilde{x}(z) = \left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right)^{-1} z\left(1-\frac{1}{z}\right)^{\alpha-1} x(0). \tag{2.29}$$

A zero of $\det\left(z\left(1-\frac{1}{z}\right)^\alpha I_n - A\right)$ represents a singular point of $\tilde{x}$. It is known that if $\tilde{f}(z)$ is the Z-transform of a sequence $f : \mathbb{N} \to \mathbb{R}$, then its radius of convergence R is given by distance from origin to an outermost (non-removable) singular point. Hence, if there is a zero z_0 with $|z| > 1$, then also the radius of convergence of at least one component x_i of $\tilde{x}$ satisfies $R > 1$. Using the Cauchy-Hadamard theorem we have

$$r = \limsup_{k\to\infty} \sqrt[k]{|x_i(k)|} > 1$$

and, consequently, $\limsup_{k\to\infty} |x_i(k)| = +\infty$ which proves that x is not bounded and thus (2.13) is not stable. $\square$

Remark 2.4. *The assertions of Theorem2.4 and Theorem2.5 describe the same stability region. But these analytical descriptions are different. In particular, the condition stated in Theorem2.5 seems to be more convenient for practical purposes, due to the explicit form of S^α.*

Example 2.7. *Consider the following system:*

$$^{C}\Delta_a^{\frac{1}{2}} x(t) = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{2}{3} & -1 \end{pmatrix} x(t), \quad t \in \mathbb{N}_{a+1-\alpha}. \tag{2.30}$$

The characteristic equation for $A = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{2}{3} & -1 \end{pmatrix}$ is $\lambda^2 + \lambda + 1 = 0$ and hence the eigenvalues of A are $\lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\lambda_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$. Since

$$|\lambda_2| = 1 > \left(2\cos\frac{|\arg\lambda_2| - \pi}{2-\frac{1}{2}}\right)^{\frac{1}{2}} = \left(2\cos\frac{-4\pi}{9}\right)^{\frac{1}{2}} = 0.589318551 \Leftrightarrow \lambda_2 \notin S^{\frac{1}{2}}.$$

Then, by Theorem2.5, the trivial solution of (2.30) is not stable.

Example 2.8. *Consider the following system:*

$$^{C}\Delta_a^{\frac{2}{3}} x(t) = \begin{pmatrix} -\frac{3}{8} & \frac{3}{2} \\ 0 & -\frac{17}{27} \end{pmatrix} x(t), \quad t \in \mathbb{N}_{a+1-\frac{2}{3}}. \tag{2.31}$$

The characteristic equation for $\begin{pmatrix} -\frac{3}{8} & \frac{3}{2} \\ 0 & -\frac{17}{27} \end{pmatrix}$ is $(\lambda + \frac{3}{8})(\lambda + \frac{17}{27}) = 0$ and hence the eigenvalues of A are $\lambda_1 = -\frac{3}{8}$ and $\lambda_2 = -\frac{17}{27}$. Since

$$|\lambda_1| = \frac{3}{8} < \left(2\cos\frac{|\arg\lambda_1| - \pi}{2-\frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_1| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_1 \in S^{\frac{2}{3}},$$

and

$$|\lambda_2| = \frac{17}{27} < \left(2\cos\frac{|\arg\lambda_2| - \pi}{2-\frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_2| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_2 \in S^{\frac{2}{3}}.$$

Then, by Theorem2.5, the trivial solution of (2.31) is asymptotically stable.

2.4.2 Stability of non-linear systems

In this section, we extend the method of the Lyapunov functions to study the stability of solutions of the **Advanced time** and **Delay time** non-linear fractional order difference systems. First, we list some definitions that will be used in study the stability properties.

Definition 2.2. *[41] A function $\phi(r)$ is said to belong to the class $\mathcal{K}$ if and only if $\phi \in C[[0,\rho),\mathbb{R}_+]$, $\phi(0) = 0$ and $\phi(r)$ is strictly monotonically increasing in r. If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi \in \mathcal{K}$, and $\lim_{r\to\infty} \phi(r) = \infty$, then ϕ is said to belong to class $\mathcal{KR}$.*

Definition 2.3. *[41] A real valued function $V(t,x)$ defined on $\mathbb{N}_a \times S_\rho$, where $S_\rho = \{x \in \mathbb{R} : \|x\| \le \rho\}$, is said to be positive definite if and only if $V(t,0) = 0$ for all $t \in \mathbb{N}_a$ and there exists $\phi(r) \in \mathcal{K}$ such that $\phi(r) \le V(t,x), \|x\| = r, (t,x) \in \mathbb{N}_a \times S_\rho$.*

Definition 2.4. *[41] A real valued function $V(t,x)$ defined on $\mathbb{N}_a \times S_\rho$, where $S_\rho = \{x \in \mathbb{R} : \|x\| \le \rho\}$, is said to be decrescent if and only if $V(t,0) = 0$ for all $t \in \mathbb{N}_a$ and there exists $\varphi(r) \in \mathcal{K}$ such that $V(t,x) \le \varphi(r), \|x\| = r, (t,x) \in \mathbb{N}_a \times S_\rho$.*

2.4.2.1 Advanced time systems

Consider the following system:

$$\begin{cases} {}^C\Delta_{t_0}^{\alpha} x(t) = f(t+\alpha-1, x(t+\alpha-1)), \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.32}$$

where $t_0 = a + n_0 \in \mathbb{N}_a$ $(n_0 \in \mathbb{N})$, $t \in \mathbb{N}_{n_0}, a = \alpha - 1$, $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \le 1$. Let $f(t,0) = 0$, for all $t \in \mathbb{N}_a$ so that the system (2.32) admits the trivial solution.

Remark 2.5. *When $\alpha = 1$, the system (2.32) become*

$$\begin{cases} \Delta x(t) = f(t, x(t)), \quad t \in \mathbb{N}_{t_0}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases}$$

For the system (2.32) we have the following theorems.

Theorem 2.6. *[41] If there exists a positive definite and decrescent scalar function $V(t,x) \in C\Big[\mathbb{N}_a \times S_\rho, \mathbb{R}_+\Big]$ such that*

$$ {}^C\Delta_{t_0}^{\alpha} V(t, x(t)) \le 0, \tag{2.33}$$

for all $t_0 \in \mathbb{N}_a$ and $(t,x) \in \mathbb{N} \times S_\rho$, then the trivial solution of (2.32) is uniformly stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of system (2.32). Since $V(t, x)$ is positive definite and decrescent, there exist $\varphi, \phi \in \mathcal{K}$ such that

$$\phi(\|x\|) \leq V(t, x) \leq \varphi(\|x\|),$$

for all $(t, x) \in \mathbb{N}_a \times S_\rho$.
For each $\epsilon > 0, 0 < \epsilon < \rho$ we choose a $\delta = \delta(\epsilon)$ such that

$$\varphi(\delta) < \phi(\epsilon).$$

For any solution $x(t)$ of (2.32) we have $\phi(\|x(t)\|) \leq V(t, x(t))$ with $\|x_0\| < \delta(\epsilon)$. Since ${}^C\Delta_{t_0}^\alpha V(t, x(t)) \leq 0$, by using Theorem1.12 we have $V(t, x(t)) \leq V(t_0, x_0)$ for all $t \in \mathbb{N}_{t_0}$. Consequently,

$$\phi(\|x(t)\|) \leq V(t, x(t)) \leq V(t_0, x_0) \leq \varphi(\|x_0\|) < \varphi(\delta) < \phi(\epsilon),$$

and thus $\|x(t)\| < \epsilon$ for all $t \in \mathbb{N}_{t_0}$. □

Theorem 2.7. *[41] If there exists a positive definite and decrescent scalar function $V(t, x) \in C\left[\mathbb{N}_a \times S_\rho, \mathbb{R}_+\right]$ such that*

$$ {}^C\Delta_{t_0}^\alpha V(t, x(t)) \leq -\psi(\|x(t + \alpha - 1\|), \quad \forall t_0 \in \mathbb{N}_a,\ (t, x) \in \mathbb{N} \times S_\rho, \tag{2.34}$$

where $\psi \in \mathcal{K}$, then the trivial solution of (2.32) is uniformly asymptotically stable.

Proof. Since all the conditions of Theorem2.6 are satisfied, the trivial solution of the system (2.32) is stable. Let $0 < \epsilon < \rho$ and $\delta = \delta(\epsilon) > 0$ correspond to stability. Choose a fixed $\epsilon_0 < \rho$ and $\delta_0 = \delta(\epsilon_0) > 0$. Now, choose $\|x_0\| < \delta_0$ and $T(\epsilon)$ large enough such that $(T + a)^{(\alpha)} \geq (\varphi(\delta_0)/\psi(\delta(\epsilon)))\Gamma(\alpha + 1)$. Such a large T can be chosen since $\lim_{T\to\infty}(\Gamma(T + \alpha)/\Gamma(T)) = \infty$. Now, we claim that $\|x(t, t_0, x_0)\| < \delta(\epsilon)$ for all $t \in [t_0, t_0 + T] \cap \mathbb{N}_{t_0}$. If this is not true, due to (2.34) and Theorem1.10, we get

$$\begin{aligned}
V(t, x(t, t_0, x_0)) &\leq V(t_0, x_0) - \frac{1}{\Gamma(\alpha)} \sum_{s=t_0+1-\alpha}^{t-\alpha} (t - s - 1)^{(\alpha-1)} \psi(\|x(s + \alpha - 1)\|) \\
&\leq \varphi(\|x_0\|) - \frac{\psi(\delta)}{\Gamma(\alpha)} \sum_{s=n_0}^{t-\alpha} (t - s - 1)^{(\alpha-1)} \\
&\leq \varphi(\delta_0) - \frac{\psi(\delta)}{\Gamma(\alpha + 1)} (t - n_0)^{(\alpha)}.
\end{aligned}$$

Substituting $t = t_0 + T$, we get

$$0 < \phi(\delta(\epsilon)) \leq V(t_0 + T, x(t_0 + T, t_0, x_0)) \leq \varphi(\delta_0) - \frac{\psi(\delta)}{\Gamma(\alpha + 1)} (T + t_0 - n_0)^{(\alpha)} \leq 0,$$

which is a contradiction. Thus, there exists a $t \in [t_0, t_0 + T]$ such that $\|x(t)\| < \delta(\epsilon)$. But in this case, since the trivial solution is uniformly stable and t is arbitrary, $\|x(t)\| < \epsilon$ for all $t \geq t_0 + T$ whenever $\|x_0\| < \delta_0$. □

Theorem 2.8. *[41] If there exists a function $V(t,x) \in C[\mathbb{N}_a \times \mathbb{R}^n, \mathbb{R}_+]$ such that*

$$\phi(\|x(t)\|) \leq V(t,x) \leq \varphi(\|x(t)\|) \quad \forall (t,x) \in \mathbb{N}_a \times \mathbb{R}^n,$$

$${}^C\Delta_{t_0}^{\alpha} V(t,x(t)) \leq -\psi(\|x(t+\alpha-1)\|) \quad \forall t_0 \in \mathbb{N}_a,\ (t,x) \in \mathbb{N} \times \mathbb{R}^n,$$

where φ, ϕ, and $\psi \in \mathcal{KR}$ hold for all $(t,x) \in \mathbb{N}_a \times \mathbb{R}^n$, then the trivial solution of (2.32) is globally uniformly asymptotically stable.

Proof. Since the conditions of Theorem2.7 are satisfied, the trivial solution of (2.32) is uniformly asymptotically stable. It remains to show that the domain of attraction of $x = 0$ is all of $\mathbb{R}^n$. Since $\lim_{r\to\infty} \phi(r) = \infty$, δ_0 in the proof of Theorem2.7 may be chosen arbitrary large and ϵ can be chosen such that it satisfies $\varphi(\delta_0) < \phi(\epsilon)$. Thus, the globally uniformly asymptotic stability of $x = 0$ is concluded. □

Remark 2.6. *Theorem2.8 gives a sufficient condition to analyze stability of (2.32). But it's not easy to construct ϕ, φ and ψ functions directly. Yet, so far there is no direct way to know the stability of system (2.32).*

2.4.2.2 Delay time systems

Now, we consider the following system:

$$\begin{cases} {}^C\Delta_a^{\alpha} x(t) = f(t+\alpha, x(t+\alpha)), & t \in \mathbb{N}_{a+1-\alpha}, \\ x(a) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.35}$$

where $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$. Let $f(t,0) = 0$, for all $t \in \mathbb{N}_a$ so that the system (2.35) admits the trivial solution.

Remark 2.7. *when $\alpha = 1$, the system (2.35) become*

$$\begin{cases} \Delta x(t) = f(t+1, x(t+1)), & t \in \mathbb{N}_a, \\ x(a) = x_0, \quad x_0 \in \mathbb{R}^n. \end{cases}$$

The result of Theorem2.8 remain correct in this case and is formulated as follows:

Theorem 2.9. *[44] If there exists a positive definite and decrescent scalar function $V(t,x)$, discrete class–$\mathcal{K}$ functions ϕ, φ and ψ such that*

$$\phi(\|x(t)\|) \leq V(t,x) \leq \varphi(\|x(t)\|), \ \ t \in \mathbb{N}_a,$$

and

$${}^C\Delta_a^\alpha V(t,x(t)) \leq -\psi(\|x(t+\alpha)\|), \quad t \in \mathbb{N}_{a+1-\alpha},$$

then, the trivial solution of (2.35) is asymptotically stable.

Now, we present a sufficient condition for stability of (2.35). We first introduce the following Lemma. In the rest, we assume the initial point $a = 0$ for simplicity.

Lemma 2.1. *[44] For $0 < \alpha \leq 1$, and any discrete time $t \in \mathbb{N}_{a+1-\alpha}$, the following inequality holds:*

$${}^C\Delta_a^\alpha x^2(t) \leq 2x(t+\alpha)^C\Delta_a^\alpha x(t). \tag{2.36}$$

Proof. We need to equivalently prove

$${}^C\Delta_a^\alpha x^2(t) - 2x(t+\alpha)^C\Delta_a^\alpha x(t) \leq 0.$$

The left hand side can be rewritten explicitly as

$$\frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}(t-s-1)^{(-\alpha)}\Delta_s\left(x^2(s) - 2x(t+\alpha)x(s)\right).$$

We add $\Delta_s x^2(t+\alpha)$

$${}^C\Delta_a^\alpha x^2(t) - 2x(t+\alpha)^C\Delta_a^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}(t-s-1)^{(-\alpha)}\Delta_s\left(x(s)-x(t+\alpha)\right)^2.$$

Using the summation by parts. We obtain

$$\begin{aligned}
{}^C\Delta_a^\alpha x^2(t) - 2x(t+\alpha)^C\Delta_a^\alpha x(t) &= \frac{-1}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}\left(x(s)-x(t+\alpha)\right)^2\Delta_s(t-s)^{(-\alpha)} \\
&+ \frac{(t-s)^{(-\alpha)}}{\Gamma(1-\alpha)}\left(x(s)-x(t+\alpha)\right)^2 |_{s=a}^{t+\alpha} \\
&= \frac{-\alpha}{\Gamma(1-\alpha)}\sum_{s=a}^{t+\alpha-1}\left(x(s)-x(t+\alpha)\right)^2(t-s-1)^{(-\alpha-1)} \\
&- \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)}\left(x(a)-x(t+\alpha)\right)^2 \leq 0.
\end{aligned}$$

This completes the proof. □

Theorem 2.10. *[44] If $x^T(t+\alpha)f(t+\alpha, x(t+\alpha)) < 0$ for any $t \in \mathbb{N}_{a+1-\alpha}$, then the trivial solution of (2.35) is asymptotically stable.*

Proof. Use a discrete Lyapunov candidate function $V(t, x(t)) = \frac{1}{2}\sum_{i=1}^{n} x_i^2(t)$.
If $x^T(t+\alpha)f(t+\alpha, x(t+\alpha)) < 0$, then we have by Lemma2.1

$$^C\Delta_a^\alpha V \leq \sum_{i=1}^{n} x_i(t+\alpha)^C\Delta_a^\alpha x_i(t) = x^T(t+\alpha)f(t+\alpha, x(t+\alpha)) < 0,$$

which means the fractional difference of $V(t, x(t))$ is negative definite.
Considering the Theorem1.10, we can obtain

$$V(t, x(t)) < V(0, x(0)), \quad \forall t \in \mathbb{N}_a,$$

or

$$\frac{1}{2}\sum_{i=1}^{n} x_i^2(t) < \frac{1}{2}\sum_{i=1}^{n} x_i^2(0).$$

According to the definition of the stability, we can determine the origin is stable.
On the other hand, the fractional difference of the V function results negative definite. The used Lyapunov function is positive definite. As a result, the equilibrium point is asymptotically stable from Theorem2.9. This completes the proof. □

Example 2.9. *Consider the linear discrete fractional equation:*

$$^C\Delta_a^\alpha x(t) = -x(t+\alpha), \quad x(0) = 0.1, \quad 0 < \alpha \leq 1, \quad t \in \mathbb{N}_{a+1-\alpha}.$$

We check that

$$x(t+\alpha)f(t+\alpha, x(t+\alpha)) = -x^2(t+\alpha) < 0, \ \ t \in \mathbb{N}_{a+1-\alpha}.$$

Hence, the trivial solution is asymptotically stable according to Theorem2.10

Example 2.10. *Consider a discrete time varying system of fractional order:*

$$\begin{cases} ^C\Delta_a^\alpha x_1(t) = -2x_1(t+\alpha) + tx_2(t+\alpha), & x_1(0) = 0.4, \quad 0 < \alpha \leq 1, \\ ^C\Delta_a^\alpha x_2(t) = -tx_1(t+\alpha) - x_2(t+\alpha), & x_2(0) = 0.8, \quad t \in \mathbb{N}_{a+1-\alpha}. \end{cases}$$

We can calculate

$$x^T(t+\alpha)f(t+\alpha, x(t+\alpha)) = -2x_1^2(t+\alpha) - x_2^2(t+\alpha) < 0, \quad \forall t \in \mathbb{N}_{a+1-\alpha}.$$

It is evident that the system is asymptotically stable from Theorem2.10.

2.5 Conclusion

In this chapter, we have presented some stability theorems of fractional order difference systems. We begin with a reminder of the stability of difference systems of integer order, then we study the stability in fractional order case. The stability of fractional difference systems within commensurate orders and other fractional discrete operators have been studied in [73, 74, 75].

Chapter 3

Discrete chaos and fractional order maps

3.1 Introduction

This chapter is meant to provide a simple and heuristic illustration of some basic features of chaos theory in nonlinear discrete dynamical systems with fractional order.
Basically, this chapter introduces the basic mathematical concepts and numerical tools to analyze such irregular geometrical entities, in order to be able to carry out the analysis and numerical simulations required for discrete fractional system in our thesis.
To this aim, the theoretical background needed for this research is presented. The definition of discrete dynamical systems will be introduced first. Next, chaos definition, characteristics of chaos and transitions to chaos will be presented. Then, we consider two systems which played a crucial role in the development of dynamical systems theory: the integer order Logistic map and the integer order Hénon map and their fractional order counterpart, to illustrate and demonstrate the methods.

3.2 Discrete time dynamical systems:

A dynamical system may be defined as a deterministic mathematical description for evolving the state of a system forward in time. Dynamical systems may be continuous time or discrete time outlined by differential equations or difference equations, respectively, or stochastic process. It is worth remarking from the outset that there is no specific difference between continuous and discrete time dynamical systems; as a continue system of dimension N can be reduced to a discrete time map of dimension $N-1$ via the Poincaré map [45] . Referring to discrete-time dynamical systems, such systems provide

accurate models for natural physical phenomena in the field of biology, chemistry and physics. Moreover, discrete-time systems can avoid the calculation error of the numerical discretization of continuous ones.
Discrete-time dynamical systems can be written as the map [46]

$$x(n+1) = f(x(n)), \tag{3.1}$$

where x_n si N-dimensional, $x(n) = (x_1(n), x_2(n), \cdots, x_N(n))$ which is a shorthand notation for

$$\begin{cases} x_1(n+1) = f_1(x_1(n), x_2(n), \cdots, x_N(n)), \\ \quad \vdots \\ x_N(n+1) = f_N(x_1(n), x_2(n), \cdots, x_N(n)), \end{cases} \tag{3.2}$$

with n denoting the discrete-time variable. Given initial values $(x_1(0), \cdots x_N(0))$ we can com-pute $(x_1(n), \cdots, x_N(n))$ successively for all positive n using (3.1). Thus, the trajectory of $x(0)$ is the sequence

$$x(0), \quad x(1) = f(x(0)), \quad x(2) = f^2(x(0), \cdots, x(n) = f^n(x(0)), \cdots$$

3.3 Discrete chaos

3.3.1 Definition of chaos

Several mathematical definitions of chaos can be found in the literature, but so far there is no universal mathematical definition of chaos. Before giving the definition of chaos, due to **R. L Devaney**, some basic definitions are necessary.
Let $(I \subset \mathbb{R}, d)$ denote a compact metric space (d is a distance), and let F be the function :

$$\begin{aligned} &F : I \to I \\ &F(x(k)) = x(k+1), x(0) \in I \end{aligned} \tag{3.3}$$

Definition 3.1. *Suppose X is a set and Y a subset of X, Y is dense in X if, for any element $x \in X$, there exists an element y in the subset Y arbitrarily close to x, that is, if the closure of Y equals X . Which amounts to saying that Y is dense in X if for all $x \in X$ we can find a sequence of points $\{y_n\} \in Y$ which converge to x.*

Definition 3.2. *F is said to have the property of sensitivity to initial conditions if there exists $\delta > 0$ such that, for $x(0) \in I$ and all $\varepsilon > 0$ there exists a point $y(0) \in I$ and an integer $j \geq 0$*

satisfying :

$$d(x(0), y(0) > \varepsilon \Rightarrow d(F^{(j)}(x(0)), F^{(j)}(y(0))) > \delta$$

Where d represents the distance and $F^{(j)}$ the $j^{ème}$ iteration of F.

Definition 3.3. *F is said to be topologically transitive if U and V being two open nonempty sets in I, there exists $x(0) \in U$ and an index $j \in \mathbb{Z}^+$, such that for $F^{(j)}(x(0)) \in V$, or equivalently, there exists an index $j \in \mathbb{Z}^+$, such that for $F^{(j)}(U) \cap V \neq \emptyset$.*
We are now in a position to state the definition of chaos, in ***Devaney****'s sense.*

Definition 3.4. *The function F of equation(3.1) is said to consist of chaotic dynamics if :*

- *F has sensitivity to initial conditions,*
- *F is topologically transitive,*
- *The set of periodic points of F is dense in I.*

Although there is no universally accepted definition of the notion of chaos, this definition remains the most interesting because the concepts on which it is based are easily observable.

3.3.2 Characteristics of Chaos

(a) Sensitivity to initial conditions: Sensitivity to initial conditions is a phenomenon discovered for the first time at the end of the **xixe** century by **Poincaré**, then was rediscovered in **1963** by **Lorenz** during his work in meteorology. This discovery led to a large number of important works, mainly in the field of mathematics. This sensitivity explains the fact that, for a chaotic system, a tiny modification of the initial conditions can lead to unpredictable results in the long term. The degree of sensitivity to the initial conditions quantifies the chaotic character of the system.

(b) Lyapunov exponents: The Lyapunov exponents are used to measure the possible divergence between two orbits resulting from neighboring initial conditions and make it possible to quantify the sensitivity to the initial conditions of a chaotic system. The number of Lyapunov exponents is equal to the dimension of the phase space.

Consider the following discrete nonlinear dynamical system :

$$x(k+1) = F(x(k)) \tag{3.4}$$

With $x(k) \in \mathbb{R}^n$. We assume that the trajectory emanating from an initial state $x(0)$ reaches an attractor. $x(k)$ is thus bounded inside the attractor.

We choose two very close initial conditions, denoted by $x(0)$ and $\acute{x}(0)$ and we look at how the resulting trajectories behave. Assuming that the two trajectories $x(k)$ and $\acute{x}(k)$ deviate exponentially, after k there comes :

$$|\acute{x}(k) - x(k)| = |\acute{x}(0) - x(0)|\, e^{\lambda k} \tag{3.5}$$

λ indicates the rate of divergence per iteration of the two trajectories whose expression is as follows :

$$\lambda = \frac{1}{k} \ln \left| \frac{\acute{x}(k) - x(k)}{\acute{x}(0) - x(0)} \right| \tag{3.6}$$

For $x(0)$ and $\acute{x}(0)$ close, if the modulus of the difference $\varepsilon = |\acute{x}(0) - x(0)|$ tends to converge to zero, we get :

$$\lambda_L = \lim_{k \to \infty} \frac{1}{k} \lim_{\varepsilon \to 0} \ln \left| \frac{\acute{x}(k) - x(k)}{\acute{x}(0) - x(0)} \right| \tag{3.7}$$

This gives :

Finally we have :

$$\lambda_L = \lim_{k \to \infty} \lim_{\varepsilon \to 0} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left| \frac{dF(x(i))}{dx(i)} \right| \tag{3.8}$$

λ_L is called the Lyapunov exponent, it measures the average rate of divergence of two distinct trajectories, from two very close initial conditions. In the case of a system of dimension $n > 1$ there exists n Lyapunov exponent $\lambda_L^{(j)}$, $(j = 1, 2, ..., n)$, each of them measures the divergence rate along one of the phase space axes. For the calculation of the Lyapunov exponent, we start from an initial point $x(0) \in \mathbb{R}^n$, to characterize the infinitesimal behavior around the point $x(k)$ by the first derived matrix $DF(x(i))$.

$$DF(x(i)) = \begin{pmatrix} \frac{\partial f_1(x(i))}{\partial x_1(i)} & \cdots & \frac{\partial f_1(x(i))}{\partial x_n(i)} \\ \vdots & \vdots\vdots\vdots & \vdots \\ \frac{\partial f_n(x(i))}{\partial x_1(i)} & \cdots & \frac{\partial f_n(x(i))}{\partial x_n(i)} \end{pmatrix} \tag{3.9}$$

Note : $J_k = DF(x(k-1))...DF(x(0))$, with : $J_0 = DF(x(0))$

The Lyapunov exponent is calculated by the following expression :

$$\lambda_L^{(j)} = \lim_{k\to\infty} \frac{1}{k} \ln|\lambda_i(J_k...J_1)|, \quad i = 1,2,...,n \tag{3.10}$$

By analyzing the Lyapunov exponents of a system, we can conclude about the type of behavior of this system as follows :

- If $\lambda_n \leq ... \leq \lambda_1 < 0$, there exist asymptotically stable fixed points.
- If $\lambda_1 = 0, \lambda_n \leq ... \leq \lambda_2 < 0$, the attractor is an asymptotically stable limit cycle.
- If $\lambda_1 = ... = \lambda_k = 0$, $\lambda_n \leq ... \leq \lambda_{k+1} < 0$,the attractor is a torus of dimension k, that is, quasi-periodic.
- If $\lambda_1 > 0, \sum_i \lambda_i < 0$, the attractor is chaotic.
- If $\lambda_1 > ... > \lambda_k > 0, \sum_i \lambda_i < 0$, the attractor is hyperchaotic.

(c) Fractal dimension: There are several types of fractal dimensions (capacity dimension, correlation dimension,$\cdots$) for chaotic attractors, among these we can mention

(i) Hausdorff dimension: The Hausdorff dimension of $M \subset \mathbb{R}^n$ is defined by :

$$D_H = sup\{d, \mu_d(M) = +\infty\} = \inf\{d, \mu_d(M) = 0\} \tag{3.11}$$

then $\mu_d(M)$ is the $d-$dimensional Hausdorff measure of the set M. This type of dimension depends only on the metric properties of the space in which the set is located (attractor or not).

(ii) Lyapunov dimension: The Lyapunov dimension is given by :

$$D_L = \frac{\sum_{i=1}^{j} \lambda_i}{\left|\lambda_{j+1}\right|} + j \tag{3.12}$$

then $\lambda_n, ..., \lambda_1$ are the Lyapunov exponents of an attractor of the dynamical system and j is the largest natural integer such that : $\sum_{i=1}^{j} \lambda_i \geq 0$. This type of dimensions takes into account the dynamics of the system.

(d) Strange attractor: The strange attractor is a geometric feature of chaos. There is no rigorous definition of a strange attractor and all the definitions found in the literature are restrictive.

3.3.3 Transitions to Chaos

There are several scenarios that describe the passage to chaos. We note in all cases that the evolution from the fixed point towards chaos is not progressive, but marked by discontinuous changes which we have already called bifurcations. One can cite three transition scenarios from a regular dynamic to a chaotic dynamic during the variation of a parameter.

(a) Cascade of period doublings: This scenario was observed in the 60 by **R. May** in population dynamics on the logistics application. This scenario is characterized by a succession of bifurcation of forks. As the stress increases, the period of a forced system is multiplied by 2, puis par 4, then by 8, ...etc. These period doublings are closer and closer, when the period is infinite, the system becomes chaotic.

(b) Intermittently: A steady periodic motion is interspersed with bursts of turbulence. As the control parameter is increased, the turbulence bursts become more and more frequent, and eventually, the turbulence dominates.

(c) Screenplay by Ruelle and Takens: This scenario via quasi-periodicity has been highlighted by the theoretical work of Ruelle and Takens. In a dynamical system with periodic behavior at only one frequency, if we change a parameter then it appears a second frequency. If the ratio between the two frequencies is rational then the behavior is periodic. But, if the relationship is irrational, the behavior is quasi-periodic. Then, we change the parameter again and a third frequency appears and so on until chaos.

3.4 Example of discrete chaotic dynamical systems

We consider two systems which played a crucial role in the development of dynamical discrete systems theory: the logistic map and the Hénon map.

3.4.1 Logistic map

The logistic map can be obtained by the discretization of logistic differential equation that was initially proposed in the population growth model by Verhulst [47]. This differential equation is

$$\frac{dx}{dt} = rx(1-x), \tag{3.13}$$

where,x denotes the total number of the population. The discretized logistic equation or logistic map:

$$x(n+1) = rx(n)(1-x(n)), \quad n = 1, 2, ... \tag{3.14}$$

This is an example of a first-order difference equation but non-invertible. Unlike its continuous version, the logistic map exhibits a complex dynamic behaviour and presents numerous applica-tions. The value ofxlies in the $[0,1]$ interval and changes over time according to the parameter $r \in (0,4]$. This map will go through a whole spectrum of possible dynamical behaviour. In particular, when $r > 3$ the logistic map (3.14) has an attracting periodic orbit of period 2^n with n tending to infinity as r tends to 3.57. When the latter value is reached there is the attractor shown in Figure3.1.
To demonstrate the sensitivity to initial conditions, we plot two very close trajectories as shown in Figure3.2 with $r = 4$ The blue curve stars from $x_1(0) = 0.1$ and the red curve stars from $x_2(0) = 0.099999$. The difference at the initial condition, 10^{-6} , is very tiny. It can be observed that at the beginning the time series are undistinguishable, but after a number of iterations, the difference between them builds up rapidly and becomes totally different. This a consequence of sensitivity, which is the characteristics of chaotic systems.

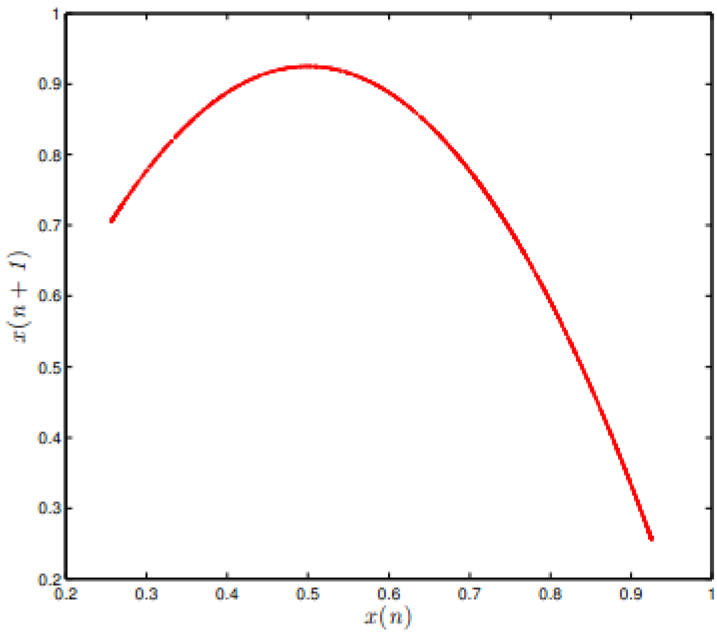


Figure 3.1: The chaotic attractor of the logistic map (3.14) with $r = 3.57$.

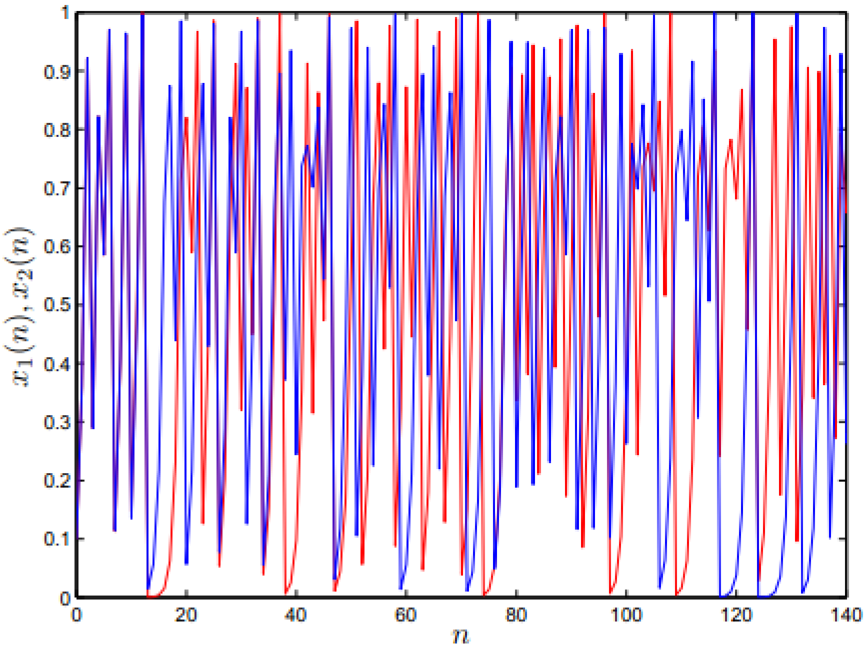


Figure 3.2: The trajectory portraits of states x_1 and x_2 starting from 0.1 and 0.10001 respectively, with $r = 4$.

3.4.2 Hénon map

On of the most commonly studied and applied discrete chaotic system is the Hénon map, which was introduced in 1976 [48] as a discretization of the Poincaré section of the famous continuous-time Lorenz system. This can be considered as a two-dimensional extension of the logistic map:

$$\begin{cases} x_1(n+1) = 1 - \alpha x_1^2(n) + x_2(n), \\ x_2(n+1) = \beta x_1(n), \end{cases} \tag{3.15}$$

where α and $|\beta| \leq 1$ are external parameters. Because of its simplicity it lends itself to computer studies and numerous investigations followed. Set $\alpha = 1.4, \beta = 0.03$ the generated attractor is presented in Figure3.3. Moreover the gently swirling boomerang-like shape of the attractor that arises through the dynamics is very appealing aesthetically. This object is now known as theHénon attractor. In fact it has become another icon of chaos theory next to the Lorenz attractor.

Remark 3.1. *[49] Up to now no one knows whether the attractor in Hénon's transformation for $\alpha = 1.4$ and $\beta = 0.3$ really is a strange attractor according to the above or a similar definition even though very extensive numerical checks have been performed which all indicate a positive answer. This underlines the incomplete state of affairs. For example we could speculate that the experimental observations are due to an attractive periodic orbit with a very long period.*

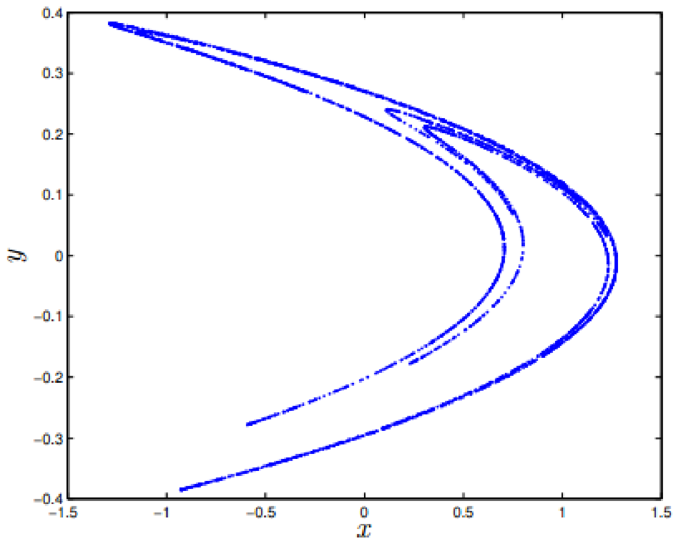


Figure 3.3: The chaotic attractor of the Hénon map(3.15)

3.5 Bifurcation Diagrams

With the discovery of chaotic dynamics, the theory has become even more important, as researchers try to find mechanisms by which systems change from simple to highly complicated behavior. Consider an n^{th}-order discrete-time system

$$x(n+1) = f(x(n), \alpha), \tag{3.16}$$

with a parameter $\alpha \in \mathbb{R}$ .As α changes, the dynamic behavior of the system is also change. Typically, a small change in α produces small quantitative changes in the states of the system. Such change in specifc behavior of the map is known as abifurcation. Accordingly, the bifurcation phenomenon describes the fundamental alteration in the dynamics of nonlinear systems under parameter variation. For this reason it is considered as a tool that help to understand equilibrium loss and its consequences for complex behavior. Moreover, bifurcation diagrams display some characteristic property of the asymptotic solution of a dynamical system as a function of a control parameter, allowing one to see at a glance where qualitative changes in the asymptotic solution occur. We call the parameter at which the dynamic behaviour changes thebifurcation parameter.
There are many types of periodic orbit local bifurcation, in the following will describe the most common bifurcations.

Example 3.1 (**Period Doubling Bifurcation**). *As a prime example, we are going to analyse in details the bifurcation diagram of the logistic map*(3.14)*. It is easy to verify that the logistic map has tow fixed point* $xf_1 = 0$ *and* $xf_2 = 1 - \frac{1}{4r}$.
The origin changes its stability with xf_2 *when it enters to the interval* $[0,1]$*. Thus, it is stable for* $0 < r < 0.25$*. The second fixed point* $xf_2 = 1 - \frac{1}{4r}$ *is stable for* $0.25 < r < 0.75$*, while it becomes unstable and spawns a stable period-two closed orbit when* r *passes through* 0.75*. Observe that the period-one fixed point still exists after the period-two orbit is created, though it has become unstable. As* r *is increased further, the period-two closed orbit becomes unstable and spawns a stable period-four closed orbit. This period- four orbit spawns a period-eight orbit and so on. This bifurcation is called theperiod-doubling bifurcation(or, sometimes, theflip bifurcation). In Figure*3.4*, the period- doubling bifurcations accumulate at a bifurcation value* r *at which the system becomes chaotic.*

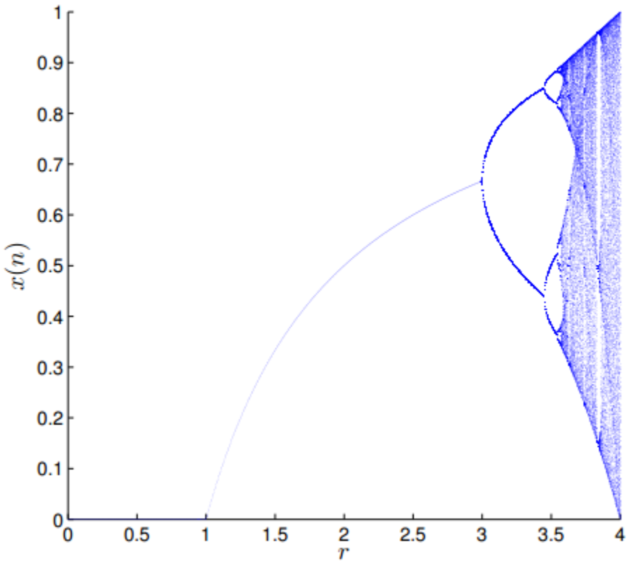


Figure 3.4: Period doubling route to chaos of the Logistic map.

Period-doubling bifurcation also occurs in the 2D Hénon map

$$\begin{cases} x_1(n+1) = 1 - \alpha x_1^2(n) + x_2(n), \\ x_2(n+1) = \beta x_1(n). \end{cases} \tag{3.17}$$

A bifurcation diagram for Hénon's map is shown in Figure3.5

Example 3.2 (**Hopf bifurcation**). *Hopf bifurcation in a discrete-time system: As a final example of the different types of bi-furcation, we present the Hopf bifurcation of a fixed point of a map. This bifurcation is the discrete-time analogue of the Hopf bifurcation of an equilibrium point. In a discrete-time Hopf bifurcation, an invariant closed curve is created as a stable fixed point loses stability when the real parts of its (complex conjugate pair of) characteristic multipliers pass through the unit circle.*

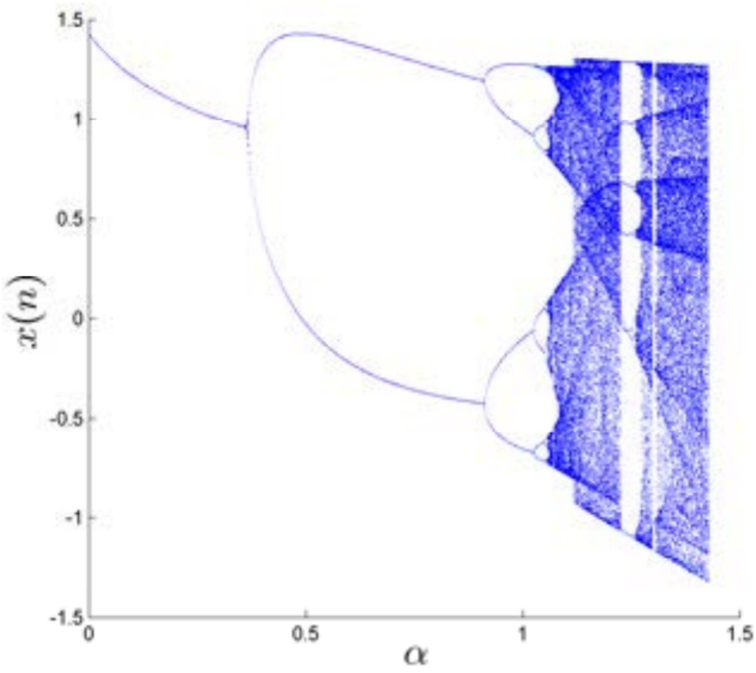


Figure 3.5: The bifurcation diagram of the Hénon map.

3.6 Fractional order discrete-time systems

Even the theory and applications of discrete fractional calculus may be considered as a novel topic, some recent contributions have been developed to deal with discrete analogues of conti-inuous fractional calculus and fractional difference equations. Very recently, some literature has studied the dynamics, including chaotic behavior of discrete-time fractional order systems using the Caputo fractional difference operator type, introduced in Section1.3.3 The first such system is the fractional-order logistic map which was proposed by Wu and Baleanu [50] where chaotic behaviors and a synchronization method of the fractional map were numerically illustrated. Over time, the dynamics of more discrete-time fractional-order systems, such as the fractional-order Hénon map have been investigated [51].

The literature seems to agree that fractional chaotic maps possess superior properties as compared to their standard counterpart. For instance, the general dynamics of fractional maps are heavily dependent on variations in the fractional order [50]. This adds new degrees of freedom to the map's states making them more suitable for data encryption, for example. In addition, Edelman [5] showed that the convergence speed as well as the convergence route depend on the initial conditions, which leads to richer dynamics. The fractional difference provides us a new powerful tool to characterize the dynamics of discrete complex systems more deeply. Recently, Peng et al. revised the fractional logistic map reported by Wu, et. al, based on Edelman's work and presented the correct simulation results for bifurcation diagrams in [52].

Motivated by the works mentioned above, we will report in our thesis different dimensional fractional order discrete-time systems with self-exited and hidden attractors. For that we start by reporting both of the fractional Logistic and Hénon maps to clarify the method used here in.

3.6.1 Fractional Logistic map

For the famous logistic map reported in Section3.4.1, as

$$x(n+1) = rx(n)(1-x(n)), \quad n = 1,2,... \tag{3.18}$$

First, we take the difference form of (3.18) to obtain

$$\Delta x(n) = rx(n)(1-x(n)) - x(n), \quad n = 1,2,... \tag{3.19}$$

We may replace the standard difference in (3.19) with the Caputo-difference operator defined in Section1.3.3, which yields

$$^{c}\Delta_{a}^{\nu}x(t) = rx(t-1+\nu)(1-x(T+\nu)) - x(t-1+\nu), \quad \text{for} \quad t \in \mathbb{N}_{a+1-\nu} \tag{3.20}$$

where a is the starting point. The case $\nu = 1$ corresponds to the non fractional scenario. To investigate the dynamics of the logistic map (3.20), we will need a discrete numerical formula that allows us to evaluate the states of the map in fractional discrete time. According to Wu et.al and other similar studies, we can obtain the following equivalent discrete integral from of equation (3.20), using Theorem1.13 reported in the first chapter as

$$x(t) = x(a) + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} (t-s-1)^{\nu-1} (rx(s-1+\nu)(1-x(s-1+\nu)) - x(s-1+\nu)), \tag{3.21}$$

for $t \in \mathbb{N}_1$. As a result, the numerical formula can be presented accordingly

$$x(n) = x(0) + \frac{1}{\Gamma(\nu)} \sum_{s=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (rx(j-1)(1-x(j-1)) - x(j-1)). \tag{3.22}$$

Where $x(0)$ is the initial condition. Compared with the map of the integer order, the fractionalized one (3.20) has a discrete kernel function. $x(n)$ depends on the past information $x(0),\cdots,x(n-1)$. As a result, the memory effects of the discrete maps means that their present state of evolution depends on all past states.

3.6.2 Fractional Hénon map

Similarly to the previous section we right the fractional Hénon map with Caputo-like difference operator. The Hénon map is given by the following pair of first-order difference equations

$$\begin{cases} x_1(n+1) = 1-\alpha x_1^2(n)+x_2(n), \\ x_2(n+1) = \beta x_1(n), \end{cases} \tag{3.23}$$

where α,β are bifurcation parameters. We can rewrite above equation:

$$\begin{cases} \delta x_1(t) &= 1-\alpha x_1^2+x_2(n)-x_1(n), \\ \Delta x_2(t) &= \beta x_1(n)-x_2(n). \end{cases} \tag{3.24}$$

From the discrete fractional calculus, we modify the standard map as a fractional one

$$\begin{cases} \Delta_a^\nu x_1(t) &= 1-\alpha x_1^2(t-1+\nu)+x_2(t-1+\nu)-x_1(t-1+\nu), \\ \Delta_a^\nu x_2(t) &= \beta x_1(t-1+\nu)-x_2(t-1+\nu). \end{cases} \tag{3.25}$$

From the previous equation, we can obtain the following discrete integral form from $0<\nu\leq 1$

$$\begin{cases} x_1(t) &= x_1(0)+\frac{1}{\Gamma(\nu)}\sum_{s=1}^{t-\nu}(t-s-1)^{\nu-1}(1-\alpha x_1^2(s-1+\nu)+x_2(s-1+\nu)-x_1(s-1+\nu)), \\ x_2(t) &= x_2(0)+\frac{1}{\Gamma(\nu)}\sum_{s=1}^{t-\nu}(t-s-1)^{(\nu-1)}(\beta x_1(s-1+\nu)-x_2(s-1+\nu)). \end{cases} \tag{3.26}$$

As a result, the numerical formula can be presented explicitly

$$\begin{cases} x_1(n) &= x_1(0)+\frac{1}{\Gamma(\nu)}\sum_{j=1}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_2(j-1)+1-\alpha x_1^2(j-1)-x_1(j-1)), \\ x_2(n) &= x_2(0)+\frac{1}{\Gamma(\nu)}\sum_{j=1}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}\beta x_1(j-1)-x_2(j-1). \end{cases} \tag{3.27}$$

where $x_1(0)$ and $x_2(0)$ are the initial states.

3.7 Conclusion

In this chapter, we have presented some basic mathematical concepts such as discrete dynamical systems, chaos theory, characteristics of chaos and transitions to chaos. Finally, we have presented two examples of discrete chaotic dynamical systems. Many examples of fractional chaotic maps with and without fixed point can be founded in . [76-80]

Chapter 4

Control theory and Synchronization types

4.1 Introduction

The phenomenon of synchronization has attracted the interest of many researchers from various fields due to its potential applications in nonlinear sciences. Various types of synchronization have been introduced in the past to synchronize dynamical systems such as full synchronization, generalized synchronization, and Q-S synchronization.
Recently, the topic of synchronization between dynamical systems described by fractional-order difference equations started to attract increasing attention [53]. Most of the research efforts have been devoted to the study of synchronization problems in commensurate FoDSs.
This chapter is an introduction aimed to give an overview about control theory, and chaos synchronization. At the end of the chapter, the different types of synchronization will be introduced.

4.2 Definition of control theory

The idea of chaos control is based on the fact that chaotic attractors have a skeleton made of an infinite number of unstable periodic orbits which are visited for short periods of time by a phase space point which follows a trajectory in the attractor. The aim of chaos control is to stabilize a previously chosen unstable periodic orbit by means of small perturbations applied to the system, so the chaotic dynamics is substituted by a periodic one chosen at will among the several available. This makes chaotic systems very interesting because they allow different uses, without performing structural changes, and employing a minimal external input [54]. Different linear and nonlinear methods to control chaos in fractional maps can be found in [81-84] .

4.3 Definition of synchronization

Synchronization is the process of controlling the output of a dynamical slave system in order to force its variables to match those of a corresponding master system in time [55]

4.4 Synchronization types:

Various types of synchronization have been introduced in the past to synchronize dynamical systems such as namely full synchronization, anti-synchronization, offset synchronization, FSHP synchronization, generalized synchronization, and QS synchronization.

4.4.1 Full synchronization

We consider a master chaotic system represented by

$$X(k+1) = F(X(k)), \tag{4.1}$$

where $X(k) \in \mathbb{R}^n$ is the state of the system (4.1) and $F : \mathbb{R}^n \to \mathbb{R}^n$. And a chaotic slave system given by

$$Y(k+1) = G(Y(k)) + U, \tag{4.2}$$

where $Y(k) \in \mathbb{R}^n$ is the state of the system (4.2) and $U = (u_i)_{1 \leq i \leq n}$ is a vector of control to be determined. We define the full synchronization error as

$$e(k) = Y(k) - X(k). \tag{4.3}$$

Thus, the complete synchronization problem is to determine the controller U such that

$$\lim_{k \to \infty} \|e(k)\| = 0. \tag{4.4}$$

hence $\|\cdot\|$ is the Euclidean norm.
and $F = G$, the relationship becomes an identical full synchronization.

and $F \neq G$, it is a non-identical full synchronization.
Complete synchronization is therefore a complete coincidence between the state variables of the two synchronized systems [56] .

4.4.2 Anti-Synchronization

Theoretically, two systems are anti-synchronized if on the one hand, the master system and the slave system have identical state vectors in absolute value but with opposite signs and that on the other hand, the sum of the state vectors of the two systems tends towards zero when the time tends to infinity. The anti-synchronization error can therefore be defined as follows

$$e(k) = Y(k) + X(k). \tag{4.5}$$

4.4.3 Offset synchronization

Researchers have found that two non-identical chaotic dynamical systems can expose a synchronization phenomenon in which the dynamic variables of the two systems become synchronized, but with a time lag [57] . We say we have a delayed (or early) synchronization if the state variables $Y(k)$ of the chaotic slave system converge towards the time-shifted state variables $X(k)$ of the chaotic master system as shown in the relationship below

$$\lim_{k\to\infty} \|Y(k) - X(k-\tau)\|0, \quad \text{ro} \quad (\lim_{k\to\infty} \|Y(k) - X(k+\tau\|), \forall x(0), \tag{4.6}$$

with τ is a very small positive number.

4.4.4 Projective synchronization

We say that we have a projective synchronization if the state variables $y_i(k)$ of the chaotic slave system $Y(k) = (u_i(k))_{1\leq i\leq n}$ synchronize with a multiple constant of the state $x_i(k)$ of master chaotic system $X(k) = (x_i(k))_{1\leq i\leq n}$, such as [58] :

$$\exists \alpha_i \neq 0, \quad \lim_{k\to\infty} |y_i(k) - \alpha_i x_i(k)| = 0, \quad \forall (x(0), y(0)), i = 1, 2, \cdots, n \tag{4.7}$$

The case where all α_i are equal to 1 represents a case of full synchronization. The case where all THE α_i are equal to 1 represents a case of complete anti-synchronization.

4.4.5 FSHP Synchronisation

We say that we have a FSHP synchronization (full state hybrid projective synchronization), if each state variable $y_i(k), 1 \leq i \leq n$ of the chaotic slave system synchronizes with a linear combination of state variables $x_i(k), 1 \leq i \leq n$, of the master chaotic system, such as :

$$\exists(\beta)_{ij} \in \mathbb{R}^{n\times n}, \lim_{k\to\infty} \left| y_i(k) - \sum_{j=1}^{n} \beta_{ij} x_i(k) \right| = 0, \quad \forall(x(0), y(0)), i = 1, 2, \cdots, n. \tag{4.8}$$

FSHP synchronization is a generalization of projective synchronization [58].

4.4.6 Generalized synchronization

Generalized synchronization is considered a generalization of synchronization complete, anti-synchronization and projective synchronization in the case of chaotic systems of different dimensions and models [59] . It is manifested by a functional relationship between the two coupled chaotic systems. We consider a couple of master-slave systems represented by

$$\begin{cases} X(k+1) = F(X(k)), \\ Y(k+1) = G(Y(k)) + U, \end{cases} \tag{4.9}$$

from where $X(k) \in \mathbb{R}^n, Y(k) \in \mathbb{R}^m$ are the states of the master and slave systems, respectively, $F : \mathbb{R}^n \to \mathbb{R}^n, G : \mathbb{R}^m \to \mathbb{R}^m$ et $U = (u_i)_{1\leq i\leq m}$ is a controller.
If there is a controller U and a function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that all the trajectories of master and slave systems, with initial conditions $x(0)$ et $y(0)$; check :

$$\lim_{k\to\infty} \|Y(k) - \phi(X(k))\| = 0, \quad \forall x(0), \forall y(0), \tag{4.10}$$

then, the master-slave systems(4.9) synchronize in the generalized sense with respect to the function ϕ . If the function ϕ is defined by $\phi(X(k)) = \Lambda X(k)$ such as $\Lambda = (\Lambda_{ij})_{m\times n}$ we say we have a full-state hybrid projective synchronization [60] .

4.4.7 Q-S Synchronisation

Q-S synchronization is considered a generalization of all previous synchronization types [61] . We say that a master, n-dimensional system, $X(k)$ and a system slave, m-dimensional, $Y(k)$ are in $Q-S$ synchronization in dimension d, if there is a controller $U = (u_i)_{1\leq i\leq m}$

$Q:\mathbb{R}^n \to \mathbb{R}^d, and two of unctions S:\mathbb{R}^m \to \mathbb{R}^d$ such as the error of synchronization

$$e(k) = Q(X(k) - S(Y(k)),) \tag{4.11}$$

checked $\lim_{k\to\infty} \|e(k)\| = 0$.

4.5 Conclusion

In this chapter, we presented the main types of chaotic synchronization proposed in the literature, and which can have a huge technological impact.
Many synchronization methods and types for discrete-time and continuous time integer-order chaotic systems have been investigated and can be found in [85, 93]

Chapter 5

Control and synchronization of fractional Grassi-Miller map based on the Caputo h-Difference operator

5.1 Introduction

Investigating dynamic properties of discrete chaotic systems with fractional order has been receiving much attention recently.This chapter provides a contribution to the topic by presenting a novel version of the fractional Grassi-Miller map, along with improved schemes for controlling and synchronizing its dynamics. By exploiting the Caputo h-difference operator, at first, the chaotic dynamics of the map are analyzed via bifurcation diagrams and phase plots. Then, a novel theorem is proved in order to stabilize the dynamics of the map at the origin by linear control laws. Additionally, two chaotic fractional Grassi-Miller maps are synchronized via linear controllers by utilizing a novel theorem based on a suitable Lyapunov function. Finally, simulation results are reported to show the effectiveness of the approach developed herein. The structure of the chapter is as follows. First, a definition of the fractional Caputo h-difference operator [5] and a novel fractional Grassi-Miller map is proposed, along with its chaotic dynamic behavior. Then, linear control laws are proposed to stabilize the dynamics of the map at the origin. In particular, a novel theorem is proved, which assures the stability condition via a suitable Lyapunov function. A master-slave system based on two chaotic Grassi-Miller maps is synchronized using linear controllers, will be presented. Finally, simulation results are reported to show the effectiveness of the control and synchronization methods developed herein.

5.2 Fractional Grassi-Miller Map Based on the Caputo h-Difference Operator

In this section, a novel version of the fractional Grassi-Miller map is presented. To this purpose, some concepts related to the Caputo h-difference operator are briefly summarized.

Troughout the rest of the chapter, we assume that $(h\mathbb{N})_a = \{a, a+h, a+2h, \cdots\}$ where h is a positive real and $a \in \mathbb{R}$. Te forward h-difference operator of a function X defined on $(h\mathbb{N})_a$ is defined as

$$\Delta_h X(t) = \frac{X(t+h) - X(t)}{h} \tag{5.1}$$

Definition 5.1. *[4] . Let $X : (h\mathbb{N})_a \to \mathbb{R}$. The fractional h-sum of positive fractional order v is defined by*

$$_h\Delta_a^{-v} X(t) = \frac{h}{\Gamma(v)} \sum_{s=(a/h)}^{(t/h)-v} (t - \sigma(sh))_h^{v-1} X(sh), \tag{5.2}$$

where $\sigma(sh) = (s+1)h, a \in \mathbb{R}$, and $t \in (h\mathbb{N})_{a+vh}. t_h^{(v)}$ is the h-falling fractional function with two real numbers t, h that can be written in the form

$$t_h^{(v)} = \frac{h^v \Gamma(t/h) + 1}{\Gamma((t/h) + 1 - v)} \tag{5.3}$$

Definition 5.2. *[38] . For $X(t)$ defined on $(h\mathbb{N})_a$ and a real order $0 < v \le 1$, the Caputo fractional h-difference operator is given by*

$$^C_h\Delta_a^v X(t) = \Delta_a^{-(n-v)} \Delta^n X(t), \quad t \in (h\mathbb{N})_{a+(n-v)h}, \tag{5.4}$$

in which $n = [v] + 1$.

Now, a theorem is briefly illustrated, with the aim to identify the stability conditions of the zero equilibrium point for the fractional nonlinear difference system written in the form

$$^C_h\Delta_a^v = F(t + vh, X(t + vh)). \tag{5.5}$$

Theorem 5.1. *The fractional nonlinear discrete system (5.5) is asymptotically stable if there exists a positive definite and decreasing scalar function $V(t, X(t))$ for the equilibrium point $x = 0$, such that $V(t, X(t)) \le 0$*

Lemma 5.1. *For every $t \in (h\mathbb{N})_{a+(n-\nu)h}$ the following in-equality holds:*

$$ {}_{h}^{C}\Delta_{a}^{\nu} X^2(t) \leq 2X(t+\nu h) \quad {}_{h}^{C}\Delta_{a}^{\nu} X(t), \quad 0 < \nu \leq 1 \tag{5.6}$$

All the details regarding the proof of Lemma5.1 can be found .

Referring to the fractional Grassi-Miller map, it was introduced in [9] using the ν-Caputo delta difference operator. The fractional map, which proved to be chaotic for proper values of the system parameters (α, β) and of the fractional order $\nu \in (0,1]$, possesses only a nonlinear term [9].

Herein, the fractional Caputo h-difference operator is considered, in order to derive a different mathematical model of the 3D Grassi-Miller map. Namely, the following equations are proposed:

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} x(t) = \alpha - y^2(t+\nu h) - \beta z(t+\nu h) - x(t+\nu h), \\ {}_{h}^{C}\Delta_{a}^{\nu} y(t) = x(t+\nu h) - y(t+\nu h), \\ {}_{h}^{C}\Delta_{a}^{\nu} z(t) = y(t+\nu h) - z(t+\nu h), \end{cases} \tag{5.7}$$

where ${}_{h}^{C}\Delta_{a}^{\nu}$ denotes the fractional h-difference operator, $t \in (h\mathbb{N})_{a+(n-\nu)h}$, a is the starting point and (α, β) are system parameters. The fractional map (5.7) can be considered a generalized model of the map introduced in [9].

The solution of the fractional Grassi-Miller map (5.7) is obtained by introducing the fractional h-sum operator. The equivalent implicit discrete formula can be written in the form:

$$\begin{cases} x(n+1) = x(0) + \frac{h^\nu}{\Gamma(\nu)} \sum\limits_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \Big(\alpha - y^2(j+1) - \beta z(j+1) - x(j+1)\Big), \\ y(n+1) = y(0) + \frac{h^\nu}{\Gamma(\nu)} \sum\limits_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (x(j+1) - y(j+1)), \\ z(n+1) = z(0) + \frac{h^\nu}{\Gamma(\nu)} \sum\limits_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (y(j+1) - z(j+1)), \end{cases} \tag{5.8}$$

where $x(0)$, $y(0)$ and $z(0)$ are the initial state values. Based on Predictor-corrector method [10], the implicit equation (5.8) is transformed into its explicit form, which can be used for investigating the dynamic behavior of the Grassi-Miller map (5.7). By taking the initial state values $x(0) = 1$, $y(0) = 0.1$ and $z(0) = 0$, with the fractional order value $\nu = 0.999$ and the system parameters $\alpha = 1$, $\beta = 0.5$, it can be shown that the map (5.7) displays the attractor reported in Figure 5.1. The computation of the bifurcation diagram and of the largest Lyapunov exponent, both reported in Figure 5.2 as a function of the system parameter β, clearly highlight the chaotic behavior of the fractional Grassi-Miller map (5.7) for $\alpha = 1$, $\beta = 0.5$ and $\nu = 0.999$. Regarding the bifurcation diagram reported in Figure 5.2, it can be noted that the map oscillates when β assumes values around 0.05. When β approaches the value of 0.1, more complex dynamic regimes appear, until β approaches the value of 0.45, when chaotic behaviours are reached. Note that the presence of chaos for $0.45 < \beta < 0.5$ is also confirmed by the positive values assumed by the maximum Lyapunov exponents Figure 5.2. Note that the evolution of states of the fractional map (5.7), which involves the adoption of the Caputo h-difference operator, are different from those of the map reported in [9], being the latter based on the ν-Caputo delta difference operator. This can be clearly seen by comparing the shapes of the chaotic attractors reported in Figure 5.1 with those of the attractors reported in [9]. Namely, the adoption of two different fractional operators has led to different shapes in the chaotic attractors as well as to different parameter values for generating chaos [9], indicating that the proposed Grassi-Miller map (5.7) provides a contribution to the topic of 3D discrete-time fractional systems.

Referring to potential applications of the proposed model (5.7), it should first be noted that 3D maps highlight a higher degree of complexity with respect to 1D and 2D maps [5, 19]. Thus, the applicability of the conceived 3D map (5.7) would mainly be in pseudo-random number generators and image encryption techniques. This makes perceive the importance of developing simple and feasible control methods, given that master-slave synchronization schemes based on the model (5.7), in combination with encryption algorithms, might be used for experimentally generating and recovering the secret keys.

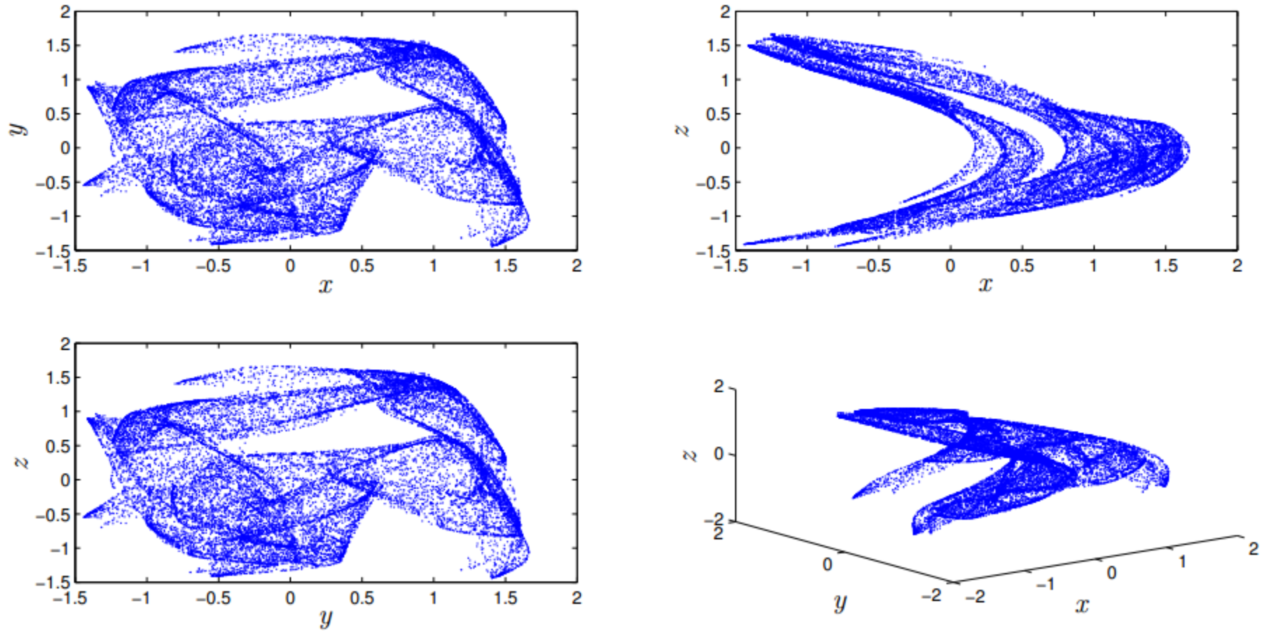


Figure 5.1: Chaotic attractor of the fractional order Grassi-Miller map for $\alpha = 1$, $\beta = 0.5$ and order $\nu = 0.999$.

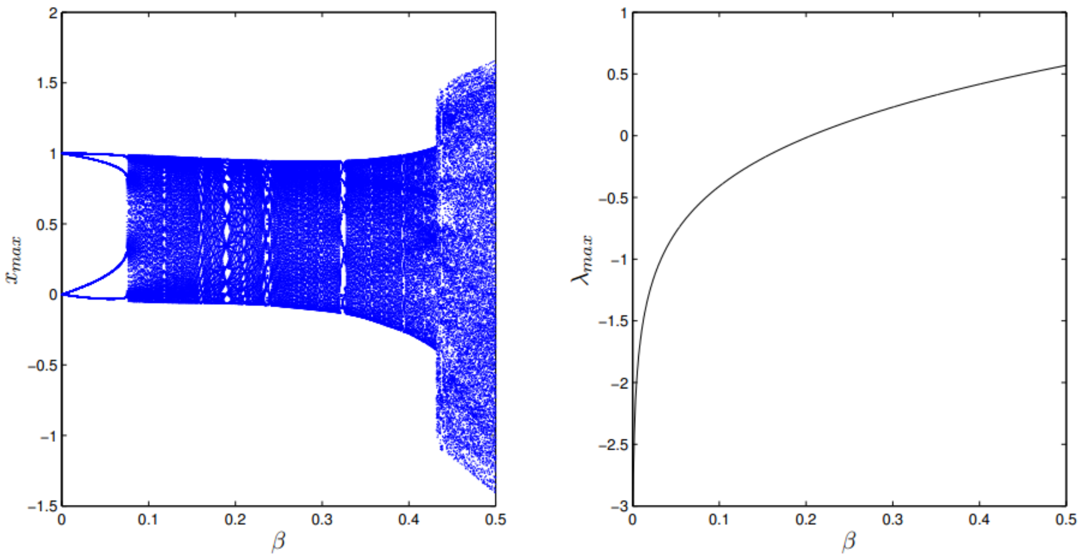


Figure 5.2: Bifurcation and largest Lyapunov exponent plots versus system parameter β for fractional order $\nu = 0.999$.

5.3 Chaos control of the new version of the Grassi-Miller map

Here, a controller is presented in order to stabilize at zero the chaotic trajectories of the state-variables in the Grassi-Miller map (5.7) with fractional order. The objective is achieved by adding two linear terms into both first and second equations of the map. Namely, the controlled fractional Grassi-Miller chaotic map is described by:

$$\begin{cases} {}_h^C\Delta_a^\nu x(t) = \alpha - y^2(t+\nu h) - \beta z(t+\nu h) - x(t+\nu h) + \mathbf{C_1}(t+\nu h), \\ {}_h^C\Delta_a^\nu y(t) = x(t+\nu h) - y(t+\nu h) + \mathbf{C_2}(t+\nu h), \\ {}_h^C\Delta_a^\nu z(t) = y(t+\nu h) - z(t+\nu h), \end{cases} \tag{5.9}$$

where $\mathbf{C_1}$ and $\mathbf{C_2}$ are suitable controllers to be determined. To this purpose, a Theorem is now given for rigorously assuring that the dynamics of (5.9) can be stabilized at zero.

Theorem 5.2. *[24] The three–dimensional fractional Grassi-Miller map* (5.9) *is controlled at the origin under the following control laws:*

$$\begin{cases} \boldsymbol{C}_1(t) = -\alpha + \beta z(t) - y(t), \\ \boldsymbol{C}_2(t) = -b_1 y(t) - z(t), \end{cases} . \tag{5.10}$$

where $|x(t)| \le b_1$, $\forall t \in (h\mathbb{N})_{a+(n-\nu)h}$.

Proof. By subtracting equations (5.10) into system (5.9), we get the following fractional difference equations:

$$\begin{cases} {}_h^C\Delta_a^\nu x(t) = -y^2(t+\nu h) - x(t+\nu h) - y(t+\nu h), \\ {}_h^C\Delta_a^\nu y(t) = x(t+\nu h) - (1+b_1)y(t+\nu h) - z(t+\nu h), \\ {}_h^C\Delta_a^\nu z(t) = y(t+\nu h) - z(t+\nu h). \end{cases} \tag{5.11}$$

By taking a Lyapunov function in the form $V = \frac{1}{2}(x^2(t)+y^2(t)+z^2(t))$, the adoption of the Caputo h-difference operator implies that

$${}_h^C\Delta_a^\nu V = \frac{1}{2}\,{}_h^C\Delta_a^\nu x^2(t) + \frac{1}{2}\,{}_h^C\Delta_a^\nu y^2(t) + \frac{1}{2}\,{}_h^C\Delta_a^\nu z^2(t). \tag{5.12}$$

By using Lemma5.1, it follows that:

$$\begin{aligned}
{}_h^C\Delta_a^\nu V &\le x(t+\nu h)\,{}_h^C\Delta_a^\nu x(t) + y(t+\nu h)\,{}_h^C\Delta_a^\nu y(t) + z(t+\nu h)\,{}_h^C\Delta_a^\nu z(t) \\
&= -x(t+\nu h)y^2(t+\nu h) - x^2(t+\nu h) - x(t+\nu h)y(t+\nu h) + y(t+\nu h)x(t+\nu h) \\
&\quad - (1+b_1)y^2(t+\nu h) - y(t+\nu h)z(t+\nu h) + z(t+\nu h)y(t+\nu h) - z^2(t+\nu h) \\
&\le |x(t+\nu h)|y^2(t+\nu h) - x^2(t+\nu h) - (1+b_1)y^2(t+\nu h) - z^2(t+\nu h) \\
&\le b_1 y^2(t+\nu h) - x^2(t+\nu h) - (1+b_1)y^2(t+\nu h) - z^2(t+\nu h) \\
&= -x^2(t+\nu h) - y^2(t+\nu h) - z^2(t+\nu h) < 0.
\end{aligned} \tag{5.13}$$

From Theorem5.1 it can be concluded that the zero equilibrium of (5.9) is asymptotically stable. As a consequence, it is proved that the dynamics of the proposed 3D Grassi-Miller map (5.7) are stabilized at the origin by the linear control laws (5.10). □

Remark 5.1. *Since all the chaotic states of the map (5.9) are bounded, it can be deduced that it is easy to find a parameter b_1 larger than the absolute value of the state variable $x(t)$, as requested by the proof of Theorem5.2. Namely, the existence of b_1 is intrinsically justified by the property of boundedness of the state $x(t)$. Thus, the value of b_1 can be easily found by looking at the plots reported in Figure 5.1, from which it is clear that $-1.6 < x(t) < 1.6$ for any t. Through the thesis, the value of b_1 has been selected as $b_1 = 1.7$. Note that the value of b_1 does not significantly affect the time for stabilizing the map dynamics.*

Now, we give the numerical simulation to prove the above theory. We select $\alpha = 1$, $\beta = 0.5$, and we give the evolution of the states and the phase-space plots as shown in Figure 5.3 for $\nu = 0.999$. These plots clearly show that the new fractional map (5.7) is driven to the origin by linear control laws in the form (5.10).

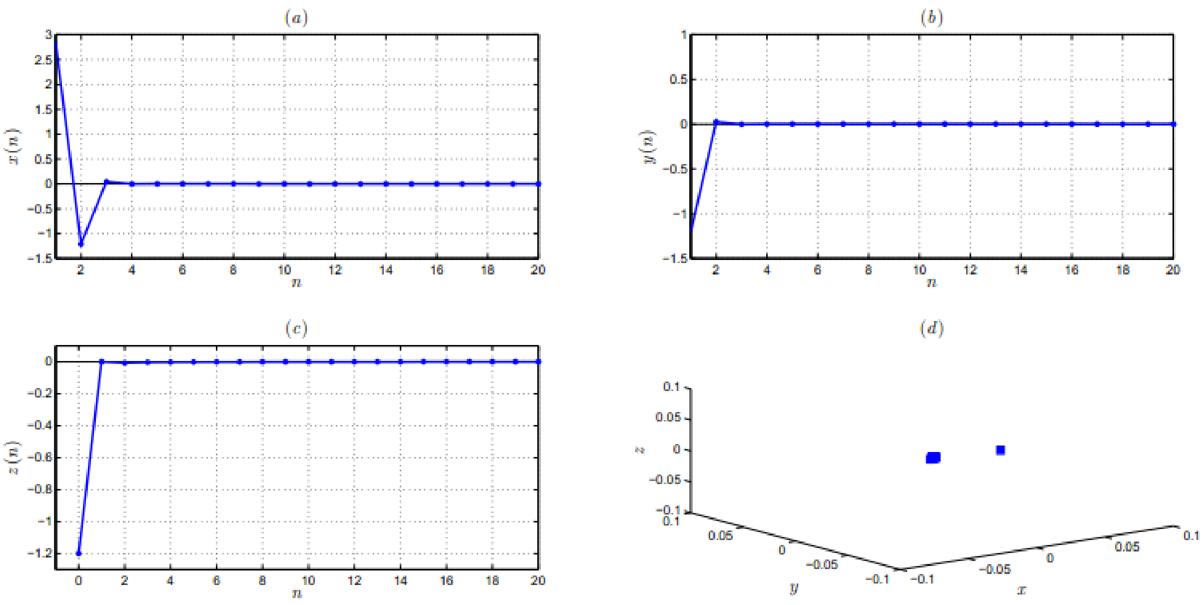


Figure 5.3: Stabilized states of the controlled fractional Grassi-Miller map (5.9) via linear control laws (5.10) with $\alpha = 1$, $\beta = 0.5$, and fractional order $\nu = 0.999$.

5.4 Synchronization of the fractional Grassi-Miller map

In this paragraph a master-slave system, based on two identical chaotic fractional Grassi-Miller maps, is synchronized using linear controllers. The dynamics of the master system can be written as:

$$\begin{cases} {}_h^C\Delta_a^\nu x_m(t) = \alpha - y_m^2(t+\nu h) - \beta z_m(t+\nu h) - x_m(t+\nu h), \\ {}_h^C\Delta_a^\nu y_m(t) = x_m(t+\nu h) - y_m(t+\nu h), \\ {}_h^C\Delta_a^\nu z_m(t) = y_m(t+\nu h) - z_m(t+\nu h). \end{cases} \tag{5.14}$$

where $x_m(t)$, $y_m(t)$ and $z_m(t)$ are the system states. The equations of the slave system are given by:

$$\begin{cases} {}_h^C\Delta_a^\nu x_s(t) = \alpha - y_s^2(t+\nu h) - \beta z_s(t+\nu h) - x_s(t+\nu h) + \mathbf{L}_1(t+\nu h), \\ {}_h^C\Delta_a^\nu y_s(t) = x_s(t+\nu h) - y_s(t+\nu h), \\ {}_h^C\Delta_a^\nu z_s(t) = y_s(t+\nu h) - z_s(t+\nu h) + \mathbf{L}_2(t+\nu h). \end{cases} \tag{5.15}$$

where $x_s(t)$, $y_s(t)$ and $z_s(t)$ are the system states, whereas $\mathbf{L_1}$ and $\mathbf{L_2}$ are suitable linear controllers to be determined. We subtract master system (5.14) from the slave system (5.15) to get the error system as

$$(e_1(t), e_2(t), e_3(t))^T = (x_s(t), y_s(t), z_s(t))^T - (x_m(t), y_m(t), z_m(t))^T. \tag{5.16}$$

Now a theorem involving a Lyapunov-based approach is proved, with the aim to synchronize the master-slave (5.14)-(5.15) via linear controllers $\mathbf{L_1}$ and $\mathbf{L_2}$.

Theorem 5.3. *[24] The master system* (5.14) *and the slave system* (5.15) *achieve synchronized dynamics, provided that the linear control laws* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are selected as:*

$$\begin{cases} \boldsymbol{L}_1(t) = \left(1 - \left(b_2 + \frac{1}{2}\right)^2\right) e_1(t), \\ \boldsymbol{L}_2(t) = \beta e_1(t) - e_2(t), \end{cases} \tag{5.17}$$

where $\left|y_m(t)\right| = \left|y_s(t)\right| \leq b_2$, $\forall t \in (h\mathbb{N})_{a+(n-\nu)h}$.

Proof. By taking into account equation (5.16), the dynamics of the error system can be written as:

$$\begin{cases} {}_h^C\Delta_a^\nu e_1(t) = y_m^2(t+\nu h) - y_s^2(t+\nu h) - \beta e_3(t+\nu h) - e_1(t+\nu h) + \mathbf{L}_1(t+\nu h), \\ {}_h^C\Delta_a^\nu e_2(t) = e_1(t+\nu h) - e_2(t+\nu h), \\ {}_h^C\Delta_a^\nu e_3(t) = e_2(t+\nu h) - e_3(t+\nu h) + \mathbf{L}_2(t+\nu h). \end{cases} \tag{5.18}$$

By substitute the control law (5.17) to error system (5.18) we get:

$$\begin{cases} {}_h^C\Delta_a^\nu e_1(t) = -(y_m(t+\nu h) + y_s(t+\nu h))e_2(t+\nu h) - \beta e_3(t+\nu h) - \left(b_2 + \frac{1}{2}\right)^2 e_1(t+\nu h), \\ {}_h^C\Delta_a^\nu e_2(t) = e_1(t+\nu h) - e_2(t+\nu h), \\ {}_h^C\Delta_a^\nu e_3(t) = \beta e_1(t+\nu h) - e_3(t+\nu h). \end{cases} \tag{5.19}$$

Now, by taking a Lyapunov function in the form $V = \frac{1}{2}(e_1^2(t) + e_2^2(t) + e_3^2(t))$, and by exploiting Lemma 5.1, it follows that

$$\begin{aligned}
h^C\Delta_a^\nu V &\le e_1(t+\nu h)\,{}_h^C\Delta_a^\nu e_1(t) + e_2(t+\nu h)\,{}_h^C\Delta_a^\nu e_2(t) + e_3(t+\nu h)\,{}_h^C\Delta_a^\nu e_3(t) \\
&= -\left(b_2+\frac{1}{2}\right)^2 e_1^2(t+\nu h) - (y_m(t+\nu h)+y_s(t+\nu h))e_1(t+\nu h)e_2(t+\nu h) \\
&\quad - \beta e_1(t+\nu h)e_3(t+\nu h) + e_2(t+\nu h)e_1(t+\nu h) - e_2^2(t+\nu h) + \beta e_1(t+\nu h) \\
&\qquad e_3(t+\nu h) - e_3^2(t+\nu h) \\
&\le -\left(b_2+\frac{1}{2}\right)^2 e_1^2(t+\nu h) + (1+|y_m(t+\nu h)+y_s(t+\nu h)|) \\
&\qquad |e_1(t+\nu h)||e_2(t+\nu h)| - e_2^2(t+\nu h) - e_3^2(t+\nu h) \\
&\le -\left(b_2+\frac{1}{2}\right)^2 e_1^2(t+\nu h) + (1+2b_2)|e_1(t+\nu h)||e_2(t+\nu h)| - e_2^2(t+\nu h) \\
&\qquad - e_3^2(t+\nu h) \\
&= -\left(\left(b_2+\frac{1}{2}\right)^2 e_1(t+\nu h) - e_2(t+\nu h)\right)^2 - e_3(t+\nu h) \le 0.
\end{aligned} \tag{5.20}$$

From Theorem5.1 it can be concluded that the dynamics of the error system (5.18) are stabilized at the origin. As a consequence, it is proved that the master system (5.14) and the slave system (5.15) achieve synchronized dynamics via linear control laws in the form (5.17). □

Remark 5.2. *It is easy to find a parameter b_2 larger than the absolute value of the variables $y_m(t) = y_s(t)$, as requested by the proof of Theorem5.3. Namely, the existence of b_2 is intrinsically justified by the property of boundedness of the chaotic states of the map (5.9). Thus, the value of b_2 can be easily found by looking at the plots reported in Figure 5.1, from which it is clear that $-1.6 < y(t) < 1.6$ for any t. Herein, in order to achieve synchronization, the value of b_2 has been selected as $b_2 = 2$. Note that the value of b_2 does not significantly affect the time for synchronizing the master-slave pair.*

In order to show the effectiveness of the proposed approach, Figure 5.4 displays the chaotic dynamics of the master system states (blue color) and of the slave system states (red color) when $\alpha = 1$, $\beta = 0.5$ and $\nu = 0.999$. These plots clearly show that two identical Grassi-Miller maps achieve chaos synchronization via linear controllers. Note that, through the manuscript, all the simulation results and the related figures have been obtained using the software MATLAB.

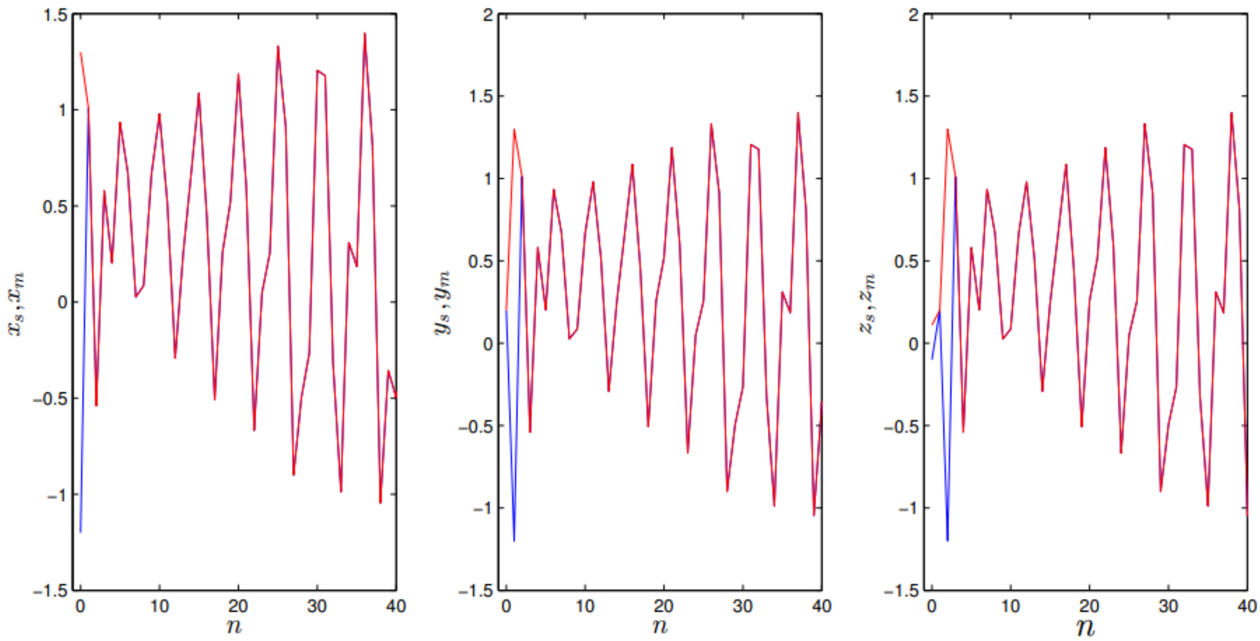


Figure 5.4: Synchronization of two identical Grassi-Miller map with $\alpha = 1$, $\beta = 0.5$ and $\nu = 0.999$: evolutions of the master system states (blue color) and of the slave system states (red color).

Now we would discuss the issue regarding the complexity of the proposed method. We would observe that the approach proposed herein is simpler than similar methods reported in literature. For example, the techniques developed in [1, 2, 9] present the drawback that very complex control laws have been exploited for controlling and synchronizing the corresponding 3D fractional maps. For example, in [1, 2] synchronization and control have been achieved using nonlinear control laws that include several nonlinear terms. This drawback also regards the Grassi-Miller map in [9], since complex nonlinear control laws have been used to achieve synchronization and control of its chaotic dynamics. Since it might be difficult to implement very complex control laws in practical applications of fractional maps, this chapter has provided a contribution to the topic by developing simple linear control laws for stabilizing and synchronizing 3D fractional maps.

Referring to synchronization issues, now comparisons are carried out with recent results regarding 3D fractional maps. The objective is to highlight the effectiveness of the conceived approach when synchronization involves 3D maps with similar degree of com-

plexity. For example, the results in [1] show that synchronization for the 3D fractional map proposed therein is achieved after more than 10 steps, whereas the map illustrated herein can be synchronized in at most 3 steps. On the other hand, the results in [2] show that synchronization for the 3D fractional map proposed therein is achieved in the same number of steps taken by our method. However, reference [2] exploits a complex control law that involves some nonlinear terms, whereas the proposed synchronization technique is simple and involves only linear terms.
Finally, the results in [24] show that synchronization for 3D fractional Grassi-Miller map proposed therein, based on the ν-Caputo delta difference, is achieved after more than 20 steps, whereas the map illustrated herein, based on the Caputo h-difference operator, achieves synchronized dynamics in at most 3 steps. These comparisons make perceive the effectiveness of the proposed synchronization strategy with respect to 3D fractional maps of similar complexity published in recent literature.
Finally, we would briefly discuss the potential applications of the conceived approach in real world. As any 3D map, the Grassi-Miller map highlights a higher degree of complexity with respect to 1D or 2D fractional maps. This increased complexity can be very useful for pseudo-random number generators in chaos-based communications systems. Moreover, since the production of images is increasing day by day in real life, confidentiality and privacy are becoming key issues when transmitting digital images using portable devices. Thus, referring to secure image transmission, the proposed discrete-time synchronization scheme could be utilized for retrieving the secrets keys at the receiver side in chaos-based image encryption systems.

5.5 Conclusion

This chapter has presented a novel version of the chaotic fractional Grassi-Miller map, based on the Caputo h-difference operator. Two novel theorems have been proved, with the aim of deriving improved schemes (with respect to those presented in [22]) for controlling and synchronizing the dynamics of the map. Namely, while synchronization and control in [22] are achieved via more complex nonlinear control laws, herein, simple linear controllers have been conceived. Finally, simulation results have been carried out to highlight the effectiveness of the proposed method.
Different results about the synchronization fractional differential continuous chaotic systems and fractional chaotic maps have been published and can be found in [94-100].

Chapter 6

Control and Synchronization of Different Dimensional Fractional Discrete Chaotic Systems

6.1 Introduction

This chapter investigates control and synchronization of fractional-order maps described by the Caputo h-difference operator. At first, two new fractional maps are introduced, i.e., the Two-Dimensional Fractional order Lorenz Discrete System (2D-FoLDS) and Three-Dimensional Fractional-order Wang Discrete System (3D-FoWDS). Then, some novel theorems based on the Lyapunov approach are proved, with the aim of controlling and synchronizing the map dynamics. In particular, a new hybrid scheme is proposed, which enables synchronization to be achieved between a master system based on a 2D-FoLDS and a slave system based on a 3D-FoWDS. Simulation results are reported to highlight the effectiveness of the conceived approach.

The chapter is organized as follows. First, we introduces the definition of fractional h-Difference Discrete Operator and useful results related to the Lyapunov stability. Second, some new versions of FoDCSs are introduced via the Caputo h-DDO, and their chaotic dynamics are analyzed in detail. Then, linear control laws are proposed to stabilize the dynamics of the considered FoDCSs at the origin. A new hybrid scheme is proposed, which enables synchronization to be achieved between a master system based on the two- dimensional fractional Lorenz map and a slave system based on the three-dimensional fractional Wang map is presented. Finally, simulation results are reported to highlight the effectiveness of the conceived approach.

6.2 Fractional h–DDOs and Lyapunov stability

As already mentioned, the DFC is considered a relatively new topic that has not yet settled. From this perspective, this section presents some preliminaries and notations related to such topic for completeness.

Definition 6.1. *[62] Let $X : (h\mathbb{N})_a \to \mathbb{R}$. For a given $\nu > 0$, the ν^{th}-order h–sum is given by:*

$$ {}_h\Delta_a^{-\nu} X(t) = \frac{h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu} (t-\sigma(sh))^{(\nu-1)} x(sh), \quad \sigma(sh) = (s+1)h, \ \ t \in (h\mathbb{N})_{a+\nu h} \tag{6.1} $$

where $a \in \mathbb{R}$ is a starting point and the h–falling factorial function is defined as:

$$ t_h^{(\nu)} = h^{\nu} \frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\nu\right)}, \qquad t, \nu \in \mathbb{R}, $$

and where $(h\mathbb{N})_{a+(1-\nu)h} = \{a+(1-\nu)h, a+(2-\nu)h, \ldots\}$.

Definition 6.2. *[63, 64] For a defined function $x(t)$ on $(h\mathbb{N})_a$, and for a given $\nu > 0$, such that $\nu \notin \mathbb{N}$; then the Caputo h-DDO is defined by:*

$$ {}_h^C\Delta_a^{\nu} X(t) = \Delta_a^{-(n-\nu)} \Delta^n X(t), \quad t \in (h\mathbb{N})_{a+(n-\nu)h}. \tag{6.2} $$

where $\Delta X(t) = \frac{X(t+h)-X(t)}{h}$ and $n = \lceil \nu \rceil + 1$.

Remark 6.1. *Using the Caputo h-difference operator is useful when dealing with applications of control theory. Namely, controllability (i.e., the possibility to transfer the considered system from a given initial state to a final state using controls from some set) and observability (i.e., the possibility of reconstruction of an initial state using control inputs and output sequences) are both readily achievable when a fractional discrete system is described via the Caputo h-difference operator [66, 67]. Examples of the usefulness in adopting the Caputo h-difference operator are illustrated in reference [68, 69, 70], regarding the controllability and the observability of fractional control systems.*

From the point of view of a significant result and a useful inequality for Lyapunov functions reported in [70], in which they are briefly illustrated below; some stability conditions of the zero equilibrium point for a nonlinear fractional-order difference discrete system will be identified later on. Such nonlinear system has the form:

$$ {}_h^C\Delta_a^{\nu} X(t) = f(t+\nu h, X(t+\nu h)), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{6.3} $$

Theorem 6.1. *[70] Let $x = 0$ be an equilibrium point of system (6.3). If there exists a positive definite and decrescent scalar function $V(t, X(t))$, such that ${}_h^C\Delta_a^\nu V(t, X(t)) \le 0$, then the equilibrium point is asymptotically stable.*

Lemma 6.1. *[70] For any discrete time $t \in (h\mathbb{N})_{a+(1-\nu)h}$, the following inequality holds:*

$$ {}_h^C\Delta_a^\nu X^2(t) \le 2X(t+\nu h) {}_h^C\Delta_a^\nu X(t), \tag{6.4} $$

where $0 < \nu \le 1$.

6.3 Some new forms of FoDCSs

In this part, two newly forms of the FoDCSs are introduced using fractional h–DDOs. The first one is associated with the Two-Dimensional Fractional-order Lorenz Discrete System (2D-FoLDS), while the second one is associated with the Three-Dimensional Fractional-order Wang Discrete System (3D-FoWDS).

6.3.1 2D-FoLDS

The earlier release of the FoLDS was established in [71] using the ν–Caputo delta DDO. It came out that this map, which possesses two nonlinear terms, is, actually, chaotic according to some proper values of its parameters $(\alpha,\ \beta)$ and some of its the fractional-order values ν, where $\nu \in (0, 1]$. Herein, a new version of 2D-FoLDS is derived using, this time, the Caputo h-DDO. In particular, the following equations are proposed:

$$ \begin{cases} {}_h^C\Delta_a^\nu x_m(t) = \alpha\beta x(t+\nu h) - \beta y(t+\nu h)x(t+\nu h), \\ {}_h^C\Delta_a^\nu y_m(t) = \beta(x(t+\nu h)^2 - y(t+\nu h)). \end{cases} \tag{6.5} $$

where ${}_h^C\Delta_a^\nu$ denotes the Caputo h-DDO, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, $a \in \mathbb{R}$ is the starting point and where α and β are the system's parameters. Map (6.5), however, can be regarded as a generalized form of the FoLDS constructed in [71]. Its solution, moreover, can be obtained via employing the fractional h-sum operator. That is, the two corresponding implicit discrete formulae of the two equations given in (6.5) are reported in [67] as follows:

$$ \begin{cases} x(n) = x(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(\alpha\beta x(j) - \beta y(j)x(j)), \\ y(n) = y(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(\beta(x^2(j) - y(j))), \end{cases} \tag{6.6} $$

subject to the given initial conditions $x(0)$ and $y(0)$.

In the light of the Predictor-corrector scheme [72], such two equations given in (6.6) could be converted into another two explicit forms which might be utilized, then, for examining the dynamic behavior of system (6.5). Anyhow, taking the initial conditions $x(0) = 0.2$ and $y(0) = 0.3$, the fractional-order value $\nu = 0.9$, and the system's parameters $\alpha = 1$ and $\beta = 0.73$ yields the attractor of map (6.5) exhibited in Figure 6.1(a). Figure 6.1(b)-(c), besides, shows the resultant calculations of both the bifurcation diagram and the Largest Lyapunov Exponent (LLE) as a function of α. Obviously, the chaotic behavior of system (6.5) has been demonstrated in those figures according when $\alpha = 0.95$, β =1 and $\nu = 0.9$.

6.3.2 3D-FoDWS

The first form of the FoDWS, which its classical case returns to Wang, was addressed and explored well by considering also the ν–Caputo DDO in [26]. Likewise the previous proposed map, the Caputo h-DDO is employed to propose the following newly 3D-FoDWS:

$$\begin{cases} {}_h^C\Delta_a^\nu x(t) = \alpha_3 y(t+\nu h) + \alpha_4 x(t+\nu h), \\ {}_h^C\Delta_a^\nu y(t) = \alpha_1 x(t+\nu h) + \alpha_2 z(t+\nu h), \\ {}_h^C\Delta_a^\nu z(t) = \alpha_7 z(t+\nu h) + \alpha_6 y(t+\nu h) z(t+\nu h) + \alpha_5, \end{cases} \tag{6.7}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$ and α_i's are the system's parameters, $i = 1,2,...,7$. Accordingly, such three equations given in (6.7) have the following equivalent numerical formulae:

$$\begin{cases} x(n) = x(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_3 y(j) - \alpha_4 x(j)), \\ y(n) = y(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_1 x(j) + \alpha_2 z(j)), \\ z(n) = z(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=1}^n \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (\alpha_7 z(j) + \alpha_6 y(j) z(j) + \alpha_5). \end{cases} \tag{6.8}$$

In Figure 6.2(b), the attractor of map (6.7) is displayed by considering the initial conditions $x(0) = 0.5, y(0) = 0.6$, $z(0) = 0.02$, and assuming $\nu = 0.9$, whereas the system's parameters are set to be as $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$. Furthermore, the resultant calculation of the bifurcation diagram as a function of α_3 is exhibited in Figure 6.2 (a). Thence, it has been clearly shown that the chaotic behavior of system (6.7) will be occure, e.g., according when $h = 0.1$, $\nu = 0.9$, and when the same values of α_i's are taken as above, where $i = 1,2,...,7$.

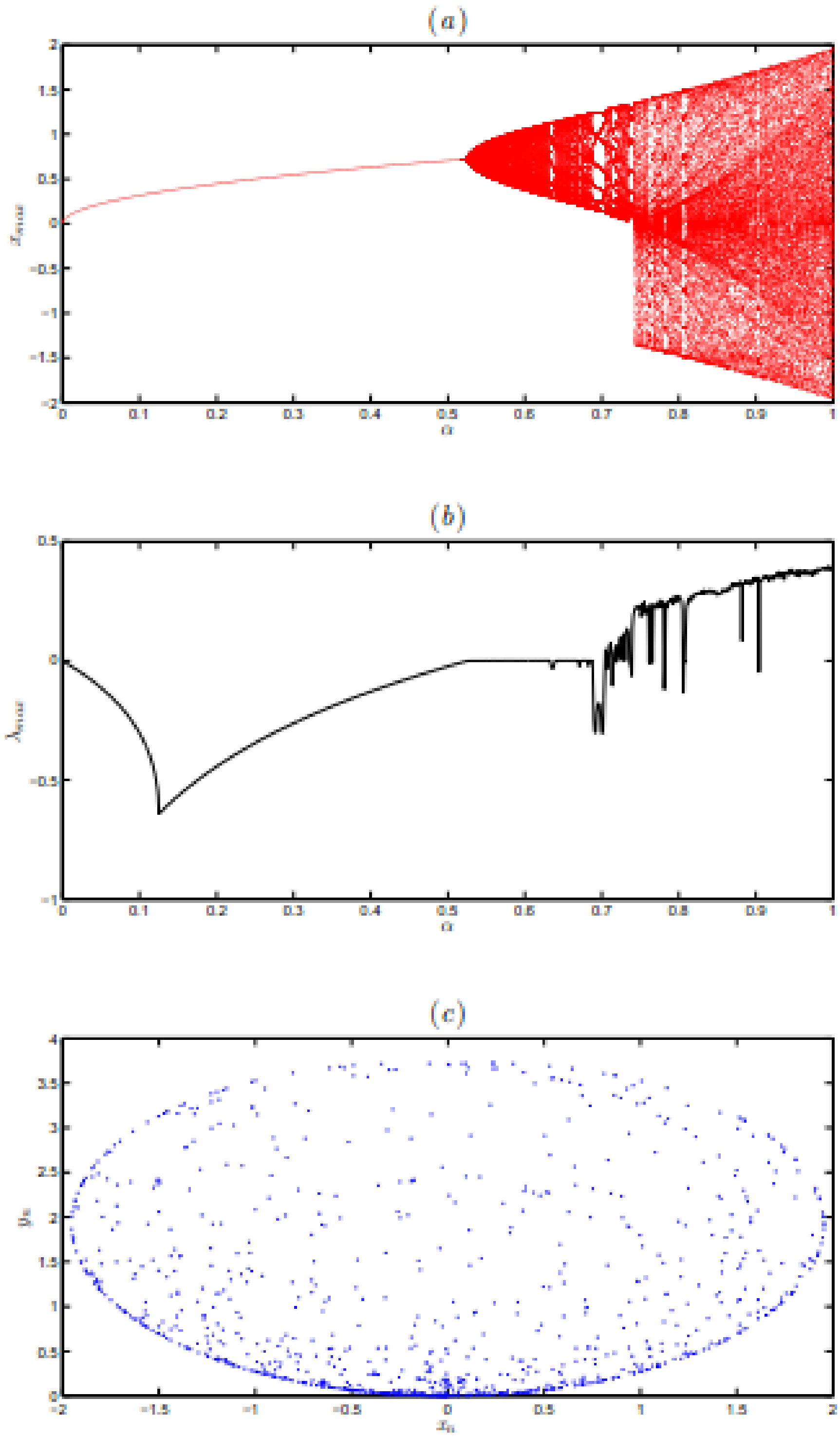


Figure 6.1: Bifurcation and the LLE diagrams of the 2D-FoLDS versus α for system's parameter $\beta = 0.73$ and initial condition $x_0 = 0.2, y_0 = 0.3$ (a) bifurcation diagram and (b) LLE. (c) Chaotic attractor of the 2D-FoLDS for $\alpha = 1, \beta = 0.75$ and $\nu = 0.9$

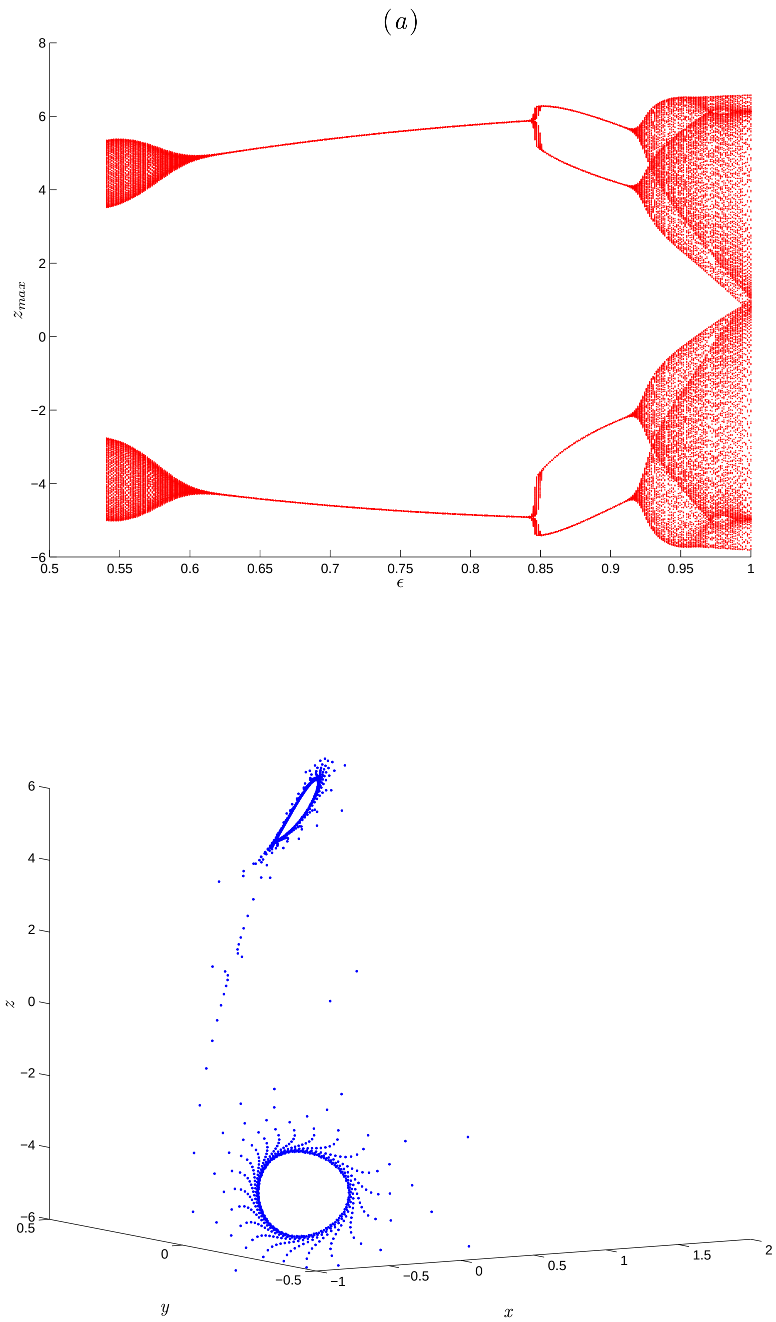


Figure 6.2: Bifurcation diagram and chaotic attractor of the 3D-FoWDS versus α_3, for system's parameter $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$ and initial condition $x_0 = 0.5$, $y_0 = 0.6$, $z_0 = 0.02$ (a) bifurcation diagram; (b) chaotic attractor.

6.4 Linear control laws

This section proposes two control laws in regards to the 2D-FoLDS and 3D-FoWDS. Besides, the Lyapunov approach is employed to establish the asymptotic convergence of such two controllers.

Theorem 6.2. *[35] The 2D-FoLDS given in (6.5) can be controlled under the following one-dimensional control law:*

$$\mathbf{C}(t) = -(1+\alpha\beta)x(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{6.9}$$

Proof. Considering (6.5) yields its controlled version, of course, under the controller given in (6.9). This version has the form:

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu}x(t) = \alpha\beta x(t+\nu h) - \beta y(t+\nu h)x(t+\nu h) + \mathbf{C}(t+\nu h), \\ {}_{h}^{C}\Delta_{a}^{\nu}y(t) = \beta(x(t+\nu h)^2 - y(t+\nu h)). \end{cases} \tag{6.10}$$

Consequently, (6.10) takes the form:

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu}x(t) = -x(t+\nu h) - \beta y(t+\nu h)x(t+\nu h), \\ {}_{h}^{C}\Delta_{a}^{\nu}y(t) = \beta(x(t+\nu h)^2 - y(t+\nu h)). \end{cases} \tag{6.11}$$

One might employ the Lyapunov approach by first letting the Lyapunov function, $V(t)$, in the form:

$$V = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t). \tag{6.12}$$

The adoption of the Caputo h-DDO yields:

$${}_{h}^{C}\Delta_{a}^{\nu}V(t) = \frac{1}{2}\,{}_{h}^{C}\Delta_{a}^{\nu}x^2(t) + \frac{1}{2}\,{}_{h}^{C}\Delta_{a}^{\nu}y^2(t). \tag{6.13}$$

Using Lemma6.1 leads us to follow the following steps:

$$\begin{aligned} {}_{h}^{C}\Delta_{a}^{\nu}V &\leq x(t+\nu h)\,{}_{h}^{C}\Delta_{a}^{\nu}x(t) + y(t+\nu h)\,{}_{h}^{C}\Delta_{a}^{\nu}y(t) \\ &= -x^2(t+\nu h) - \beta y(t+\nu h)x^2(t+\nu h) \\ &+ \beta y(t+\nu h)x(t+\nu h)^2 - \beta y^2(t+\nu h) \\ &= -x^2(t+\nu h) - \beta y^2(t+\nu h) < 0, \qquad \text{(That is because } \beta = 0.73\text{)}. \end{aligned}$$

Hence, it can be deduced, based on Theorem6.1, that the equilibrium point at zero of system (6.11) is asymptotically stable. Therefore, it has been, indeed, shown that the dynamics of the proposed 2D-FoLDS given in (6.5) can be stabilized by controller (6.9). □

In order to highlight the potency of the proposed controller, we illustrate the evolution of all the states and the phase-space plots of the controlled system (6.10) in Figure6.3 according when $\alpha = 1$, $\beta = 0.73$ and $\nu = 0.9$. Obviously, all these plots clearly reflect that the proposed 2D-FoLDS has been completely controlled. Next, an additional control law related to the 3D-FoWDS given in (6.7) is, however, established in identical fashion of the preceding discussion.

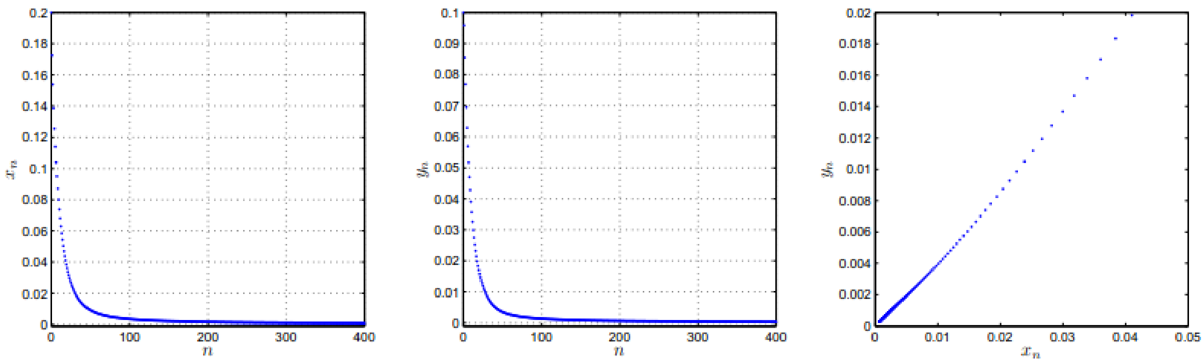


Figure 6.3: The states and phase space of the controlled 2D-FoLDS.

Theorem 6.3. *[35] The 3D-FoWDS given in (6.7) can be controlled under the following two-dimensional control law:*

$$\begin{cases} \mathbf{L}_1(t) = -(\alpha_3 + \alpha_1)y(t) - (\alpha_4 + 1)x(t), \\ \mathbf{L}_2(t) = -\alpha_5 - (|\alpha_6 b|)z(t) - \alpha_2 y(t), \end{cases} \tag{6.14}$$

where $|y(t)| \leq b$ and $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Proof. By considering both (6.7) and (6.14), the following controlled map will be deduced:

$$\begin{cases} {}_h^C\Delta_a^\nu x(t) = \alpha_3 y(t+\nu h) + \alpha_4 x(t+\nu h) + \mathbf{L}_1, \\ {}_h^C\Delta_a^\nu y(t) = \alpha_1 x(t+\nu h) + \alpha_2 z(t+\nu h), \\ {}_h^C\Delta_a^\nu z(t) = \alpha_7 z(t+\nu h) + \alpha_6 y(t+\nu h) z(t+\nu h) + \alpha_5 + \mathbf{L}_2. \end{cases} \tag{6.15}$$

Substituting (6.14) into (6.15) yields the following system:

$$\begin{cases} {}_h^C\Delta_a^\nu x(t) = -\alpha_1 y(t+\nu h) - x(t+\nu h), \\ {}_h^C\Delta_a^\nu y(t) = \alpha_1 x(t+\nu h) + \alpha_2 z(t+\nu h), \\ {}_h^C\Delta_a^\nu z(t) = (\alpha_7 - |\alpha_6| b) z(t+\nu h) + \alpha_6 y(t+\nu h) z(t+\nu h) - \alpha_2 y(t+\nu h). \end{cases} \tag{6.16}$$

Now, assume the Lyapunov function has the form:

$$V = \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t) + \frac{1}{2}z^2(t). \tag{6.17}$$

This implies ${}_h^C\Delta_a^\nu V ={}_h^C \Delta_a^\nu x^2(t) +{}_h^C \Delta_a^\nu y^2(t) +{}_h^C \Delta_a^\nu z^2(t)$, and then by applying Lemma6.1, one obtains:

$$\begin{aligned}
{}_h^C\Delta_a^\nu V &\leq x(t+\nu h){}_h^C\Delta_a^\nu x(t+\nu h) + y(t+\nu h){}_h^C\Delta_a^\nu y(t) + z(t+\nu h){}_h^C\Delta_a^\nu z(t) \\
&= -\alpha_1 x(t+\nu h)y(t-1+\nu) - x^2(t+\nu h) + \alpha_1 y(t-1+\nu)x(t+\nu h) \\
&+ \alpha_2 y(t+\nu h)z(t+\nu h) + (\alpha_7 - |\alpha_6|b)z^2(t+\nu h) + \alpha_6 y(t+\nu h)z^2(t+\nu h) \\
&- \alpha_2 z(t+\nu h)y(t+\nu h) \\
&\leq -x^2(t+\nu h) + (\alpha_7 - |\alpha_6|b)z^2(t+\nu h) + \alpha_6 y(t+\nu h)z^2(t+\nu h) \\
&\leq -x^2(t+\nu h) + (\alpha_7 - |\alpha_6|b)z^2(t+\nu h) + \left|\alpha_6 y(t+\nu h)\right| z^2(t+\nu h) \\
&\leq -x^2(t+\nu h) + (\alpha_7 - |\alpha_6|b)z^2(t+\nu h) + |\alpha_6| b z^2(t+\nu h) \\
&= -x^2(t+\nu h) + \alpha_7 z^2(t+\nu h) < 0, \qquad (\text{ because } \alpha_7 = -1.9).
\end{aligned}$$

Again, it can be concluded, based also on Theorem6.1, that the equilibrium point at zero of system (6.16) is asymptotically stable. In this case as well, it has been shown that the dynamics of the other proposed 3D-FoWDS given in (6.7) could be stabilized by controller (6.14). □

Remark 6.2. *The existence of the upper-bound constant b, identified in Theorem6.3, is justified by the boundedness property that characterizes all chaotic maps' states.*

With aim of showing some findings associated with Theorem 6, a numerical simulation has been made as shown in Figure6.4. In this figure, the evolution of all the states and the phase-space plots of the controlled system (6.15) have been exhibited according when $\nu = 0.9$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$. Based on such simulation, one can definitely observe that the 3D-FoWDS has been also completely controlled.

Remark 6.3. *Observe that the two controllers established in this section demand a minor control effort due to their linearity.*

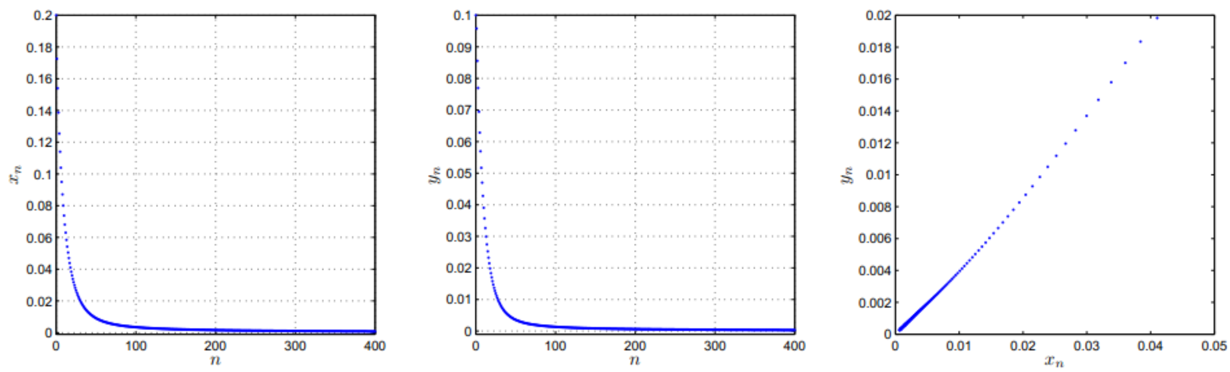


Figure 6.4: The states of the controlled 3D-FoWDS

6.5 Hybrid synchronization scheme

In this part, the two fractional-order maps (2D-FoLDS and 3D-FoWDS) will be endeavored, despite their various dimensions, in view of possible still valid to be synchronized according to suitable synchronization scheme within a certain time. One might suppose the 2D-FoLDS as a master system, and indicate to its states by typing the subscript, m, for each of them. That is;

$$\begin{cases} {}^C_h\Delta^\nu_a x_m(t) = \alpha\beta x_m(t+\nu h) - \beta y_m(t+\nu h)x_m(t+\nu h), \\ {}^C_h\Delta^\nu_a y_m(t) = \beta(x_m(t+\nu h)^2 - y_m(t+\nu h)). \end{cases} \tag{6.18}$$

At the same time, the 3D-FoWDS is supposed to be a slave system and all its states are indicated by typing another subscript, say s, for each of them, i.e.

$$\begin{cases} {}^C_h\Delta^\nu_a x_s(t) = \alpha_3 y_s(t+\nu h) + \alpha_4 x_s(t+\nu h) + \mathbf{U}_1, \\ {}^C_h\Delta^\nu_a y_s(t) = \alpha_1 x_s(t+\nu h) + \alpha_2 z_s(t+\nu h) + \mathbf{U}_2, \\ {}^C_h\Delta^\nu_a z_s(t) = \alpha_7 z_s(t+\nu h) + \alpha_6 y_s(t+\nu h)z_s(t+\nu h) + \alpha_5 + \mathbf{U}_3, \end{cases} \tag{6.19}$$

where $\mathbf{U}_1, \mathbf{U}_2$ and $\mathbf{U}_3$ are the synchronization controllers that needed to be established. Actually, the process of picking up an adaptive control law $(\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3)^T$ aims to compel the following synchronization errors:

$$\begin{cases} e_1 = x_s - x_m, \\ e_2 = y_s + y_m, \\ e_3 = z_s - (x_m + y_m), \end{cases} \tag{6.20}$$

to be asymptotically tended to the origin, i.e.

$$\lim_{t\to+\infty} |e_i(t)| = 0, \quad \text{for } i = 1,2,3. \tag{6.21}$$

Remark 6.4. *In light of the error system (6.20), it is apparent that the two states x_s and x_m are completely synchronized, while the state y_s is anti-synchronized with its corresponding state y_m, and finally the state z_s is appeared as a full-state synchronized with two other states; x_m and y_m. Such three types of synchronization (complectly, anti-, and full-state synchronizations) show co-existing between the master and slave systems given in (6.18) and (6.19), respectively.*

For highlighting other significant results related to the proposed synchronization scheme, we state the following theorem which is considered one of the most main results of this work.

Theorem 6.4. *[35] The two systems, master and slave ones given in (6.18) and (6.19) respectively, achieve synchronized dynamics under the following control law:*

$$\begin{cases} \mathbf{U}_1(t) = -\beta y_m(t)x_m(t) - \alpha_3 y_s(t) + (\alpha\beta - \alpha_4)\, x_m(t), \\ \mathbf{U}_2(t) = -\beta x_m^2(t) - \alpha_2 z_s(t) - \alpha_1 x_s(t) - \beta y_s(t), \\ \mathbf{U}_3(t) = -(\alpha_7+1)\, z_s(t) - \alpha_6 y_s(t) z_s(t) - \beta y_m(t)x_m(t) - \alpha_5 + \beta x_m^2(t) \\ \qquad\qquad + (\beta+1)\, y_m(t) + (\alpha\beta+1)\, x_m(t), \end{cases} \tag{6.22}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$.

Proof. For the purpose of establishing an asymptotic convergence of the synchronization errors, given in (6.20), to zero according to (6.22), we start applying the Caputo h-DDO on (6.20), which yields:

$$\begin{cases} {}_h^C\Delta_a^\nu e_1 = \alpha_3 y_s(t+\nu h) + \alpha_4 x_s(t+\nu h) - \alpha\beta x_m(t+\nu h) + \beta y_m(t+\nu h)x_m(t+\nu h) + \mathbf{U}_1, \\ {}_h^C\Delta_a^\nu e_2 = \alpha_1 x_s(t+\nu h) + \alpha_2 z_s(t+\nu h) + \beta x_m^2(t+\nu h) - \beta y_m(t+\nu h) + \mathbf{U}_2, \\ {}_h^C\Delta_a^\nu e_3 = \alpha_7 z_s(t+\nu h) + \alpha_6 y_s(t+\nu h) z_s(t+\nu h) + \alpha_5 - \alpha\beta x_m(t+\nu h) \\ \qquad\qquad + \beta y_m(t+\nu h)x_m(t+\nu h) - \beta x_m^2(t+\nu h) - \beta y_m(t+\nu h) + \mathbf{U}_3. \end{cases} \tag{6.23}$$

Substituting the proposed control law given in (6.22) into (6.23) leads to the following new discrete system:

$$\begin{cases} {}_h^C\Delta_a^\nu e_1 = \alpha_4 e_1, \\ {}_h^C\Delta_a^\nu e_2 = -\beta e_2, \\ {}_h^C\Delta_a^\nu e_3 = -e_3. \end{cases} \tag{6.24}$$

Now, letting $V = \frac{1}{2}e_1^2(t) + \frac{1}{2}e_2^2(t) + \frac{1}{2}e_3^2(t)$ implies ${}_h^C\Delta_a^\nu V =_h^C \Delta_a^\nu e_1^2(t) +_h^C \Delta_a^\nu e_2^2(t) + {}_h^C\Delta_a^\nu e_3^2(t)$, and by using Lemma6.1, we obtain:

$$\begin{aligned} {}_h^C\Delta_a^\nu V &\leq e_1(t+\nu h){}_h^C\Delta_a^\nu e_1(t+\nu h) + e_2(t+\nu h){}_h^C\Delta_a^\nu e_2^2(t) + e_3(t+\nu h){}_h^C\Delta_a^\nu e_3(t) \\ &= \alpha_4 e_1^2 - \beta e_2^2 - e_3^2 < 0. \end{aligned}$$

In the light of Theorem6.1, it can be deduced that the dynamics of the error system (6.20) have been stabilized at the origin. As a consequence, the master and the slave systems, given in (6.18) and (6.19) respectively, have achieved the synchronized dynamics via non-control laws. □

In order to show the effectiveness of the proposed approach, Figure 6.5 displays the synchronization errors. These plots clearly show that the two fractional-order maps achieve hybrid synchronization.

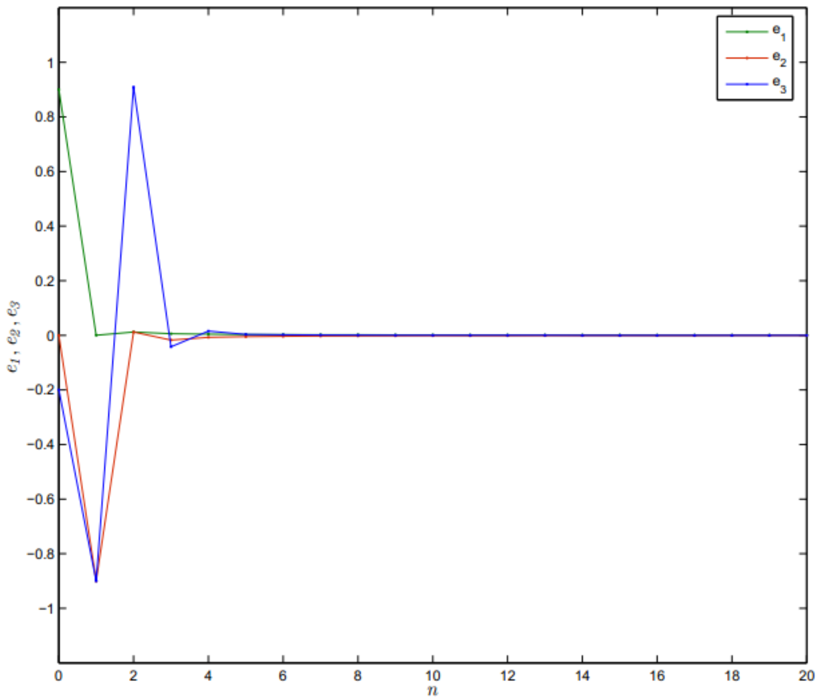


Figure 6.5: Synchronization error of the 2D-FoLDS and 3D-FoWDS

6.6 Conclusion

This chapter has established two new versions of the Factional-order Discrete Chaotic Systems , namely the Two-Dimensional Fractional-order Lorenz Discrete System and Three-Dimensional Fractional-order Wang Discrete System . Using the Caputo h-Difference Discrete Operator, all the states of such two versions have been demonstrated to contain chaos. Despite all this, these states could still be controlled through quite simple linear controllers as is demonstrated in some parts of this chapter. Besides, we have constructed a suitable synchronization scheme which has allowed us to establish a proper controller that has the ability to synchronize the two fractional-order maps under consideration. It has been further shown that all the trajectories of such two maps, together with their proposed controller, converge asymptotically to zero using Lyapunov approach. Finally, several numerical simulations have been performed to highlight the potency of all proposed theoretical findings.

General Conclusion and perspectives

This thesis presents two new contributions in the field of fractional order dynamical systems, Namely, the first is that, it has presented a novel version of the chaotic fractional Grassi-Miller map, based on the Caputo h-difference operator. Two novel theorems have been proved, with the aim of deriving improved schemes for controlling and synchronizing the dynamics of the map.

The second is that, it has established two new versions of the Factional-order Discrete Chaotic Systems , namely the Two-Dimensional Fractional-order Lorenz Discrete System and Three-Dimensional Fractional-order Wang Discrete System. Using the Caputo h-Difference Discrete Operator, all the states of such two versions have been demonstrated to contain chaos. Despite all this, these states could still be controlled through quite simple linear controllers as is demonstrated in some parts of this work. Besides, we have constructed a suitable synchronization scheme which has allowed us to establish a proper controller that has the ability to synchronize the two fractional-order maps under consideration. It has been further shown that all the trajectories of such two maps, together with their proposed controller, converge asymptotically to zero using Lyapunov approach. Finally, several numerical simulations have been performed to highlight the potency of all proposed theoretical findings.

Our future work will focus on the alpha which was between 0 and 1 it will be variable i. e. it varies according to t.

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