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Sur quelques méthodes spectrales pour la résolution de l'équation de transport neutronique.

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Introduction:

L'objet de ces travaux est la contribution à l'étude de la résolution de l'équation de transport en dimension 2, la recherche de la solution analytique par des méthodes spectrales et l'étude de la convergence spectrale.

Les méthodes utilisées pour la résolution sont des méthodes spectrales, la première étant le développement en polynômes de Tchebychev, et la deuxième est le développement en fonctions de Walsh, combinées avec les transformations de Sumudu.

La convergence spectrale est prouvée et la vitesse de convergence est estimée grâce à une règle de quadrature spéciale dite quadrature de Gauss.

Contents

Part I

INTRODUCTION

In recent years, a new class of equations has acquired great significance in mathematical physics in connection with the rapid development of neutron physics and its associated studies. These are the so-called kinetic (transport) equations which describe the process of neutron transport in a substance. These equations are linear, integro-differential equations in partial derivatives of the first order. Much of the efforts in transport theory are devoted to searching for methods that generate accurate results. In the stationary case they have the form

$$\sum_{i=1}^{3} v_i \frac{\partial N}{\partial x_i} + \alpha(P, \mid v \mid) N = \frac{1}{4\pi} \int \theta(P, v, v') N(v', P') dv' + F(v, P), \tag{1.1}$$

where the unknown function $N(v, P)$ is the density of neutrons moving with velocity $v = (v_1, v_2, v_3)$ at the point $P = (x_1, x_2, x_3)$.

With equation (1.1) we associate the boundary conditions

$$N(v, P) = F_1(v, P), \qquad P \in \Gamma,\ (v, n) < 0. \tag{1.2}$$

For simplicity we assume that the region G where neutron transport occurs is convex and bounded by a piece-wise smooth surface Γ. In (1.2) n is the outwards drawn normal vector at the point P to the boundary Γ.

The transport equations (1.1) describe different physical process in particle transport. Besides the above mentioned process of neutron scattering in a substance we have also such processes as the dispersion of light in the atmosphere, the passage of γ-rays through a dispersive medium, the transport of radiation in stellar atmospheres, etc.

Thus these equations have wide application in physics, geophysics and astrophysics.

A detailed solution of the equations for neutron transport can be found, for example, in an article by Davison [22].

Equation (1.1) is, in substance, Boltzmann's linearized transport equation for the distributed of molecules.

Since the problem (1.1) and (1.2) has an extremely complex structure different approximations to it have acquired importance as simplifications. Among the approximation theories we have, for example, multigroup approximation, age theory, the diffusion approximation, etc. However, approximate methods are not always sufficiently precise in practice. Therefore it is useful to examine the accuracy of the various approximations, particularly those constructed directly by the use of modern computer techniques. The construction of methods of solution and their substantiation demand, however, a preliminary qualitative study of the problem and, first of all, of such aspects of it as existence, uniqueness, the continuous dependence of the solution on the data of the problem, spectral properties, in particular, the properties of eigenvalues and eigenfunctions, variational principles, etc. Moreover, these things are mathematically interesting in themselves, since they are connected with the new class of problems which describe much more complex physical processes.

Thus the question arises of constructing a rigorous mathematical theory for transport equations. Among the mathematical works dealing with transport equations, the method proposed by Chandrasekhar [20] solves analytically the discrete equations , (S_N equations), the spherical harmonics method [24] expands the angular flux in Legendre polynomials, the F_N method [29] transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain [27], [28], the SGF method [7], [8] is a numerical nodal method that generates numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method [54] solve analytically the S_N equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the LTS_N method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as LTS_N [5], LTP_N [56], LTW_N [16], $LTCh_N$ [17], LTA_N [18], LTD_N [9].

The analytical character of this solution, in the sense that no approximation is made along its derivation, constitutes its main feature. The idea encompassed is threefold: application of the Laplace Transform to the set of ordinary equations resulting from the approximation, analytical solution of the resulting linear system depending on the complex parameter s and inversion of the transformed angular flux by the Heaviside expansion technique.

We remark that the second step was accomplished by the application of the procedures that we shall describe further ahead. For the LTS_N approach, exploiting the structure of the corresponding matrix, the inversion was performed by employing the definition of matrix inversion [5]. On the other hand, for the remaining approaches, the matrix inversion was performed by the *Trzaska's* method [52].

The series expansions method has been largely used in the solution of the differential equation. In particular, Legendre Polynomials [24] and the Walsh function [49] expansion have been employed to solve the one-dimensional linear transport.

During the following ten to fifteen years much effort, both native and foreign, has been expended on theoretical-physical and mathematical, particularly numerical, methods of approximate solution of the transport equations[1] especially an important work has been done in the context of multidimensional transport problems, based on analytical and numerical approaches, Fourier transforms [58] or the discrete-ordinates method and the transverse-integrated equations [44]. Even commercial codes are available [48]. However, we consider it still today a big challenge in the particle-transport theory in the sense of obtaining procedures that can be applied to a wide range of problems as well as getting high quality computational results As a rule the problems considered were of a practical nature and it was necessary to get answers quickly, albeit crudely. The methods used were either of minor importance or were not properly examined. Thus, pursuing the objective of attaining solutions, based on analytical procedures, for the multidimensional

[1] A detailled bibliographical index of publications related to these subjects can be found in the monographs [41], [22], [20].

transport problems, in this work we use the spectral method [30] to decompose the multidimensional problem into a set of one-dimensional problems, whose can then be solved by one of the well-known methods such as the P_N method [22], the F_N method [29], the discrete-ordinates method [20], [44], the LTS_N method and the other ones employ the Laplace transform [55] and so on.

According to Gottlieb and Orszag [30], spectral methods involve representation the solution to a problem as a truncated series of known functions of the independent variables. The determination of the expansion coefficients is, of course, a fundamental issue in this method and we can then recall some approximation to this end. But in regard to that, one should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties.

The purpose of this thesis is to meet this problem to some extent, using the spectral method .

The principal results contained in this work, aside from those of part II and III, were published in our earlier papers [1]. We note also the works of Cardona [17] and Vilhena [53], which concern spectral investigations and the examination of solutions to stationary problems. These articles have some points in common with ours, however none overlap.

An outline of this thesis is as follows: In the part I of this thesis we present a new approximation for the one-group linear transport equation with anisotropic scattering in a slab, using Walsh functions combined with Sumudu transform [38]. To this end, the angular flux is expanded in a truncated series of Walsh functions in the angular variable. Replacing this expression in the transport equation and taking moments like in the P_N method [24], leads to a new approximation. The resultant first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique.

The inversion of the transformed coefficients is performed also analytically, using *Trzaska's* method and the heaviside technique.

In part II of this work is devoted to study a convergence of a combined spectral and (S_N) discrete approximation for a multidimensional, steady state, linear transport problem with isotropic scattering. The procedure is based on expansion of the angular flux in a truncated series of the Chebyshev polynomials in spatial variables that results in the transformation of the multidimensional problems into a set of one-dimensional problems. The convergence of this approach is studied in the context of the discrete-ordinates equations based on a special quadrature rule for the scattering integral. The discrete-ordinates and quadrature errors are expanded in truncated series of Chebyshev polynomials of degree$\leq L$, and the convergence is derived assuming $L \leq \sigma_t - 4\pi\sigma_s$ where σ_t and σ_s are total- and scattering cross-sections respectively.

Appendix I is devoted to certain properties of the Chebyshev and Legendre polynomials that are frequently used in this thesis, in Appendix II we derive the spectral equations in three dimensional setting.

For the convenience of the reader most of the ideas and results of Sumudu transform an *Trzaska's* method are consolidated in Appendix III and IV.

Part II

A combined Walsh function and Sumudu transform for solving the two-dimensional neutron transport equation.

Chapter 2

Introduction

The neutron transport equation is a linear case of the Boltzmann equation with wide applications in physics and engineering.

As is well known, the study of a given transport equation is a quite important and interesting in transport theory. Various methods have been developed to investigate, and special attention has been given to the task of searching methods that generate accurate results to transport problems in the context of deterministic methods based on analytical procedures, for the multidimensional transport problems, one of the effective methods to treat linear transport equation is the spectral method [43, 41, 31], etc..., whose basic goals is to find exact solution for approximations of the transport equation, several approaches have been suggested. Among them, the method proposed by Chandrasekhar [20] solves analytically the discrete equations, (S_N equations), the spherical harmonics method [24] expands the angular flux in Legendre polynomials, the F_N method [29] transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain [27, 28], the SGF method [7, 8], is a numerical nodal method that generates numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method [54] solve analytically the S_N equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the LTS_N method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as LTS_N [5], LTP_N [55], LTW_N [16], $LTCh_N$ [17], LTA_N [18], LTD_N [9].

The analytical character of this solution, in the sense that no approximation is made along its derivation, constitutes its main feature. The idea encompassed is threefold: application of the Laplace Transform to the set of ordinary equations resulting from the approximation, analytical solution of the resulting linear system depending on the complex parameter s and inversion of the transformed angular flux by the Heaviside expansion technique.

We remark that the second step was accomplished by the application of the procedures that we shall describe further ahead. For the LTS_N approach, exploiting the structure of the corresponding matrix, the inversion was performed by employing the definition of matrix inversion [5]. On the other hand, for the remaining approaches, the matrix inversion was performed by the *Trzaska's* method [52].

The series expansions method has been largely used in the solution of the differential equation. In particular, Legendre polynomials [24] the Sumudu transform [**?**] and the Walsh function [16, 49] expansion have been employed to solve the one-dimensional linear transport the Chebyshev polynomials have been employed to solve the two-dimensional linear transport [1, 34] and for three dimensional case [35, 36].

According to Gottlieb [30], spectral method involve representation the solution to a problem as a truncated series of known functions of the independent variables, of course there exist other method to determine the coefficients of expansion, but in regard to that, we should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties. Thereby, the orthogonal functions and polynomial series have received considerable attention in dealing with various problem. The main characteristic of this technique is that reduces this problems to those of solving a system of algebraic equations, thus greatly simplifying the problem and making it computational plausible.

In the present paper we present a new approximation for the two-dimensional transport equation, using Walsh function combined with the Sumudu transform. The approach is based on expansion of the angular flux in a truncated series of Walsh function in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique [10].

The inversion of the transformed coefficients is obtained using *Trzaska's* method [52] and the Heaviside expansion technique. In our knowledge, the combination of the Walsh function and the Sumudu Transform to solve the two-dimensional transport equation, in this setting, is not considered in the literature.

Chapter 3

Walsh Function

The Walsh functions have many properties similar to those of the trigonometric functions. For example they form a complete, total collection of functions with respect to the space of square Lebesgue integrable functions. However, they are simpler in structure to the trigonometric functions because they take only the values 1 and -1. They may be expressed as linear combinations of the Haar functions [33], so many proofs about the Haar functions carry over to the Walsh system easily. Moreover, the Walsh functions are Haar wavelet packets; see [59]. For a good account of the properties of the Haar wavelets and other wavelets. We use the ordering of the Walsh functions due to Paley [46]. Any function $f \in L^2\,[0,1]$ can be expanded as a series of Walsh functions

$$f(x) = \sum_{i=0}^{\infty} c_i W_i(x) \quad \text{where } c_i = \int_0^1 f(x) W_i(x). \tag{3.1}$$

Fine [26] discovered an important property of the Walsh Fourier series: the $m = 2^n$ th partial sum of the Walsh series of a function f is piece-wise constant, equal to the L^1 mean of f, on each subinterval $[(i-1)/m, i/m]$. For this reason, Walsh series in applications are always truncated to $m = 2^n$ terms. In this case, the coefficients c_i of the

Walsh (-Fourier) series are given by

$$c_i = \sum_{j=0}^{m-1} \frac{1}{m} W_{ij} f_j, \tag{3.2}$$

where f_j is the average value of the function $f(x)$ in the jth interval of width $1/m$ in the interval $(0, 1)$, and W_{ij} is the value of the ith Walsh function in the jth subinterval. The order m Walsh matrix, $\mathcal{W}_m$, has elements W_{ij} .

Let $f(x)$ have a Walsh series with coefficients c_i and its integral from 0 to x have a Walsh series with coefficients of b_i: $\int_0^x f(t)dt = \sum_{i=0}^{\infty} b_i W_i(x)$. If we truncate to $m = 2^n$ terms and use the obvious vector notation, then integration is performed by matrix multiplication $\mathbf{b} = P_m^T \mathbf{c}$ where

$$P_m^T = \begin{bmatrix} P_{m/2} & \frac{1}{2m} I_{m/2} \\ -\frac{1}{2m} I_{m/2} & O_{m/2} \end{bmatrix}, \; P_2^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix}, \tag{3.3}$$

and I_m is the unit matrix, O_m is the zero matrix (of order m), see [21].

Chapter 4

The Two-Dimensional Spectral Solution

Consider the two-dimensional linear, steady state, transport equation given by

$$\mu\frac{\partial}{\partial x}\Psi(x,y,\mu,\phi)+\sqrt{1-\mu^2}\cos\phi\frac{\partial}{\partial y}\Psi(x,y,\mu,\phi)+\sigma_t\Psi(x,y,\mu,\phi)$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi(x,y,\mu^{'},\phi^{'})d\phi^{'}d\mu^{'}+S(x,y,\mu,\phi) \tag{4.1}$$

in the domain $\Omega=\{\mathbf{x}:=(x,y):\ \ 0\le x\le 1,\quad -1\le y\le 1\}$ and the direction in $D=\{(\mu,\theta):-1\le\mu\le 1,\quad 0\le\theta\le 2\pi\}$. Here $\Psi(\mathbf{x},\mu,\phi)$ is the angular flux, σ_t and σ_s denote the total and the differential cross section, respectively, $\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)$ describes the scattering from an assumed pre-collision angular coordinates $(\mu^{'},\theta^{'})$ to a post-collision coordinates (μ,θ) and S is the source term. See [44] for the details.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the z variable. In this way the actual spatial domain may be assumed to be a cylinder with the cross-section Ω and the axial symmetry in z. Then D will correspond to the projection of the points on the unit sphere (the "speed") onto the unit disc (which coincides with D.) See, [2] for the details.

Given the functions $f_1(y,\mu,\phi)$ and $f_2(x,\mu,\phi)$, describing the incident flux, we seek for a solution of (4.1) subject to the following boundary conditions:

For $0 \leq \theta \leq 2\pi$, let

$$\Psi(x,y,\mu,\theta) = \begin{cases} f_1(y,\mu,\phi), \text{ for } x=0; \quad 0<\mu\leq 1, \\ 0, \text{ for } x=1; \quad -1\leq\mu<0. \end{cases} \tag{4.2}$$

For $-1<\mu<1$, let

$$\Psi(x,y=\pm 1,\mu,\theta) = \begin{cases} f_2(y,\mu,\phi), \text{ for } y=-1, \quad 0<\cos\theta\leq 1, \\ 0, \text{ for } y=1, \quad -1\leq\cos\theta<0. \end{cases} \tag{4.3}$$

Theorem 4.1. *Consider the integro-differential equation (4.1) under the boundary conditions (4.2) and (4.3), then the function $\Psi(x,y,\mu,\theta)$ satisfy the following first-order linear differential equation system for the spatial component $\Psi_k(x,\mu,\theta)$*

$$\mu\frac{\partial}{\partial x}\Psi_k(x,\mu,\theta)+\sigma_t\Psi_k(x,\mu,\theta)$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\rightarrow\mu,\phi)\Psi_k(x,\mu^{'},\phi^{'})d\theta^{'}d\mu^{'}+G_k(x,\mu,\theta)$$

where

$$G_k(x,\mu,\theta)=S_k(x,\mu,\theta)-\sqrt{1-\mu^2}\cos\theta\sum_{i=k+1}^{I}A_i^k\Psi_k(x,\mu,\theta)$$

and

$$A_i^k=\frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(T_i(y))\frac{T_k(y)}{\sqrt{1-y^2}}dy$$

$$S_k(x,\mu,\theta)=\frac{2}{\pi}\int_{-1}^{1}S(x,y,\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy.$$

proof:

Expanding the angular flux $\Psi(\mathbf{x},\mu,\theta)$ in terms of the Chebyshev polynomials in the

y variable, leads to

$$\Psi(\mathbf{x},\mu,\theta)=\sum_{i=0}^{I}\Psi_i(x,\mu,\theta)T_i(y). \tag{4.4}$$

Below we determine the first component, i.e., $\Psi_0(x,\mu,\theta)$ explicitly, whereas the other components, $\Psi_i(x,\mu,\theta)$, $i=1,...I$, will appear as the unknowns in I one dimensional transport equations: We start to determine $\Psi_0(x,\mu,\theta)$, by inserting (4.4) into the boundary conditions (4.3) at $y=\pm 1$, to find that:

$$\Psi_0(x,\mu,\theta)=f_2(x,\mu,\phi)-\sum_{i=1}^{I}(-1)^i\Psi_i(x,\mu,\theta), \qquad 0<\cos\theta\leq 1, \tag{4.5}$$

$$\Psi_0(x,\mu,\theta)=-\sum_{i=1}^{I}\Psi_i(x,\mu,\theta), \qquad -1\leq\cos\theta<0. \tag{4.6}$$

where $-1\leq x\leq 1$, $-1<\mu<1$, and we have used the fact that for the Chebyshev polynomials $T_0(x)\equiv 0$, $T_i(1)\equiv 1$ and $T_i(-1)\equiv(-1)^i$.

If we now insert Ψ from (4.4) into (4.1), multiply the resulting equation by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$, and integrate over y we find that the components $\Psi_k(x,\mu,\theta)$, $k=1,...,I$, satisfy the following I one-dimensional equations:

$$\mu\frac{\partial}{\partial x}\Psi_k(x,\mu,\theta)+\sigma_t\Psi_k(x,\mu,\theta)$$

$$\int_{-1}^{1}\int_0^{2\pi}\sigma_s(\mu',\phi'\to\mu,\phi)\Psi_k(x,\mu',\phi')d\theta' d\mu'+G_k(x,\mu,\theta) \tag{4.7}$$

The same procedure with the boundary condition (4.2) at $x=0$ and (4.4) yields

$$\Psi(0,y,\mu,\theta)=f_1(y,\mu,\phi)=\sum_{i=0}^{I}\Psi_i(0,\mu,\theta)T_i(y). \tag{4.8}$$

Now multiply (4.8) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$, and integrate over y we find that

$$\Psi_k(0, \mu, \theta) = \frac{2}{\pi} \int_{-1}^{1} f_1(y; \mu, \theta) \frac{T_k(y)}{\sqrt{1-y^2}} dy. \tag{4.9}$$

Similarly, (note the sign of μ below), the boundary condition at $x = 1$ is written as

$$\sum_{i=0}^{I} \Psi_i(0, -\mu, \theta) T_i(y) = 0 \qquad 0 < \mu \leq 1. \tag{4.10}$$

Multiplying (4.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 1, ..., I$ and integrating over y, we get

$$\Psi_k(0, -\mu, \theta) = 0 \qquad 0 < \mu \leq 1, \qquad 0 \leq \theta \leq 2\pi. \tag{4.11}$$

We can easily check that G_k in (4.7) is written as

$$G_k(x, \mu, \theta) = S_k(x, \mu, \theta) - \sqrt{1-\mu^2} \cos\theta \sum_{i=k+1}^{I} A_i^k \Psi_k(x, \mu, \theta) \tag{4.12}$$

where

$$A_i^k = \frac{2}{\pi} \int_{-1}^{1} \frac{d}{dy}(T_i(y)) \frac{T_k(y)}{\sqrt{1-y^2}} dy \tag{4.13}$$

and

$$S_k(x, \mu, \theta) = \frac{2}{\pi} \int_{-1}^{1} S(x, y, \mu, \theta) \frac{T_k(y)}{\sqrt{1-y^2}} dy. \tag{4.14}$$

Note that the solutions to the one-dimensional problems given through the equation (4.7)-(4.14) define the components $\Psi_k(x, \mu, \theta)$, for $k = 1, ..., I$, in this decreasing order to avoid the coupling of the equations. Once this is done, the angular flux given by (4.4) is completely determined. Here we have used the convention $\sum_{i=I+1}^{I} ... = 0$. Hence the starting $G_I(x, \mu, \theta) \equiv S_I(x, \mu, \theta)$. Note also that although the solution, developed in here, rely on specific boundary conditions the procedure is quite general in the sense that the expression for the first component, $\Psi_0(x, \mu, \theta)$, keeps the information from the boundary conditions in the y variable, while the other components are derived based on the boundary conditions in x.

Chapter 5

Analysis

Now we would like to solve the first-order linear differential equation system with isotropic scattering, i.e., $\sigma_s(\mu^{'}, \phi^{'} \to \mu, \phi) \equiv \sigma_s =$ constant. Assuming isotropic scattering, the equation (4.7) is written as

$$\mu \frac{\partial}{\partial x} \Psi_k(x, \mu, \theta) + \sigma_t \Psi_k(x, \mu, \theta)$$

$$\sigma_s \int_{-1}^{1} \int_{0}^{2\pi} \Psi_k(x, \mu^{'}, \phi^{'}) d\theta^{'} d\mu^{'} + G_k(x, \mu, \theta) \tag{5.1}$$

for $\mathbf{x} \in \Omega := \{(x, y) : 0 \leq x \leq 1, -1 \leq y \leq 1\}$ $\mu \in [-1, 1]$ and $\theta \in [0, 2\pi]$.

Subject to the following boundary conditions:

For $0 \leq \theta \leq 2\pi$, let

$$\Psi_k(x, \mu, \theta) = \begin{cases} f_1(\mu, \phi), \text{ for } x = 0; & 0 < \mu \leq 1, \\ 0, \text{ for } x = 1; & -1 \leq \mu < 0. \end{cases}$$

The study of the problem with the anisotropic scattering is a rather involved task. See, e.g., [3] for an approach involving anisotropic scattering.

For this problem we expand the angular flux in terms of the Walsh function in the angular variable with its domain extended into the interval $[-1, 1]$. To this end, the

Walsh function $W_n(\mu)$ are extended in an even and odd fashion as follows [16]:

$$W_n^e(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \geq 0 \\ W_n(-\mu), & \text{if } \mu < 0 \end{cases}, \tag{5.2}$$

$$W_n^o(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \geq 0 \\ -W_n(-\mu), & \text{if } \mu < 0 \end{cases}, \tag{5.3}$$

for $n = 0, 1, ..., N$. The important feature of this procedure relies the fact that a function $f(\mu)$ defined in the interval $[-1, 1]$ can be expanded in terms of these extended functions in the manner:

$$f(\mu) = \sum_{n=0}^{\infty} \left[a_n W_n^e(\mu) + b_n W_n^o(\mu)\right], \tag{5.4}$$

where the coefficients a_n and b_n are determined as:

$$a_n = \frac{1}{2}\int_{-1}^{1} f(\mu) W_n^e(\mu) d\mu, \tag{5.5}$$

$$b_n = \frac{1}{2}\int_{-1}^{1} f(\mu) W_n^o(\mu) d\mu, \tag{5.6}$$

So, in order to use the Walsh function for the solution of the problem (4.1), the angular flux is approximated by the truncated expansion:

$$\Psi_k(x, \mu, \theta) = \sum_{n=0}^{N} \left[\alpha_{nk}(x, \theta) W_n^e(\mu) + \beta_{nk}(x, \theta) W_n^o(\mu)\right] \tag{5.7}$$

Replacing this expansion into the linear transport equation (5.1), it turns out:

$$\sum_{n=0}^{N} \left[\left\{\mu \frac{\partial \alpha_{nk}}{\partial x}(x, \theta) + \sigma_t \alpha_{nk}(x, \theta)\right\} W_n^e(\mu) + \left\{\mu \frac{\partial \beta_{nk}}{\partial x}(x, \theta) + \sigma_t \beta_{nk}(x, \theta)\right\} W_n^o(\mu)\right]$$

$$\sum_{n=0}^{N} \sigma_s \left[\int_{-1}^{1}\int_{0}^{2\pi} \alpha_{nk}(x, \theta') W_n^e(\mu') d\theta' d\mu' + \int_{-1}^{1}\int_{0}^{2\pi} \alpha_{nk}(x, \theta') W_n^o(\mu') d\theta' d\mu'\right] + G_k(x, \mu, \theta) \tag{5.8}$$

Multiplying equation (5.8) by W_m^e , $m = 0., ..., N$ and integrating into the interval $[-1, 1]$, results:

$$\sum_{n=0}^{N}\left[\frac{\partial\beta_{nk}}{\partial x}(x,\theta)\int_{-1}^{1}\mu W_n^o(\mu)W_n^e(\mu)d\mu+\sigma_t\alpha_{nk}(x,\theta)\int_{-1}^{1}W_n^e(\mu)W_m^e(\mu)d\mu\right]=$$

$$\sum_{n=0}^{N}\sigma_s\left[\int_0^{2\pi}\alpha_{nk}(x,\theta')d\theta'\int_{-1}^{1}W_n^o(\mu^{'})W_n^o(\mu^{'})d\mu^{'}\right]+\int_{-1}^{1}G_k(x,\mu,\theta)W_n^e(\mu)d\mu \tag{5.9}$$

Similarly, multiplying equation (5.8) by W_m^0 , $m = 0., ..., N$ and integrating yields:

$$\sum_{n=0}^{N}\left[\frac{\partial\alpha_{nk}}{\partial x}(x,\theta)\int_{-1}^{1}\mu W_n^o(\mu)W_n^e(\mu)d\mu+\sigma_t\beta_{nk}(x,\theta)\int_{-1}^{1}W_n^0(\mu)W_m^0(\mu)d\mu\right]=$$

$$\sum_{n=0}^{N}\sigma_s\left[\int_0^{2\pi}\beta_{nk}(x,\theta')d\theta'\int_{-1}^{1}W_n^o(\mu^{'})W_n^o(\mu^{'})d\mu^{'}\right]+\int_{-1}^{1}G_k(x,\mu,\theta)W_n^0(\mu)d\mu \tag{5.10}$$

The integrals appearing in equations (5.9) (5.10) are known and are given [19] as

$$D_{n,m}=\frac{1}{2}\int_{-1}^{1}\mu W_n^o(\mu)W_n^e(\mu)d\mu=\int_0^1\mu W_{(n+m)\bmod 2}(\mu) \tag{5.11}$$

or

$$D_{n,m}=\begin{cases}1/2\ ,\ \text{ if } n=m\\ -2^{-(k+2)}\ ,\ \text{if } (n+m)\bmod 2=2^k,\ k \text{ natural}\\ 0,\ \text{at another case}\end{cases} \tag{5.12}$$

where the notation $(n + m) \bmod 2$ denotes the $\bmod 2$ sum of the binary digits n and m [?]

Now, following the idea of applying the Sumudu transform (cf. [10] Theorem 2.2 p. 107) to equations (5.9) and (5.10) using (5.11), we obtain an algebraic linear system

$$\sum_{n=0}^{N}D_{n,m}p\overline{\beta}_{n,k}(p,\theta)-\sigma_s\sum_{n=0}^{N}p\overline{\alpha}_{n,k}(p,\theta)+\sigma_t\overline{\alpha}_{n,k}(p,\theta)=$$

$$\int_{-1}^{1} \overline{G}_k(x,\mu,\theta) W_n^e(\mu) d\mu + \sum_{n=0}^{N} D_{n,m} \beta_{n,k}(0,\theta) \tag{5.13}$$

$$\sum_{n=0}^{N} D_{n,m} p \overline{\alpha}_{n,k}(p,\theta) - \sigma_s \sum_{n=0}^{N} p \overline{\beta}_{n,k}(p,\theta) + \sigma_t \overline{\beta}_{n,k}(p,\theta) =$$

$$\int_{-1}^{1} \overline{G}_k(x,\mu,\theta) W_n^o(\mu) d\mu + \sum_{n=0}^{N} D_{n,m} \alpha_{n,k}(0,\theta) \tag{5.14}$$

The equations (5.13) and (5.14) can be recast in the following matricial form:

$$\begin{bmatrix} (\sigma_t - p\sigma_s)\mathbf{I} & p\mathfrak{D} \\ p\mathfrak{D} & (\sigma_t - p\sigma_s)\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \overline{\alpha}(p,\theta) \\ \overline{\beta}(p,\theta) \end{bmatrix} = \begin{bmatrix} \mathfrak{D}\alpha(0,\theta) \\ \mathfrak{D}\beta(0,\theta) \end{bmatrix} + \begin{bmatrix} \overline{\mathbf{G}}_1 \mathbf{I} \\ \overline{\mathbf{G}}_2 \mathbf{I} \end{bmatrix}, \tag{5.15}$$

where $\mathbf{I}$ is the identity of order $N+1$. The matrix $\mathfrak{D}$ has its elements defined by equation (5.11) and

$$\overline{\mathbf{G}}_1 = \int_{-1}^{1} \overline{G}_k(x,\mu,\theta) W_n^e(\mu) d\mu \tag{5.16}$$

$$\overline{\mathbf{G}}_2 = \int_{-1}^{1} \overline{G}_k(x,\mu,\theta) W_n^o(\mu) d\mu \tag{5.17}$$

with $\overline{\alpha}(p,\theta)$, $\overline{\beta}(p,\theta)$ and $\overline{G}_k(x,\mu,\theta)$ denoting the Sumudu transform of the column vectors $\alpha(x,\theta) = [\alpha_{1,k}(x,\theta) ... \alpha_{N,k}(x,\theta)]^{\mathrm{T}}$, $\beta(x,\theta) = \left[\beta_{1,k}(x,\theta) ... \beta_{N,k}(x,\theta)\right]^{\mathrm{T}}$ and $G_k(x,\mu,\theta)$, $k = 1...N$ respectively.

The vectors $\overline{\alpha}(p,\theta)$ and $\overline{\beta}(p,\theta)$ are determined solving equation (5.15) by the method of Trzaska [52] and are given as:

$$\begin{bmatrix} \alpha(x,\theta) \\ \beta(x,\theta) \end{bmatrix} = \left\{ \sum_{n=0}^{N} \exp(\xi_k x,\theta) \begin{bmatrix} \mathbf{\Delta}_k^1 & \mathbf{\Delta}_k^2 \\ \mathbf{\Delta}_k^3 & \mathbf{\Delta}_k^4 \end{bmatrix} \right\} \cdot \begin{bmatrix} \mathfrak{D}\beta(0,\theta) \\ \mathfrak{D}\alpha(0,\theta) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \mathbf{I} \\ \mathbf{G}_2 \mathbf{I} \end{bmatrix}, \tag{5.18}$$

where ξ_k are the roots of the characteristic polynomial of the matrix appearing on the left hand side of the equation (5.15) and $\mathbf{\Delta}_k^i$, $i = 1,2,3,4$, are matrices resulting from the application of the *Trzaska's* method.

The unknown vectors $\alpha(0,\theta)$ and $\beta(0,\theta)$ are determined using the boundary conditions (4.2) and (4.3), after its expansion in terms of the extended Walsh functions. This procedure leads to the following linear system:

$$\alpha(0,\theta)+\beta(0,\theta)=\mathfrak{F} \tag{5.19}$$

$$\alpha(1,\theta)-\beta(1,\theta)=\mathbf{0} \tag{5.20}$$

where $\mathfrak{F}$ is the vector with $(N+1)$ components, expressed as:

$$\mathfrak{F}_k=\int_0^1 f_1(\mu,\theta)W_k(\mu)d\mu \tag{5.21}$$

Replacing the equations (5.19) and (5.20) into equation (5.18), it turns out the following linear system for the unknown vectors $\alpha(0,\theta)$ and $\beta(0,\theta)$

$$\begin{bmatrix}\alpha(0,\theta)\\ \beta(0,\theta)\end{bmatrix}=\left\{\sum_{n=0}^{N}\exp(0,\theta)\begin{bmatrix}\mathbf{\Delta}_k^1 & \mathbf{\Delta}_k^2\\ \mathbf{\Delta}_k^3 & \mathbf{\Delta}_k^4\end{bmatrix}\right\}\cdot\begin{bmatrix}\mathfrak{D}\beta(0,\theta)\\ \mathfrak{D}\alpha(0,\theta)\end{bmatrix}+\begin{bmatrix}\mathbf{G}_1\mathbf{I}\\ \mathbf{G}_2\mathbf{I}\end{bmatrix}, \tag{5.22}$$

$$\begin{bmatrix}\alpha(1,\theta)\\ \beta(1,\theta)\end{bmatrix}=\left\{\sum_{n=0}^{N}\exp(\xi_k,\theta)\begin{bmatrix}\mathbf{\Delta}_k^1 & \mathbf{\Delta}_k^2\\ \mathbf{\Delta}_k^3 & \mathbf{\Delta}_k^4\end{bmatrix}\right\}\cdot\begin{bmatrix}\mathfrak{D}\beta(0,\theta)\\ \mathfrak{D}\alpha(0,\theta)\end{bmatrix}+\begin{bmatrix}\mathbf{G}_1\mathbf{I}\\ \mathbf{G}_2\mathbf{I}\end{bmatrix}, \tag{5.23}$$

we rewrite linear system for the unknown vectors $\alpha(0,\theta)$ and $\beta(0,\theta)$ as

$$\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\cdot\begin{bmatrix}\alpha(0,\theta)\\ \beta(0,\theta)\end{bmatrix}+\begin{bmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{bmatrix}=\begin{bmatrix}\mathfrak{F}\\ \mathbf{0}\end{bmatrix} \tag{5.24}$$

where

$$\mathbf{A}_1=\sum_{n=0}^{N}\exp(0,\theta)\left[\mathbf{\Delta}_k^2+\mathbf{\Delta}_k^4\right]\ \mathfrak{D} \tag{5.25}$$

$$\mathbf{A}_2=\sum_{n=0}^{N}\exp(0,\theta)\left[\mathbf{\Delta}_k^1+\mathbf{\Delta}_k^3\right]\ \mathfrak{D} \tag{5.26}$$

$$\mathbf{A}_3 = \sum_{n=0}^{N} \exp(\xi_k, \theta) \left[\mathbf{\Delta}_k^2 - \mathbf{\Delta}_k^4 \right] \ \mathfrak{D} \tag{5.27}$$

$$\mathbf{A}_4 = \sum_{n=0}^{N} \exp(\xi_k, \theta) \left[\mathbf{\Delta}_k^1 - \mathbf{\Delta}_k^3 \right] \ \mathfrak{D} \tag{5.28}$$

$$\mathfrak{B}_1 = (\mathbf{G}_1 + \mathbf{G}_2)\mathbf{I} \tag{5.29}$$

$$\mathfrak{B}_2 = (\mathbf{G}_1 - \mathbf{G}_2)\mathbf{I} \tag{5.30}$$

So, the vectors $\alpha(0, \theta)$ and $\beta(0, \theta)$ are determined solving equation (5.24). Therefore the functions $\alpha(x, \theta)$ and $\beta(x, \theta)$ given by equation (5.18) are completely determined. Consequently, an analytical formulation is obtained for angular flux in term of the Walsh function, given by equation (5.7)

Chapter 6

Conclusion

The Walsh function combined with Sumudu transform should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper, although we have not investigated this idea thoroughly. We will be considering more complicated geometries in future studies, during which we will ascertain this method's usefulness for larger spatial dimensional problems. In preparation for these problems, we are currently investigating the effectiveness of spectral methods combined with Sumudu transform in solving the linear system of differential equation analytically.

An adaptation of the method for the convergence of the spectral solution within the framework of the analytical solution to study and prove convergence by using the discrete ordinates method is relatively new. The methods employing Sumudu transforms combined with Walsh function represent very interesting new ideas for studying the convergence of many numerical methods and can be extended easily to general linear transport problems. In fact only some preliminary results have been obtained. In this context we intend to study the existence and uniqueness of its solution by using C_0 *semigroup approach*. Our attention will be focused in this direction.

Part III

CHEBYSHEV SPECTRAL-S_N METHOD FOR THE NEUTRON TRANSPORT EQUATION

Chapter 7

Introduction

In this part we develop spectral approximation for two and three dimensional, steady state, linear transport equation with isotropic scattering, in bounded domain. The procedure is based on the expansion of the angular flux in a truncated series of Chebyshev polynomials in the spatial variables. We study the convergence of this method in two dimensional case, where we use a special quadrature rule to discretise in the angular variables, approximating the scalar flux. The similarity of the spectral method to the finite element method is evident: the bases functions have a constant norm and the procedure is to represent the approximate solution as a linear combination of finite number of basis functions (truncated series of Chebyshev polynomials) and then use a variational formulation. The main difference is that: the finite element bases functions are locally supported, whereas the Chebyshev polynomials are global functions. See also [12] for further details.

In [53] this approach, with no convergence rate analysis, is considered for a truncated series of general orthogonal polynomials. The detailed study in [53] is carried out for the Legendre polynomials, where an index mix caused that a significant drift term is argued to be of lower and therefore its contribution is not included in the estimates.

We apply this procedure using Chebyshev polynomials with, e.g., the advantage of having constant weighted-L_2 norms, and give a full convergence study including estimates of the contribution from the whole drift term. The final estimations via an inverse iterative/induction argument, based on an estimate derived from some elementary properties of Chebyshev polynomials in Appendix I. In our knowledge convergence rate analysis, in this setting, is not considered in the literature.

Related problems, in different setting, are studied in the nuclear engineering literature, see, e.g., reference in Vilhena et al [53]. Barros and Larsen [6] carried out a spectral nodal method for certain discrete-ordinates problems. Chebyshev spectral methods for radiative transfer problems are studied, e.g., by Kim and Ishimaru in [39] and by Kim and Moscoso [40]. In, e.g., astrophysical aspects, spectral methods are considered for relativistic gravitation and gravitational radiation by Bonazzola et al [12]. A multi-domain spectral method is studied by Grangclément et al [32], for scalar and vectorial Poisson equation. C++ software library, developed for multi-domain, is available in public domain (GPL), **http://www.lorene.obspm.fr.** For more detailed study on Chebyshev spectral method and also approximations by the spectral methods we refer the reader to monographs by Boyd [13] and Bernardi and Maday [11].

An outline of this part is as follows: In Section 8 we derive the truncated spectral equations in 2 dimensions. In Section 9 we prove that a certain weighted-L_2 norm for the error in the discrete-ordinates approximation of the spectral solution is dominated by that of a quadrature approximation. In Section 10 we construct a special quadrature rule and derive convergence rates for the quadrature error. Combining the results of Section 9 and 10, we conclude the convergence of the discrete-ordinates for the spectral method. Appendix I is devoted to certain properties of the Chebyshev polynomials, that are frequently used in the paper, and also the proof of a crucial estimate used in the approximation of the contribution from the drift term. Finally in Appendix II we derive the spectral equations in a three dimensional setting.

Chapter 8

The Two-Dimensional Spectral Solution.

Consider the two-dimensional linear, steady state, transport equation given by

$$\mu\frac{\partial}{\partial x}\Psi(x,y,\mu,\phi)+\sqrt{1-\mu^2}\cos\phi\frac{\partial}{\partial y}\Psi(x,y,\mu,\phi)+\sigma_t\Psi(x,y,\mu,\phi)$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi(x,y,\mu^{'},\phi^{'})d\phi^{'}d\mu^{'}+S(x,y,\mu,\phi) \tag{8.1}$$

in the rectangular domain $\Omega=\{\mathbf{x}:=(x,y): \quad -1\leq x\leq 1, \quad -1\leq y\leq 1\}$ and the direction in $D=\{(\mu,\theta): \quad -1\leq\mu\leq 1, \quad 0\leq\theta\leq 2\pi\}$. Here $\Psi(\mathbf{x},\mu,\phi)$ is the angular flux, σ_t and σ_s denote the total and the differential cross section, respectively, $\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)$ describes the scattering from an assumed pre-collision angular coordinates $(\mu^{'},\theta^{'})$ to a post-collision coordinates (μ,θ) and S is the source term. See [44] for the details.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the z variable. In this way the actual spatial domain may be assumed to be a cylinder with the cross-section Ω and the axial symmetry in z. Then D will correspond to the projection of the points on the unit sphere (the "speed') onto the unit disc (which coincides with D.) See, [2] for the details.

Given the functions $f_1(y,\mu,\phi)$ and $f_2(x,\mu,\phi)$, describing the incident flux, we seek for a solution of (8.1) subject to the following boundary conditions:

For $0 \le \theta \le 2\pi$, let

$$\Psi(x=\pm 1, y, \mu, \theta) = \begin{cases} f_1(y,\mu,\phi), \; x=-1, & 0<\mu\le 1, \\ 0, \; x=1, & -1\le\mu<0. \end{cases} \tag{8.2}$$

For $-1<\mu<1$, let

$$\Psi(x, y=\pm 1, \mu, \theta) = \begin{cases} f_2(y,\mu,\phi), \; y=-1, & 0<\cos\theta\le 1, \\ 0, \; y=1, & -1\le\cos\theta<0. \end{cases} \tag{8.3}$$

Expanding the angular flux $\Psi(\mathbf{x},\mu,\theta)$ in terms of the Chebyshev polynomials in the y variable, leads to

$$\Psi(\mathbf{x},\mu,\theta) = \sum_{i=0}^{I} \Psi_i(x,\mu,\theta) T_i(y). \tag{8.4}$$

Below we determine the first component, i.e., $\Psi_0(x,\mu,\theta)$ explicitly, whereas the other components, $\Psi_i(x,\mu,\theta)$, $i=1,...I$, will appear as the unknowns in I one dimensional transport equations: We start to determine $\Psi_0(x,\mu,\theta)$, by inserting (8.4) into the boundary conditions (8.3) at $y=\pm 1$, to find that:

$$\Psi_0(x,\mu,\theta) = f_2(x,\mu,\phi) - \sum_{i=1}^{I} (-1)^i \Psi_i(x,\mu,\theta), \quad 0<\cos\theta\le 1, \tag{8.5}$$

$$\Psi_0(x,\mu,\theta) = -\sum_{i=1}^{I} \Psi_i(x,\mu,\theta), \quad -1\le\cos\theta<0. \tag{8.6}$$

where $-1\le x\le 1$, $-1<\mu<1$, and we have used the fact that for the Chebyshev polynomials $T_0(x)\equiv 0$, $T_i(1)\equiv 1$ and $T_i(-1)\equiv(-1)^i$. See Appendix I.

If we now insert Ψ from (8.4) into (8.1), multiply the resulting equation by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$, and integrate over y we find that the components $\Psi_k(x,\mu,\theta)$, $k=1,...,I$, satisfy the following I one-dimensional equations:

$$\mu\frac{\partial}{\partial x}\Psi_k(x,\mu,\theta)+\sigma_t\Psi_k(x,\mu,\theta)$$

$$\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\phi^{'}\to\mu,\phi)\Psi_k(x,\mu^{'},\phi^{'})d\theta^{'}d\mu^{'}+G_k(x,\mu,\theta) \qquad (8.7)$$

The same procedure with the boundary condition (8.2) at $x=-1$, and (8.4) yields

$$\Psi(-1,y,\mu,\theta)=f_1(y,\mu,\phi)=\sum_{i=0}^{I}\Psi_i(-1,\mu,\theta)T_i(y). \qquad (8.8)$$

Now multiply (8.8) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$, and integrate over y we find that

$$\Psi_k(-1,\mu,\theta)=\frac{2}{\pi}\int_{-1}^{1}f_1(y;\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy. \qquad (8.9)$$

Similarly, (note the sign of μ below), the boundary condition at $x=1$ is written as

$$\sum_{i=0}^{I}\Psi_i(1,-\mu,\theta)T_i(y)=0 \qquad 0<\mu\leq 1. \qquad (8.10)$$

Multiplying (8.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k=1,...,I$ and integrating over y, we get

$$\Psi_k(1,-\mu,\theta)=0 \qquad 0<\mu\leq 1, \qquad 0\leq\theta\leq 2\pi. \qquad (8.11)$$

We can easily check that G_k in (8.7) is written as

$$G_k(x,\mu,\theta)=S_k(x,\mu,\theta)-\sqrt{1-\mu^2}\cos\theta\sum_{i=k+1}^{I}A_i^k\Psi_k(x,\mu,\theta) \qquad (8.12)$$

where

$$A_i^k=\frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(T_i(y))\frac{T_k(y)}{\sqrt{1-y^2}}dy \qquad (8.13)$$

and

$$S_k(x,\mu,\theta) = \frac{2}{\pi}\int_{-1}^{1} S(x,y,\mu,\theta)\frac{T_k(y)}{\sqrt{1-y^2}}dy. \tag{8.14}$$

Note that the solutions to the one-dimensional problems given through the equation (8.7)-(8.14) define the components $\Psi_k(x,\mu,\theta)$, for $k = 1, ..., I$, in this decreasing order to avoid the coupling of the equations. Once this is done, the angular flux given by (8.4) is completely determined. Here we have used the convention $\sum_{i=I+1}^{I} ... = 0$. Hence the starting $G_I(x,\mu,\theta) \equiv S_I(x,\mu,\theta)$. Note also that although the solution, developed in here, rely on specific boundary conditions the procedure is quite general in the sense that the expression for the first component, $\Psi_0(x,\mu,\theta)$, keeps the information from the boundary conditions in the y variable, while the other components are derived based on the boundary conditions in x.

Chapter 9

Convergence of the Spectral Solution.

In the sequel we focus on the two dimensional, steady state linear transport process with isotropic scattering, i.e., $\sigma_s(\mu', \phi' \to \mu, \phi) \equiv \sigma_s =$ constant. For this problem we show, using a weighted-L_2 norm, convergence of the spectral solution defined for the spatial variables. More specifically we show that: in a certain weighted-L_2 norm, the (truncated) *discrete ordinate* approximation error for the spectral solution is dominated by that of a special quadrature approximation error. The study of convergence of this quadrature approximation is the matter of the next section.

Assuming isotropic scattering, the equation (8.1) is written as

$$\mu\frac{\partial}{\partial x}\Psi(\mathbf{x},\mu,\theta) + \sqrt{1-\mu^2}\cos\theta\frac{\partial}{\partial y}\Psi(\mathbf{x},\mu,\theta) + \sigma_t\Psi(\mathbf{x},\mu,\theta)$$

$$= \sigma_s\int_{-1}^{1}\int_{0}^{2\pi}\Psi(\mathbf{x},\mu',\theta')d\theta' d\mu' + S(\mathbf{x},\mu,\theta) \tag{9.1}$$

for $\mathbf{x} \in \Omega := \{(x,y)\colon \quad -1 \le x \le 1, \quad -1 \le y \le 1\}$, $\mu \in [-1,1]$ and $\theta \in [0,2\pi]$. the study of the problem with the anisotropic scattering is a rather involved task. See, e.g., [4] for an approach involving anisotropic scattering. Consider now the discrete ordinate

(S_N) approximation of the equation (9.1): for $m = 1, ..., M$, let

$$\mu_m \frac{\partial}{\partial x}\Psi_m(\mathbf{x}) + \eta_m \frac{\partial}{\partial y}\Psi_m(\mathbf{x}) + \sigma_t \Psi_m(\mathbf{x}) = \sigma_s \sum_{n=1}^{M} \omega_n \Psi_n(\mathbf{x}) + S_m(\mathbf{x}), \tag{9.2}$$

where

$$\eta_m = \sqrt{1-\mu_m^2}\cos\theta_m \tag{9.3}$$

and $\Psi_m(\mathbf{x}) := \Psi_m(x, y)$ is the angular flux in the directions defined by μ_m and η_m and associated with the quadrature weights ω_m. Finally $S_m(\mathbf{x})$ is the corresponding inhomogeneous source term defined in the discrete directions $(\mu_m, \eta_m) \in [-1, 1]^2$.

We assume a quadrature mesh $(\mu_m, \eta_m) \neq (0, 0)$,

$$\begin{cases} \mu_1 < \mu_2 < .... < \mu_M, \\ \eta_1 < \eta_2 < ... < \eta_{M,} \end{cases} \tag{9.4}$$

satisfying the following conditions:

$$\omega_m \sim 4\pi/M, \qquad \sum_{m=1}^{M} \omega_m \sim 4\pi, \qquad m = 1, ..., M \tag{9.5}$$

Further, we assume that the discrete-ordinates equation (9.2) satisfy the same boundary conditions, in the discrete directions, as the continuous one, i.e., (9.1) (as stated in Section 8). We shall prove that, under certain assumptions, the solution of the equation (9.2) would converge to that of the equation (9.1) as $N \to \infty$.

To this approach we define the error in the *approximate flux* by

$$\epsilon_m(\mathbf{x}) = \Psi(\mathbf{x}, \mu_m, \eta_m) - \Psi_m(\mathbf{x}), \qquad m = 1, ..., M, \tag{9.6}$$

and the *truncation error in the quadrature formula* as

$$\tau(\mathbf{x}) = \int_{-1}^{1}\int_{0}^{2\pi} \Psi(\mathbf{x}; \mu', \theta')d\mu' d\theta' - \sum_{n=1}^{M} \omega_n \Psi(\mathbf{x}, \mu_m, \eta_m). \tag{9.7}$$

Subtracting the discrete-ordinates equation (9.2) from the continuous equation (9.1) in the discrete directions, for each $m = 1, ..., M$, an equation relating the discrete-ordinates approximation error to the quadrature error, viz,

$$\mu_m \frac{\partial \epsilon_m(\mathbf{x})}{\partial x} + \eta_m \frac{\partial \epsilon_m(\mathbf{x})}{\partial y} + \sigma_t \epsilon_m(\mathbf{x}) = \sigma_s \sum_{n=1}^{M} \omega_n \epsilon_m(\mathbf{x}) + \sigma_s \tau(\mathbf{x}). \tag{9.8}$$

We expand both the approximation and the quadrature errors in a truncated series of Chebyshev polynomials in y,

$$\epsilon_m(x,y) = \sum_{l=0}^{L} \epsilon_m^l(x) T_l(y), \tag{9.9}$$

$$\tau(x,y) = \sum_{l=0}^{L} \tau^l(x) T_l(y) \tag{9.10}$$

and define the $l-th$ moments of the errors by

$$\|\epsilon^l\| = \left[\frac{2-\delta_{l,0}}{\pi} \int_{-1}^{1} \sum_{m=1}^{M} \omega_m (\epsilon_m^l(x))^2 dx \right]^{1/2} \tag{9.11}$$

$$\|\tau^l\| = \left[\frac{2-\delta_{l,0}}{\pi} \int_{-1}^{1} (\tau^l(x))^2 dx \right]^{1/2}. \tag{9.12}$$

Remark. Note that (9.9) and (9.10) involve further, truncated, approximations of $\tau(\mathbf{x})$, in (9.7) and the solution $\epsilon(\mathbf{x})$ of (9.6). We keep using the same notation as before the truncation. Also, despite the recent truncation in y, we use equalities in (9.9), (9.10), as well as in the subsequent relation below.

The main result of this paper is as follows:

Theorem 9.1. *Let $L = \mathcal{O}(\sigma)$ where $\sigma = \sigma_t - 4\pi\sigma_s$, then for $l = 0, 1, ..., L$*

$$\|\epsilon^l\| \to 0, \qquad as \quad M \to \infty.$$

In the remaining part of this section we show that, for $\omega_m \sim 4\pi/M$, $m = 1, ..., M$ the L_2 norm of the truncated spectral error $\left\|\epsilon^l\right\|$, counted in a reverse order on $l = L, L-1, ..., 0$, is dominated by that quadrature error $\left\|\tau^l\right\|$.

The next section is devoted to proof of the following result:

Theorem 9.2. *For $\omega_m \sim 4\pi/M$, $m = 1, ..., M$, if $\Psi \in L_2(\mu, \theta)$, then*

$$\left\|\tau^l\right\| \to 0, \qquad \text{as} \quad M \to \infty.$$

To prepare for the proof of the Theorem 9.1, we substitute (9.9) and (9.10) into the equation (9.8) to get

$$\mu_m \sum_{l=0}^{L} \frac{d\epsilon_m^l(x)}{dx} T_l(y) + \eta_m \sum_{l=0}^{L} \epsilon_m^l(x) \frac{dT_l}{dy}(y) + \sigma_t \sum_{l=0}^{L} \epsilon_m^l(x) T_l(y)$$

$$= \sigma_s \sum_{n=1}^{M} \omega_n \sum_{l=0}^{L} \epsilon_n^l(x) T_l(y) + \sigma_s \sum_{l=0}^{L} \tau^l(x) T_l(y), \tag{9.13}$$

Multiplying (9.13) by $\frac{T_j(y)}{\sqrt{1-y^2}}$, $j = 0, ..., L$ and integrating over y yields

$$\frac{\pi}{2-\delta_{j,0}} \mu_m \frac{d\epsilon_m^j(x)}{dx} + \eta_m \sum_{l=0}^{L} \gamma_j(l) \epsilon_m^l(x) + \frac{\pi}{2-\delta_{j,0}} \sigma_t \epsilon_m^j(x) =$$

$$\frac{\pi}{2-\delta_{j,0}} \sigma_s \sum_{n=1}^{M} \omega_n \epsilon_m^l(x) + \frac{\pi}{2-\delta_{j,0}} \sigma_s \tau^j(x), \tag{9.14}$$

where

$$\gamma_j(l) = \int_{-1}^{1} \frac{dT_l(y)}{dy}(y) \frac{T_j(y)}{\sqrt{1-y^2}} dy. \tag{9.15}$$

Finally, we multiply the equation (9.14) by $\epsilon_m^j(x)$ and integrate over x to obtain

$$\frac{\pi}{2-\delta_{j,0}} \mu_m \int_{-1}^{1} \epsilon_m^j(x) \frac{d\epsilon_m^j(x)}{dx} dx + \eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x) \epsilon_m^l(x) dx$$

$$+\frac{\pi}{2-\delta_{j,0}}\sigma_t \int_{-1}^{1} \left[\epsilon_m^j(x)\right]^2 dx \tag{9.16}$$

$$= \frac{\pi}{2-\delta_{j,0}}\sigma_s \sum_{n=1}^{M} \omega_n \int_{-1}^{1} \epsilon_m^j(x)\epsilon_n^j(x)dx + \frac{\pi}{2-\delta_{j,0}}\sigma_s \int_{-1}^{1} \epsilon_m^j(x)\tau^j(x)dx$$

Now we rewrite the first term in equation (9.16) as

$$\mu_m \int_{-1}^{1} \epsilon_m^j(x)\frac{d\epsilon_m^j(x)}{dx}dx = \frac{\mu_m}{2}\left[(\epsilon_m^j(-1))^2 - (\epsilon_m^j(1))^2\right]. \tag{9.17}$$

Note that $(\mu_m(\epsilon_m^j(1))^2 - (\epsilon_m^j(-1))^2) > 0$. Indeed, for $\mu_m > 0$, using the boundary condition $\epsilon_m(-1, y) = 0$ and the identity

$$\epsilon_m^j(x) = \frac{2-\delta_{j,0}}{\pi} \int_{-1}^{1} \epsilon_m(x,y)T_j(y)\frac{1}{\sqrt{1-y^2}}dy, \tag{9.18}$$

we find that $\epsilon_m^j(-1) = 0$. The same is valid for $x = 1$, when $\mu_m < 0$. Consequently,

$$\frac{2-\delta_{j,0}}{\pi}\eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x)\epsilon_m^l(x)dx + \sigma_t \int_{-1}^{1} \left[\epsilon_m^j(x)\right]^2 dx$$

$$\leq \sigma_s \sum_{n=1}^{M} \omega_n \int_{-1}^{1} \epsilon_m^j(x)\epsilon_n^j(x)dx + \sigma_s \int_{-1}^{1} \epsilon_m^j(x)\tau^j(x)dx \tag{9.19}$$

To proceed we multiply the inequality (9.19) by ω_m and sum over m to obtain

$$\sigma_t \int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x)\right]^2 dx \leq \sigma_s \int_{-1}^{1} \left[\sum_{n=1}^{M} \omega_n \epsilon_m^j(x)\right]^2 dx$$

$$+\sigma_s \int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x)\right] \tau^j(x)dx \tag{9.20}$$

$$-\frac{2-\delta_{j,0}}{\pi} \sum_{m=1}^{M} \omega_m \left[\eta_m \sum_{l=0}^{L} \gamma_j(l) \int_{-1}^{1} \epsilon_m^j(x)\epsilon_m^l(x)dx\right]$$

$$:= I + II + III.$$

The crucial part is now to estimate the $\gamma-$term III using the elementary properties of the Chebyshev polynomials. We start with the simpler terms I and II :

Lemma 9.3. *With $\omega_m \sim 4\pi/M$, $m = 1, ..., M$, we have, for $j = 0, ..., L$, that*

$$\mid I \mid \leq 4\pi\sigma_s \frac{2-\delta_{j,0}}{\pi} \left\|\epsilon^j(x)\right\|^2$$

$$\mid II \mid \leq \sqrt{4\pi}\sigma_s \frac{2-\delta_{j,0}}{\pi} \left\|\epsilon^j(x)\right\| \left\|\tau^j\right\| \tag{9.21}$$

Proof. We use the elementary relation

$$(a_1 + a_2 + ... + a_M)^2 \leq M\left(a_1^2 + a_2^2 + ... + a_M^2\right),$$

to write

$$\left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x)\right]^2 \leq M \max_{1\leq m\leq M} \mid \omega_m \mid \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x)\right]^2. \tag{9.22}$$

integrating (9.22) over x and using $\omega_m \sim 4\pi/M$ we get

$$\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x)\right]^2 dx \leq 4\pi \int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left[\epsilon_m^j(x)\right]^2 dx, \tag{9.23}$$

and hence the first estimate follows recalling (9.11). As for the second estimate, applying the Cauchy-Schwarz inequality, (9.23), (9.11) and (9.12) we get

$$\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x)\right] \tau^j(x) dx$$

$$\leq \left(\int_{-1}^{1} \left[\sum_{m=1}^{M} \omega_m \epsilon_m^j(x)\right]^2 dx\right)^{1/2} \times \left(\int_{-1}^{1} \left|\tau^j(x)\right|^2 dx\right)^{1/2} \tag{9.24}$$

$$\sqrt{4\pi}\left[\int_{-1}^{1} \sum_{m=1}^{M} \omega_m \left(\epsilon_m^j(x)\right)^2 dx\right]^{1/2} \times \sqrt{\frac{\pi}{2-\delta_{j,0}}} \left\|\tau^j\right\|$$

$$\leq \sqrt{4\pi}\frac{\pi}{2-\delta_{j,0}}\left\|\epsilon^j\right\|\left\|\tau^j\right\|,$$

which gives the desired estimate for II and the proof is complete.

Next using the proposition 3 from the Appendix I we estimate the contribution from the γ-term III and derive the following key estimate:

Proposition 9.4. *For $k = 0, 1, 2, ..., L$, we have the recursive estimates*

$$\left\|\epsilon^{L-k}\right\| \leq \sum_{j=0}^{k} \frac{\left(1-(-1)^{j+k}\right)}{\sigma}(L-j)\left\|\epsilon^{L-j}\right\| + \frac{\sqrt{4\pi}\sigma_s}{\sigma}\left\|\epsilon^{L-k}\right\|. \tag{9.25}$$

Hence, in particular the starting estimate, for $k = 0$, is:

$$\left\|\epsilon^L\right\| \leq \frac{\sqrt{4\pi}\sigma_s}{\sigma}\left\|\tau^L\right\|. \tag{9.26}$$

With these estimates we can easily prove our main result:

Proof of Theorem 9.1. Proposition 9.4 and Theorem 9.2 give the desired result.

Proof of Theorem 9.4. By the Proposition 3 (see Appendix I) we have that

$$\gamma_j(l) = 0, \qquad \text{for} \quad j \geq l, \tag{9.27}$$

whereas for $j \leq l$,

$$\gamma_j(l) = \begin{cases} 0 & \text{for } j+l \quad \text{even} \\ l\pi & \text{for } j+l \quad \text{odd.} \end{cases} \tag{9.28}$$

Therefore if we start with $j = L$, then $\gamma_j(L) = 0$ and hence (9.20) combined with the definition (9.11) and Lemma 9.3 yields

$$\sigma_t\frac{\pi}{2}\left\|\epsilon^L\right\|^2 \leq 4\pi\sigma_s\frac{\pi}{2}\left\|\epsilon^L\right\|^2 + \sqrt{4\pi}\left\|\epsilon^L\right\|\left\|\tau^L\right\|. \tag{9.29}$$

Now rearranging the terms and recalling that $\sigma := \sigma_t - 4\pi\sigma_s$ we obtain (9.26).

The proof of (9.25) is a reversed inductive argument as follows:

For $j = L-1$ we have that $\gamma_j(L) = \gamma_{L-1}(L) = L\pi$, whereas $\gamma_{L-1}(l) = 0$, for $l < L$.

Hence, using (9.27) we get

$$\sum_{l=0}^{L} \gamma_j(l)\epsilon_m^l(x) = \sum_{l=0}^{L} \gamma_{L-1}(l)\epsilon_m^l(x) = \gamma_{L-1}(L)\epsilon_m^L(x) = L\pi\epsilon_m^L(x). \tag{9.30}$$

Thus using the Cauchy-Shwarz inequality

$$\begin{aligned}
\mid III \mid = \mid & -\frac{2-\delta_{j,0}}{\pi}\sum_{m=1}^{M}\omega_m\left[\eta_m\int_{-1}^{1}\sum_{l=0}^{L}\gamma_{L-1}(l)\epsilon_m^l(x)\epsilon_m^{L-1}(x)dx\right] \mid \\
& \le \frac{2}{\pi}L\pi\int_{-1}^{1} \mid \sum_{m=1}^{M}\eta_m\omega_m\epsilon_m^L(x)\epsilon_m^{L-1}(x) \mid dx \\
& \le 2L(\max_m \mid \eta_m \mid)\left[\int_{-1}^{1}\sum_{m=1}^{M}\omega_m\left[\epsilon_m^L(x)\right]^2 dx\right]^{1/2} \times \qquad (9.31) \\
& \left[\int_{-1}^{1}\sum_{m=1}^{M}\omega_m\left[\epsilon_m^{L-1}(x)\right]^2 dx\right]^{1/2} \\
& 2L\sqrt{\frac{\pi}{2}}\left\|\epsilon^L\right\|\sqrt{\frac{\pi}{2}}\left\|\epsilon^{L-1}\right\| = L\pi\left\|\epsilon^L\right\|\left\|\epsilon^{L-1}\right\|.
\end{aligned}$$

Inserting in (9.20) and using also (9.11) and Lemma 9.3, with $j = L-1$, we get

$$\begin{aligned}
\sigma_t\frac{\pi}{2}\left\|\epsilon^{L-1}\right\|^2 \le 4\pi\sigma_s\frac{\pi}{2}\left\|\epsilon^{L-1}\right\|^2 &+ \sqrt{4\pi}\sigma_s\frac{\pi}{2}\left\|\epsilon^{L-1}\right\|\left\|\tau^{L-1}\right\| \\
&+ L\pi\left\|\epsilon^L\right\|\left\|\epsilon^{L-1}\right\|,
\end{aligned} \tag{9.32}$$

or equivalently using the notation $\sigma = \sigma_t - 4\pi\sigma_s$,

$$\sigma\left\|\epsilon^{L-1}\right\| \le 2L\left\|\epsilon^L\right\| + \sqrt{4\pi}\sigma_s\left\|\epsilon^{L-1}\right\|. \tag{9.33}$$

The same procedure applied to $j = L-2$ yields $\gamma_j(L) = \gamma_{L-2}(L) = 0$, (note that here

$j+L$ is even), $\gamma_{L-2}(L-1)=(L-1)\pi$ and $\gamma_{L-2}(l)=0$, for $l<L-1$. Thus

$$\sum_{l=0}^{L}\gamma_{L-2}(l)\epsilon_m^l(x)=\gamma_{L-2}(L-1)\epsilon_m^{L-1}(x)=(L-1)\pi\epsilon_m^{L-1}(x), \tag{9.34}$$

so that, as in the previous step

$$\sigma\left\|\epsilon^{L-2}\right\|\leq 2(L-1)\left\|\epsilon^{L-1}\right\|+\sqrt{4\pi}\sigma_s\left\|\tau^{L-2}\right\|. \tag{9.35}$$

Similarly since for $j=L-3$; we have $\gamma_{L-3}(L)=L\pi$, $\gamma_{L-3}(L-1)=0$, $\gamma_{L-3}(L-2)=(L-2)\pi$ and $\gamma_{L-3}(l)=0$ for $l<L-2$, we get

$$\sum_{l=0}^{L}\gamma_{L-3}(l)\epsilon_m^l(x)=\gamma_{L-3}(L-2)\epsilon_m^{L-2}(x)+\gamma_{L-3}(L)\epsilon_m^L(x)$$

$$=2(L-2)\epsilon_m^{L-2}(x)+2L\epsilon_m^L(x), \tag{9.36}$$

which using the same procedure as before yields

$$\sigma\left\|\epsilon^{L-3}\right\|\leq 2L\left\|\epsilon^{L}\right\|+2(L-2)\left\|\epsilon^{L-2}\right\|+\sqrt{4\pi}\sigma_s\left\|\tau^{L-3}\right\|. \tag{9.37}$$

Now the formula (9.25) is proved by an induction argument.

Chapter 10

The Quadrature Rule and Proof of Theorem 9.2.

In this section we construct a special quadrature mesh satisfying the conditions in (9.5) and prove the Theorem 9.2 in this setting. This would provide us the remaining step in the proof of the Theorem 9.1 and complete the convergence analysis. We also derive convergence rates for the quadrature error (9.7) where we identify the angular domain

$$D = \{(\mu, \theta): \quad -1 \leq \mu \leq 1, \quad 0 \leq \theta \leq 2\pi\}, \tag{10.1}$$

by

$$\widetilde{D} = \Big\{(\mu, \eta): \quad -1 \leq \mu, \eta \leq 1, \quad \eta = \sqrt{1-\mu^2}\cos\theta.\Big\} \tag{10.2}$$

Then the quadrature (cubature) rule, for the multiple integral in (9.1) can be constructed using (10.2) as in (9.7), see [25]. To derive convergence rates, below we construct an equivalent rule, directly discretizing D given by (10.1), and with a somewhat general features:

$$\int_0^{2\pi}\int_{-1}^{1} \Psi(\mathbf{x}, \mu, \theta) d\mu d\theta \sim \sum_{\Delta} \omega_{kj} \Psi(\mathbf{x}, \mu, \theta), \tag{10.3}$$

where $\Delta := \{(\mu_k, \theta_j),\ k = 1, ..., K \text{ and } j = 1, ..., J,\ J \sim K\} \subset D$ is a $M = JK$, discrete set of points in D consisting of the Gauss quadrature points $\mu_k \in [-1, 1]$ associated with the equally spaced $\theta_j = \frac{2\pi}{J}$, $j = 1, ..., J$, and weights $\omega_{kj} = A_k W_j$ where $W_j = \frac{2\pi}{J}$, $j = 1, ..., J$, and A_k are given below. Thus the error in (10.3) can be split into two decoupled quadrature error:

$$
\begin{aligned}
\mid e_M(\Psi) \mid :=& \mid \int_0^{2\pi} \int_{-1}^{1} \Psi(\mathbf{x}, \mu, \theta) d\mu d\theta - \sum_{\Delta} \omega_{kj} \Psi(\mathbf{x}, \mu_k, \theta_j) \mid \\
\leq& \int_0^{2\pi} \mid \int_{-1}^{1} \Psi(\mathbf{x}; \mu, \theta) d\mu - \sum_{k=1}^{K} A_k \Psi(\mathbf{x}, \mu_k, \theta) \mid d\theta \\
&+ \sum_{k=1}^{K} A_k \left[\mid \int_0^{2\pi} \Psi(\mathbf{x}, \mu_k, \theta) d\theta - \sum_{j=1}^{J} W_j \Psi(\mathbf{x}, \mu_k, \theta_j) \mid \right] \\
:=& \int_0^{2\pi} \mid e_K [\Psi(\mathbf{x}; \theta)] \mid d\theta + \sum_{k=1}^{K} A_k \mid e_J [\Psi(\mathbf{x}, \mu_k)] \mid,
\end{aligned}
\tag{10.4}
$$

with the obvious notations for the two quadrature errors:

$$
e_J [\Psi(\mathbf{x}; \mu)] := \int_0^{2\pi} \Psi(\mathbf{x}, \mu, \theta) d\theta - \sum_{j=1}^{J} W_j \Psi(\mathbf{x}, \mu, \theta_j), \tag{10.5}
$$

$$
e_K [\Psi(\mathbf{x}; \mu)] := \int_{-1}^{1} \Psi(\mathbf{x}; \mu, \theta) d\mu - \sum_{k=1}^{K} A_k \Psi(\mathbf{x}, \mu_k, \theta), \tag{10.6}
$$

Below we derive error estimates for the quadrature rules (10.5) and (10.6), with optimal convergence rates with respect to the assumed regularity of Ψ in μ and θ.

Lemma 10.1. *Let $e_J [\Psi]$ denote the error in (10.5), with J equally spaced quadrature points $\theta_j \in [0, 2\pi]$. Suppose that $\mid \frac{\partial^r \Psi(\mathbf{x}, \mu, \theta)}{\partial \theta^r} \mid$ is integrable on $[0, 2\pi]$, then*

$$
\mid e_J [\Psi] \mid \leq \frac{C_r}{J^r} \int_0^{2\pi} \mid \frac{\partial^r \Psi(\mathbf{x}, \mu, \theta)}{\partial \theta^r} \mid d\theta, \tag{10.7}
$$

where C_r is independent of J and Ψ.

Lemma 10.2. *Let $e_K[\Psi]$ denote the error on K-point Gaussian quadrature approximation of the integral of Ψ on $\mu \in [-1,1]$. Suppose that $(1-\mu^2) \mid \frac{\partial^r \Psi(\mathbf{x},\mu,\theta)}{\partial \theta^r} \mid$ is integrable on $[-1,1]$, then*

$$\mid e_K[\Psi] \mid \leq \frac{C_s}{K^s} \int_{-1}^{1} \mid \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} \mid .(1-\mu^2)^{s/2} d\mu, \tag{10.8}$$

where C_s is independent of K and Ψ.

We postpone the proofs of these lemmas and first derive of the proof of Theorem 9.2 from them. For the transport equation (9.1), in polygonal domains, the regularity requirements in the lemmas 10.1 and 10.2 are proved for $r = s = 1$ in [2]:

Proposition 4.3. *Let $\frac{\partial \Psi}{\partial \theta} \in L_1[0,2\pi]$ and $\frac{\partial \Psi}{\partial \mu} \in L_1^{\widetilde{\omega}}[0,2\pi]$, where $\widetilde{\omega} := (1-\mu^2)^{1/2}$. Then for the quadrature error $\tau(\mathbf{x})$ of the approximation (4.3) we have,*

$$\|\tau\|_{L_2(\Omega)} \leq C \left(\frac{1}{J} + \frac{1}{K} \right) \|g\|_{H(\Omega)}, \tag{10.9}$$

where g is the right hand side of (9.1), i.e. $g = \sigma_s \widetilde{\Psi} + S$ with $\widetilde{\Psi} = \int_{-1}^{1} \int_0^{2\pi} \Psi$, and $H^1(\Omega)$ is the usual L_2-based Sobolev space of order one on Ω.

Now we are ready to derive our final error estimate:

Proof of Thorem 9.2. We multiply (9.10) by $\frac{T_k(y)}{\sqrt{1-y^2}}$, $k = 0, ..., L$ integrate over $y \in [-1,1]$ and use the Cauchy-Shwarz inequality to get for $l = 0, ..., L$,

$$\tau^l(x) = \frac{2-\delta_{l,o}}{\pi} \int_{-1}^{1} \tau(\mathbf{x}) \frac{T_l(y)}{\sqrt{1-y^2 dy}}$$

$$\leq \frac{2-\delta_{l,o}}{\pi} \left[\int_{-1}^{1} \tau(\mathbf{x})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}} \right]^{1/2} \left[\int_{-1}^{1} \tau(\mathbf{y})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}} \right]^{1/2} \tag{10.10}$$

$$= \left[\frac{2-\delta_{l,o}}{\pi} \int_{-1}^{1} \tau(\mathbf{x})^2 \frac{T_l(y)}{\sqrt{1-y^2 dy}} \right]^{1/2}.$$

Now recalling (9.12) it follows that

$$\|\tau\| \leq \frac{2-\delta_{l,o}}{\pi}\left[\int_{-1}^{1}\int_{-1}^{1}\tau(\mathbf{x})^2\frac{dy}{\sqrt{1-y^2dy}}dx\right]^{1/2} \leq C\,\|\tau\|_{L_2(\Omega)}\,. \tag{10.11}$$

Combining with (10.9), recalling also $M \sim J^{1/2} \sim K^{1/2}$ we get the desired result.

Remark. The convergence rate in the Lemmas 10.1 and 10.2, as well as the rates in Proposition 10.3, can be improved up to the optimal order $\mathcal{O}(J^{2-\epsilon}) \sim \mathcal{O}(K^{2-\epsilon})$, ϵ arbitrarily small, for the neutron transport equation, in polygonal domains using, e.g., post processing procedure cf. Asadzadeh [3].

Now it remains to verify the estimates in Lemmas 10.1-10.2.

Proof of Lemma 10.1. We assume that Ψ is 2π-periodic in θ and in the quadrature formula

$$\int_0^{2\pi}\Psi(\mathbf{x},\mu,\theta)d\theta \sim \sum_{j=1}^{J}W_j\Psi(\mathbf{x},\mu,\theta_j), \tag{10.12}$$

approximate Ψ by trigonometric polynomials in θ. Then we can easily check that: on matter how we choose the quadrature points θ_j and weights W_j, the formula (10.12) can not be exact for trigonometric polynomials of degree J, (see, e.g., [42] for the details). It turns out that the highest degree of precision $J-1$ is achieved just for our simplest quadrature formula: equally spaced nodes $\theta_j = \frac{2\pi j}{J}$ and constant weights $W_j = \frac{2\pi}{J}$, $j = 1, 2, ..., J$. Thus we have

$$\int_0^{2\pi}\Psi(\theta)d\theta \sim \frac{2\pi}{J}\sum_{j=1}^{J}\Psi\left((j-1)\frac{2\pi}{J}\right). \tag{10.13}$$

We can easily verify (10.13) is exact for the functions $e^{imx}, m = 0, 1, ..., J-1$. Further a trigonometric polynomial of degree J, with the Fourier series expansion

$$T_j(x) \equiv \frac{a_0}{2} + \sum_{j=1}^{J}(a_j\cos jx + b_j\sin jx), \tag{10.14}$$

having $2J+1$ degrees of freedom $(a_0, a_j, b_j, j = 1, ..., J)$ corresponds to an algebraic polynomials of degree $2J$. Thus (10.13) is exact for algebraic polynomials of degree $2J-1$, so that for $\Psi \in C^r[0, 2\pi]$, $r = 2J$, (Ψ is $2J$ times continuously differentiable in θ), using Taylor expansion up to degree $2J-1$, in both sides of (10.12), we obtain the desired result.

Lemma 10.2 is a special case of the classical result due to DeVore and Scott (Theorem 3 in [23], Proposition 10.4 below): Consider, for positive integer s, the function space

$$\Psi \in Y_\omega^s := \left\{ u \in L^1_{loc}(]-1,1[) : \|u\|_{\omega,s} < \infty \right\} \tag{10.15}$$

with ω being a weight function and

$$\|u\|_{\omega,s} = \int_{-1}^{1} \left[| u(\mu) | + | u^{(s)}(\mu) | (1-\mu^2)^s \right] \omega(\mu) d\mu, \tag{10.16}$$

where $u^{(s)}$ is interpreted as a weak derivative.

Proposition 10.4. (DeVore and Scott). *Let $e_K[\Psi]$ denote the error in K-point Gaussian quadrature approximation of the integral of Ψ on $[-1,1]$. Suppose that $(1-\mu^2)^s \mid \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} \mid$ (weak derivative) is integrable on $[-1,1]$, i.e., $\Psi \in Y_1^s$, where s is any positive integer such that $1 \leq s \leq 2K$. Then*

$$| e_K[\Psi] | \leq C_s \int_{-1}^{1} | \frac{\partial^s \Psi(\mathbf{x},\mu,\theta)}{\partial \mu^s} | \min \left\{ \left[\frac{\sqrt{1-\mu^2}}{K} \right]^s, (1-\mu^2)^s \right\} d\mu, \tag{10.17}$$

where C_s is independent of K and Ψ.

Proof of Lemma 10.2. This follows, evidently, from the Proposition 10.4.

Below we review a procedure, based on analyzing the Peano kernel for the quadrature error (10.6), and establish the bound (10.8) for $s = 1$, see [2] or [23]. This would suffices to justify the use of Proposition 10.3. The full proof of (10.8), or (10.17), for $s \geq 1$ is treated as in [23]. Consider the Gauss quadrature rule

$$\int_{-1}^{1} \Psi(\mathbf{x};\mu,\theta)d\mu \sim \sum_{k=1}^{K} A_k \Psi(\mathbf{x},\mu_k,\theta), \tag{10.18}$$

where

$$\mu_k := -\cos\alpha_k, \quad \alpha_k \in \left[\frac{(2k-1)\pi}{2K+1}, \frac{2k\pi}{2K+1},\right], \quad k = 1,...K, \tag{10.19}$$

are zeros of Legendre polynomials and

$$A_k := \int_{-1}^{1} \prod_{l\neq k} \frac{x - x_l}{x_k - x_l} dx, \quad k = 1,...K, \tag{10.20}$$

are the integrals of the associated Lagrange interpolation polynomials. Now using the Peano kernel theorem we can write

$$e_K[\Psi] = \int_{-1}^{1} \Lambda(\zeta)\Psi'(\zeta)d\zeta, \tag{10.21}$$

where $\Lambda(\zeta) = e_K[H_\zeta]$, $\mid \zeta \mid \leq 1$ and H_ζ is the Heaviside function

$$H_\zeta(\mu) := \begin{cases} 0, & \mu < \zeta, \\ 1, & \mu \geq \zeta \end{cases} \tag{10.22}$$

it follows that

$$\Lambda(\zeta) = 1 - \zeta - \sum_{\mu_k > \zeta} A_k = \sum_{\mu_k < \zeta} A_k - \zeta - 1. \tag{10.23}$$

By the Chebyshev-Markov-Stieltjes (cf. [51] p. 50) inequality we have

$$1 + \mu_k \leq \sum_{k=1}^{K} A_i \leq 1 + \mu_{k+1}, \quad k = 1,...,K. \tag{10.24}$$

Thus with $-1 = \mu_0 < \mu_1 < ... < \mu_K < \mu_{K+1} = 1$ we get for $k = 1, ..., K$ that

$$\mu_{k-1} - \mu_k \leq \Lambda(\mu_{k^-}) \leq 0 \leq \Lambda(\mu_{k^+}) \leq \mu_{k+1} - \mu_k. \tag{10.25}$$

Since Λ vanishes on each interval $\left[\mu_{k-1}, \mu_k\right]$ and has the slope one almost everywhere, we have

$$\max\left\{| \Lambda(\mu) | : \mu \in \left[\mu_{k-1}, \mu_k\right]\right\} \leq \mu_k - \mu_{k-1}, \qquad k = 1, ..., K. \tag{10.26}$$

To bound $\mu_k - \mu_{k-1}$, we define $I_k := [\alpha_{k-1}, \alpha_k]$, then

$$\mu_k - \mu_{k-1} = \cos\alpha_{k-1} - \cos\alpha_k = \int_{\alpha_{k-1}}^{\alpha_k} \sin\alpha d\alpha$$

$$\leq (\alpha_k - \alpha_{k-1}) \max_{\alpha\in I_k} \{\sin\alpha\} \leq \frac{3\pi}{2K} \max_{\alpha\in I_k} \{\sin\alpha\}. \tag{10.27}$$

Now since $(\sin\alpha)/\alpha$ is decreasing in $[0, \pi]$, using (10.19) we get

$$\sin\alpha \leq \left[\frac{\alpha}{\alpha_{k-1}}\right] \sin\alpha_{k-1} \leq \left[\frac{\alpha_k}{\alpha_{k-1}}\right] \sin\alpha_{k-1} \leq 4\sin\alpha_{k-1}, \qquad \alpha \in I_k, \tag{10.28}$$

for $k = 2, ..., K$. By the symmetry properties of α_j (cf. [51]) we also get

$$\sin\alpha \leq 4\sin\alpha_{k-1}, \qquad \alpha \in I_k, \qquad k = 2, ..., K. \tag{10.29}$$

Thus for $k = 2, ..., K$,

$$\max_{\alpha\in I_k} \{\sin\alpha\} \leq 4 \min_{\alpha\in I_k} \{\sin\alpha_{k-1}\} \equiv 4 \min_{\alpha\in I_k} \left\{\sqrt{1 - \cos^2\alpha}\right\}. \tag{10.30}$$

Hence, combining (10.27) and (10.30), and using (10.19) we have for $k = 2, ..., K$,

$$\mu_k - \mu_{k-1} = \frac{6\pi}{K} \min_{\alpha\in I_k} \left\{\sqrt{1 - \cos^2\alpha}\right\} = \frac{6\pi}{K} \min_{\alpha\in I_k} \left\{\sqrt{1 - \mu^2}\right\}. \tag{10.31}$$

Thus, by (10.26), for $\mu \in [\mu_1, \mu_N]$,

$$| \Lambda(\mu) | \leq \frac{6\pi\sqrt{1-\mu^2}}{K}. \tag{10.32}$$

The corresponding estimate for $\mu \in [-1, \mu_1]$ and $\mu \in [\mu_1, 1]$ is (see [23]):

$$| \Lambda(\mu) | \leq \frac{\pi\sqrt{1-\mu^2}}{\sqrt{2}K}. \tag{10.33}$$

Summing up we have shown

$$| e_K [\Psi] | \leq \frac{6\pi}{K} \int_{-1}^{1} | \frac{\partial \Psi}{\partial \mu} | . \sqrt{1-\mu^2} d\mu. \tag{10.34}$$

This proves (10.8) for $s = 1$. For further details we refer to [2] and [23].

Chapter 11

Concluding remarks.

We believe that the idea of using the spectral method for searching solutions to the multidimensional transport problems, in addition to leads us to a solution for all values of the independent variables, is promising for two reasons: first the proposed decompositions reduce the solution of the multidimensional problem into a set of one-dimensional ones that have well established deterministic solutions. Furthermore, in the framework of the analytical solution it may be possible to study and to prove the convergence, that implies numerical stability, and the estimation of the error for the proposed solution. Of course the question left to be answered concerns the investigation of the approximating basis functions to be considered in the expansion as well as other aspects like computational implementations and performances. This is an important issue to be investigated from now. In regard to that, just some preliminary results were obtained by application of the spectral method.

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Part IV

APPENDIX I: Elementary properties of Chebyshev polynomials and Legendre polynomials

Chebyshev polynomials are weighted orthogonal polynomials defined by

$$T_n(x) = \cos(n \ \arccos(x)), \tag{11.1}$$

with the weight function $\omega(x) = \frac{1}{\sqrt{1-x^2}}$. Thus Chebyshev polynomials are a subclass of Jacobi polynomials, where the Jacobi weights $\omega_J = (1+x)^a(1-x)^b$, $a, b > -1$ are restricted to $a = b = -1/2$. It follows that

$$\int_{-1}^{1} T_i(x)T_j(x)\omega(x)dx = \begin{cases} 0 & i \neq j \\ \pi/2 & i = j \neq 0 \\ \pi & i = j = 0. \end{cases} \tag{11.2}$$

Hence

$$\| T_i \|_\omega = \frac{\pi}{2 - \delta_{i,0}}, \qquad i = 0, 1, ... \tag{11.3}$$

$T_n(x)$ is a polynomial of degree n, orthogonal to all polynomials of degree$\leq n-1$. On differentiating $T_n(x) = \cos n\beta$ with respect to $x(= \cos\beta)$ we obtain a polynomial of degree $n-1$ called the *Chebyshev polynomials of second kind:*

$$U_{n-1} = \frac{1}{n}T_n'(x) = \frac{\sin n\beta}{\sin\beta}, \qquad x = \cos\beta. \tag{11.4}$$

Further we can easily verify the following properties (see [47] for the details):

For even (odd) n only even (odd) powers of x occur in $T_n(x)$.

$$T_n(-x) = (-1)^n T_n(x). \tag{11.5}$$

$$\frac{1}{2} + T_2(x) + T_4(x) + ... + T_{2k}(x) = \frac{U_{2k}(x)}{2}, \qquad k = 0, 1, ..., \tag{11.6}$$

$$T_1(x) + T_3(x) + ... + T_{2k+1}(x) = \frac{U_{2k+1}(x)}{2}, \qquad k = 0, 1, ..., . \tag{11.7}$$

We have also the recurrence relations for the Chebyshev polynomials.

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \tag{11.8}$$

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0 \tag{11.9}$$

and for the Legendre polynomials

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) - \left[xP_n(x) - P_{n-1}(x)\right]/(n+1) \tag{11.10}$$

Below we formulate and prove the formulae (4.10), (4.11), (4.12).

Proposition 1 *Let*

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \tag{11.11}$$

and

$$P_{l+1}(x) = 2xP_l(x) - P_{l-1}(x) - \left[xP_l(x) - P_{l-1}(x)\right]/(l+1) \tag{11.12}$$

we have for $l > 2$ *and* $k = 2$

$$\alpha^2_{n,l+1} := \frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l} + \alpha^2_{n-1,l}\right] - \frac{l}{l+1}\alpha^2_{n,j-1} \tag{11.13}$$

where

$$\alpha^2_{n,l+1} := \int_{-1}^{1} T_n(\mu)P_{l+1}(\mu)d\mu \tag{11.14}$$

and for $l > 2$ *and* $k = 3$

$$\alpha^3_{n,l+1} = \frac{2l+1}{2l+2}\left[\alpha^3_{n+1,l} + \alpha^3_{n-1,l}\right] - \frac{l}{l+1}\alpha^3_{n,j-1} \tag{11.15}$$

where

$$\alpha^3_{n,l+1} := \int_{-1}^{1} \frac{T_n(\mu)P_{l+1}(\mu)}{\sqrt{1-\mu^2}}d\mu \tag{11.16}$$

proof. for $k = 2$

by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu)+P_l(\mu)T_{n-1}(\mu)\right]-\frac{l}{2\mu\,(l+1)}\left[P_{l-1}(\mu)T_{n+1}(\mu)+P_{l-1}(\mu)T_{n-1}(\mu)\right] \tag{11.17}$$

we can rewrite this equation as

$$\frac{2l+1}{2l+2}\left[P_l(\mu)T_{n+1}(\mu)+P_l(\mu)T_{n-1}(\mu)\right]-\frac{l}{2\mu\,(l+1)}P_{l-1}(\mu)\left[T_{n+1}(\mu)+T_{n-1}(\mu)\right] \tag{11.18}$$

it is known that

$$T_{n+1}(\mu)+T_{n-1}(\mu)=2\mu T_n(\mu) \tag{11.19}$$

after doing some algebraic manipulations and integrating over $\mu \in [-1,1]$ on the resulting equation we get

$$\alpha^2_{n,l+1}=\frac{2l+1}{2l+2}\left[\alpha^2_{n+1,l}+\alpha^2_{n-1,l}\right]-\frac{l}{l+1}\alpha^2_{n,j-1} \tag{11.20}$$

The case $k=3$ is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1,1]$ we get the desired result.

Below we formulate and prove the property that has been essential in deriving the basic estimate in section 9 (proposition 9.3):

Proposition 2 *Let*

$$\gamma_j(l)=\int_{-1}^{1}\frac{dT_l(y)}{dy}(y).\frac{T_j(y)}{\sqrt{1-y^2}}dy, \tag{11.21}$$

we have that

$$\gamma_j(l)=0 \qquad for \quad j\geq l, \tag{11.22}$$

and for $j<l$,

$$\gamma_j(l)=\begin{cases} 0 & j\ +l \quad even \\ l\pi & j+l \quad odd. \end{cases} \tag{11.23}$$

Proof. The first assertion is a trivial consequence of the fact that T_j is orthogonal to all polynomials of degree $\leq j-1$. As for the second assertion we note that

$$T_j'(x) = lU_{l-1}(x).$$

Thus if l is odd then $l-1$ is even, say $l-1=2k$, hence using (11.6)

$$\gamma_j(l) = 2l\int_{-1}^{1}\left[\frac{1}{2}+T_2(x)+T_4(x)+...+T_{l-1}(x)\right].\frac{T_j(y)}{\sqrt{1-y^2}}dy$$

$$= \begin{cases} 0 & j \quad \textit{odd}, \quad \textit{i.e.}, \ j+l \quad \textit{even} \\ 2l\frac{\pi}{2-\delta_{j,0}} & j \quad \textit{even}, \quad \textit{i.e.}, \ j+l \quad \textit{odd}. \end{cases} \tag{11.24}$$

The case l is even is treated similarly and using (11.7) and the proof is complete.

Part V

APPENDIX II: the three-dimensional spectral solution.

We extend now the approach presented in Section 8 to the transport process in three dimension,

$$\mu\frac{\partial}{\partial x}\Psi(\mathbf{x},\mu,\theta)+\sqrt{1-\mu^2}\left[\cos\theta\frac{\partial}{\partial y}\Psi(\mathbf{x},\mu,\theta)+\sin\theta\frac{\partial}{\partial z}\Psi(\mathbf{x},\mu,\theta)\right]$$

$$+\sigma_t\Psi(\mathbf{x},\mu,\theta)=\int_{-1}^{1}\int_{0}^{2\pi}\sigma_s(\mu^{'},\theta^{'}\to\mu,\theta)\Psi(\mathbf{x},\mu^{'},\theta^{'})d\theta^{'}d\mu^{'}+S(\mathbf{x},\mu,\theta) \qquad (11.25)$$

where we assume that the spatial variable $\mathbf{x}:=(x,y,z)$ varies in the cubic domain $\Omega:=\{(x,y,z):-1\leq x,y,z\leq 1\}$, and $\Psi(\mathbf{x},\mu,\theta):=\Psi(x,y,z,\mu,\theta)$ is the angular flux in the direction defined by $\mu\in[-1,1]$ and $\theta\in[0,2\pi]$,

We seek for a solution of (11.25) satisfying the following boundary conditions:

For the boundary terms in x; for $0\leq\theta\leq 2\pi$,

$$\Psi(x=\pm 1,y,z,\mu,\theta)=\begin{cases} f_1(y,z,\mu,\theta),\ x=-1, & 0<\mu\leq 1,\\ 0,\ x=1, & -1\leq\mu<0.\end{cases} \qquad (11.26)$$

For the boundary terms in y and for $-1\leq\mu<1$,

$$\Psi(x,y=\pm 1,z,\mu,\theta)=\begin{cases} f_2(x,z,\mu,\theta),\ y=-1, & 0<\cos\theta\leq 1,\\ 0,\ y=1, & -1\leq\cos\theta<0.\end{cases} \qquad (11.27)$$

Finally, for the boundary terms in z; for $-1\leq\mu<1$,

$$\Psi(x,y,z=\pm 1,\mu,\theta)=\begin{cases} f_3(x,y,\mu,\theta),\ z=-1, & 0\leq\theta<\pi,\\ 0,\ z=1, & \pi<\theta\leq 2\pi.\end{cases} \qquad (11.28)$$

Here we assume that $f_1(y,z,\mu,\phi)$, $f_2(x,z,\mu,\phi)$and $f_3(x,y,\mu,\phi)$are given function.

Expanding the angular flux $\Psi(x, y, z, \mu, \phi)$in a truncated series of Chebyshev polynomials $T_i(y)$and $R_j(z)$ leads to

$$\Psi(x, y, z, \mu, \theta) = \sum_{i=0}^{I} \sum_{j=0}^{J} \Psi_{i,j}(x, \mu, \theta) T_i(y) R_j(z). \tag{11.29}$$

We repeat the procedure in Section 8 and insert $\Psi(x, y, z, \mu, \theta)$ given by (11.29) into the boundary condition in (11.27), for $y = \pm 1$. Multiplying the resulting expressions by $\frac{R_j(z)}{\sqrt{1-z^2}}$ and integrating over z, we get the components $\Psi_{0,j}(x, \mu, \theta)$ for $j = 0, ...J$:

$$\Psi_{0,j}(x, \mu, \theta) = f_2^j(x, \mu, \theta) - \sum_{i=1}^{I} (-1)^j \Psi_{i,j}(x, \mu, \theta); \quad 0 < \cos\theta \leq 1, \tag{11.30}$$

and

$$\Psi_{0,j}(x, \mu, \theta) = -\sum_{i=1}^{I} \Psi_{i,j}(x, \mu, \theta); \quad -1 \leq \cos\theta < 0. \tag{11.31}$$

Similarly, we substitute $\Psi(x, y, z, \mu, \theta)$ from (11.29) into the boundary conditions for $z = \pm 1$, multiply the resulting expression by $\frac{T_i(y)}{\sqrt{1-y^2}}$, $i = 0, ...I$ and integrating over y, to define the components $\Psi_{i,0}(x, \mu, \theta)$: For $-1 \leq x \leq 1$, $-1 < \mu < 1$,

$$\Psi_{i,0}(x, \mu, \theta) = f_3^i(x, \mu, \theta) - \sum_{j=1}^{J} (-1)^j \Psi_{i,j}(x, \mu, \theta); \quad 0 \leq \theta < \pi, \tag{11.32}$$

$$\Psi_{i,0}(x, \mu, \theta) = -\sum_{j=1}^{J} \Psi_{i,j}(x, \mu, \theta); \quad \pi < \theta \leq 2\pi, \tag{11.33}$$

where

$$f_2^\beta(x, \mu, \theta) = \frac{2 - \delta_{0,j}}{\pi} \int_{-1}^{1} f_2(x, z, \mu, \theta) \frac{R_j(z)}{\sqrt{1 - z^2}} dz \tag{11.34}$$

$$f_3^i(x, \mu, \theta) = \frac{2 - \delta_{i,0}}{\pi} \int_{-1}^{1} f_3(x, y, \mu, \theta) \frac{T_i(y)}{\sqrt{1 - y^2}} dy. \tag{11.35}$$

To determine the components $\Psi_{i,j}(x, \mu, \theta)$, $i = 1, ...I$, and $j = 1, ...J$, we substitute $\Psi(x, \mu, \theta)$, from (11.29) into (11.25) and the boundary conditions for $x = \pm 1$. Multiplying the resulting expressions by $\frac{T_i(y)}{\sqrt{1-y^2}} \times \frac{R_j(z)}{\sqrt{1-z^2}}$, and integrating over y and z we obtain $I \times J$ one-dimensional transport problems, viz

$$\mu \frac{\partial \Psi_{i,j}}{\partial x}(x,\mu,\theta) + \sigma_t \Psi_{i,j}(x,\mu,\theta) = G_{i,j}(x;\mu,\theta)$$

$$\int_{-1}^{1}\int_{-1}^{1} \sigma_s(\mu',\theta' \to \mu,\theta)\Psi_{i,j}(x,\mu',\theta')d\theta' d\mu' \tag{11.36}$$

with the boundary conditions

$$\Psi_{i,j}(-1,\mu,\eta) = f_1^{i,j}(\mu,\theta), \tag{11.37}$$

where

$$f_1^{i,,j}(\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}} f_1(y,z,\mu,\theta)dzdy, \tag{11.38}$$

and

$$\Psi_{i,j}(1,-\mu,\theta) = 0, \tag{11.39}$$

for $0 < \mu \leq 1$, and $0 \leq \theta \leq 2\pi$. Finally

$$G_{i,j}(x;\mu,\theta) = S_{i,j}(x,\mu,\theta) -$$

$$\sqrt{1-\mu^2}\left[\cos\theta \sum_{k=i+1}^{I} A_i^k \Psi_{k,j}(x,\mu,\theta) + \sin\theta \sum_{l=j+1}^{J} B_j^l \Psi_{i,l}(x,\mu,\theta)\right], \tag{11.40}$$

with

$$S_{i,j}(x,\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}} S(\mathbf{x},\mu,\theta)dzdy, \tag{11.41}$$

$$A_i^k = \frac{2}{\pi}\int_{-1}^{1} \frac{d}{dy}(T_k(y))\frac{T_i(y)}{\sqrt{1-y^2}}dy \tag{11.42}$$

$$B_j^l = \frac{2}{\pi}\int_{-1}^{1} \frac{d}{dy}(R_l(y))\frac{R_j(z)}{\sqrt{1-z^2}}dz. \tag{11.43}$$

Now, starting from the solution of the problem given by equations (11.36)-(11.43) for $\Psi_{I,J}(x,\mu,\theta)$, we then solve the problems for the other components, in the decreasing order in i and j. Recall that $\sum_{i=I+1}^{I} \cdots = \sum_{j=J+1}^{J} \equiv 0$. Hence, solving $I \times J$ one-dimensional problems, the angular flux $\Psi(\mathbf{x},\mu,\theta)$ is now completely determined through (11.29).

Remark: If we have to deal with different type of boundary conditions, we have to keep in mind that the first components $\Psi_{i,0}(x,\mu,\theta)$ and $\Psi_{0,j}(x,\mu,\theta)$ for $i = 1, ..., I$ and $j = 1, ..., J$ will satisfy one-dimensional transport problems subject to the same of boundary conditions of the original problem in the variable x.

Part VI

APPENDIX III: the SUMUDU transform.

The Sumudu transform is a new integral transform [57], which is a little known and not widely used whose defined for the functions of exponential order. So we consider functions in the set A, defined by

$$A = \left\{ f(t) \mid \exists M, \tau_1, \text{ and/or } \tau_2 > 0, \text{such that } \mid f(t) \mid < Me^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\} \tag{11.44}$$

For a given function in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist, and each may be infinite. Instead of being used as a power to the exponential as in the case of the Laplace transform, the variable u in the Sumudu transform is used to factor the variable t in the argument of the function f. Specifically, for $f(t)$ in A, the Sumudu transform is defined by

$$G(u) = \mathbf{S}\left[f(t)\right] = \begin{cases} \int_0^\infty f(ut)e^{-t}dt, & 0 \leq u \leq \tau_2, \\ \\ \int_0^\infty f(ut)e^{-t}dt, & -\tau_1 \leq u \leq 0. \end{cases} \tag{11.45}$$

Albeit similar in expression, the two parts in the previous definition arise because in the domain of f, the variable t may not change sign. For further details and properties of Sumudu transform we refer to [10] and [57].

Theorem 3 *Let $n \geq 1$, and let $G_n(u)$ and $F_n(u)$ be the Sumudu and Laplace transform of the nth derivative of $f^{(n)}(t)$, of the function $f(t)$, respectively. Then*

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} \tag{11.46}$$

proof. By definition, the Laplace transform for $f^{(n)}(t)$ is given by

$$F_n(s) = s^n F(u) - \sum_{k=0}^{n-1} s^{n-(k+1)} f^{(k)}(0). \tag{11.47}$$

Therefore

$$F_n\left(\frac{1}{u}\right) = \frac{F(\frac{1}{u})}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-(k+1)}}. \tag{11.48}$$

Now, since $G_k(u) = F_k(1/u)/u$, *for* $0 \leq k \leq m$, *we have*

$$G_n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} = \frac{1}{u^n}\left[G(u) - \sum_{k=0}^{n-1} u^k f^{(k)}(0)\right] \tag{11.49}$$

In particular, this means that the Sumudu transform of the second derivative of the function f *is given by*

$$G_2(u) = \mathbf{S}\left[f^{''}(t)\right] = \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f^{'}(0)}{u^2}. \tag{11.50}$$

Part VII

APPENDIX IV: the TRZASKA'S method.

Trzaska's method consist to compute the inverse of regular matrix $M(s) = sA_0 - A$ called the linear matrix pencil [14] where only A_0 is singular, or both A_0 and A are singular. Nonsingular systems are considered as a particular case of singular systems.

We expand the linear matrix pencil inverse as follows

$$M^{-1}(s) = \frac{D(s)}{d(s)} = \frac{P_1}{s - p_1} + \frac{P_2}{s - p_2} + ... + \frac{P_{n-1}}{s - p_{n-1}} + P_n \tag{11.51}$$

where $d(s) = \det M(s)$ and $D(s)$ =adj $M(s)$ denote the determinant and the adjoint matrix of regular matrix pencil $M(s)$, respectively and $P_1, P_2, ..., P_n$ called the partial matrices.

To develop an efficient formula for the determination of the matrix $D(s)$ and characteristic polynomial $d(s)$, we apply the Cayley-Hamilton theorem to $M(s)$, so that we have

$$M^n(s) + a_1(s)M^{n-1}(s) + ... + a_{n-1}(s)M(s) + a_n(s)I = 0 \tag{11.52}$$

where I and 0 denote the $n \times n$ unit matrix and zero matrix respectively.

It follows from Eq. (11.52) that

$$I = -\frac{1}{a_n(s)} \left[M^n(s) + a_1(s)M^{n-1}(s) + ... + a_{n-1}(s)M(s)\right] \tag{11.53}$$

Premultiplying both sides of Eq .(11.53) by $M^{-1}(s)$, gives

$$M^{-1}(s) = -\frac{1}{a_n(s)} \left[M^{n-1}(s) + a_1(s)M^{n-2}(s) + ... + a_{n-1}(s)I\right] \tag{11.54}$$

This equation states that the inverse of the linear matrix pencil $M(s)$ can be expressed in terms of its successive integer powers of $n - k$ $(k = 1, 2, ..., n)$ orders premutiplied by the corresponding coefficients $a_{k-1}(s), a_0(s) = 1$. The coefficients $a_k(s)$, $(k = 0, 1, ..., n)$ can be represented in the following form

$$a_k(s) = a_{k,k}s^k + a_{k,k-1}s^{k-1} + ... + a_{k,1}s + a_{k,0} \tag{11.55}$$

where $a_{k,l}$ are real numbers with $l = 0, 1, ..., k$ such that

$$a_{k,k} = \frac{1}{k}\text{trace}\left[M_{k,k} + \sum_{l=1}^{k-1} a_{l,l} M_{k-l,k-l}\right], k = 1, 2, ..., n \tag{11.56}$$

$$a_{k,0} = -\frac{1}{k}\text{trace}\left[M_{k,0} + \sum_{l=1}^{k-1} a_{l,0} M_{k-l,0}\right], k = 1, 2, ..., n \tag{11.57}$$

$$a_{k,l} = -\frac{1}{k}\text{trace}\left[M_{k,l} + \sum_{\substack{h=1, j<h \\ q=1, r<q}}^{k-1} a_{h,j} M_{q,r}\right], k = 1, 2, ..., n \tag{11.58}$$

with $h + q = k$ and $j + r = 1 < k$.

The matrix $M_{k,l}$ will be compute by using the Matrix Pascal Triangle [52].

For example, the coefficients of the polynomials $a_3(s)$ can be computed by applying the above rules as follows

$$a_{3,3} = -\frac{1}{3}\text{trace}\left(M_{3,3} + a_{1,1} M_{2,2} + a_{2,2} M_{1,1}\right) \tag{11.59}$$

$$a_{3,2} = -\frac{1}{3}\text{trace}\left(M_{3,2} + a_{1,0} M_{2,2} + a_{1,1} M_{2,1} + a_{2,1} M_{1,1} + a_{2,2} M_{1,0}\right) \tag{11.60}$$

$$a_{3,1} = -\frac{1}{3}\text{trace}\left(M_{3,1} + a_{1,0} M_{2,1} + a_{1,1} M_{2,0} + a_{2,0} M_{1,1} + a_{2,1} M_{1,0}\right) \tag{11.61}$$

$$a_{3,0} = -\frac{1}{3}\text{trace}\left(M_{3,0} + a_{1,0} M_{2,0} + a_{2,0} M_{1,0}\right). \tag{11.62}$$

Moreover the kth power of the linear matrix pencil $M(s)$ can be expressed in the following manner

$$M^k(s) = s^k M_{k,k} + s^{k-1} M_{k,k-1} + ... + s M_{k,1} + M_{k,0} \tag{11.63}$$

tacking into account Eqs. (11.55) and (11.56) we can state that

$$D(s) = s^{n-1} D_{n-1} + s^{n-2} D_{n-2} + ... + s D_1 + D_0 \tag{11.64}$$

and D_k $(k = 0, 1, 2, ..., n-1)$ is a $n \times n$ constant matrix determined by

$$D_k = M_{n-1,k} + \sum_{\substack{h=1, j<h \\ q=1, r<q}}^{k-1} a_{h,j} M_{q,r} \tag{11.65}$$

with $k = 0, 1, 2, ..., n-1$ and $h + q = k$.

The partial matrices $P_1, P_2, ..., P_n$ in expression (11.51) are independent of s and are expressed by

$$P_k = q_k \left[p_k^{n-1} D_{n-1} + p_k^{n-2} D_{n-2} + ... + p_k D_1 + D_0 \right] \tag{11.66}$$

where

$$q_k = - \left[\dot{a}_n \left(p_k \right) \right]^{-1} \qquad \text{with } k = 1, 2, ..., n-1 \tag{11.67}$$

and

$$\dot{a}_n \left(p_k \right) = \left[\frac{d}{ds} a_n(s) \right]_{s=p_k} \tag{11.68}$$

Thus knowing $D(s)$ and $d(s)$ we can easily find the matrices $P_1, P_2, ..., P_n$.

For $k = n$, we have

$$P_n = D_{n-1} \tag{11.69}$$

Thus for all matrices, $P_1, P_2, ..., P_n$ we give the following fundamental equation:

$$\begin{bmatrix} P_1 \\ P_2 \\ . \\ . \\ . \\ P_{n-1} \end{bmatrix} = \begin{bmatrix} q_1 & & & & 0 \\ 0 & q_2 & & & \\ & & . & & \\ & & & . & \\ & & & . & 0 \\ 0 & & & & q_{n-1} \end{bmatrix} \times$$

$$\times \begin{bmatrix} I & p_1 I & . & . & . & p_1^{n-1} I \\ I & p_2 I & . & . & . & p_2^{n-1} I \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ I & p_{n-1} I & & & . & p_{n-1}^{n-1} I \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ . \\ . \\ D_{n-1} \end{bmatrix} \tag{11.70}$$

$$P_n = D_{n-1} \tag{11.71}$$

or in more compact form

$$[P] = [\text{diag } q_k]_1^{n-1} \, [V] \, [D] \, , \;\; P_n = D_{n-1} \tag{11.72}$$

where $[V]$ denotes the Kronecker product of the Vandermonde and unit matrices of appropriate dimensions. For further details we refer to [52].

Part VIII

Illustrative Examples

EXAMPLE 1

Consider the matrices

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

the linear matrix pencil $M(s) = sA_0 - A$ (is regular).

$$M^{-1}(s) = -\frac{1}{a_3(s)} \left[s^2 D_2 + s^1 D_1 + D_0 \right]$$

where

$$a_3(s) = a_{3,3}s^3 + a_{3,2}s^2 + a_{3,1}s + a_{3,0}.$$

Applying (11.60) and the rule of the Matrix Pascal Triangle we obtain

$$D_2 = M_{2,2} + a_{1,1}M_{1,1} + a_{2,2}I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_1 = M_{2,1} + a_{1,1}M_{1,0} + a_{1,0}M_{1,1} + a_{2,1}I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D_0 = M_{2,0} + a_{1,0}M_{1,0} + a_{2,0}I = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & -2 \end{bmatrix}$$

and

$$a_{3,3} = 0, \quad a_{3,2} = 1, \quad a_{3,1} = -3 \quad a_{3,0} = 1,$$

so that

$$a_3(s) = s^2 - 3s + 1 = -d(s).$$

Evaluating the zeros of $a_3(s)$, we obtain $p_1 = \frac{1}{2}(3 + \sqrt{5})$ and $p_2 = \frac{1}{2}(3 - \sqrt{5})$. Now using expression (8.1), we obtain the following partial, fraction expansion of the linear matrix pencil inverse:

$$M^{-1}(s) = \frac{P_1}{s - p_1} + \frac{P_2}{s - p_2} + P_3$$

where by Eqs. (11.70), we have

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \begin{bmatrix} I & p_1 I & p_1^2 I \\ I & p_2 I & p_2^2 I \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} & -\frac{3}{2} - \frac{\sqrt{5}}{2} & -1 \\ 1 & \frac{1}{2} + \frac{\sqrt{5}}{2} & -\frac{1}{2} + \frac{\sqrt{5}}{2} \\ -1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{1}{2} - \frac{\sqrt{5}}{2} & \frac{3}{2} - \frac{\sqrt{5}}{2} & 1 \\ -1 & -\frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 & \frac{1}{2} - \frac{\sqrt{5}}{2} & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$$

and

$$P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 2.

This example deals with a singular system with the following singular matrices A and B

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

The matrix pencil takes the form

$$M(s) = s \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

this matrix pencil is regular, so that by applying (11.55), (11.56), (11.57), (11.58) and (11.54), we obtain

$$a_2(s) = s \quad \text{and} \quad M^{-1}(s) = \frac{P_1}{s} + P_2$$

where

$$P_1 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

EXAMPLE 3.

Consider the nonsingular system where

$$A = I, \qquad B = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

the matrix pencil takes the form

$$M(s) = s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

so that

$$\begin{aligned} M^{-1}(s) &= \frac{1}{d(s)} \left[s^2 D_2 + s D_1 + D_0 \right] \\ &= \frac{P_1}{s - p_1} + \frac{P_2}{s - p_2} + \frac{P_3}{s - p_3} \end{aligned}$$

where $-d(s) = a_3(s) = a_{3,3}s^3 + a_{3,2}s^2 + a_{3,1}s + a_{3,0}$ applying (11.55) and the rule of the Matrix Pascal Triangle we obtain

$$a_{3,3} = -1, \qquad a_{3,2} = 5, \qquad a_{3,1} = -8, \qquad a_{3,0} = 4,$$

Evaluating the zeros of $a_3(s) = -s^3+5s^2-8s+4 = 0 \Longleftrightarrow a_3(s) = -(s-1)(s^2-4s+4) = 0 \Longleftrightarrow s_1 = s_2 = 2;\ s_3 = 1$ so $p_1 = p_2 = 2$ and $p_3 = 1$

$$D_2 = M_{2,2} + a_{1,1}M_{1,1} + a_{2,2}I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D_1 = M_{2,1} + a_{1,1}M_{1,0} + a_{1,0}M_{1,1} + a_{2,1}I = \begin{bmatrix} -2 & -1 & 1 \\ 2 & -5 & 1 \\ 1 & -1 & -3 \end{bmatrix},$$

$$D_0 = M_{2,0} + a_{1,0}M_{1,0} + a_{2,0}I = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 5 & -1 \\ -2 & 2 & 2 \end{bmatrix},$$

we compute the partial fraction matrices P_1, P_2 and P_3 by expression (11.72), we get

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ P_3 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Conclusion:

Nous avons tout d'abord présenté une introduction générale, où nous avons passé en revue les différentes méthodes existantes pour la résolution de l'équation de transport et les différentes interprétations physiques de cette équation.

Notre premier travail fût de résoudre l'équation de transport en dimension 2, dans des milieux anisotropiques, et à l'état stationnaire, moyennant les polynômes de Tchebychev.

Dans un deuxième temps, nous nous sommes intéressés à l'application d'autres méthodes spectrales, où la décomposition du flux angulaire se fait par troncature du développement en série de fonction, nous avons choisi les fonctions de Walsh, combinées avec les transformations de Sumudu pour résoudre analytiquement l'équation de transport en dimension2 et dans des milieux isotropiques.

Nous avons pu voir que nos résultats étaient semblables à ceux existant dans la littérature.

Possédant donc deux méthodes de résolution de l'équation de transport, on s'est intéressé à la convergence spectrale de la solution de l'équation en dimension 2.

Nous avons prouvé cette convergence et obtenu des résultats sur la vitesse de convergence. Nous avons choisi de nous inspirer d'une décomposition spéciale dite quadrature de Gauss. Cette approche est basée sur la décomposition de l'erreur en deux erreurs, où chacune d'elle est majorée, en utilisant les résultats de DeVore et Scott.

Perspectives

Du point de vue des perspectives ouvertes par ces travaux il nous parait raisonnablement envisageable de développer d'autres méthodes de résolutions de l'équation de transport, moyennant les méthodes spectrales et aussi d'envisager le cas multidimensionnel ($n \geq 3$).

Outre cela, nous pouvons envisager le cas numérique, ou encore de travailler sur des domaines à géométrie complexe ou dans des milieux non uniformes.

Les perspectives de telles méthodes sont donc nombreuses et variées.

<u>Résumé.</u>
Il y a beaucoup de littérature disponible qui traite la résolution l'équation de transport stationnaire monodimensionnelle et bidimensionnelle à tel point qu’il nous est pratiquement impossible de les mentionner toutes. Dans leur récent travail Kadem et Asadzadeh ont prouvé la convergence spectrale de la solution de l’équation de transport en dimension 2 puis généralisé en dimension 3 par Kadem moyennant les polynômes de Chebyshev combinée avec une règle de quadrature spéciale. Dans cette thèse nous concentrons notre attention dans cette direction, où nous étudions l'expansion spectrale en polynôme de Walsh combinée avec les transformations de Sumudu pour résoudre analytiquement l'équation de transport neutronique dans des milieux isotropiques en dimension 2.

Mots et phrases clés : équation de transport linéaire, dispersion isotopique, méthode spectrale de Walsh.
Classification AMS 1991: 65N35, 65D32, 82D75, 40A10, 41A50.

<u>Abstract.</u>
There is much literature available regarding the subject of solving the one and two-dimensional steady-state transport equation that it would be impossible to mention all of them. In their recent work Kadem and Asadzadeh are proved the spectral convergence of the solution of transport equation in two-dimensional case and extended by Kadem in three dimensional case using the Chebyshev polynomials combined with special quadrature rule. In this thesis we focus our attention in this direction, we study the spectral Walsh polynomial expansion combined with the Sumudu transform leading to solve, analytically, the neutron transport equation in isotropic two-dimensional media.

Key words and phrases: Linear transport equation, isotropic scattering, Walsh spectral method.
1991 AMS Subject Classification: 65N35, 65D32, 82D75, 40A10, 41A50.

<u>ملخص.</u>
كثيرا من الأبحاث تطرقت و بشكل وافي عن حلول معادلة التنقل ذات البعد1 و2 إلى درجة أنه لا يمكننا سردها كلها.
لقد تطرق قادم و أسدزاده خلال أبحاثهم إلى دراسة تقارب الحلول و تقدير الخطأ باستعمال قاعدة تربيعية خاصة ثم عممت هده النتيجة من طرف قادم إلى البعد 3. في هده الأطروحة نركز أاهتمامنا في هدا الاتجاه حيث نتطرق إلى حل معادلة التنقل ذات البعد 2 تحليليا باستعمال نشر كثير حدود والش مع تحويلات سومودو.

كلمات و جمل فاتحة : تقارب تحليلي- معادلة تنقل خطية- تبديد متجانس- الطريقة الطيفية لواش - طريقة ترتيب الغير المتصل.
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