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Introduction

Various phenomena of modern natural science often lead to nonlocal problems on mathematical modeling, and nonlocal models turn out to be often more precise that local conditions; see [51]. Nonlocal problems form a relatively new division of differential equations theory and generate a need in developing some new methods of research [50]. Nowadays various nonlocal problems for partial differential equations are actively studied and one can find a lot of papers dealing with them; see [53, 59, 63, 30, 67] and references therein. We focus our attention on nonlocal problems with integral conditions for hyperbolic, parabolic and elliptic equations which have been studied in [52, 54, 55, 56, 62, 59, 66, 68, 70, 72, 73, 1, 2, 27, 19, 82, 25, 26, 46]. Systematic studies of nonlocal problems with integral conditions originated with the papers by Cannon [60] and Kamynin [65]. These and further investigations of nonlocal problems show that classical methods most widely used to prove solvability of initial-boundary problems break down when applied to nonlocal problems. Nowadays several methods have been devised for overcoming the difficulties arising because of nonlocal conditions. It appears that conditions for the existence and uniqueness of a solution to the nonlocal problem are closely related to the notion of regular boundary conditions [57, 58, 74]. It is known that the system of root functions of an ordinary differential operator with strongly regular boundary conditions form a Riesz basis in $L_2(0,1)$. This property is particularly useful for obtaining results on solvability of boundary problems. Ordinary differential equations with integral boundary conditions represent a very interesting and important class of problems. They arise in many areas of applied mathematics and physics. Ordinary differential equations on an interval with non localacal conditions relating the solution values at various points of the interval were studied in [75, 83] and other papers. For a detailed survey of the literature on the topic, see [78, 82, 83]. Nonlocal conditions are usually represented in the form of a Stieltjes integral. When considering such conditions, one usually assumes that this integral contain an atomic measure at the endpoints of the interval, which allows one to construct the adjoint operator [77, 78, 81]. In other papers, either the measure was required to be constant sign on some subintervals [79], or some restrictions were imposed on the asymptotic behavior of the weight functions in the integral conditions [80]. The methods of the theory of nonlocal elleptic problems [84] allow one to aviod such restrictions. This approach was used for the firest time in [82].

The manuscript is subdivided into five chapters in addition to an introduction containing a fairly brief presentation and bibliographical review of the work already undertaken on the topic addressed. A plan of the manuscript is also inserted. In the first chapter, we present some necessary concepts related to the half-group theory, which we will use in the second and third chapters. The second chapter is devoted to the study of the boundary problems for an ordinary differential equations of the second order with non-regular conditions, namely the conditions of integral type. We study a second- order ordinary differential operator with a spectral parameter and with integral conditions. On certain conditions imposed on the weight functions appearing in the integral conditions, we obtain a priori estimates for the solutions and prove the unique solvability of the problem for sufficiently large parameter values . In addition, we prove the Fredholm solvability and the spectrum of the posed problem. The tird chapter deal with a coupled system of differential equations with nonlocal conditions. We reduce the problem posed to an abstract first-order differential equation, whose coeficient near the derivative is a generally non-invertible operator. Using the results of chapter one, we establish existence and uniqueness results for the reduced problem. Thus of the studied problem. In the fourth chapter we study a boundary problem for parabolic equation with integral conditions. Such a condition is not regular. The study method is based on the reduction of the posed problem to an equivalent problem for abstract differential equation with integral condition. Thus, the operator induced by the problem posed has a non-dense domain, and not self-adjoint. On certain conditions imposed on the data of the problem posed, one shows that the operator coefficient generates a semigroup with singularity. Then one exploiting the results of the first chapter, one establishes the solvability of the reduced problem, thus the solvability of the treated problem. In the fifth chapter, we are interested in the problem of boundary values with integral condition for a first order abstract differential equation. We use ideas from [10] the goal is the construction of an approximation algorithm of a solution of the problem posed. We present the algorithm as a general approximation scheme, which includes finite element methods and finite difference methods and projection methods.

Chapter 1

Semigroups with non-densely defined generating operator

1.1 Statement of problem

We consider the Cauchy problem

$$u^{'}(t)+Au(t)=f(t) \qquad (0<t\leq T),u(0)=u_0 \tag{1.1.1}$$

for a linear first order differential equation in a Banach space E. Here A is a given linear operator with domain $D\subset E$, $f(t)$, $t>0$ is a given continuous ranging in E and u_0 is a given element of E.

This problem was studied by many authors (see.[47, 48] and references therein) and investigated by semigroup methods. As a rule, strongly continuous and analytic semigroups were used in this case. The operator A for these semigroups turns out to be densely defined, and its resolvent $(A+\lambda I)^{-1}$ satisfies

$$\left\|(A+\lambda I)^{-1}\right\|\leq c|\lambda|^{-r} \tag{1.1.2}$$

with $r=1$ in the complex half-plane $\operatorname{Re} A>\omega>0$ the case of an analytic semigroup.

However, in applications, there are situations when these conditions turn out to be strict. Namely, the operator A may have a domain of definition that is not dense in E, and its resolvent may admit a worse estimate for decay ($r < 1$).

For example, in [18] examples of differential operators in space of continuous functions. If these operators satisfy some boundary conditions, then their domains of definition are not dense in the space under consideration.

In [49] an example is given of the differentiation operator $A = \frac{(-1)^k d^{2k}}{dx^{2k}}$ in the space $L_p(0,1)$ with irregular boundary conditions. In particular, conditions are considered in which some moments of functions from the domain of the operator vanish. In this case, the domain of definition of the operator A is not a dense set in $L_p(0,1)$, and estimate (1.1.2) of the norm of its resolvent holds for $r < 1$, and the corresponding semigroup has a singularity at zero.

1.2 Semigroups with singularities

In this section we present other examples of non-densely defined linear operators A and the corresponding semigroups $U(t)$, these semigroups and their derivatives admit a singularity at zero.

$$\|U(t)\| \leq Mt^{-\alpha} \exp(-\omega t), \qquad \left\|U'(t)\right\| \leq Mt^{-\beta} \exp(-\omega t). \tag{1.2.1}$$

for some $\alpha \geq 0, \beta \geq 1, \omega > 0$ with β is not rigidly connected with α. This distinguishes this class of semigroups of previously studied classes (eg,in [47, 18, 21, 35]). In this paper we consider some properties of such semigroups. We also study the fractional powers of A, and finally considered problem (1.1.1) and the theorem of its solvability. Note that in this theorem the parameter β can be any integer greater than or equal to $1+\alpha, 0 \leq \alpha < 1$. This distinguishes this theory from the corresponding result [49], where the more general case of variable operator coefficient $A = A(t)$, but $\beta < 2$.

Let us give the definition [49] of the class of semigroups used. We denote by $L(E)$ the space of linear bounded operators acting from E to E. We assume that there is an operator function $U(t)$ $(t>0)$ with the following properties:

1- $U(t)$ for every $t>0$-bounded linear operator acting from E to D;

2- $U(t)U(s)=U(t+s) \qquad (t,s>0)$;

3- $U(t)$ is differentiable for $t>0$ in the norm of $L(E)$ and $\frac{d}{dt}U(t)=-AU(t)$;

4- $U(t)Au=AU(t)u$ for $u\in D$, $t>0$;

5- $\lim_{t\to 0^+}U(t)u=u$ for $u\in D$;

6- the estimates (1.2.1) for some $\alpha\geq 0,\beta\geq 1,\omega>0$.

Definition 1.2.1. *An operator-valued function $U(t)$, possessing the properties $1-6$, listed above will be called the semigroup (of class $A(\alpha,\beta)$) generated by operator A.*

Note that the semigroup $U(t)$ is uniquely determined by the operator A. This follows from the fact that the Cauchy problem $u^{'}+Au=0, u(0)=0$ has only the trivial solution in the class of functions $u=u(t)$, admitting for large t the estimate

$$\|u(t)\|\leq M\exp(\omega t)$$

. The indices α and β in (1.2.1) are necessarily related by the inequality $\alpha+1\leq\beta$. Indeed, property 3 °of the semigroup implies the formula

$$U(t)=U(\tau)+\int_t^\tau AU(s)\,ds \qquad (0<t\leq\tau)$$

.

Therefore

$$\begin{aligned}\|U(t)\| &\leq M\tau^{-\alpha}\exp(-\omega\tau)+M\int_t^\tau s^{-\alpha}\exp(-\omega s)\,ds\\ &\leq M\tau^{-\alpha}+\frac{M}{\beta-1}\left[t^{-(\beta-1)}-\tau^{-(\beta-1)}\right]\leq ct^{-(\beta-1)}.\end{aligned}$$

By the condition

$$\|U(t)\| \le Mt^{-\alpha}$$

for small t. Hence, $\alpha \le \beta - 1$. If $\beta = 1$, then $\alpha = 0$, and in what follows we will assume, as before, that the condition $\alpha + 1 \le \beta$.

We introduce the set

$$D_0 = \left\{ x \in E : \exists \lim_{t \to 0^+} \frac{U(t) - I}{t} x \right\}$$

and on this set define the operator

$$A_0 x = \lim_{t \to 0^+} \frac{U(t) - I}{t} x,$$

which is called the generating operator of the semigroup. Use the equality

$$U(t)x - U(s)x = -\int_s^t AU(\tau)x d\tau \qquad (t, s > 0).$$

If $x \in D$, then you can pass to the limit as $s \to 0$:

$$U(t)x - x = -\int_0^t U(\tau)Axd\tau.$$

Therefore, the expression

$$\frac{U(t) - I}{t} x = -\frac{1}{t}\int_0^t U(\tau)Axd\tau$$

has a limit as $t \to 0$, and hence $x \in D_0$, $D \subset D_0$ and $A_0 x = -Ax$ for $x \in D$.

Now let $x \in D_0$. Then equality

$$U(t)\frac{U(s) - I}{s} x = -\frac{1}{s}\int_t^{t+s} AU(\tau)xd\tau$$

One can pass to the limit as $s \to 0$: $U(t)A_0 x = -AU(t)x$. Let the operator A have a bounded inverse operator A^{-1}. Then $U(t)A^{-2}A_0 x = -U(t)A^{-1}x$. One can pass to the limit as $t \to 0$. As a result, we obtain equation $x = -A^{-1}A_0 x$. This means that $x \in D$, $D_0 \subset D$ and $A = -A_0$.

1.3 Examples

We consider some examples of semigroups of class $A(\alpha, \beta)$.

1. If D dense in E $\left(\overline{D} = E\right)$, and $\alpha = 0, \beta = 1$, then $U(t)$-analytical semigroup.

2. Let the operator A acts in the space $L_2(-\infty, +\infty) \times L_2(-\infty, +\infty)$ and given by the formula

$$Au = -D^2 u_1 - iD^k u_1, Dl u_1 - D^2 u_2 - iD^k u_2,$$

where $D = i\frac{d}{dx}, u = \{u_1, u_2\}, k \geq 2, l \geq 2$. Such an operator generates a system of differential equations parabolic in the sense of Shilov [47]. If we apply the Fourier transform with respect to the variable x to both parts of the system and denote the image of the function $u(t, x)$ by $\widetilde{u}(t, p)$, then we come to the system of ordinary differential equations

$$\frac{d\widetilde{u}}{dt} + A(p)\widetilde{u} = 0$$

where

$$A(p) = \begin{bmatrix} p^2 - ip^k & 0 \\ -p^l & p^2 - ip^k \end{bmatrix}.$$

In this case

$$U(t, p) = \begin{bmatrix} \exp\left(\left(-p^2 + ip^k\right)t\right) & 0 \\ tp^l \exp\left(\left(-p^2 + ip^k\right)t\right) & \exp\left(\left(-p^2 + ip^k\right)t\right) \end{bmatrix}$$

is a semigroup of class $A\left(\frac{l}{2} - 1, \frac{l+k}{2} - 1\right)$, since

$$\left|p^l \exp\left(-p^2 t\right)\right| \leq ct^{\frac{-l}{2}}.$$

Resolvent of $A(p)$ has the form

$$(A(p) + \lambda I)^{-1} = \begin{bmatrix} \frac{1}{\lambda + p^2 - ip^k} & 0 \\ \frac{p^l}{\left(\lambda + p^2 - ip^k\right)^2} & \frac{1}{\lambda + p^2 - ip^k} \end{bmatrix}.$$

This shows that for $l > 2k$ operator A has no regular points. If $l \leq 2k$ estimate (1.1.2) holds with $r = \frac{2k - l}{k}$.

3. Let E be a set of sequences $u=\{x_n,y_n\}_1^\infty$, for which the norm

$$\|u\|=\sum_{n=1}^{\infty}\left(n^{\frac{1}{2}}|x_n|+|y_n|\right)$$

. We introduce a subspace $L=\{u\in E: x_1=y_1=0\}$ and define the operator function in E

$$U(t)u=\{0,0;(x_n\cos nt-y_n\sin nt)\exp\left(in^pt-nt\right),(x_n\sin nt+y_n\cos nt)\exp\left(in^pt-nt\right)\}_2^\infty.$$

Here $p\geq 1$ is a parameter. Operator-valued function $U(t)$ is a semigroup of class $A\left(\frac{1}{2},\frac{1}{2}+p\right)$ for any $p\geq 1$. It is generated by the operator A acting according to the rule

$$Au=\{0,0;-(in^p-n)x_n+ny_n,-(in^p-n)y_n-nx_n\}_2^\infty$$

with domain

$$D(A)=\left\{u\in L:\sum_{n=2}^{\infty}\left[n^{\frac{1}{2}}|(in^p-n)x_n-ny_n|+|(in^p-n)y_n+nx_n|\right]<\infty\right\}.$$

For this operator in (1.1.2) $r=2-\frac{3}{2p}$ for $p<\frac{3}{2}$ and $r=1$ for $p\geq\frac{3}{2}$.

4. Let $E=L_p(0,\infty)\cap L_1(0,\infty)$ with the norm $\|u\|=\|u\|_{L_p}+\|u\|_{L_1}$ and A -differential operator $A=\frac{-d^2}{dx^2}$ with domain

$$D(A)=\left\{u(x)\in E,\ \exists u'(x)\in E, \int_0^\infty u(x)\,dx=0\right\}.$$

Then the semigroup generated by this operator is given by the formula

$$U(t)\varphi(x)=\frac{1}{2\sqrt{\pi t}}\int_0^\infty\left[\exp\left(-\frac{(s+x)^2}{4t}\right)+\exp\left(-\frac{(s-x)^2}{4t}\right)\right]\varphi(s)\,ds$$
$$-\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{\pi t}}\int_0^\infty\varphi(s)\,ds.$$

It belongs to the class $A\left(\frac{1}{2}-\frac{1}{2p},\frac{3}{2}-\frac{1}{2p}\right)$, and in (1.2.1) component decreasing resolvent $r=\frac{1}{2}+\frac{1}{2p}$.

Remark 1.3.1. *Note that in Examples 2 and 3 the parameters α and β and r are related to each other by some rigid equalities, in Examples 3 and 4 the domains of definition of the operators under consideration are not dense subsets of E, and in Example 2 the operator has no regular points (for$l > 2k$). Therefore, in what follows we do not use the properties of the resolvent of the operator A, the parameters α and β are not related by any equalities.*

Remark 1.3.2. *Estimate (1.2.1) allow one to construct negative fractional powers of the operator A.*

Suppose that there exists a bounded inverse operator A^{-1}. denote by $D\left(A^{-\rho}\right)$ the set of elements $u \in E$, for which improperly (at zero) the integral $\int\limits_0^\infty s^{\rho-1}U(s)\,ds \qquad (\rho > 0)$. For such u, we put

$$A^{-\rho}u = \frac{1}{\Gamma(\rho)}\int\limits_0^\infty s^{\rho-1}U(s)\,udt. \tag{1.3.1}$$

From this definition it follows easily continuous embedding $D \subset D\left(A^{-\rho}\right)$. If $\rho > \alpha$, then $D\left(A^{-\rho}\right) = E$, and the operators $A^{-\rho}$ are bounded.. For $\rho < \alpha$, as shows an example of [47], these operators may not be bonded.

We establish some properties of fractional powers.

Lemma 1.3.1. *The following equality:*

$$A^{-p}A^{-q} = A^{-(p+q)}, p > 0, q > \alpha; \tag{1.3.2}$$

$$A^{-p}A^{-q} = A^{-q}A^{-p} = A^{-(p+q)}, p, q > \alpha; \tag{1.3.3}$$

$$A^{-p}A^{-q}u = A^{-(p+q)}u = A^{-q}A^{-p}u, p > \alpha, q > 0, u \in D\left(A^{-q}\right). \tag{1.3.4}$$

Proof. Consider the equality

$$\frac{1}{\Gamma(p)}\int_{\varepsilon}^{\infty}\tau^{p-1}U(\tau)\left\{\frac{1}{\Gamma(q)}\int_{\delta}^{\infty}\tau^{q-1}U(s)\,ud\right\}d\tau$$

$$=\frac{1}{\Gamma(p)\Gamma(q)}\int_{\varepsilon+\delta}^{\infty}\left\{\int_{\frac{\varepsilon}{\sigma}}^{1-\frac{\delta}{\sigma}}s^{p-1}(1-s)^{q-1}ds\right\}\sigma^{p+q-1}U(\sigma)\,ud\sigma \tag{1.3.5}$$

In the last expression the inner integral is the limit of the $\delta\to0^{+}$ uniformly in $\sigma\geq\varepsilon>0$. Therefore, (1.3.5) can be put $\delta=0$. We show now that the right-hand side of this equation $(\delta=0)$ there is a limit for $\varepsilon\to0^{+}$. Let $p,q<1$. Then

$$\left\|\frac{1}{\Gamma(p)\Gamma(q)}\int_{0}^{\infty}\left\{\int_{0}^{1}s^{p-1}(1-s)^{q-1}ds\right\}\sigma^{p+q-1}U(\sigma)\,ud\sigma\frac{1}{\Gamma(p)\Gamma(q)}\int_{\varepsilon}^{\infty}\left\{\int_{\frac{\varepsilon}{\sigma}}^{1}s^{p-1}(1-s)^{q-1}ds\right\}\sigma^{p+q-1}U(\sigma)\,ud\sigma\right\|$$

$$=\frac{1}{\Gamma(p)\Gamma(q)}\left\|\int_{0}^{\varepsilon}\left\{\int_{0}^{1}s^{p-1}(1-s)^{q-1}ds\right\}\sigma^{p+q-1}U(\sigma)\,ud\sigma+\int_{\varepsilon}^{\infty}\left\{\int_{0}^{\frac{\varepsilon}{\sigma}}s^{p-1}(1-s)^{q-1}ds\right\}\sigma^{p+q-1}U(\sigma)\,ud\sigma\right\|$$

$$\leq\frac{1}{\Gamma(p+q)}\int_{0}^{\varepsilon}\frac{Md\sigma\|u\|}{\sigma^{1-p-q+\alpha}}+\frac{1}{\Gamma(p)\Gamma(q)}\int_{\varepsilon}^{\infty}\left\{\left(1-\frac{\varepsilon}{\sigma}\right)^{q-1}\int_{0}^{\frac{\varepsilon}{\sigma}}s^{p-1}ds\right\}\frac{M\exp(-\omega\sigma)d\sigma\|u\|}{\sigma^{1-p-q+\alpha}}$$

$$\leq c_1\varepsilon^{p+q-\alpha}\|u\|+c_2\varepsilon^{p}\int_{\varepsilon}^{\infty}\frac{\exp(-\omega\sigma)d\sigma\|u\|}{\sigma^{\alpha}(\sigma-\varepsilon)^{1-q}}\to_{\varepsilon\to0}0,$$

Since the last integral does not exceed

$$\int_{0}^{\infty}\frac{\exp(-\omega\sigma)\,d\sigma}{\sigma^{1-q+\alpha}}.$$

From these estimated follows that the right side of (1.3.5) as $\varepsilon\to0^{+}$tends to $A^{-(p+q)}u$, when $p,q<1$. The case $p<1$, $q\geq1$, or $p\geq1, q<1$, or $p,q\geq1$ are considered analogously. Equation (1.3.2) is proved. Now, if $p>\alpha$, then, interchanging p and q, we obtain the equality $A^{-q}A^{-p}=A^{-(p+q)}$. It follows from (1.3.3). Referring to (1.3.4). If $u\in D\left(A^{-q}\right)$, $p>\alpha$, then the left-hand side of equation (1.3.5) for $\varepsilon\to0^{+}$ there is a limit, when $\delta=0$. Therefore, there is a limit and at the right side. Hence, we obtain the first equality in (1.3.4). The second equality (1.3.4) follows from (1.3.2). The lemma is proved. □

Let $\rho=n$ $(n=1,2,....)$. Then the formula

$$A_n u=\frac{1}{\Gamma(n)}\int_{0}^{\infty}s^{n-1}U(s)\,vds\qquad(\alpha<1)$$

defines a left inverse to the operator A^n, that is $A_n A^n u = u$ for items $u \in D(A^n)$. Since it is assumed the existence of A^{-1}, the left A_n inverse is also a right inverse. Thus, when $\rho = n$; equation (1.3.1) gives the operator inverse to A^n, and $A^{-n} = A_n$. If $\overline{D(A)} = E$, A_n operator that is the right inverse of A^n, and $A^{-n} = A_n$ without assuming the existence of a bounded operator A^{-1}. We define the positive fractional powers of the operator A. Let $A^\rho = \left(A^{-\rho}\right)^{-1}$ $\rho > 0$. This definition is correct. Indeed, let $u \in D(A^\rho)$ and $A^{-\rho}u = 0$. Then, using Lemma 1, we obtain the equality

$$A^{-(2+[\rho])}u = A^{-(2+[\rho]-\rho)}A^{-\rho}u = 0$$

($[\rho]$ - integer part of ρ). It follows that $u = 0$. This implies the existence of operator $\left(A^{-\rho}\right)^{-1}$. For the elements $u \in D\left(A^2\right)$ we have the representation

$$A^\rho u = \frac{1}{\Gamma(1-\rho)} \int_0^t s^{-\rho} AU(s)\, v ds \qquad (0 < \rho < 1). \tag{1.3.6}$$

We will demonstrate this. Let A_0 operator given right-hand side of equation (1.3.6). Similarly to Lemma 1 proves that

$$A^{-\rho}A_0 u = u, A_0 A^{-\rho} u = u \left(u \in D\left(A^2\right)\right)$$

The first one means that $A_0 u = A^\rho u$ for $u \in D\left(A^2\right)$. Consequently, $D\left(A^2\right) \subset D(A^\rho)$. Formula (1.3.6) is mounted.

Lemma 1.3.2. *The following inequality moments:*

$$\left\|A^{-\rho}u\right\| \le c \left\|A^{-1}u\right\|^{\rho-\alpha} \|u\|^{1+\alpha-\rho}, \text{ if } \rho \in (\alpha, 1] \tag{1.3.7}$$

$$\|A^\rho u\| \le c \|Au\|^{\rho+\alpha} \|u\|^{1-\rho-\alpha}, \text{ if } \rho \in (0, 1-\alpha), u \in D(A). \tag{1.3.8}$$

Proof. For $\rho > \alpha$ the operator $A^{-\rho}$ is bounded and

$$\left\|A^{-\rho}u\right\| \le \frac{1}{\Gamma(\rho)} \int_0^N s^{\rho-1} \|U(s)\| \|u\| ds + \frac{1}{\Gamma(\rho)} \left\| \int_N^\infty s^{\rho-1} AU(s) A^{-1} v ds \right\|$$
$$\le c_1 N^{\rho-\alpha} \|u\| + c_2 N^{\rho-\alpha-1} \left\|A^{-1}u\right\|.$$

Minimizing this expression for $N > 0$, we obtain (1.3.7). Putting there $u_1 = A^{-1}u$ and using Lemma 1, we obtain the second inequality (1.3.8). □

Lemma 1.3.3. *The inequality*

$$\|A^\rho U(t)\| \leq ct^{-\beta\rho-\alpha(1-\rho)-\delta}\exp(-\omega t), \qquad \rho \in (0,1).$$

Here $\delta > 0$, if $\beta > 1+\alpha$, and $\delta = 0$, if $\beta = 1+\alpha$. If further estimated

$$\left\|A^2 U(t)\right\| \leq Mt^{-\gamma}\exp(-\omega t), \qquad 1+\beta \leq \gamma, \omega > 0,$$

then the inequality

$$\left\|A^{1+\rho}U(t)\right\| \leq ct^{-\gamma\rho-\beta(1-\rho)-\delta}\exp(-\omega t), \qquad \rho \in (0,1)$$

Here $\delta > 0$, if $\gamma > 1+\beta$, and $\delta = 0$, if $\gamma = 1+\beta$.

Lemma 1.3.4. *The following limit relations:*

a)

$$\lim_{t \to 0^+} U(t)A^{-\rho} = A^{-\rho}$$

at $\rho > \min\left\{2\alpha, \frac{\beta-1}{\beta-\alpha}\right\}$;

b) *if* $\overline{D} = E$, *then*

$$\lim_{t \to 0^+} U(t)A^{-\rho} = A^{-\rho}$$

for $\rho > \alpha$.

Proof. Proof of Lemmas 3 and 4 is given in [49]. □

Remark 1.3.3. *When $\rho \leq \alpha$ expression and $A^{-\rho}(U(t) - I)$ can not aspire to zero. This is evidenced by the example 2 and. 3. Indeed, in this example,*

$$(U(t,p) - I)A^{-\rho}(p) = \begin{bmatrix} \frac{\exp\left(ip^k t - p^2 t\right) - 1}{\left(ip^k - p^2\right)^\rho} & 0 \\ \frac{tp^l \exp\left(ip^k t - p^2 t\right)}{\left(ip^k - p^2\right)^\rho} + \frac{\rho p^l\left[\exp\left(ip^k t - p^2 t\right) - 1\right]}{\left(ip^k - p^2\right)^{1+\rho}} & \frac{\exp\left(ip^k t - p^2 t\right) - 1}{\left(ip^k - p^2\right)^\rho} \end{bmatrix}.$$

If $\rho \le \frac{l}{k} - 1$, *then* $\rho \le \alpha$, *and since* $k \ge 2$, *there is no such district that*

$$\left| \frac{p^l \left[\exp\left(ip^k t - p^2 t \right) - 1 \right]}{\left(ip^k - p^2 \right)^{1+\rho}} \right| \ge c > 0,$$

where $|p| \ge R > 0$.

Lemma 1.3.5. *If* $0 < \rho < \frac{1-\alpha}{\beta-\alpha}$, *then*

$$A^{\rho} \int_0^t U(t-s) f(s) \, ds = \int_0^t A^{\rho} U(t-s) f(s) \, ds. \tag{1.3.9}$$

Here, $f(s)$ *continuous for* $s > 0$, *the function for which* $\|f(s)\| \le cs^{-\mu}$, $\mu < 1$.

Proof. The integral J on the right in (1.3.9) exists, since the estimate

$$\|A^{\rho} U(t-s) f(s)\| \le c(t-s)^{-\beta\rho - \alpha(1-\rho) - \delta} s^{-\mu}$$

and $\beta\rho + \alpha(1-\rho) < 1$. We show that $J \in D\left(A^{-\rho}\right)$. For this we consider equality

$$\frac{1}{\Gamma(\rho)} \int_a^{\infty} \tau^{\rho-1} U(\tau) \left\{ \int_0^t A^{\rho} U(t-s) f(s) \, ds \right\} d\tau$$
$$= \frac{1}{\Gamma(\rho)} \int_0^t \tau^{\rho-1} U(\tau) \left\{ \int_a^{\infty} \tau^{\rho-1} A^{\rho} U(\tau + t - s) \, d\tau \right\} f(s) \, ds.$$

Here in the last expression the inner integral is the limit as well $a \to 0^+$, since

$$\|A^{\rho} U(\tau + t - s)\| \le c(t-s)^{-\beta\rho - \alpha(1-\rho) - \delta} \exp(-\omega(\tau + t - s)).$$

Therefore, there is a limit of the left side of the last equality in and $a \to 0^+$. This means that $J \in D\left(A^{-\rho}\right)$. Let

$$w = \int_0^t U(t-s) f(s) \, ds.$$

Then

$$A^{-1} w = \int_0^t A^{-1} A^{-\rho} A^{\rho} U(t-s) f(s) \, ds = A^{-1} A^{-\rho} \int_0^t A^{\rho} U(t-s) f(s) \, ds.$$

It follows from this equality

$$w = A^{-\rho} \int_0^t A^{\rho} U(t-s) f(s)\, ds,$$

which means that $w \in D(A^{\rho})$. formula "Equation" (1.3.9) is proved. □

1.4 Solvability of the problem

5. Let us turn to problem (1.1.1). By its solution we mean a continuous function $u = u(t)$ for $t \geq 0$, for which there exist and are continuous for $t > 0$ functions $u'(t)$, $Au(t))$ and relations (1.1.1) are satisfied. Such a solution is called "weakened" in [47].

Consider the homogeneous problem

$$u' + Au = 0 \qquad (t > 0), u(0) = u_0. \tag{1.4.1}$$

If there exists a semigroup $U(t)$ of class $A(\alpha, \beta)$ generated by A, then the solution of (1.4.1) is given by the formula $u(t) = U(t) u_0$, if $u_0 \in D(A^{\rho})$, where $\rho > \min\left\{2\alpha, \frac{\beta-1}{\beta-\alpha}\right\}$ or$\rho > \alpha$ in the case of $\overline{D} = E$ (Lemma 4). In the general case, it is impossible to improve ρ, as is stated in the remark to Lemma 4. Note that there are no restrictions on the parameters α and β, and the solution to this problem can exist in the of operator A does not have regular points. Let us turn to the inhomogeneous problem (1.1.1). Its solution $u(t) = U(t) u_0 + g(t)$, where

$$g(t) = \int_0^t U(t-s) f(s)\, ds$$

inhomogeneous particular solution of equation (1.1.1). The properties of operation function $g(t)$ were studied in [49]. Note that the condition $\beta < 2$ arose [49] in the construction of of the resolving operator of the problem with the variable operator coefficient. In the case of constant operator coefficients this condition is absent. therefore, we have

Theorem 1.4.1. *Suppose that conditions*

1- *there exists a bounded inverse operator* A^{-1}*;*

2- *there exists a semigroup* $U(t)$ *Class* $A(\alpha,\beta)$*, and in some* $\omega > 0$*,* $0 \leq \alpha < 1, \alpha + 1 \leq \beta$*;*

3-

$$\|f(t+\Delta t) - f(t)\| \leq ct^{-\mu} |\Delta t|^{\varepsilon}$$

for some $\varepsilon \in \left(\frac{\beta-1}{\beta-\alpha}, 1\right], \mu \in [0,1)$*;*

4- $u_0 \in D(A^{\rho})$ *for some* $\rho \in \left(\min\left\{2\alpha, \frac{\beta-1}{\beta-\alpha}\right\}, 1\right]$ *if* $\overline{D} = E$*, then* $u_0 \in D(A^{\rho})$ *at* $\rho \in (\alpha, 1]$.

Then the problem (1.1.1) has a unique solution $u(t)$*, is given by the formula*

$$u(t) = U(t)u_0 + \int_0^t U(t-s) f(s) ds.$$

Chapter 2

On a second-order ordinary differential operator with integral conditions

2.1 Introduction

Ordinary differential equations with integral boundary conditions represent a very interesting and important class of problems. They arise in many areas of applied mathematics and physics, and have been much studied by many authors [1, 2, 4]. In [1], the equation

$$Au + \lambda^2 u = -a_0(t)u''(t) + a_1(t)u'(t) + a_2(t)u(t) + \lambda^2 u(t) = f_0(t) \qquad \text{for } t \in (0,1),$$

is considered with the integral boundary conditions

$$B_i u = \int_0^1 e_i(t)u(t)dt = f_i \quad \text{ for } i = 1, 2.$$

E. I. Galakhov, and A.L. Skubachevskii [1] obtained a priori estimate of the solutions for sufficiently large values of the spectral parameter λ. Moreover, they proved the Fredholm solvability of the problem. The idea was based on the method devellopped in

[82]. In [2], K. A. Darovskaya, and A. L. Skubachevskii asume the same problem but by changing the unknown functions in the integral boundary conditions by its derivatives i.e :

$$B_i u = \int_0^1 e_i(t)u'(t)dt = f_i \quad \text{for } i = 1, 2.$$

This chapter is devoted to a problem inspired from [1, 2], where the integral boundary conditions are homogeneous, containing the unknown functions and its derivatives.

2.2 Statement of the problem

We consider the following problem

$$Au + \lambda^2 u = -a_0(t)u''(t) + a_1(t)u'(t) + a_2(t)u(t) + \lambda^2 u(t) = f(t) \qquad \text{for } t \in (0,1), \quad (2.2.1)$$

with the integral condition

$$\begin{cases} B_1 u = \int_0^1 e_1(t)u(t)dt = 0, \\ B_2 u = \int_0^1 e_2(t)u'(t)dt = 0. \end{cases} \qquad (2.2.2)$$

Where a_i $(i = 0, 1, 2)$ are real-valued functions belong in the space $C[0,1]$, $a_0(t) \geq k > 0$ for $t \in [0,1]$, $f \in L^2(0,1)$ is a complex-valued function, $\lambda \in \mathbb{C}$ is a spectral parameter, and e_i $(i = 1, 2)$ are linearly independent real-valued functions such that $e_i \in C[0,1]$. Our results obtained concern the Fredholm solvability of the problem and a priori estimate of its solutions for sufficiently large values of the spectral parameter. This study based on the technique of the works [1, 2].

In the next section, we present some preliminaries and introduce an auxiliary problem with some proprties. The main results are then stated and proved in Section 3.

2.3 Preliminaries

In this section, we will establish a result that is useful to the proof of our main results.

Let $W^1_\infty(a,b)$ be the space of absolutely continuous functions $u(t)$, $t \in [a,b]$ such that $u' \in L^\infty(a,b)$.

Set,

$$W^1_{\infty,\beta}(0,1) = \Big\{u \in C[0,1] \ : \ u \in W^1_\infty(0,\beta), \ u \in W^1_\infty(1-\beta,1)\Big\}$$

where $0 < \beta < \frac{1}{2}$. Throug this paper, we assume that $a_0 \in W^1_{\infty,\beta}(0,1)$.

In the Sobolev space $W^2(0,1)$ we introduce the following equivalent norm with the parameter λ :

$$|||u|||_{W^2(0,1)} = \left(\|u\|^2_{W^2(0,1)} + |\lambda|^4 \|u\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}.$$

Let $W[0,1] = L^2(0,1) \times \mathbb{C} \times \mathbb{C}$, and consider the bounded linear operator

$$L(\lambda) : W^2(0,1) \to W[0,1]$$

given by

$$L(\lambda)u = (Au + \lambda^2 u, 0, 0).$$

We also define the unbounded operator

$$\mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L^2(0,1) \to L^2(0,1)$$

with the domain

$$\mathcal{D}(\mathcal{A}_B) = \Big\{u \in W^2(0,1) : B_i u = 0, \ i = 1,2\Big\},$$

by $\mathcal{A}_B u = Au$. Note that the operator $\mathcal{A}_B$ is not densely defined in $L^2(0,1)$.

In order to get our main results, we study the following problem, say the model problem

2.4 Model problem

$$A_0 u + \lambda^2 u = -pu''(t) + \lambda^2 u(t) = f(t) \qquad \text{for } t \in (0,1), \tag{2.4.1}$$

with

$$\begin{cases} e_1(0)\int_0^{\frac{1}{2}} u(t)dt + e_1(1)\int_{\frac{1}{2}}^1 u(t)dt = 0, \\ e_2(0)\int_0^{\frac{1}{2}} u'(t)dt + e_2(1)\int_{\frac{1}{2}}^1 u'(t)dt = 0, \end{cases} \tag{2.4.2}$$

where $p > 0$ is a constant, $f \in L^2(0,1)$. Consider the set

$$\omega_{\epsilon,q} = \{\lambda \in \mathbb{C} : |\arg\lambda| \le \epsilon \text{ or } |\arg\lambda - \pi| \le \epsilon\} \cap \{\lambda \in \mathbb{C} : |\lambda| \ge q\},$$

where $\epsilon \in (0, \frac{\pi}{2})$, and let $\Delta_e = e_1(0)e_2(1) + e_1(1)e_2(0)$.

Lemma 2.4.1. *Let $\Delta_e \neq 0$. Then for any $\epsilon \in (0, \frac{\pi}{2})$, there exists $q_0 > 1$ such that for each $\lambda \in \omega_{\epsilon,q_0}$ the problem (2.4.1), (2.4.2) has a unique solution $u \in W^2(0,1)$ and*

$$|||u|||_{W^2(0,1)} \le K|\lambda|^{\frac{1}{2}} \|f\|_{L^2(0,1)}, \tag{2.4.3}$$

where $K > 0$ does not depend on λ and u.

Proof. 1. Let $q_0 > 1$ sufficiently large, we consider problem (2.4.1), (2.4.2) with $f(t) = e^{i2\pi st}$ $(s \neq 0)$, and $\lambda \in \omega_{\epsilon,q_0}$. Then we have

$$-pu_s''(t) + \lambda^2 u_s(t) = e^{i2\pi st} \qquad \text{for } t \in (0,1), \tag{2.4.4}$$

with

$$\begin{cases} e_1(0)\int_0^{\frac{1}{2}} u_s(t)dt + e_1(1)\int_{\frac{1}{2}}^1 u_s(t)dt = 0, \\ e_2(0)\int_0^{\frac{1}{2}} u_s'(t)dt + e_2(1)\int_{\frac{1}{2}}^1 u_s'(t)dt = 0. \end{cases} \tag{2.4.5}$$

Obviously, the function

$$u_s(t) = c_{1s}e^{\mu t} + c_{2s}e^{-\mu t} + \frac{e^{i2\pi st}}{4p\pi^2 s^2 + \lambda^2} \qquad \text{for } t \in (0,1), \tag{2.4.6}$$

where $\mu = \sqrt{\frac{\lambda^2}{p}}$, $\Re\mu > 0$, is a solution of Eq. (2.4.4).

Substituting (2.4.6) into the integral conditions (2.4.5), we obtain

$$MC_s = B_s, \tag{2.4.7}$$

where

$$M=\begin{pmatrix} \frac{e_1(0)\left(e^{\frac{\mu}{2}}-1\right)+e_1(1)\left(e^{\mu}-e^{\frac{\mu}{2}}\right)}{\mu} & \frac{e_1(0)\left(1-e^{-\frac{\mu}{2}}\right)+e_1(1)\left(e^{-\frac{\mu}{2}}-e^{-\mu}\right)}{\mu} \\ e_2(0)\left(e^{\frac{\mu}{2}}-1\right)+e_2(1)\left(e^{\mu}-e^{\frac{\mu}{2}}\right) & e_2(0)\left(e^{-\frac{\mu}{2}}-1\right)+e_2(1)\left(e^{-\mu}-e^{-\frac{\mu}{2}}\right) \end{pmatrix}$$

and

$$C_s=(c_{1s},c_{2s})^T,\ B_s=\left(\frac{(1-(-1)^s)(e_1(0)-e_1(1))}{i2\pi s(4p\pi^2s^2+\lambda^2)},\frac{(1-(-1)^s)(e_2(0)-e_2(1))}{(4p\pi^2s^2+\lambda^2)}\right)^T.$$

We set

$$g(\lambda)=\begin{vmatrix} \frac{e_1(1)e^{\mu}}{\mu} & \frac{e_1(0)}{\mu} \\ e_2(1)e^{\mu} & -e_2(0) \end{vmatrix}=-\frac{\Delta_e e^{\mu}}{\mu},$$

and $\Delta(\lambda)=\det M$. An easy computation gives

$$\Delta(\lambda)=g(\lambda)+\frac{1}{\mu}\left[(2e^{\frac{\mu}{2}}-e^{-\mu}+2e^{-\frac{\mu}{2}}-2)\Delta_e+(4-2e^{\frac{\mu}{2}}-2e^{-\frac{\mu}{2}})(e_1(1)e_2(1)+e_1(0)e_2(0))\right],$$

and

$$\Delta(\lambda)=g(\lambda)+o(g(\lambda)) \qquad \text{as } |\lambda|\to\infty,\ \ \lambda\in\omega_{\epsilon,q_0}. \tag{2.4.8}$$

Therefore, if $q_0>1$ is sufficiently large, then $\Delta(\lambda)\neq 0$ for $\lambda\in\omega_{\epsilon,q_0}$. Thus, from Cramer's rule, the system of linear algebraic equations (2.4.7) has a unique solution :

$$c_{js}=\frac{\Delta_{js}(\lambda)}{\Delta(\lambda)} \qquad (j=1,2), \tag{2.4.9}$$

where $\Delta_{js}(\lambda)=0$ for $s=2m$ and

$$\Delta_{js}(\lambda)=g_{js}(\lambda)+o(g_{js}(\lambda)) \qquad \text{as } |\lambda|\to\infty,\ \ \lambda\in\omega_{\epsilon,q_0},\ \text{for } s=2m+1; \tag{2.4.10}$$

with

$$g_{js}(\lambda)=\frac{(\alpha_j\mu+\beta_j s)e^{(j-1)\mu}}{i\pi s(4p\pi^2s^2+\lambda^2)\mu}, \qquad \alpha_j,\beta_j\in\mathbb{C}\ \ (j=1,2).$$

Combining(2.4.8), (2.4.9),(2.4.10), we obtain

$$\begin{aligned} c_{js}&=-\tfrac{(\alpha_j\mu+\beta_j s)e^{(j-2)\mu}+o(\mu e^{(j-2)\mu})}{i\pi s(4p\pi^2s^2+\lambda^2)\Delta_e} \quad \text{as } |\lambda|\to\infty,\ \ \lambda\in\omega_{\epsilon,q_0} \quad (j=1,2,\ s=2m+1).\\ c_{js}&=0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=1,2,\ s=2m). \end{aligned} \tag{2.4.11}$$

Substituting (2.4.11) into (2.4.6), we obtain

$$u_s(t) = v_s(t) + w_s(t) \qquad \text{as } |\lambda| \to \infty, \;\; \lambda \in \omega_{\epsilon,q_0},$$

where

$$\begin{aligned} v_s(t) &= -\frac{\left[(\alpha_1\mu+\beta_1 s)e^{-\mu}+o(\mu e^{-\mu})\right]e^{\mu t}+[(\alpha_2\mu+\beta_2 s)+o(\mu)]e^{-\mu t}}{i\pi s(4p\pi^2 s^2+\lambda^2)\Delta_e} \qquad &\text{for } s = 2m+1, \\ v_s(t) &= 0 &\text{for } s = 2m, \end{aligned} \tag{2.4.12}$$

$$w_s(t) = \frac{e^{i2\pi st}}{4p\pi^2 s^2 + \lambda^2}. \tag{2.4.13}$$

For $\lambda \in \omega_{\epsilon,q_0}$, it is easy to see that

$$\frac{|\lambda|\cos\epsilon}{\sqrt{p}} \leq \Re\mu, \tag{2.4.14}$$

$$4p\pi^2 s^2 \sin 2\epsilon \leq \left|4p\pi^2 s^2 + \lambda^2\right|, \tag{2.4.15}$$

$$|\lambda|^2 \sin 2\epsilon \leq \left|4p\pi^2 s^2 + \lambda^2\right|. \tag{2.4.16}$$

Then, from (2.4.12), (2.4.14), and (2.4.16), it follows that

$$\|v_s\|_{L^2(0,1)} \leq k_1 \left|4p\pi^2 s^2 + \lambda^2\right|^{-1} |s|^{-1} (|\mu| + |s|) \left(\frac{1 - e^{-2\Re\mu}}{\Re\mu}\right)^{\frac{1}{2}} \leq k_2 |\lambda|^{-\frac{3}{2}} |s|^{-1},$$

$$\left\|v_s'\right\|_{L^2(0,1)} \leq k_1 \left|4p\pi^2 s^2 + \lambda^2\right|^{-1} |s|^{-1} |\mu| (|\mu| + |s|) \left(\frac{1 - e^{-2\Re\mu}}{\Re\mu}\right)^{\frac{1}{2}} \leq k_2 |\lambda|^{-\frac{1}{2}} |s|^{-1},$$

$$\left\|v_s''\right\|_{L^2(0,1)} \leq k_1 \left|4p\pi^2 s^2 + \lambda^2\right|^{-1} |s|^{-1} |\mu|^2 (|\mu| + |s|) \left(\frac{1 - e^{-2\Re\mu}}{\Re\mu}\right)^{\frac{1}{2}} \leq k_2 |\lambda|^{\frac{1}{2}} |s|^{-1}.$$

Here and further $k_1, \ldots, k_4 > 0$ do not depend on λ and s.

Therefore,

$$|||v_s|||_{W^2(0,1)} \leq k_3 |\lambda|^{\frac{1}{2}} |s|^{-1} \qquad (\lambda \in \omega_{\epsilon,q_0}, \; s \neq 0). \tag{2.4.17}$$

Similarly, from (2.4.13), (2.4.14), and (2.4.15), we have

$$|||w_s|||_{W^2(0,1)} \leq k_4 \qquad (\lambda \in \omega_{\epsilon,q_0},\ s \neq 0). \tag{2.4.18}$$

2. Now, we consider problem (2.4.1), (2.4.2) with $f(t) = 1$, and $\lambda \in \omega_{\epsilon,q_0}$, where $q_0 > 1$ is sufficiently large. Then this problem takes the form

$$-pu_0''(t) + \lambda^2 u_0(t) = 1 \qquad \text{for } t \in (0,1), \tag{2.4.19}$$

with

$$\begin{cases} e_1(0)\int\limits_0^{\frac{1}{2}} u_0(t)dt + e_1(1)\int\limits_{\frac{1}{2}}^{1} u_0(t)dt = 0, \\ e_2(0)\int\limits_0^{\frac{1}{2}} u_0'(t)dt + e_2(1)\int\limits_{\frac{1}{2}}^{1} u_0'(t)dt = 0. \end{cases}$$

The strong solution of eqution(2.4.19) is given by the formula

$$u_0(t) = c_1 e^{\mu t} + c_2 e^{-\mu t} + \frac{1}{\lambda^2} \qquad \text{for } t \in (0,1),$$

where $\mu = \sqrt{\frac{\lambda^2}{p}}$, $\Re\mu > 0$.

In a similar manner to Part 1 of the proof, we obtain

$$c_j = (-2\lambda^2\Delta_e)^{-1}\left[\gamma_j \mu e^{(j-2)\mu} + o(\mu e^{(j-2)\mu})\right] \quad \text{as } |\lambda| \to \infty,\ \ \lambda \in \omega_{\epsilon,q_0} \quad (j = 1,2),$$

where $\gamma_j \in \mathbb{C}$ $(j = 1,2)$. and $u_0(t) = v_0(t) + w_0(t)$, where

$$v_0(t) = (-2\lambda^2\Delta_e)^{-1}\left[(\gamma_1 \mu e^{-\mu} + o(\mu e^{-\mu}))e^{\mu t} + (\gamma_2\mu + o(\mu))e^{-\mu t}\right] \quad \text{as } |\lambda| \to \infty,\ \ \lambda \in \omega_{\epsilon,q_0},$$

and $w_0(t) = \frac{1}{\lambda^2}$.

Obviously,

$$|||v_0|||_{W^2(0,1)} \leq k_5 |\lambda|^{\frac{1}{2}} \qquad (\lambda \in \omega_{\epsilon,q_0}), \tag{2.4.20}$$

$$|||w_0|||_{W^2(0,1)} \equiv 1 \qquad (\lambda \in \omega_{\epsilon,q_0}), \tag{2.4.21}$$

where $k_5 > 0$ does not depend on λ.

3. Finally, we consider problem (2.4.1), (2.4.2) for $f(t) = F_N(t)$ and $\lambda \in \omega_{\epsilon,q_0}$ ($q_0 > 1$ is sufficiently large), where

$$F_N(t) = \sum_{|s|\le N} f_s e^{i2\pi st}, \qquad f_s = \int_0^1 f(t)e^{-i2\pi st}dt.$$

Since $\Delta(\lambda) \neq 0$ ($\lambda \in \omega_{\epsilon,q_0}$), it follows from the Part 1, and Part 2 of the proof that

$$U_N(t) = \sum_{|s|\le N} f_s u_s(t),$$

is a unique solution of problem (2.4.1), (2.4.2).

Using inequalities (2.4.18), (2.4.19), and (2.4.20), (2.4.21), and the orthogonality of the functions w_s in $W^2(0,1)$, we obtain

$$\begin{aligned} |||U_N|||_{W^2(0,1)} &\le \left|\left|\left|\sum_{|s|\le N} f_s v_s\right|\right|\right|_{W^2(0,1)} + \left|\left|\left|\sum_{|s|\le N} f_s w_s\right|\right|\right|_{W^2(0,1)} \\ &\le \sum_{|s|\le N} |f_s|\cdot |||v_s|||_{W^2(0,1)} + \left(k_6 \sum_{|s|\le N} |f_s|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{|s|\le N} |f_s|^2\right)^{\frac{1}{2}} \left(\sum_{|s|\le N} |||v_s|||^2_{W^2(0,1)}\right)^{\frac{1}{2}} + k_6^{\frac{1}{2}} \|f\|_{L^2(0,1)} \\ &\le k_7 |\lambda|^{\frac{1}{2}} \|f\|_{L^2(0,1)} + k_6^{\frac{1}{2}} \|f\|_{L^2(0,1)} \le k_8 |\lambda|^{\frac{1}{2}} \|f\|_{L^2(0,1)} \qquad (\lambda \in \omega_{\epsilon,q_0}), \end{aligned} \tag{2.4.22}$$

where $k_6, k_7 > 0$ do not depend on λ and f, $k_8 = k_7 + k_6^{\frac{1}{2}}$.

Similarly to the proof of inequality (2.4.22), one can show that $\{U_N\}$ is a Cauchy sequence in $W^2(0,1)$. Since $W^2(0,1)$ is complete and the limit is unique, and since $\Delta(\lambda) \neq 0$ ($\lambda \in \omega_{\epsilon,q_0}$), the problem (2.4.1), (2.4.2) has aunique solution $u \in W^2(0,1)$ for any $f \in L^2(0,1)$ and we have $|||U_N - u|||_{W^2(0,1)} \to 0$ as $N \to \infty$. By passing to the limit in inequality (2.4.22), we obtain the estimate

$$|||u|||_{W^2(0,1)} \le k_8 |\lambda|^{\frac{1}{2}} \|f\|_{L^2(0,1)} \qquad (\lambda \in \omega_{\epsilon,q_0}).$$

□

2.5 Main results

In the following, we shall give the main results of this chapter.

Theorem 2.5.1. *Let $\Delta_e \neq 0$. Then for any $\epsilon \in (0, \frac{\pi}{2})$, there exists $q > 1$ such that for any $\lambda \in \omega_{\epsilon,q}$ and $u \in W^2(0,1)$:*

$$|||u|||_{W^2(0,1)} \leq C\,|\lambda|^{\frac{1}{2}}\,\|f\|_{L^2(0,1)} \tag{2.5.1}$$

where $C > 0$ does not depend on λ and u.

Proof. 1. First, we transform the equation (2.2.1) to the form in which the coefficient of the second derivative becoms equal to a constant in a sufficiently small neighborhood of 0 and 1. To do so, we introduce the new variable $\tau = d^{-1} \int\limits_0^t b^{-1}(\xi)d\xi$, where

$$d = \int\limits_0^1 b^{-1}(\xi)d\xi,$$

$$b(\xi) = \begin{cases} \sqrt{a_0(\xi)} & \text{for } \xi \in [0,\beta] \cup [1-\beta,1], \\ \frac{(1-\xi-\beta)\sqrt{a_0(\beta)}+(\xi-\beta)\sqrt{a_0(1-\beta)}}{1-2\beta} & \text{for } \xi \in (\beta, 1-\beta). \end{cases}$$

Since $a_0 \in W^1_{\infty,\beta}(0,1)$ and $a_0(t) \geq k > 0$, we have

$$0 < k_1 \leq b^{-1}(t) \leq k^{-\frac{1}{2}} \quad \text{and} \quad b^{-1} \in W^1_\infty(0,1),$$

Therefore, the function $\tau = \tau(t)$ maps the interval $[0,1]$ into itself and

$$\frac{d\tau}{dt} = (db(t))^{-1} \geq k_1 k^{\frac{1}{2}} > 0, \qquad \frac{d\tau}{dt} \in W^1_\infty(0,1).$$

The transformation $\tau = \tau(t)$ reduce problem (2.2.1), (2.2.2) to the form

$$-\bar{a}_0(\tau)\bar{u}''(\tau) + \bar{a}_1(\tau)\bar{u}'(\tau) + \bar{a}_2(\tau)\bar{u}(\tau) + \lambda^2\bar{u}(\tau) = \bar{f}(\tau) \qquad \text{for } \tau \in (0,1) \tag{2.5.2}$$

with

$$\begin{cases} \int\limits_0^1 \bar{e_1}(\tau)\bar{u}(\tau)d\tau = 0 \\ \int\limits_0^1 \bar{e_2}(\tau)\bar{u}'(\tau)d\tau = 0 \end{cases}, \tag{2.5.3}$$

where

$$\bar{u}(\tau) = u(t(\tau)), \quad \bar{f}(\tau) = f(t(\tau)), \quad \bar{a}_0(\tau) = a_0(t(\tau))(db(t(\tau)))^{-2} \in W^1_{\infty,\beta_0}(0,1),$$

$$\bar{a}_1(\tau) = \frac{a_1(t(\tau))}{db(t(\tau))} + \frac{a_0(t(\tau))b'(t(\tau))}{d^2b^3(t(\tau))} \in L^\infty(0,1), \quad \bar{a}_2(\tau) = a_2(t(\tau)) \in C[0,1],$$

$$\bar{e_i}(\tau) = e_i(t(\tau))(db(t(\tau)))^{2-i} \in C[0,1] \quad (i=1,2), \text{ and}$$

$$\beta_0 = \min\left\{ d^{-1}\int\limits_0^\beta b^{-1}(\xi)d\xi, 1-d^{-1}\int\limits_0^{1-\beta} b^{-1}(\xi)d\xi, \right\},$$

Moreover,

$$\bar{a}_0(\tau) = \frac{1}{d^2}, \qquad \text{for } \tau \in [0,\beta_0]\cup[1-\beta_0,1].$$

Note that the inequality(2.5.1) is invariant with respect to the substitution $\tau = \tau(t)$. We have

$$\Delta_{\bar{e}} = \bar{e_1}(0)\bar{e_2}(1) + \bar{e_1}(1)\bar{e_2}(0) = d\sqrt{a_0(0)}e_1(0)e_2(1) + d\sqrt{a_0(1)}e_1(1)e_2(0).$$

It is easy to see that $\Delta_{\bar{e}} \neq 0$.

Therefore, without loss of generality, we can assume that

$$a_0 \in W^1_{\infty,\beta}(0,1), \quad a_i \in L^\infty(0,1) \quad (i=1,2), \quad e_i \in C[0,1] \quad (i=1,2),$$

$$a_0(t) \geq k > 0, \quad a_0(t) = a_0(0) \quad \text{for } t \in [0,\beta]\cup[1-\beta,1], \text{ and } \Delta_e \neq 0.$$

2. We obtain an a priori estimate of solutions of problem(2.2.1), (2.2.2) on the closed interval $[\frac{\delta}{3}, 1-\frac{\delta}{3}]$, where $\delta = \delta(\lambda) > 0$.

We consider a truncating function $\zeta \in C^\infty(\mathbb{R})$ such that

$$0 \leq \zeta(t) \leq 1, \ \ \zeta(t) = 1 \text{ for } |t| < \frac{1}{4}, \ \ \zeta(t) = 0 \text{ for } |t| > \frac{1}{3}.$$

For each $\lambda \in \omega_{\epsilon,q_1}$, we introduce the function

$$\eta(t) = \zeta(\frac{t}{\delta}) + \zeta(\frac{t-1}{\delta}),$$

where $q_1 = \max\left\{q_0, \beta^{-3}\right\}$, $\delta = |\lambda|^{-\frac{1}{3}-2\gamma}$ and $0 < \gamma < \frac{1}{12}$.

It is easy to see that

$$\left|\eta^{(j)}(t)\right| \leq k_1\delta^{-j} \qquad \text{for } t \in \mathbb{R},\ (j = 1, 2), \tag{2.5.4}$$

where $k_1 > 0$ does not depend on t and δ. Since $|\lambda| \geq q_1$, we obtain $\delta < q_1^{-\frac{1}{3}} \leq \beta$.

By virtue of the a priori estimate for solution of "local" boundary value problems with a parameter [3], for any $\epsilon > 0$, there exists $q_2 \geq q_1$ such that any solution $u \in W^2(0,1)$ of problem (2.2.1) (2.2.2) for $\lambda \in \omega_{\epsilon,q_2}$ satisfies the inequality

$$|||(1-\eta)u|||_{W^2(0,1)} \leq k_2 \left\|(A+\lambda^2 I)((1-\eta)u)\right\|_{L^2(0,1)},$$

Obviously,

$$(A+\lambda^2 I)((1-\eta)u) = (1-\eta)(A+\lambda^2 I)u - a_0(1-\eta)''u - 2a_0(1-\eta)'u' + a_1(1-\eta)'u$$

Then, since a_i $(i = 0, 1, 2)$ are bounded and using the inequality (3.4) and the Leibnitz formula, we obtain

$$\begin{aligned} |||(1-\eta)u|||_{W^2(0,1)} &\leq k_2 \left\|(A+\lambda^2 I)((1-\eta)u)\right\|_{L^2(0,1)} \\ &\leq k_3 \left[\left\|(A+\lambda^2 I)u\right\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|u\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \left(\|u\|_{L^2(0,1)} + \left\|u'\right\|_{L^2(0,1)}\right)\right] \\ &\leq k_4 \left(\left\|(A+\lambda^2 I)u\right\|_{L^2(0,1)} + |\lambda|^{\frac{2}{3}+4\gamma} \|u\|_{L^2(0,1)} + |\lambda|^{\frac{1}{3}+2\gamma} \|u\|_{W^1(0,1)}\right), \end{aligned}$$

where $k_2, k_3, k_4 > 0$ do not depend on λ and u.

Using the known interpolation inequality

$$|\lambda| \|u\|_{W^1(0,1)} \leq c\left(\|u\|_{W^2(0,1)} + |\lambda|^2 \|u\|_{L^2(0,1)}\right),$$

we find that

$$\begin{aligned} |||(1-\eta)u|||_{W^2(0,1)} &\leq k_5 \left[\left\|(A+\lambda^2 I)u\right\|_{L^2(0,1)} + |\lambda|^{-\frac{2}{3}+2\gamma} \|u\|_{W^2(0,1)} + \left(|\lambda|^{\frac{2}{3}+4\gamma} + |\lambda|^{\frac{4}{3}+2\gamma}\right) \|u\|_{L^2(0,1)}\right] \\ &\leq k_5 \left(\left\|(A+\lambda^2 I)u\right\|_{L^2(0,1)} + \right. \\ &\left. |\lambda|^{-\frac{2}{3}+2\gamma} \|u\|_{W^2(0,1)} + |\lambda|^2 \left(|\lambda|^{-\frac{4}{3}+4\gamma} + |\lambda|^{-\frac{2}{3}+2\gamma}\right) \|u\|_{L^2(0,1)}\right) \\ &\leq k_6 \left[\left\|(A+\lambda^2 I)u\right\|_{L^2(0,1)} + |\lambda|^{-\frac{2}{3}+2\gamma} \left(\|u\|_{W^2(0,1)} + |\lambda|^2 \|u\|_{L^2(0,1)}\right)\right]. \end{aligned}$$

It follows that

$$|||(1-\eta)u|||_{W^2(0,1)} \le k_6 \left(\|f\|_{L^2(0,1)} + |\lambda|^{-\sigma} |||u|||_{W^2(0,1)} \right), \tag{2.5.5}$$

where $\sigma = \frac{2}{3} - 2\gamma > 0$ and $k_5, k_6 > 0$ do not depend on λ and u.

3. Now, we obtain an a priori estimate for solution of problem (2.2.1) , (2.2.2) near the end-points of the interval $(0,1)$.

We introduce the operators A_0 and A_1 by

$$A_0 u = -a_0(0)u''(t) \quad \text{and} \quad A_1 u = -a_0(t)u''(t) \qquad \text{for } t \in (0,1).$$

We set $p = a_0(0)$ in equation (2.1). Then, applying Lemma 1, we obtain

$$|||\eta u|||_{W^2(0,1)} \le K|\lambda|^{\frac{1}{2}} \left\|(A_0 + \lambda^2 I)(\eta u)\right\|_{L^2(0,1)} \le K|\lambda|^{\frac{1}{2}} \Big(\left\|(A + \lambda^2 I)(\eta u)\right\|_{L^2(0,1)} + \left\|a_1(\eta u)'\right\|_{L^2(0,1)} + \|a_2(\eta u)\|_{L^2(0,1)} + \|(A_0 - A_1)(\eta u)\|_{L^2(0,1)} \Big).$$

Similarly to the proof of (2.5.5), we obtain

$$|||\eta u|||_{W^2(0,1)} \le k_7 |\lambda|^{\frac{1}{2}} \left(\|f\|_{L^2(0,1)} + |\lambda|^{-\sigma} |||u|||_{W^2(0,1)} + \|(A_0 - A_1)(\eta u)\|_{L^2(0,1)} \right).$$

It follows from the definition of $\eta(t)$ and inequality $\delta < \beta$ that $\sup p\eta \subset [0,\beta] \cup [1-\beta, 1]$. But $a_0(t) = a_0(0)$ for $t \in [0,\beta] \cup [1-\beta, 1]$. Therefore

$$\|(A_0 - A_1)(\eta u)\|_{L^2(0,1)} = 0.$$

Thus, we have

$$|||\eta u|||_{W^2(0,1)} \le k_7 \left(|\lambda|^{\frac{1}{2}} \|f\|_{L^2(0,1)} + |\lambda|^{-\chi} |||u|||_{W^2(0,1)} \right), \tag{2.5.6}$$

where $\chi = \frac{1}{6} - 2\gamma > 0$.

Therefore, choosing $q > q_2$ so that $(k_6 + k_7)q^{-\chi} < 1$, we derive the inequality (2.5.1) from (2.5.5) and (2.5.6). The proof is complete. □

Corollaire 2.5.1. *Let $\Delta_e \neq 0$. Then the following assertions are true. a) The operators*

$$L(\lambda) : W^2(0,1) \to W[0,1] \text{ and } \mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L^2(0,1) \to L^2(0,1),$$

possess the Fredholm property and $indL(\lambda) = ind\mathcal{A}_B = 0$ for all $\lambda \in \mathbb{C}$.

b) The spectrum $\sigma(\mathcal{A}_B)$ is discrete.

c) For $\mu \notin \sigma(\mathcal{A}_B)$, the resolvent

$$R(\mu, \mathcal{A}_B) = (\mu I - \mathcal{A}_B)^{-1} : L^2(0,1) \to L^2(0,1),$$

is a compact operator.

d) For any $0 < \delta < \pi$, all points of the spectrum $\sigma(\mathcal{A}_B)$, except, possibly, finitly many of them, belong to the angle $|\arg\mu| \leq \delta$ of the complex plane.

Proof. 1. Consider problem (2.2.1), (2.2.2) with $f(t) = 0$ and $\lambda \in \omega_{\epsilon,q}$. Then from inequality (2.5.1), we have $u = 0$. Therefore, the eigenvalues of $\mathcal{A}_B$ do not belong to the set

$$\Omega_{\delta,r} = \{\mu \in \mathbb{C} : |\arg\mu| \geq \delta,\ \ |\mu| \geq r\},$$

where $\delta = \pi - 2\epsilon$, $r = q^2$.

It is easy to see that if $\mu = -\lambda^2$ is not an eigenvalues of $\mathcal{A}_B$, then problem (2.2.1), (2.2.2) has a unique solution for any $f \in L^2(0,1)$. This fact, together with the inequality (2.5.1), implies that the operator $L(\lambda) : W^2(0,1) \to W[0,1]$ admits a bounded inverse $L^{-1}(\lambda) : W[0,1] \to W^2(0,1)$ for $\lambda \in \omega_{\epsilon,q}$. Therefore, the operator

$$\mu I - \mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L^2(0,1) \to L^2(0,1),$$

has a bounded inverse

$$(\mu I - \mathcal{A}_B)^{-1} : L^2(0,1) \to W^2(0,1),$$

for $\mu \in \Omega_{\delta,r}$. Since $W^2(0,1)$ is compactly embeded in $L^2(0,1)$, it follows that the spectrum $\sigma(\mathcal{A}_B)$ is discrete. Note that

$$\sigma(\mathcal{A}_B) \subset \{\mu \in \mathbb{C} : |\arg\mu| < \delta\} \cup \{\mu \in \mathbb{C} : |\mu| < r\}.$$

2. Let $\lambda_0 \in \omega_{\epsilon,q}$. Then, we have

$$L(\lambda) = \left[I + (L(\lambda) - L(\lambda_0))L^{-1}(\lambda_0)\right] L(\lambda_0)$$

for each $\lambda \in \mathbb{C}$, where I is the identity operator in $W[0,1]$. It is easy to see that

$$(L(\lambda) - L(\lambda_0))u = ((\lambda^2 - \lambda_0^2)u, 0, 0).$$

Since $W^2(0,1)$ is compactly embeded in $L^2(0,1)$, the operator

$$(L(\lambda) - L(\lambda_0))L^{-1}(\lambda_0) : W[0,1] \to W[0,1]$$

is a compact operator. Thus, it follows that $L(\lambda) : W^2(0,1) \to W[0,1]$ is a Fredholm operator and $indL(\lambda) = 0$Similarly, we can prove that the operator $\mathcal{A}_B : \mathcal{D}(\mathcal{A}_B) \subset L^2(0,1) \to L^2(0,1)$ is a Fredholm operator and $ind\mathcal{A}_B = 0$. □

Chapter 3

On a coupled system of differential equations with nonlocal conditions

3.1 Introduction

This chapter is devoted to the study of the system

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} - z &= 0, \\ \frac{\partial u}{\partial x} + v &= 0, \\ \frac{\partial^2 z}{\partial x^2} &= f(t,x), \end{aligned}$$

where $t \in [0,1]$ and $x \in [0,1]$, $u = u(t,x)$, $v = v(t,x)$, $z = z(t,x)$ are unknown functions, and $f(t,x)$ is a given function. This system is supplemented with the mixed nonlocal boundary conditions

$$u(t,0) = 0, \int_0^1 u(t,x)dx = 0, \text{ and } z(t,0) = 0, \int_0^1 z(t,x)dx = 0,$$

and the condition

$$\frac{\partial^2 u(0,x)}{\partial x^2} = u_0(x)$$

at $t = 0$, where $u_0(x)$ is a given function.

We consider this problem in the Banach space $E = L^p(0,1) \times L^p(0,1) \times L^p(0,1)$ (with respect to x) of elements $w = (u,v,z)$ for some p, $1 \leq p < \infty$.

The case of regular boundary conditions was studied by Yu. T. Silchenko [19, 20].

We study a system of coupled differential equations with mixed nonlocal boundary conditions. We transform this system to an abstract Cauchy problem, and we obtain the existence and uniquness solution of this problem with the use of an infinitely differentiable semigroup that has a singularity at zero.

3.2 Formulation of the problem

In Banach space $E = L^p(0,1) \times L^p(0,1) \times L^p(0,1)$ $(1 \leq p < \infty)$, we consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} - z = 0, \tag{3.2.1}$$

$$\frac{\partial u}{\partial x} + v = 0, \tag{3.2.2}$$

$$\frac{\partial^2 z}{\partial x^2} = f(t,x), \tag{3.2.3}$$

with the integral boundary conditions

$$u(t,0) = 0, \int_0^1 u(t,x)dx = 0, \text{ and } z(t,0) = 0, \int_0^1 z(t,x)dx = 0, \tag{3.2.4}$$

and the condition at $t = 0$

$$\frac{\partial^2 u(0,x)}{\partial x^2} = u_0(x). \tag{3.2.5}$$

Here $u = u(t,x)$, $v = v(t,x)$, $z = z(t,x)$ $(0 < t \leq 1, 0 \leq x \leq 1)$ are unknown functions, and $f(t,x)$, $u_0(x)$ are given functions.

Note that for $w=(u,v,z)\in E, \|w\|=\|u\|+\|v\|+\|z\|$ and we associate to problem (3.2.1)-(3.2.5) the operators

$$A=\begin{pmatrix} \frac{d^2}{dx^2} & -\frac{d}{dx} & 1 \\ -\frac{d}{dx} & -1 & 0 \\ 0 & 0 & -\frac{d^2}{dx^2} \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C=\frac{d^2}{dx^2}$$

with domains

$$D=D(A)=\left\{ w\in W_p^2\times W_p^1\times W_p^2,\ \ w=(u,v,z),\ u(0)=\int_0^1 u(x)dx=0,\ z(0)=\int_0^1 z(x)dx=0 \right\},$$

$D_B=D(B)=E$, and $D(C)=\left\{ z\in W_p^2(0,1),\ z(0)=0 \text{ and } \int_0^1 z(x)dx=0 \right\}$. Then problem (3.2.1)-(3.2.5) can be transformed to the abstract Cauchy problem

$$Bw_t'-Aw=F(t), \qquad Bw(0)=w_0, \tag{2.6}$$

where $F(t)=(0,0,f(t,.))$ and $w_0=\left(C^{-1}u_0,0,0\right)$.

After the elimination of z, this problem take the form

$$Bw_t'-Aw=BA^{-1}\Phi(t), \qquad Bw(0)=BA^{-1}w_1 \tag{3.2.6}$$

in the subspace $L^p(0,1)\times L^p(0,1)$ of elements $w=(u,v)^T$ with norm $\|w\|=\|u\|+\|v\|$, where

$$A=\begin{pmatrix} \frac{d^2}{dx^2} & -\frac{d}{dx} \\ -\frac{d}{dx} & -1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with the corresponding domains, $\Phi(t)=(2f(t,.),0)^T$, and $w_1=(2u_0,0)^T$.

3.3 Statement of the Main Result

In order to formulate the conditions that should be satisfied for the matrix operators A and B, we consider the estimate

$$\left\|B\left(\lambda B-A\right)^{-1}\right\|\leq M\left(1+|\lambda|\right)^{-r}. \tag{3.3.1}$$

Problem (3.2.6) is studied in this paper with the use of an infinitely differentiable semigroup that has a singularity at zero, to the construction of this semigroup, we make the main assumption : for each λ in the sector $S = \{\lambda \in \mathbb{C} : |\arg\lambda| < \pi - \delta\}$ and for some $\delta \in \left(0, \frac{\pi}{2}\right)$,

$$\begin{cases} \text{the generalized resolvent } \lambda B - A \text{ has a bounded inverse} \\ \text{the operators } B(\lambda B - A)^{-1} \text{ is bounded, and its norm satisfies condition (3.3.1) for some } r > 0. \end{cases} \tag{3.3.2}$$

Theorem 3.3.1. *Let the function $f(t,x)$ satisfy the Hőlder condition*

$$\|f(t+\Delta t,.) - f(t,.)\|_{L^P} \leq C\,|\Delta t|^{\nu}$$

with respect to t for some $\nu \in (0,1]$. Let $u_0 \in L^P$. Then problem (3.2.1)-(3.2.5) has a unique solution.

Theorem 3.3.2. *Let the operator A and the closed operator B satisfy condition (3.3.2) with some $r \in (0,1]$, and let the function $\Phi(t)$ satisfy the Hőlder condition*

$$\|\Phi(t+\Delta t,.) - \Phi(t,.)\| \leq C\,|\Delta t|^{\nu}$$

with some $\nu \in (0,1]$. Then problem (3.2.6) has a unique solution for each $w_1 \in E$.

3.4 Existence of solutions of the abstract Cauchy problem

To prove Theorem(3.3.2), we first study the homogeneous problem

$$Bw_t' - Aw = 0, \qquad 0 < t \leq 1, \tag{3.4.1}$$

$$Bw(0) = BA^{-1}w_1. \tag{3.4.2}$$

Lemma 3.4.1. *Let the operators A and B satisfy condition (2.5.2). Then for $t>0$, there exists an operators function $T(t)\in L(E)$ with the following properties:*

(1) $T(t):E\to D$;

(2) $T(t+s)=T(t)AT(s),t,s>0$;

(3) *if $t>0$, then the operator function $T(t)$ is continuously differentiable in the norm of the space $L(E)$ and $BT'(t)=AT(t)$;*

(4) $\lim BT(t)=BA^{-1}$ *in the norm of the space $L(E)$ as $t\to 0^{+}$;*

(5) $\|T(t)\|\le M_2t^{r-1}$, $\|T'(t)\|\le M_2t^{r-2}$, $\|BT'(t)\|\le M_2t^{r-1}$, *and* $\|AT(t)\|\le M_2t^{r-1}$.

Proof. Let $\Gamma=\Gamma_1\cup\Gamma_2$ the contour lying in the sector S such that

$$\Gamma_1=\left\{\lambda\in\mathbb{C}:\lambda=|\lambda|\,e^{-i\varphi}\text{ or }\lambda=|\lambda|\,e^{i\varphi},|\lambda|\ge R\right\},\Gamma_2=\left\{\lambda\in\mathbb{C}:\lambda=Re^{i\Psi},|\Psi|\le\varphi\right\},$$

for some $\varphi\in\left(\frac{\pi}{2},\pi-\delta\right)$, and we set

$$U(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}B\,(\lambda B-A)^{-1}\,d\lambda, \tag{3.4.3}$$

Obviously, from the estimate (2.5.1), we have the inequality

$$\left\|B\,(\lambda B-A)^{-1}\right\|\le M\,|\lambda|^{-r}$$

is valid on the contour Γ. And the integral in (3.4.3) admits the estimate

$$\left\|e^{\lambda t}B\,(\lambda B-A)^{-1}\right\|\le Me^{|\lambda|t\cos\varphi}\,|\lambda|^{-r},\qquad \cos\varphi<0.$$

Therefore, we obtain the absolute convergence of the integral (3.4.3) at infinity and

$$\|U(t)\|\le 2M\int\limits_{R}^{\infty}e^{-|\lambda|t|\cos\varphi|}\,|\lambda|^{-r}\,|d\lambda|+\int\limits_{\Gamma_2}e^{|\lambda|t\cos\varphi}\,|\lambda|^{-r}\,|d\lambda|.$$

The second integral is bounded by some constant M_1. And for $r<1$, we set $|\lambda|\,t\,|\cos\varphi|=s$. Then

$$\|U(t)\|\le 2Mt^{r-1}\int\limits_{0}^{\infty}e^{-s}s^{-r}ds+M_1\le M_2t^{r-1}$$

. If $r \geq 1$, we have

$$\|U(t)\| \leq 2M \left(t^{r-1} \int\limits_{Rt|\cos\varphi|}^{\infty} e^{-s} s^{-r} ds + \int\limits_{0}^{|\cos\varphi|} e^{-Rt|\cos\Psi|} R^{1-r} d\Psi \right)$$

. For small t, one can take $R = \frac{1}{t}$. Then

$$\|U(t)\| \leq 2M \left(t^{r-1} \int\limits_{|\cos\varphi|}^{\infty} e^{-s} s^{-r} ds + t^{r-1} \right) \leq M_2 t^{r-1}.$$

Similarly with the preceding considerations, we obtain the existence of the integral

$$U'(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} \lambda e^{\lambda t} B (\lambda B - A)^{-1} d\lambda,$$

and the estimate $\|U'(t)\| \leq M_2 t^{r-2}$. Obviously, the operator function $U(t)$ is infinitely differentiable for $t > 0$.

With the use of the generalized resolvent identity

$$(\lambda B - A)^{-1} B (\mu B - A)^{-1} = \frac{1}{\mu - \lambda} \left((\lambda B - A)^{-1} - (\mu B - A)^{-1} \right),$$

we can prove that the operator function $U(t)$ has the semigroup property $U(t)U(s) = U(t+s)$, $t, s > 0$.

Now, with the use of the identity

$$\lambda B (\lambda B - A)^{-1} = I + A (\lambda B - A)^{-1}, \tag{3.4.4}$$

we obtain

$$U'(t) = \frac{1}{2\pi i} \left(\int\limits_{\Gamma} e^{\lambda t} d\lambda I + \int\limits_{\Gamma} e^{\lambda t} A (\lambda B - A)^{-1} d\lambda \right) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{\lambda t} A (\lambda B - A)^{-1} d\lambda.$$

By applying the bounded operator BA^{-1} to both sides of the last relation, we obtain

$$BA^{-1} U'(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{\lambda t} B (\lambda B - A)^{-1} d\lambda = U(t).$$

Since A^{-1} is bounded, we have

$$B\left[A^{-1}U(t)\right]' = U(t).$$

By setting $A^{-1}U(t) = T(t)$, we have $BT'(t) = AT(t)$.

With the use of identity (3.4.4) and to clarify the behavior of the operator function $U(t)$ at zero, we transform the integral in (4.3) :

$$U(t) = \frac{1}{2\pi i}\left(\int_{\Gamma} \lambda^{-1} e^{\lambda t} d\lambda I + \int_{\Gamma} e^{\lambda t} \lambda^{-1} A\left(\lambda B - A\right)^{-1} d\lambda\right).$$

Here the first integral is equal to unity. Therefore

$$U(t) = I + \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t} \lambda^{-1} A\left(\lambda B - A\right)^{-1} d\lambda.$$

By applying the bounded operator BA^{-1} on the left and by passing to the limit as $t \to 0^{+}$, we obtain

$$\lim_{t \to 0^{+}} BA^{-1}U(t) = BA^{-1} + \frac{1}{2\pi i}\int_{\Gamma} \lambda^{-1} B\left(\lambda B - A\right)^{-1} d\lambda,$$

since the last integral is absolutely convergent. Moreover, it is zero. Indeed, consider the circle $\sigma^2 + \tau^2 = \rho^2 \quad (\rho > R)$. By Γ_ρ we denote the part of this circle lying in the sector S; we have $\lambda = \rho e^{i\Psi}$, $|\Psi| \le \varphi$, on this circle. By $\Gamma_{0\rho}$ we denote the part of the contour Γ lying inside the circle of radius ρ. Then the integral of the function $\lambda^{-1}B\left(\lambda B - A\right)^{-1}$ over the closed contour $\Gamma_\rho \cup \Gamma_{0\rho}$ is zero. Now we consider this closed contour as $\rho \to \infty$. Then

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_\rho} \lambda^{-1} B\left(\lambda B - A\right)^{-1} d\lambda \right\| \le \frac{1}{2\pi}\int_{-\varphi}^{\varphi} \frac{1}{\rho}\frac{M}{\rho^r}\rho d\Psi \le \frac{4}{\rho^r},$$

since $\varphi \in \left(\frac{\pi}{2}, \pi - \delta\right)$ by construction. The last expression tends to zero as $\rho \to \infty$. It follows that the above-mentioned integral is zero and $\lim BA^{-1}\left[U(t) - I\right] = 0$ as $t \to 0^{+}$.

Finaly, from the corresponding properties of the operator function $U(t)$ and the boundedness of the operator A^{-1}, we obtain The properties of the operator function $T(t)$ mentioned in the lemma. □

The item (3) and (4) in Lemma1 implies that the function $w(t) = T(t)w_1$ for some $w_1 \in E$ is a solution of the homogeneous problem (3.4.1)-(3.4.2).

Lemma 3.4.2. *Let condition(2.5.2) be valid for the operators A and B. If B is a closed operator, then for each $w_1 \in E$ problem (3.4.1)-(3.4.2) has a unique solution.*

Proof. We have the existence of solution from the preceding considerations. Let us prove the uniqueness. By Lemma 1, $\|w(t)\| \leq Mt^{r-1}$ $(r > 0)$. We introduce the function

$$g_\epsilon(\lambda) = \int_\epsilon^{\frac{1}{\epsilon}} e^{-\lambda t} w(t)dt \quad \text{for} \quad \operatorname{Re}\lambda > 0, \quad \epsilon > 0$$

, where $w(t)$ is a solution of problem (3.4.1)-(3.4.2) with $w_1 = 0$. Obviosly, $g_\epsilon(\lambda)$ has the limit

$$g(\lambda) = \int_0^\infty e^{-\lambda t} w(t)dt \quad \text{as} \quad \epsilon \to 0^+$$

. And

$$Ag_\epsilon(\lambda) = \int_\epsilon^{\frac{1}{\epsilon}} e^{-\lambda t} Aw(t)dt = \int_\epsilon^{\frac{1}{\epsilon}} e^{-\lambda t} Bw'(t)dt$$

$$= e^{-\frac{\lambda}{\epsilon}} Bw(\frac{1}{\epsilon}) - e^{-\lambda\epsilon} Bw(\epsilon) + \lambda \int_\epsilon^{\frac{1}{\epsilon}} e^{-\lambda t} Bw(t)dt$$

By passing to the limit as $\epsilon \to 0$, we obtain

$$Ag(\lambda) = \lambda B \int_0^\infty e^{-\lambda t} w(t)dt = \lambda Bg(\lambda)$$

. Therefore, $Ag(\lambda) = \lambda Bg(\lambda)$ and $(\lambda B - A)g(\lambda) = 0$. Since λ belongs to the generalized resolvent set, it follows that $g(\lambda) = 0$ for $\operatorname{Re}\lambda > 0$. This is possible only if $w(t) = 0$. □

Now, consider the the abstract Cauchy problem (3.2.6)Let

$$h(t) = \int_0^t T(t-s)f(s)ds,$$

where $f(t)$ is a function defined on $[0,1]$ and satisfying the Hőlder condition

$$\|f(t+\Delta t)-f(t)\| \leq c|\Delta t|^{\nu} \tag{3.4.5}$$

for some $\nu \in (0,1]$. Let $r \in (0,1]$. We set $h_\epsilon(t) = \int\limits_0^{t-\epsilon} T(t-s)f(s)ds$, $\epsilon > 0$. Obviously, $h_\epsilon(t) \to h(t)$ uniformly (with respect to $t > 0$) as $\epsilon \to 0$; it is continuously differentiable, and

$$h'_\epsilon(t) = \int\limits_0^{t-\epsilon} T(t-s)f(s)ds + T(\epsilon)f(t-\epsilon). \tag{3.4.6}$$

Since B is a closed operator and the Hőlder condition (3.4.5) we can prove the existence of the integral $\int\limits_0^{t} BT'(t-s)f(s)ds$ with the use of methods in [2,3]. Then by applying the operator B to the relation (3.4.6), and let $\epsilon \to 0$ on the right-hand side, we have $Bh'_\epsilon(t)$ converges uniformly with respect to t to $Bh'(t)$ as $\epsilon \to 0$. Consequensly,

$$Bh'(t) = \int\limits_0^{t} BT'(t-s)f(s)ds + BA^{-1}f(t).$$

Since $BT'(t) = AT(t)$, it follows that

$$Bh'(t) = \int\limits_0^{t} AT(t-s)f(s)ds + BA^{-1}f(t) = Ah(t) + BA^{-1}f(t).$$

This implies that the function $h(t)$ satisfies the nonhomogeneous equation

$$Bh'(t) = Ah(t) + BA^{-1}f(t),$$

and $h(t) \to 0$ as $t \to 0^+$. Therefore, the function

$$w(t) = T(t)w_1 + \int\limits_0^{t} T(t-s)f(s)ds \tag{18}$$

is a solution of problem (3.2.6) for $\Phi(t) = f(t)$.

3.5 Proof of main result

Consider the generalized resolvent equation $(\lambda B - A)w = \Phi$, $\Phi = (f,g)^T$, $w = (u,v)^T$, by the construction of the operators A and B, this equation take the form

$$-u''_{xx} + \lambda u + v'_x = f(x), \qquad u'_x + v = g(x), \qquad x \in [0,1],$$
$$u(0) = 0, \int\limits_0^1 u(x)dx = 0.$$

We have $v(x) = g(x) - u'_x$ and $v'(x) = g'(x) - u''_x$, then we obtain the problem

$$-2u''_{xx} + \lambda u = f(x) - g'(x), \tag{3.5.1}$$

$$u(0) = 0, \int\limits_0^1 u(x)dx = 0. \tag{3.5.2}$$

The strong solution of equation (3.5.1) is given by the formula

$$u(x) = \frac{1}{4\rho}\left(\int\limits_0^x e^{(s-x)\rho} f(s)ds + \int\limits_x^1 e^{(x-s)\rho} f(s)ds\right) + \frac{1}{4}\left(\int\limits_0^x e^{(s-x)\rho} g(s)ds - \int\limits_x^1 e^{(x-s)\rho} g(s)ds\right)$$
$$+\frac{1}{4\rho}\left(g(0)e^{-\rho x} - g(1)e^{\rho(x-1)}\right). \tag{3.5.3}$$

Substituting (3.5.3) into the condition (3.5.2), we obtain

$$g(0) = \frac{\int\limits_0^1 \left(2e^{-\rho} - e^{\rho(s-2)} - e^{-\rho s}\right) f(s)ds - \rho\int\limits_0^1 \left(e^{\rho(s-2)} - e^{-\rho s}\right) g(s)ds}{\left(e^{-\rho} - 1\right)^2}$$

$$g(1) = \frac{\int\limits_0^1 \left(2 - e^{\rho(s-1)} + e^{-\rho(s+1)} - 2e^{-\rho s}\right) f(s)ds - \rho\int\limits_0^1 \left(e^{-\rho(s+1)} + e^{\rho(s-1)} - 2e^{-\rho s}\right) g(s)ds}{\left(e^{-\rho} - 1\right)^2}.$$

Therfore, for the functions u and v, we have the closed-form expressions

$$u(x) = \frac{1}{4\rho}\left(\int\limits_0^x e^{(s-x)\rho} f(s)ds + \int\limits_x^1 e^{(x-s)\rho} f(s)ds\right) + \int\limits_0^1 G_1(x,s,\rho) f(s)ds$$
$$+\frac{1}{4}\left(\int\limits_0^x e^{(s-x)\rho} g(s)ds - \int\limits_x^1 e^{(x-s)\rho} g(s)ds\right) + \int\limits_0^1 G_2(x,s,\rho) g(s)ds,$$
$$v(x) = g(x) - u'_x = g(x) - G_3(x,\rho).$$

Where $G_1(x,s,\rho), G_2(x,s,\rho)$, and $G_3(x,\rho)$ are some functions, and $\rho = \sqrt{\frac{\lambda}{2}}$. Further, we note that $(\lambda B - A)^{-1}(f,g)^T = (u,v)^T$. Therfore, by virtue of the structure of the matrix B,

$$B(\lambda B - A)^{-1}(f,g)^T = (u,0)^T$$

. Since the function $u(x)$ is represented in closed form, by straightforward computations we can obtain the estimate $\left\|B(\lambda B - A)^{-1}\right\| \leq c|\lambda|^{-\frac{1}{2}}$ in the sector $|\arg\lambda| < \pi$.

Let us now return to the the abstract Cauchy problem (3.2.6). We note that the matrix operators A and B satisfy the assumptions of theorem 2.

Further, by simple computations, we obtain

$$BA^{-1}\begin{pmatrix} 2f(t,x) \\ 0 \end{pmatrix} = \begin{pmatrix} \int_0^x (x-s)f(t,s)ds - x\int_0^1 (s-1)^2 f(t,s)ds \\ 0 \end{pmatrix} = \begin{pmatrix} C^{-1}f(t,x) \\ 0 \end{pmatrix}. \tag{3.5.4}$$

Therefore, by matching (3.5.4) in (3.2.6), we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} = C^{-1}f(t,x)$$
$$\frac{\partial u}{\partial x} + v = 0.$$

Now, by setting $z = C^{-1}f$, we find that the functions u, v, and z satisfy equationsqs (3.2.1), (3.2.2) and (3.2.3). Similarly, one can verity the validity of condition (3.2.5). The proof is complete.

Chapter 4

On the solvability of the parabolic equation with a nonlocal conditions

4.1 Introduction

In this chapter, for a linear parabolic equation with the principal part in divergence form, a boundary value problem with nonlocal(irregular) conditions of integral type is considered. Sufficient conditions of the unique solvability are found for the above mentioned problem.

We consider a boundary value problem for one dimensional parabolic equation with integral conditions as boundary conditions and instead of the initial value condition.we imposed the nonlocal condition of integral type. Boundary value problems for parabolic equations in which one local classical condition is replaced by an integral condition were studied in.[30] by various methods. Integral conditions are much wider used as overdetermination conditions in the analysis of inverse problems for parabolic equations [36]. The problems with integral conditions for partial differential equations were considered in numerous works in the recent years, since such problems have important practical applications. The integral conditions are used, in particular, in models of heat conduction [38, 30, 60] and humidity transfer [39], in demographic models [43], and in inverse models of the theory of heat conduction [36].

The models with integral conditions for ordinary differential equations and partial differential equations in bounded and unbounded domains are conditionally well posed. Therefore, the determination of conditions for the proper solvability of such problems arose the interest of many researchers (see, e.g., works [44, 45, 40, 42, 37] and references therein).The idea of using two integral conditions instead of boundary conditions was suggested in [36]. The study of this type of boundary value problems goes back to [34]. [31]. In this paper we obtain sufficient conditions for the unique solvability of the considered problem. in L_p $(1 \leq p < \infty)$ space. Note that a similar problem was considered in [49] in L_2 space with local classical condition and integral conditions as boundary conditions and the nonlocal condition of integral type.

4.2 Statement of problem

Consider the parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b(x)\,u(x) + f(t,x), \tag{4.2.1}$$

where $t \in (0,1]$, $x \in [0,1]$, $b(x)$, and $f(t,\ x)$ are given functions, and $b(x) \leq -b_0$ for some b_0 sufficiently large. We seek a solution $u = u(t,x)$ to (4.2.1) satisfying the boundary conditions

$$\int_0^1 \varphi_1(x)\,u(t,x)\,dx = 0, \int_0^1 \varphi_2(x)\,u(t,x)\,dx = 0, \tag{4.2.2}$$

and the condition

$$\int_0^1 \Phi_1(t)\,u(t,x) + \Phi_2(t)\,u'(t,x)\,dt = v_0(x), \tag{4.2.3}$$

where $\varphi_i(x), i = 1,2$ are given functions linear independent, $\Phi_1(x), \Phi_2(x)$, and $v_0(x)$ are given functions. Note that conditions (4.2.2) are not regular. The heat equation under similar conditions (for $\varphi_i(x) = 1$ and Dirichlet conditions) was considered in [60] and by [36]. The nonlocal conditions (4.2.3) is imposed on the function $u(t,x)$ instead of the initial value condition. Generally speaking , the problem with such a condition is not well defined [31, 34]. In the present paper, we study problem (4.2.1)-(4.2.3) and establish sufficients conditions of the unique solvability of the problem.

4.3 Reducing problem

The above mentioned problem is considered in the space L_p(with respect to $x \in [0,1]$). A function $u(t,x)$ is siad to be a solution of the problem if all of the terms of equation (4.2.1) belong to L_p, the differentiation with respect to t beig understood in the norm of L_p for $t > 0$, and conditions 4.2.2)-(4.2.3) are fulfilled. Introduce the following operators acting in L_p :

$$Au(x) = -\frac{d^2u}{dx^2} - b(x)u, \tag{4.3.1}$$

with the domain

$$D(A) = \left\{ u(x) \in W_p^2(0,1) : \int_0^1 \varphi_1(x)\, u(t,x)\, dx = \int_0^1 \varphi_2(x)\, u(t,x)\, dx = 0 \right\},$$

and the operator

$$\Phi(t)\, u(x) = \Phi_1(t)\, u(x) + \Phi_2(t)\, u'(x), \tag{4.3.2}$$

with the domain

$$D(\Phi(t)) = W_p^1(0,1).$$

Thus, the problem (4.2.1)-(4.2.3) is reduced to the abstract problem

$$\frac{du}{dt} + Au = f(t), 0 < t \leq 1, \tag{4.3.3}$$

$$\int_0^1 \Phi(t)\, u(t)\, dt = u_0 \tag{4.3.4}$$

in the space $E = L_p$, where A and $\Phi(t)$ are given operator valued functions, $f(t)$ is a given function taking values in E, and u_0 is a given element from E. A similar problem in the case of the homogeneous equation (4.3.3), a constant operator coefficient A, and a scalar function $\Phi(t)$ is considered in [34]. Note that here we use a class of semigroups (different from the one in [34]) that allows us to work with operators having nondense domains.

4.4 Solvability of reducing problem

We investigate the solvability of the problem (4.3.3)-(4.3.4) in arbitrary Banach space E. Let us make fundamental assumptions (for brevity, they are not too general) and cite some facts needed for the sequel. Suppose that the operator A has a bounded inverse A^{-1}, and for every t, there exist the operator function $U(t) = \exp(-tA)$ satisfying the following conditions:

1. $U(t)$ is a bounded linear operator from E to D, $t \in (0, \infty)$,
2. $U(t)U(s) = U(t+s), t, s > 0$,
3. $\lim\limits_{t \to +0} U(t)v = v$ for $v \in D$,
4. $U(t)$ is differentiable with respect to $t > 0$ and $\frac{dU(t)}{dt} = -AU(t)$,
5. $U(t)$ commutes with A on D,
6. the estimates

$$\|U(t)\| \le Mt^{-\alpha} \exp(-wt), \left\|U'(t)\right\| \le Mt^{-\beta} \exp(-wt) \tag{4.4.1}$$

   are valid for some $M > 0, w > 0, \alpha \ge 0, \beta \ge 1$.

The operator function $U(t)$ is called the semigroup (of class $A(\alpha, \beta)$) generated by the operator A. If $\overline{D} = E, \alpha = 0$, and $\beta = 1$, then this the calss of analytical semigroups.

Theorem 4.4.1. *Suppose that the conditions*

1. *there exists a bounded inverse operator* A^{-1},

2. *there exists a semigroup* $U(t)$ *of class* $A(\alpha,\beta)$ *generated by* A, *estimate (4.4.1) are satisfied and for some* $\omega > 0$, $0 \le \alpha < 1, 1+\alpha \le \beta$,

3. $u_0 \in D(A)$.

Then the homogeneous problem (4.3.3)-(4.3.4) has a unique solution $u(t)$, *is given by the formula* $u(t) = U(t)u_0$.

Note that under the conditions of Theorem 4.4.1, we have the limit relation

$$\lim_{t\to 0} A^{-1}U(t)v = A^{-1}v, v \in E. \tag{4.4.2}$$

In the right hand side, $f(t)$ of (4.3.3) satisfies the Holder condition

$$\|f(t+\triangle t) - f(t)\| \le c|\triangle t|^{\varepsilon} \tag{4.4.3}$$

for some $\varepsilon \in \left(\frac{\beta-1}{\beta-\alpha}, 1\right]$, then [49] the function $g(t) = \int\limits_0^t U(t,s) f(s)\,ds$ belongs to D and the estimate

$$\|Ag(t)\| \le ct^{-\alpha}\|f\|_{\varepsilon} \tag{4.4.4}$$

is valid. Here

$$\|f\|_{\varepsilon} = \max_{0\le t\le 1}\|f(t)\| + \sup_{0\le t\prec t+\triangle t\le 1} \frac{\|f(t+\triangle t) - f(t)\|}{(\triangle t)^{\varepsilon}}.$$

We also assume that the operator $\Phi(t)$ is subordinate to the operator A for all t, i.e.,

$$\|\Phi(t)v\| \le c\|Av\| \text{ as } v \in D_t,$$

where the positive constant c does not depend on t. In particular, this means that $D \subset D(\Phi(t))$ and the operators $\Phi(t)A^{-1}$ are bounded for all t. We assume that the operators $\Phi(t)A^{-1}$ are continuously differentiable in the norm of the space E and the operator $\Phi(0)$ has a bounded inverse $\Phi^{-1}(0)$. Turn to equation (4.3.3) with condition (4.3.4). the solution of this equation is written in the form

$$v(t) = U(t)v_0 + g(t) \tag{4.4.5}$$

with $v_0 = v(0)$ to be determined and $g(t) = \int_0^t U(t-s) f(s)\, ds$. In order to find v_0, we use Condition (4.3.4) and integrate equality (4.4.5) with respect to t from 0 to 1 after applying the operator $\Phi(t)$ to it. We get

$$v_1 = \int_0^1 \Phi(t)\, v(t)\, dt = \int_0^1 \Phi(t)\, U(t)\, v_0 dt + \int_0^1 \Phi(t)\, g(t)\, dt. \tag{4.4.6}$$

By virtue of estimates (4.4.1) and (4.4.4) with $\alpha < 1$, the latter integral exists (it can be written in the form $\int_0^1 \Phi(t)\, A^{-1} A g(t)\, dt$). Further, using property (4) of the semigroup operator, we integrate by parts the first term on the right hand-side of formula (4.4.6), substituting (4.4.2) for the lower limit

$$\int_0^1 \Phi(t)\, U(t)\, v_0 dt = -\int_0^1 \Phi(t)\, A^{-1} U'(t)\, v_0 dt = \left(\Phi(0)\, A^{-1} - \Phi(1)\, A^{-1} U(1)\right) v_0 + \int_0^1 \left(\Phi(t)\, A^{-1}\right)' U(t)\, v_0 dt.$$

Since the operator $\left(\Phi(t)\, A^{-1}\right)'$ is bounded, the integral on the right hand side exists; therefore, the integral (understod as an improper integral) on the left hand-side exists. Hence,

$$v_1 = \left(\Phi(0)\, A^{-1} - \Phi(1)\, A^{-1} U(1)\right) v_0 + \int_0^1 \left(\Phi(t)\, A^{-1}\right)' U(t)\, v_0 dt. + \int_0^1 \Phi(t)\, g(t)\, dt. \tag{4.4.7}$$

Let the operator $\Phi(0)\, A^{-1} \Phi(t)\, A^{-1}$ and its derivative be bounded (uniformly with respect to t)

$$\left\| \Phi(0)\, A^{-1} \Phi(t)\, A^{-1} \right\| \leq q, \quad \left\| \Phi(0)\, A^{-1} \left(\Phi(t)\, A^{-1}\right)' \right\| \leq p. \tag{4.4.8}$$

Then the operator $A\Phi^{-1}(0)$ can be applied to the right hand sides in (4.4.7). Hence, v_1 belongs to $D\left(A\Phi^{-1}(0)\right)$ and

$$A\Phi^{-1}(0)\, v_1 = \left(I - A\Phi^{-1}(0)\, \Phi(1)\, A^{-1} U(1)\right) v_0$$
$$+ A(0)\, \Phi^{-1}(0) \int_0^1 \left(\Phi(t)\, A^{-1}(t)\right)' U(t)\, v_0 dt. + A\Phi^{-1}(0) \int_0^1 \Phi(t)\, g(t)\, dt.$$

Introduce the bounded operators

$$K = A\Phi^{-1}(0)\,\Phi(1)\,A^{-1}U(1), \quad L = A\Phi^{-1}(0)\int_0^1 \left(\Phi(t)\,A^{-1}\right)' U(t)\,dt.$$

In terms of these operators, the latter relation can be rewritten as

$$A\Phi^{-1}(0)\,v_1 = (I-K)\,v_0 + Lv_0 + A\Phi^{-1}(0)\int_0^1 \Phi(t)\,g(t)\,dt. \tag{4.4.9}$$

Estimate the norms of the operators K and L, using inequalities (4.4.1) and (4.4.8)

$$\|K\| \le q\,\|U(1)\| \le Mq\exp(-\omega),$$
$$\|L\| \le p\int_0^1 \|U(t)\|\,dt \le p\int_0^1 Mt^{-\alpha}\exp(-\omega t)\,dt \le pM\int_0^\infty t^{-\alpha}\exp(-\omega t)\,dt = pM\frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}},$$

where Γ is the Gamma function. If $qM\exp(-\omega) < 1$, then the operator $I-K$ is continously invertible. Therefore, it follows from (4.4.9) that

$$(I-K)^{-1}A\Phi^{-1}(0)\,v_1 = v_0 + (I-K)^{-1}Lv_0 + (I-K)^{-1}A\Phi^{-1}(0)\int_0^1 \Phi(t)\,g(t)\,dt.$$

Estimate the norm of the operator $(I-K)^{-1}L$

$$\left\|(I-K)^{-1}L\right\| \le \frac{1}{1-Mq\exp(-\omega)}pM\frac{\Gamma(1-\alpha)}{\omega^{1-\alpha}}.$$

Take

$$\omega > \max\left\{\ln(2qM),\,(2pM\Gamma(1-\alpha))^{\frac{1}{1-\alpha}}\right\}. \tag{4.4.10}$$

Then

$$\left\|(I-K)^{-1}L\right\| < 1$$

and the expression written before v_0 can be inverted. Thus,

$$v_0 = \left(I + (I-K)^{-1}L\right)^{-1}(I-K)^{-1}A\Phi^{-1}(0)\left(v_1 - \int_0^1 \Phi(t)\,g(t)\,dt\right).$$

This relation defines the element v_0 uniquely. Substituting the obtained espression for v_0 in (4.4.5), we get the solution of the original problem

$$v(t) = U(t)\left(I + (I-K)^{-1}L\right)^{-1}(I-K)^{-1}A\Phi^{-1}(0)$$
$$\times\left(v_1 - \int_0^1 \Phi(t)\int_0^t U(t-s)f(s)\,dsdt\right) + \int_0^t U(t-s)f(s)\,ds. \quad (4.4.11)$$

Thus, we have established the following theorem.

Theorem 4.4.2. *Let the following conditions be fulfilled:*

1. *the operator A has a bounded inverse A^{-1},*
2. *the the operator function $\Phi(t)$ is subordinate to the operator A, there exists the bounded operator $\Phi^{-1}(0)$, the operator function $\Phi(t)A^{-1}$ is continuously differentaible, and condition (4.4.8) hold,*
3. *the operator A generates a semigroup of class $A(\alpha,\beta)$ with $0 \le \alpha < 1, \quad \alpha + 1 \le \beta$,*
4. *the number ω in estimates (4.4.1) satisfies condition (4.4.10), where q and p obeys inqualities (4.4.8),*
5. *the function $f(t)$ satisfies the Hölder condition (4.4.3) with some $\varepsilon \in \left(\frac{\beta-1}{\beta-\alpha}, 1\right]$,*
6. *$v_1 \in D\left(A\Phi^{-1}(0)\right)$.*

Then problem (4.3.3), (4.3.4) has a unique solution given by (4.4.11) and the solution of the probblem is estimated by

$$\|v(t)\| \le Mt^{-\alpha}\exp(-\omega t)\left(\|v_1\| + \|f\|_\varepsilon\right).$$

Let us verify that the operator A and $\Phi(t)$ introduced in Sec.2 and acting in the space L_p satisfy the conditions of Theorem 4.4.2.

4.5 An example

Consider the resolvent equation for the operator A in the simplest case where $b(x) = -b_0$, and $\varphi_1(0)\varphi_2(1) - \varphi_1(1)\varphi_2(0) \neq 0$ (for example $\varphi_1(x) = 1, \varphi_2(x) = x$). We have

$$-v'' + b_0 v + \lambda v = f(x)$$

with the conditions

$$\int_0^1 v(x)\,dx = 0, \int_0^1 x v(x)\,dx = 0.$$

The solution of this problem can be written out explicity

$$\begin{aligned}
v(x) = {} & \frac{-2\exp -\rho(x+1)}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 f(s)\,ds + \frac{\exp\rho(x-1)}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 f(s)\,ds \\
& + \frac{\exp(-\rho(x+1))(\exp(\rho x)-1)}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 \exp(-\rho s) f(s)\,ds \\
& - \frac{\exp(\rho(x-1))(1-\exp(-\rho))}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 \exp(-\rho s) f(s)\,ds \\
& + \frac{\exp(-\rho(x+1))}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 (\exp(-\rho s)+\exp(-\rho(1-s))) f(s)\,ds \\
& - \frac{\exp(\rho(x-1))}{\Delta(\rho)(1-\exp(-\rho))^2}\int_0^1 (\exp(-\rho s)+\exp(-\rho(1-s))) f(s)\,ds \\
& - \frac{1}{\Delta(\rho)}\int_0^x \exp(-\rho(x-s)) f(s)\,ds - \frac{1}{\Delta(\rho)}\int_x^1 \exp(-\rho(s-x)) f(s)\,ds,
\end{aligned}$$

where$\rho=\sqrt{\lambda+b_0}$ and $\Delta(\rho)=-2\rho$.

$$
\begin{aligned}
v(x) &= \exp(-\rho x)\frac{2(\exp(\rho)-1)}{\Delta(\rho)K(\rho)\rho^2}\int_0^1 sf(s)\,ds-\exp(-\rho x)\frac{(\exp(\rho)-1)}{\Delta(\rho)K(\rho)\rho}\int_0^1 \exp(\rho(s-1))f(s)\,ds\\
&+\exp(-\rho x)\frac{(\exp(\rho)-1)}{\Delta(\rho)K(\rho)\rho^3}\int_0^1(\exp(-\rho s)-\exp(\rho(s-1)))f(s)\,ds-\exp(-\rho x)\frac{2\frac{\exp(\rho)}{\rho}+2\frac{(1-\exp(\rho))}{\rho^2}}{\Delta(\rho)K(\rho)}\int_0^1 f(s)\,ds\\
&+\exp(-\rho x)\frac{\frac{\exp(\rho)}{\rho}+\frac{(1-\exp(\rho))}{\rho^2}}{\Delta(\rho)K(\rho)}\int_0^1(\exp(-\rho s)+\exp(-\rho(1-s)))f(s)\,ds\\
&+\exp(\rho x)\frac{-2\frac{\exp(-\rho)}{\rho}+2\frac{(1-\exp(-\rho))}{\rho^2}}{\Delta(\rho)K(\rho)}\int_0^1 f(s)\,ds-\exp(\rho x)\frac{-\frac{\exp(-\rho)}{\rho}+\frac{(1-\exp(-\rho))}{\rho^2}}{\Delta(\rho)K(\rho)}\int_0^1(\exp(-\rho s)+\exp(-\rho(1-s\\
&+\exp(\rho x)\frac{2(\exp(-\rho)-1)}{\Delta(\rho)K(\rho)\rho^2}\int_0^1 sf(s)\,ds-\exp(\rho x)\frac{2(\exp(-\rho)-1)}{\Delta(\rho)K(\rho)\rho^2}\int_0^1\exp(-\rho(1-s))f(s)\,ds\\
&+\exp(\rho x)\frac{(\exp(-\rho)-1)}{\Delta(\rho)K(\rho)\rho^2}\int_0^1(\exp(-\rho s)-\exp(-\rho(1-s)))f(s)\,ds\\
&-\frac{1}{\Delta(\rho)}\int_0^x\exp(-\rho(x-s))f(s)\,ds-\frac{1}{\Delta(\rho)}\int_x^1\exp(-\rho(s-x))f(s)\,ds,
\end{aligned}
$$

where

$$K(\rho)=\frac{-\rho(\exp(\rho)-\exp(-\rho))+2(\exp(\rho)+\exp(-\rho)-2)}{\rho^3}$$

while $v_1(x)$ joins the other terms of the expression for the solution. Calculating the norm of this function, we obtain the estimate

$$\left\|(A+\lambda I)^{-1}\right\|\leq C\frac{1}{|\lambda+b_0|^{\frac{1}{2}+\frac{1}{2p}}},\mathrm{Re}\,\lambda>-b_0.$$

in the general case, this estimate can be established by methods of [27, 33, 35]. The estimate allows one to construct see [35] the corresponfing semigroup of operators. For this semigroup, one has

$$\alpha=\frac{1}{2}-\frac{1}{2p},\beta=\frac{3}{2}-\frac{1}{2p},$$

and $\omega\geq b_0$ in estimates (4.4.1). Show that the operator A has a bouded inverse. Consider the equation

$$-\frac{d^2v}{dx^2}-b(x)v=f(x)$$

with the conditions

$$\int_0^1\varphi_1(x)v(x)\,dx=0,\int_0^1\varphi_2(x)v(x)\,dx=0.$$

For simplicity, suppose that $b(x)=-b_0, \varphi_1(x)=1$ and $\varphi_2(x)=x$. Put $z(x)=\int\limits_0^x (x-s)\,u(s)\,ds$ then we come to the problem

$$z'''' - b_0 z'' = -f(x),$$
$$z(0)=0, z(1)=0, z'(0)=0, z'(1)=0$$

for the function $z(x)$. Since the boundary value conditions are regular [71], and the unique solution of the homogeneous problem is zero, then there exists the green function $G(x,s)$, which allows us to persent the solution of this problem as follows:

$$z(x)=\int\limits_0^1 G(x,s)\,f(s)\,ds.$$

Hence,

$$v(x)=A^{-1}f=z''(x)=\int\limits_0^1 G''_{xx}(x,s)\,f(s)\,ds,\ v'(x)=\int\limits_0^1 G'''_{xxx}(x,s)\,f(s)\,ds.$$

Consider the operator $\Phi(t)\,A^{-1}$. From formula (4.3.2), we get

$$\begin{aligned}\Phi(t)\,A^{-1}f(x) &= \Phi_1(t)\,v(x)+\Phi_2(t)\,v'(x)\\ &= \Phi_1(t)\int\limits_0^1 G''_{xx}(x,s)\,f(s)\,ds+\Phi_2(t)\int\limits_0^1 G'''_{xxx}(x,s)\,f(s)\,ds.\end{aligned}$$

Therefore, $\left\|\Phi(t)\,A^{-1}f\right\| \le c\,\|f\|$ for any function $f(x)\in L_p$, which implies that the operator $\Phi(t)$ is subordinate to the operator A. Now we are ready to formulate conditions ensuring the solvability of problem(4.2.1), (4.2.3).

Theorem 4.5.1. *Let the following conditions be fulfilled:*

1. *the function $b(x)$ is continuous, $b(x)\le -b_0$ for sufficiently large $b_0>0$ (chosen according to the data such that (4.4.10) holfs),*

2. *the functions $\Phi_1(t),\Phi_2(t)$, and $\varphi_1(x),\varphi_2(x)$ are continuous and $\varphi_1(0)\,\varphi_2(1)-\varphi_1(1)\,\varphi_2(0)\neq 0$,*

3. *the function* $v_1(x)$ *belongs to* D,

4. *the function* $f(t,x)$ *satisfies the Holder condition in the norm of the space* L_p *with* $p \in [1,\infty[: \|f(t+\triangle t,.) - f(t,.)\| \leq c|\triangle t|^{\varepsilon}$ *for some* $\epsilon \in \left(\frac{1}{2} - \frac{1}{2p}, 1\right]$.

Then problem (4.2.1), (4.2.3) has a unique solution.

Chapter 5

Approximation of astract first order differential equation with integral condition

5.1 Introduction

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B; by $\rho(B)$ the resolvent set of B; by $\mathcal{N}(B)$ the null space of B and by $\mathcal{R}(B)$ the range of B.

Let A be a generator of analytic C_0-semigroup $U(t)$, defined on a Banach space E, which means that $A : D(A) \subseteq E \to E$ is a closed linear operator, such that

$$\left\|(\lambda I - A)^{-1}\right\|_{B(E)} \leq \frac{1}{1+|\lambda|}, \text{ for any } \operatorname{Re}\lambda \geq 0. \tag{5.1.1}$$

Consider in a Banach space E the equation

$$u'(t) = Au(t), t \in [0,T] \tag{5.1.2}$$

Definition 5.1.1. *The vector fonction $u(t) = U(t) f$; $0 \le t \le T$, corresponding to some element $f \in E$ is called a generalized solution of (5.1.2). If, in addition, $f \in D(A)$, then the solution $u(t) = U(t) f$ is said to be classical.*

Remark 5.1.1. *In the case, when $f \in D(A)$ obviously f coincides with the initial state $u(0)$ of the corresponding solution $u(t)$.*

Suppose that the initial state f is unknown, and consider the additional relation

$$\int_0^T w(t) u(t) \, dt = g; \tag{5.1.3}$$

where $g \in E$ is a given element in E and $w(t)$ is scalar measurable function of bounded variation on the segment $[0, \ T]$

Remark 5.1.2. *The integral occurring in (5.1.3) is well-defined in the sense of Bochner for any function $u(t) = U(t) f$.*

Definition 5.1.2. *A generalized solution of the problem (5.1.2), (5.1.3) is defined to be a function $u(t) = U(t) f$; $0 \le t \le T$, corresponding to some element $f \in E$ and reducing relation (5.1.3) to a valid identity. If, in addition $f \in D(A)$, then the corresponding solution $u(t) = U(t) f$ of the problem (5.1.2), (5.1.3) is called a classical solution.*

From Definition 1, the solution of (5.1.2) is given in the form $u(t) = U(t) f$. Therefore, the function $u(t) = U(t) f$ satisfies the condition (5.1.3) if and only if f satisfies the equation

$$\int_0^T w(t) U(t) f dt = g. \tag{5.1.4}$$

So, for $f \in E$, we have the operator equation $Bf = g$, where,

$$Bf = \int_0^T w(t) U(t) f dt.$$

Lemma 5.1.1. *([25]). The operator B maps E into $D(A)$.*

Remark 5.1.3. *If $g \in E \backslash D(A)$, then the problem (5.1.2), (5.1.3) is unsolvable in the sense of the Definition 5.1.2.*

Now applying the operator A in (5.1.4) and integrating by parts, we get the Fredholm second order equation in the form

$$(I-K)f = G, \tag{5.1.5}$$

where,

$$Kf = \left(\frac{w(T)}{w(0)} U(T) + \frac{1}{w(0)} \int_0^T U(t)\, d(-w(t)) \right) f, \tag{5.1.6}$$

and,

$$G = -\frac{1}{w(0)} Ag. \tag{5.1.7}$$

In such settings one would say that the problem (5.1.2), (5.1.3) is well posed if the element g is given in the space $D(A)$ and the unknow element f is considered as an element from the space E. From (5.1.1) it follows that the resolvent $(\lambda I - A)^{-1}$ exists for $\lambda = 0$ and is positive operator, and therefore A^{-1} exists, which implies equivalence of the problem (5.1.2), (5.1.3) to the Fredholm second order equation (5.1.5). Using ideas from [10] the aim of this paper is the construction of an algorithm for the approximation of an element f, which solves the problem (5.1.2), (5.1.3) or, in other words, we want to solve equation (5.1.5). We present the algorithm as a general approximation scheme, which includes finite element methods and finite difference methods and projection methods.

The main question is a solvability of the problem (5.1.2), (5.1.3). It is clear that in the case of compact operator K the operator $(I-K)$ is Fredholm operator of index 0. Most of the results on the existence of solution of the problem (5.1.2), (5.1.3) are concerned to compactness or positivity property of resolvent of operator A. So the existence of bounded inverse operator $(I-K)^{-1}$ follows practically from condition $\mathcal{N}(I-K) = \{0\}$

and compact convergence of resolvent, see Theorem 5.2.2 and Step 4 of the proof of Theorem 5.5.1. There are some theorems proved, say, in [25, 26], which guarantee that condition $\mathcal{N}(I-K)=\{0\}$ holds. Namely, let us list some results which could be applied here.

Consider in a Banach space E the problem of finding an element f from relations

$$u^{'}(t)=Au(t)\,, t\in[0,\ T]\,, \tag{5.1.8}$$

with

$$\int_0^T w(t)\,u(t)\,dt=g, \tag{5.1.9}$$

where $g\in E$ is a given element in E and $w(t)$ is scalar measurable function of bounded variation on the segment $[0,\ T]$.

Theorem 5.1.1. *([25, 26]). Let $w(t)$ be a nonnegative non increasing function for $t\in[0,T]$ such that $w(t)>0$ as $t\to 0^+$, and let the semigroup $U(t)$ generated by the operator A satisfy the estimate $\|U(t)\|\leq M\exp(-\beta t)$ with constants $M\geq 1,\ \beta>0$. Then the problem (5.1.8)-(5.1.9) is well-posed.*

If E is a Banach lattice. We recall that an order set $(E,\preceq)$ is called a lattice if for any pair of elements $x,y\in E$ the elements $\sup(x,y)$ and $\inf(x,y)$ exist in E. Moreover, for any $x\in E$ we define $x^+=\sup(x,0),\ x^-=\inf(-x,0)$ which called positive and negative parts, respectively. The following relation is valid, $x=x^+-x^-$.

Definition 5.1.3. *Let B be a linear operator on E. The operator B is called positive if $Bx\succeq 0$ for all $x\succeq 0$.*

Definition 5.1.4. *A C_0- semigroup $\exp(tA)\,, t\geq 0$, is called positive in a Banach space with a cone E^+ if $\exp(tA)\,E^+\subseteq E^+$ for any $t\geq 0$.*

Definition 5.1.5. *A C_0- semigroup* $\exp(tA), t \geq 0$, *is positive iff resolvent* $(\lambda I - A)^{-1} E^{+} \subseteq E^{+}$ *for any* $\lambda > w(A)$.

Definition 5.1.6. *A linear* $A : D(A) \subseteq E \to E$ *is said to have the positive off-diagonal (**POD**) property if* $\langle Au, \phi \rangle \geq 0$ *whenever* $0 \preceq u \in D(A)$ *and* $0 \preceq \phi \in E^{*}$ *with* $\langle u, \phi \rangle = 0$.

Theorem 5.1.2. *([25, 26]). Let* $w(t)$ *be a nonnegative non increasing function for* $t \in [0, T]$ *such that* $w(t) > 0$ *as* $t \to 0^{+}$, *and let the semigroup* $U(t)$ *generated by the operator* A *be positive and compact for* $t > 0$. *Assume that the spectrum of* A *lies in the half-plane* $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$. *Then the problem (5.1.8)-(5.1.9) is well-posed.*

5.2 General approximation scheme

Now we give the algorithm on general approximation scheme which includes finite element and finite difference methods and projection methods.

The general approximation scheme, due to [7, 15, 17] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E; E_n), n \in IN = \{1, 2, ...\}$, with the property: $\|p_n x\|_{E_n} \to \|x\|_E$ as $n \to \infty$ for any $x \in E$.

Definition 5.2.1. *The sequence of elements* $\{x_n\}, x_n \in E_n, n \in IN$; *is said to be* $\mathcal{P}$-*convergent to* $x \in E$ *iff* $\|x_n - p_n x\|_{E_n} \to 0$ *as* $n \to \infty$ *and we write this* $x_n \xrightarrow{\mathcal{P}} x$.

Definition 5.2.2. *The sequence of bounded linear operators* $B_n \in B(E_n), n \in IN$, *is said to be* $\mathcal{PP}$-*convergent to the bounded linear operator* $B \in B(E)$ *if for every* $x \in E$ *and for every sequence* $\{x_n\}, x_n \in E_n, n \in IN$; *such that* $x_n \xrightarrow{\mathcal{P}} x$ *one has* $Bx_n \xrightarrow{\mathcal{P}} Bx$. *We write then* $B_n \xrightarrow{\mathcal{PP}} B$.

For general examples of notions of $\mathcal{P}$-convergence see for instance [14].

Remark 5.2.1. *If we put $E_n = E$ and $p_n = I$ for each $n \in IN$, where I is the identity operator on E, then Definition 5.2.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \to B$.*

Definition 5.2.3. *A sequence of elements $\{x_n\}, x_n \in E_n, n \in IN$, is said to be $\mathcal{P}$-compact if for any $IN^{'} \subset IN$ there exist $IN^{''} \subset IN^{'}$ and $x \in E$ such that $x_n \xrightarrow{\mathcal{P}} x$, as $n \to \infty$ in $IN^{''}$.*

Definition 5.2.4. *A system $\{p_n\}$ is said to be discrete order preserving if for all sequences $\{x_n\}, x_n \in E_n$, and any element $x \in E$, the following implication holds:*

$$x_n \xrightarrow{\mathcal{P}} x \text{ implies } x_n^+ \xrightarrow{\mathcal{P}} x^+.$$

It is know [8] that $\{p_n\}$ preserves the order iff $\left\|p_n x^+ - (p_n x)^n\right\|_{E_n} \to 0$ as $n \to \infty$ for any $x \in E$. If $B_n \xrightarrow{\mathcal{PP}} B$ and $B_n \succeq 0$ for $n \geq n_0$ and the system $\{p_n\}$ is order preserving, then [12] $B \succeq 0$. However, the inverse statement does not hold in general and we need to assume positiveness of $B_n \succeq 0$.

Definition 5.2.5. *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in IN$, converges compactly to an operator $B \in B(E)$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds:*

$$\|x_n\|_{E_n} = O(1), \{B_n x_n\} \text{ is } \mathcal{P} - \text{compact.}$$

Let us mention that the last implication could be writtten as $\mu(\{B_n x_n\}) = 0$ as $\|x_n\| \leq$constant for mesure of noncompactness $\mu(.)$. The main property of $\mu(.)$ is that$\mu(\{y_n\}) = 0$ iff $\{y_n\}$ is $\mathcal{P}$-compact. It is also easy to check that $\mu(\{x_n + y_n\}) \leq \mu(\{x_n\}) + \mu(\{y_n\})$ and $\mu(\{D_n x_n\}) \leq \overline{\lim}_{n\to\infty} \|D_n\| \|x_n\|$ for any operators $D_n \in B(E_n)$ and any sequences $\{x_n\}, \{y_n\}$.

Definition 5.2.6. *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in IN$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ and $\left\|B_n^{-1}\right\|_{E_n} = O(1), n \to \infty$. We will write this as: $B_n \xrightarrow{\mathcal{PP}} B$ stably.*

Definition 5.2.7. *A sequence of operators* $\{B_n\}, B_n \in B(E_n)$, $n \in IN$, *is called regularly convergent to the operator* $B \in B(E)$ *iff* $B_n \overset{\mathcal{PP}}{\to} B$ *and the following implication holds:* $\|x_n\|_{E_n} = O(1)$ *and* $\{B_n x_n\}$ *is* $\mathcal{P}$*-compact,* $\{x_n\}$ *is* $\mathcal{P}$*-compact. We write this as:* $B_n \overset{\mathcal{PP}}{\to} B$ *regularly.*

Theorem 5.2.1. *([17]). Let* $C_n, Q_n \in B(E_n), C, Q \in B(E)$ *and* $R(Q) = E$.
Assume also that $C_n \overset{\mathcal{PP}}{\to} C$ *compactly and* $Q_n \overset{\mathcal{PP}}{\to} Q$ *stably. Then* $Q_n + C_n \overset{\mathcal{PP}}{\to} Q + C$ *converge regularly.*

Theorem 5.2.2. *([17]). For* $B_n \in B(E_n)$ *and* $B \in B(E)$ *the following conditions are equivalent.*

(i) $B_n \overset{\mathcal{PP}}{\to} B$ *regularly,* B_n *are Fredholm operators of index* 0 *and* $\mathcal{N}(B) = \{0\}$*;*

(ii) $B_n \overset{\mathcal{PP}}{\to} B$ *stably and* $\mathcal{R}(B) = E$*;*

(iii) $B_n \overset{\mathcal{PP}}{\to} B$ *stably and regularly;*

If one of conditions (i)–(iii) holds, then there exist $B_n^{-1} \in B(E_n)$, $B^{-1} \in B(E)$, *and* $B_n^{-1} \overset{\mathcal{PP}}{\to} B^{-1}$ *regularly and stably.*

Definition 5.2.8. *The region of stability* $\Delta_s = \Delta_s(\{A_n\}), A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \rho(A_n)$ *for almost all* n *and such that the sequence* $\left\{\left\|(\lambda I_n - A_n)^{-1}\right\|\right\}_{n \in \mathbb{N}}$ *is bounded. The region of convergence* $\Delta_c = \Delta_c(\{A_n\}), A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \Delta_s(\{A_n\})$ *and such that the sequence of operators* $\left\{(\lambda I_n - A_n)^{-1}\right\}_{n \in \mathbb{N}}$ *is* $\mathcal{PP}-$ *convergent to some operator* $S(\lambda) \in B(E)$.

Definition 5.2.9. *A sequence of operators* $\{L_n\}, L_n \in \mathcal{C}(E_n)$, *is said regularly compatible with an operator* $L \in \mathcal{C}(E)$ *if* (L_n, L) *are compatible and, for any bounded sequence* $\|x_n\|_{E_n} = O(1)$ *such that* $x_n \in D(L_n)$ *and* $\{L_n x_n\}$ *is* $\mathcal{P}$*-compact, it follows that* $\{x_n\}$ *is* $\mathcal{P}$*-compact, and the* $\mathcal{P}$*- convergence of* $\{x_n\}$ *to some element* x *and convergence of* $\{L_n x_n\}$ *to some element* y *as* $n \to \infty$ *in* $\mathbb{N}' \subseteq \mathbb{N}$ *imply that* $x \in D(L)$ *and* $Lx = y$.

Definition 5.2.10. *The region of regularity* $\Delta_r = \Delta_r(\{A_n\}, A)$*, is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $(L_n(\lambda), L(\lambda))$ *are regularly compatible, where* $L_n(\lambda) = \lambda I_n - A_n$ *and* $L(\lambda) = \lambda I - A$.

The relationships between these regions are given by the following statement.

Proposition 5.2.1. *([16]). Suppose that* $\Delta_c \neq \emptyset$ *and* $\mathcal{N}(S(\lambda)) = \{0\}$ *at least for one point* $\lambda \in \Delta_c$*, so that* $S(\lambda) = (\lambda I - A)^{-1}$*. Then* (A_n, A) *are compatible and*

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

Definition 5.2.11. *The region of compact convergence of resolvent,*
$\Delta_{cc} = \Delta_{cc}(A_n, A)$*, where* $A_n \in \mathcal{C}(E_n)$ *and* $A \in \mathcal{C}(E)$ *is defined as the set of all* $\lambda \in \Delta_c \cap \rho(A)$ *such that* $(\lambda I_n - A_n)^{-1} \overset{\mathcal{PP}}{\to} (\lambda I - A)^{-1}$ *compactly.*

Theorem 5.2.3. *([6]). Assume that* $\Delta_{cc} \neq \emptyset$*. Then for any* $\mu \in \Delta_s$ *the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \text{ and } \|(\mu I_n - A_n)x_n\|_{E_n} = O(1) \Rightarrow \{A_n\} \text{ is } \mathcal{P}-\text{compact} \qquad (5.2.1)$$

Conversely, if for some $\mu \in \Delta_c \cap \rho(A)$ *implication (5.2.1) holds, then* $\Delta_{cc} \neq \emptyset$.

Corollaire 5.2.1. *([6]). Assume that* $\Delta_{cc} \neq \emptyset$*. Then* $\Delta_{cc} = \Delta_c \cap \rho(A)$.

Theorem 5.2.4. *([6]). Assume that* $\Delta_{cc} \neq \emptyset$*. Then* $\Delta_r = \mathbb{C}$.

In the case of unbounded operators, and we know in general infinitesimal generators are unbounded, we consider the notion of compatibility.

Definition 5.2.12. *The sequence of closed linear operators* $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in IN$*, are said to be compatible with a closed linear operator* $A \in \mathcal{C}(E)$ *iff for each* $x \in D(A)$ *there is a sequence* $\{x_n\}$*,* $x_n \in D(A_n) \subseteq E_n, n \in IN$*, such that* $x_n \overset{\mathcal{P}}{\to} x$ *and* $A_n x_n \overset{\mathcal{P}}{\to} Ax$*. We write* (A_n, A) *are compatible.*

Note, that (A_n, A) are compatible if resolvent converge $(\lambda I_n - A_n)^{-1} \stackrel{\mathcal{PP}}{\to} (\lambda I - A)^{-1}$. Usually in practice Banach spaces E_n are finite dimensional, although, in general, say for the case of a closed operator A, we have $dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

5.3 Discretization of semigroups

Let us consider the well-posed Cauchy problem in the Banach space E with operator $A \in \mathcal{C}(E)$

$$\begin{cases} u'(t) = Au(t); t \in [0; \infty), \\ u(0) = u^0 \in E, \end{cases} \tag{5.3.1}$$

where operator A generates C_0-semigroup $U(t)$. It is well-known that the C_0-semigroup gives the solution of (5.3.1) by the formula $u(t) = U(t) u^0$ for $t \geq 0$. The theory of well-posed problems and numerical analysis of these problems have been developed extensively, see [6, 9]. Let us consider on the general discretization scheme the semidiscrete approximation of the problem (5.3.1) in the Banach spaces E_n,

$$\begin{cases} u_n'(t) = A_n u_n(t); t \in [0; \infty), \\ u_n(0) = u_n^0 \in E_n, \end{cases} \tag{5.3.2}$$

with the operators $A_n \in \mathcal{C}(E_n)$, such that they generate C_0-semigroups, which are consistent with the operator $A \in \mathcal{C}(E)$ and $u_n^0 \stackrel{\mathcal{P}}{\to} u^0$.

5.4 The simplest discretization schemes

We have the following version of Trotter-Kato's Theorem on general approximation scheme.

Theorem 5.4.1. *([14, Theorem ABC]). Assume that $A \in C(E); A_n \in C(E_n)$ and they generate C_0-semigroups. The following conditions (A) and (B) are equivalent to condition*

(C).

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ *such that the resolvents converge* $(\lambda I_n - A_n)^{-1} \overset{\mathcal{PP}}{\rightarrow} (\lambda I - A)^{-1}$;

(B) *Stability. There are some constants* $M \geq 1$ *and* ω; *which are not depending on* n *and such that* $\|U_n(t)\| \leq M \exp(\omega t)$ *for* $t \geq 0$ *and any* $n \in \mathbb{N}$;

(C) *Convergence. For any finite* $T > 0$ *one has* $\max\limits_{t \in [0;T]} \left\|U_n(t) u_n^0 - p_n U(t) u^0\right\| \to 0$ *as* $n \to \infty$; *whenever* $u_n^0 \overset{\mathcal{P}}{\rightarrow} u^0$ *for any* $u_n^0 \in E_n$; $u^0 \in E$.

Remark 5.4.1. *The condition* (A) *in the contents of these Theorems is equivalent to compatibility of operators* (A_n, A).

Theorem 5.4.2. *([6]) Let operators* A *and* A_n *generate analytic* C_0*-semigroup. The following conditions* (A) *and* (B_1) *are equivalent to condition* (C_1).

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ *such that the resolvents converge* $(\lambda I_n - A_n)^{-1} \overset{\mathcal{PP}}{\rightarrow} (\lambda I - A)^{-1}$;

($\mathbf{B}_1$) *Stability. There are some constants* $M_1 \geq 1$ *and* ω_1 *independent of* n *such that for any* $Re\lambda > \omega_1$, $\left\|(\lambda I_n - A_n)^{-1}\right\| \leq \frac{M_1}{|\lambda - \omega_1|}$ *for all* $n \in \mathbb{N}$;

($\mathbf{C}_1$) *Convergence. For any finite* $\mu > 0$ *and some* $0 < \theta < \frac{\pi}{2}$ *we have* $\max\limits_{\eta \in \Sigma(\theta,\mu)} \left\|U_n(\eta) u_n^0 - p_n U(\eta) u^0\right\| \to 0$ *as* $n \to \infty$ *whenever* $u_n^0 \overset{\mathcal{P}}{\rightarrow} u^0$.

Here $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ *and* $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$.

Definition 5.4.1. *An element* $e \in E^+$ *is said to be an order unit in a Banach lattice* E *if for every* $x \in E$ *there exists* $0 \leq \lambda \in \mathbb{R}$ *such that* $-\lambda e \preceq x \preceq \lambda e$. *For* $e \in intE^+$ *we can define the order unit norm by*

$$\|x\|_e = \inf\{\lambda \geq 0 : -\lambda e \preceq x \preceq \lambda e\}.$$

An order Banach space E is callad an order unit space if there exists $e \in intE^{+}$ such that $\|.\|_E = \|.\|_e$.

The following version of the Trotter-Kato's Theorem for positive C_0- semigroup holds.

Theorem 5.4.3. *([12]) Let the operators A_n and A from (5.3.1) and (5.3.2) be compatible, let E, E_n be order unit spaces, and let $e_n \in D(A_n) \cap intE_n^{+}$. Assume that the operators A_n have the POD property and $A_n e_n \preceq 0$ for sufficiently large n. Then $\exp(tA_n) \overset{\mathcal{PP}}{\rightarrow} \exp(tA)$ uniformly in $t \in [0,T]$.*

We can assume that conditions (A) and (B) for the corresponding C_0- semigroups case are satisfied without any restriction of generality if any discretization processes in time are considered.

We denote by $T_n(.)$ a family of discrete semigroups as in [9], i.e. $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = \left[\frac{t}{\tau_n}\right]$, as $n \to 0$, $n \to \infty$. The generator of discrete semigroups is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and so $T_n(t) = \left(I_n + \tau_n \breve{A}_n\right)^{k_n}$; where $t = k_n \tau_n$.

Theorem 5.4.4. *([14, Theorem ABC- discr]). The following conditions (A) and (B_0) are equivalent to condition (C_0): (A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho\left(\breve{A}_n\right)$ such that the resolvents converge $\left(\lambda I_n - \breve{A}_n\right)^{-1} \overset{\mathcal{PP}}{\rightarrow} (\lambda I - A)^{-1}$, (B_0) Stability. There are some constants $M_2 \geq 1$ and $\omega_2 \in \mathbb{R}$ such that $\|T_n(t)\| \leq M_2 \exp(\omega_2 t)$ for $t \in IR+$, $n \in IN$, (C_0) Convergence. For any finite $T > 0$ one has $\max\limits_{t \in [0;T]} \left\|T_n(t) u_n^0 - p_n \exp(tA) u^0\right\| \to 0$ as $n \to \infty$, whenever $u_n^0 \overset{\mathcal{P}}{\rightarrow} u^0$ for any $u_n^0 \in E_n$, $u^0 \in E$.*

Theorem 5.4.5. *([14]). Assume that $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ and they generate C_0-semigroup. Assume also that conditions (A) and (B) of Theorem 5.4.1 holds. Then the implicit difference scheme*

$$\frac{\overline{U_n}(t+\tau_n) - \overline{U_n}(t)}{\tau_n} = A_n \overline{U_n}(t+\tau), \overline{U_n}(0) = u_n^0, \tag{5.4.1}$$

is stable, i.e. $\left\|(I_n - \tau_n A_n)^{-k_n}\right\| \le M_2 \exp(\omega_2 t)$, $t = k_n\tau_n \in IR+$; *and gives an approximation to the solution of the problem (5.3.1), i.e.* $\overline{U_n}(t) \equiv (I_n - \tau_n A_n)^{-k_n} u_n^0 \xrightarrow{\mathcal{P}} \exp(tA) u_n^0$ $\mathcal{P}$*-converges uniformly with respect to* $t = k_n\tau_n \in [0;T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $n \to \infty$, $k_n \to \infty$, $\tau_n \to 0$.

Theorem 5.4.6. *([6]). Assume that conditions* (A) *and* (B_1) *of Theorem 5.4.2 hold and condition*

$$\|\tau_n A_n\| \le \frac{1}{(M+2)}, n \in IN, \tag{5.4.2}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t+\tau_n) - U_n(t)}{\tau_n} = A_n U_n(t), U_n(0) = u_n^0, \tag{5.4.3}$$

is stable and gives an approximation to the solution of the problem (5.1.2), i.e. $U_n(t) \equiv (I_n + \tau_n A_n)^{k_n} u_n^0 \xrightarrow{\mathcal{P}} u(t)$ *discretely* $\mathcal{P}$*-converge uniformly with respect to* $t = k_n\tau_n \in [0;T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $n \to \infty$, $k_n \to \infty$, $\tau_n \to 0$.

Let us introduce the following equivalent conditions:

(B_1') Stability. There are constants $M^{'}$, $\omega^{'}$ such that

$$\|\exp(tA_n)\| \le M^{'} \exp\left(\omega^{'} t\right), \|A_n \exp(tA_n)\| \le \frac{M^{'}}{t} \exp\left(\omega^{'} t\right), t \in IR+.$$

(B_1'') Stability. There are constants $M^{'^{'}}$, $\omega^{'^{'}}$ and $\tau^* > 0$ such that

$$\left\|(I_n - \tau_n A_n)^{-k}\right\| \le M^{'^{'}} \exp\left(\omega^{'^{'}} k\tau_n\right),$$
$$\left\|k\tau_n A_n (I_n - \tau_n A_n)^{-k}\right\| \le M^{'^{'}} \exp\left(\omega^{'^{'}} k\tau_n\right) \text{ for } 0 < \tau_n < \tau^*, n, k \in IN.$$

Theorem 5.4.7. *The conditions* (A) *and* (B_1') *are equivalent to the condition* (C_1).

Proof. See ([14]). □

Remark 5.4.2. *Conditions* (B_1), (B_1') *and* (B_1'') *are equivalent, see ([13])*

5.5 Main results

Let A_n be a generator of compact analytic C_0-semigroup $U_n(t)$. Consider in a Banach space E_n the equations

$$u_n^{'}(t) = A_n u_n(t), t \in [0,T] \tag{5.5.1}$$

with the integral conditions

$$\int_0^T w_n(t) u_n(t) \, dt = g_n. \tag{5.5.2}$$

The solution of the problem (5.5.1), (5.5.2) is given by the formula $u_n(t) = U_n(t) f_n$, where $f_n = (I - K_n)^{-1} G_n$ and corresponding second order Fredholm equation can be written in the form:

$$(I_n - K_n) f_n = G_n, \tag{5.5.3}$$

where

$$K_n f_n = \left(\frac{w_n(T)}{w_n(0)} U_n(T) + \frac{1}{w_n(0)} \int_0^T U_n(t) \, d(-w_n(t)) \, t \right) f_n, \tag{5.5.4}$$

and

$$G_n = -\frac{1}{w_n(0)} A_n g_n$$

Before we formulate our main results just recall that condition $\mathcal{N}(I-K) = \{0\}$ could be obtained from Theorems in Section 2.

Theorem 5.5.1. *Let $w(t)$ be a nonnegative non increasing function for $t \in [0,T]$ such that $w(t) > 0$ as $t \to 0^+$, $w_n(t)$ be a nonnegative non increasing function for $t \in [0,T]$ such that $w_n(t) > 0$ as $t \to 0^+$, and they converge $w_n(t) \to w(t)$ uniformly in $t \in [0;T]$. Let conditions (A); (B_{01}) be satisfied and $G_n \to G$. Assume also that $\mathcal{N}(I-K) = \{0\}$; operator $(\lambda I - A)^{-1}$ is compact and $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly. Then solutions of the problems (5.5.3) exist and converge to the solution of the problem (5.1.5); i.e. $f_n \to f$.*

Proof. The proof is done in four steps.

Step 1. First, let us show that the com

pact convergence of resolvents $R(\lambda; A_n) \to R(\lambda; A)$ is equivalent to the compact convergence of C_0-semigroups $U_n(t) \to U(t)$ for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n U_n(t)\| \le \frac{M}{t}\exp(\omega t)$; which is exactly condition (B_1'), we obtain the boundedness in n of the sequence $\{(A_n - \lambda I_n) U_n(t) x_n\}$ for any fixed $t > 0$. Because of the compact convergence of resolvent, we obtain the compactness of the sequence $\{U_n(t) x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(.)$ (for the definition, see [17]), we establish that $\mu\left(\left\{(\lambda I_n - A_n)^{-1} x_n\right\}\right) = 0$ for any $\|x_n\| = O(1)$.We have

$$
\begin{aligned}
\mu\left(\left\{(\lambda I_n - A_n)^{-1} x_n\right\}\right) &= \mu\left(\left\{(\int_0^\infty \exp(-t\lambda) U_n(t) x_n dt\right\}\right) \\
&\le \mu\left(\left\{(\int_0^q \exp(-t\lambda) U_n(t) x_n dt\right\}\right) \\
&+ \mu\left(\left\{(\int_p^\infty \exp(-t\lambda) U_n(t) x_n dt\right\}\right) \\
&+ \mu\left(\left\{U_n(\varepsilon)\int_q^p \exp(-t\lambda) U_n(t-\varepsilon) x_n dt\right\}\right).
\end{aligned}
$$

Two first terms can be made less than ε by the choice of q, p. The last term is equal to zero because of the compact convergence $U_n(\varepsilon) \to U(\varepsilon)$ for any $0 < \varepsilon < q$.

Step 2. Consider the operators K and K_n defined by (5.1.6) and (5.5.4) on the spaces E and E_n. The operator K defined by (5.1.6) is compact in E. Indeed, we obtain that the

$$
K_\varepsilon = \left(\frac{w(T)}{w(0)} U(T) + \frac{1}{w(0)} \int_\varepsilon^T U(t)\, d(-w(t))\right)
$$

is a product of compact and bounded operators. Moreover $\|K_\varepsilon - K\| \le C\varepsilon$, where

$$K=\left(\frac{w(T)}{w(0)}U(T)+\frac{1}{w(0)}\int_0^T U(t)\,d(-w(t))\right)$$

and $\varepsilon>0$. Then it follows that the operator $K:E\to E$ is compact.

Step 3. It is easy to see that $K_n\to K$. To show that $K_n\to K$ compactly, we assume that $\|f_n\|_{E_n}=O(1)$. Now $\{K_nf_n\}$ is $\mathcal{P}$-compact because of representation

$$K_{\varepsilon,n}=\left(\frac{w_n(T)}{w_n(0)}U_n(T)+\frac{1}{w_n(0)}\int_\varepsilon^T U_n(t)\,d(-w_n(t))\right)$$

and one can easy verify the vanishing of the noncompactness measure $\mu(\{K_nf_n\})=0$ for all $n\in\mathbb{N}$, taking into an account that $\|K_{\varepsilon,n}-K_n\|\le C\varepsilon$.

Step 4. Now $I_n\to I$ stably and $K_n\to K$ compactly. Hence it follows from Theorem 5.2.1 that $I_n-K_n\to I-K$ regularly. Moreover, the nullspace $\mathcal{N}(I-K)=\{0\}$ and the operators I_n-K_n are Fredholm of index zero. Then it follows from Theorem 5.2.2 that $I_n-K_n\to I-K$ stably, i.e. $(I_n-K_n)^{-1}\to(I-K)^{-1}$.

Since $G_n\to G$, one gets $f_n=(I_n-K_n)^{-1}G_n\to(I-K)^{-1}G=f$. The Theorem is proved. □

One can find that solution of the problem (5.5.3) according to Theorem 5.5.2, and under the assumption that functions $w_n(t),w(t)\in C^1([0;T])$ and they converge $w_n(t)\to w(t)$ uniformly in $t\in[0;T]$.

Theorem 5.5.2. *Let C_0-semigroups $U_n(t)$ be positive and compact for $t>0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$ and $w_n(t)\ge0,w_n(0)>0;w_n^{'}(t)\le0$ for any $t\in[0;T]$. Define the operator K_n as in (5.5). Then $r(K_n)<1$.*

We are recalling that $r(A)$ is the spectral radius of $A\in B(E)$. The spectral radius of A, denoted by $r(A)$, is the radius of the smallest disk centered at zero that contains $\sigma(A)$,

$$r(A)=\{|\lambda|:\lambda\in\sigma(A)\}.$$

It is well known that for every $A \in B(E)$, we have

$$r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}, \text{ and } r(A) \le \|A\|.$$

Proof. The proof of the Theorem 5.5.2 is similar to the proof of the Theorem 5.5.4. □

As a consequence of the Theorem 5.5.2, we have the following

Theorem 5.5.3. *Let C_0-semigroup $U_n(t)$ be positive and compact for $t > 0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ and $w_n(t) \ge 0, w_n(0) > 0; w_n^{'}(t) \le 0$ for any $t \in [0;T]$. Then for any $g \in D(A_n)$, there is unique solution of the problem (5.5.1), (5.5.2).*

Since $r(K_n) < 1$, could be organized as follows

$$f_{n,j+1} = K_n f_{n,j} - \frac{1}{w_n(0)} A_n g_n, \quad n, j = 0; 1..., \tag{5.5.5}$$

with initial condition $f_{n,0} = 0$. The value $K_n f_{n,j}$ is nothing else as a solution of Cauchy problem

$$v_n'(t) = A_n v_n(t) - \frac{w_n^{'}(T-t)}{w_n(0)} f_{n;j}, \quad v_n(0) = \frac{w_n(T)}{w_n(0)} f_{n;j}$$

at the point T; i.e.

$$K_n f_{n,j} = v_n(T, f_{n,j}) = \frac{w_n(T)}{w_n(0)} U_n(T) f_{n,j} + \frac{1}{w_n(0)} \int_0^T -w_n^{'}(t) U_n(t) f_{n,j} dt.$$

So (5.5.5) could be written in the form, starting from

$$f_{n,0} = 0, \quad f_{n,j+1} = v_n(T, f_{n,j}) - \frac{1}{w_n(0)} A_n g_n, \quad n, j = 0; 1; ......$$

Moreover, $f_{n,j} \to f_n$ as $j \to \infty$ since $r(K_n) < 1$.

There are different ways how one can calculate $v_n(T, f_{n,j})$ One can use directly Theorems 5.4.5, 5.4.6 or maybe some higher order difference schemes for approximation of $U_n(T)$;

say as in [6, 11], and then apply some quadrature formula for approximation the term

$$\frac{1}{w_n(0)}\int_0^T -w_n^{'}(t)\,U_n(t)\,f_{n,j}dt$$

In this paper we consider just the simplest way which comes from Theorem 5.4.5. In case of Theorem 5.4.6 we have to assume stability condition, but the other considerations are the same. So following the scheme (5.4.1) we consider approximation of the equation (1) by

$$\frac{\overline{U_n}(t+\tau_n)-\overline{U_n}(t)}{\tau_n} = A_n\overline{U_n}(t+\tau),$$

and approximation of the condition (5.5.2) by

$$\sum_{j=0}^{k-1} w_n(j\tau_n)\,u_n(j\tau_n+\tau_n)\,\tau_n = g_n. \tag{5.5.6}$$

The solution of the scheme (5.5) can be written in the form

$$\overline{U_n}(t) = (I_n-\tau_n A_n)^{-k}\,u_n^0; t = k\tau_n$$

To construct approximation of operator K_n in (5.5.4),we just consider the simplest formula ($T = k_n\tau_n$):

$$\breve{K}_n = (I_n-\tau_n A_n)^{-k_n}\frac{w_n(T)}{w_n(0)} - \frac{1}{w_n(0)}\sum_{l=0}^{k_n-1}(I_n-\tau_n A_n)^{-l}\,\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n.$$

Theorem 5.5.4. *Let C_0-semigroup $U_n(t)$ be positive and compact for $t>0$. Assume that the spectrum of A_n lies in the half-plane $\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda<0\}$ and $w_n(t)\geq 0, w_n(0)>0; w_n^{'}(t)\leq 0$ for any $t\in[0;T]$. Define the operator $\breve{K}_n$ as in (5.5). Then $r\left(\breve{K}_n\right)<1$.*

Proof. The operator $\breve{K}_n$ is positive and compact, so by Krein-Rutman Theorem there are $\lambda_0 \geq 0$ and $0 \leq f_n^0 \neq 0$ such that $\breve{K}_n\ f_n^0 = \lambda_0 f_n^0$, and moreover, $r\left(\breve{K}_n\right) = \lambda_0$. Assume now in contradiction that $\lambda_0 \geq 1$.
Substutting $\overline{U_n}(t) = (I_n - \tau_n A_n)^{-k} f_n$ in

$$\sum_{l=0}^{k-1} w_n(l\tau_n)\, u_n(l\tau_n + \tau_n)\, \tau_n. \tag{5.5.7}$$

with $f_n = f_n^0$. One gets that

$$\sum_{l=0}^{k-1} w_n(l\tau_n)\,(I_n - \tau_n A_n)^{-l-1} f_n^0 \tau_n. \tag{5.5.8}$$

is positive for positive f_n^0. Putting

$$\varphi_n = \sum_{l=0}^{k-1} w_n(l\tau_n)\,(I_n - \tau_n A_n)^{-l-1} f_n^0 \tau_n. \tag{5.5.9}$$

So, applying the operator A_n to (5.5), and using the formula of summation by parts. $\tau_n \sum_{l=0}^{k-1} \frac{v_{l+1}-v_l}{\tau_n} y_l = (y_k v_k - y_0 v_0) - \tau_n \sum_{l=0}^{k-1} v_{l+1} \frac{y_{l+1}-y_l}{\tau_n}$. We obtain that $A_n \varphi_n = -w_n(0) f_n^0 + w_n(0) \widetilde{K_n} f_n^0 = -w_n(0) f_n + w_n(0) \lambda_0 f_n = w_n(0)(\lambda_0 - 1) f_n^0 \geq 0$, since $w_n(0) > 0$, $\lambda_0 \geq 1$, and $f_n^0 \geq 0$. So, if we apply $(-A_n)^{-1}$; then because of positiveness of C_0-semigroup $U_n(t)$, the resolvent $(-A_n)^{-1}$ is also positive and $(-A_n)^{-1} A_n \varphi_n \geq 0$; which means that $0 \geq \varphi_n$. From the other hand from (5.5) it follows that $\varphi_n \geq 0$ for $f_n^0 \geq 0$. This means that $\varphi_n = 0$; which means that $w_n(l\tau_n)(I_n - \tau_n A_n)^{-l-1} f_n = 0$ for all $l = 0, ...k-1$, in particular for $l = 0$ we have $w_n(0)(I_n - \tau_n A_n)^{-1} f_n = 0$, because $Ker\,(I_n - \tau_n A_n)^{-1} = \{0\}$, and $w_n(0) \neq 0$, one gets that f_n^0. But this contradicts to $f_n^0 \neq 0$. The Theorem is proved. □

From Theorem 5.5.4 it follows that one can organize the process$\breve{f}_{n,j+1} = \breve{K}_n \breve{f}_{n,j} - \frac{1}{w_n(0)} A_n g_n$, $n; j = 0; 1$, which converges $\breve{f}_{n,j} \to \breve{f}_n$ as $j \to \infty$; where $\breve{f}_n$ is a solution of the problem $\breve{f}_n = \breve{K}_n \breve{f}_n - \frac{1}{w_n(0)} A_n g_n$.

Theorem 5.5.5. *Let C_0-semigroups $U_n(t)$ be positive and analytic. Assume also that functions $w_n(t), w(t) \in C^1([0;T])$ and they converge $w_n^{'}(t) \to w^{'}(t)$ uniformly in $t \in [0;T]$.*

Let conditions (A); (B_{01}) be satisfied and $G_n \to G$. Assume also that $\mathcal{N}(I-K)=\{0\}$, operator $(\lambda I-A)^{-1}$ is compact and $(\lambda I_n-A_n)^{-1}\to(\lambda I-A)^{-1}$ compactly and $w_n(t)\in C^3([0;T])$ and $\left|w_n^{'''}(t)\right|\le$constant; $t\in[0;T]$. Then solutions of the problems (5.5) exist and converge to the solution of the problem (5.1.5); i.e. $\breve{f}_n\to f$ as $n\to\infty$.

Proof. If $\breve{K}_n\to K$ compactly, then the statement of the Theorem 5.5.5 follows the same way as in the **Step 4** of Theorem 5.5.1. So, we are going to show that $\breve{K}_n\to K$ compactly. To do this it is enough to prove that $\left\|\breve{K}_n-K\right\|\to 0$ as $n\to\infty$; since the statement $K_n\to K$ compactly is already proved in Theorem 5.5.1. One can write

$$K_n-\breve{K}_n=\frac{w_n(T)}{w_n(0)}U_n(T)-(I_n-\tau_nA_n)^{-k_n}\frac{w_n(T)}{w_n(0)}+\frac{1}{w_n(0)}\int_0^T-w_n^{'}(t)\,U_n(t)\,dt-$$
$$\frac{1}{w_n(0)}\sum_{l=0}^{k_n-1}(I_n-\tau_nA_n)^{-l-1}\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n$$

where $k_n\tau_n=T$. In [5], it is proved under condition (B_1) that

$$\left\|U_n(t)-(I_n-\tau_nA_n)^{-k_n}\right\|\le C\frac{\tau_n}{t}\exp(\omega t)$$

as $k_n\to\infty$ and $t=k\tau_n$.Let us consider now the difference

$$\sum_{l=0}^{k_n-1}\frac{1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}-w_n^{'}(t)\,U_n(t)\,dt$$
$$-\frac{1}{w_n(0)}\sum_{l=0}^{k_n-1}(I_n-\tau_nA_n)^{-l-1}\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n.$$

To finish with the demonstration we have to use

$$\pm\sum_{l=0}^{k_n-1}\frac{-1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}U_n(t)\,\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n dt$$

terms.Indeed, it is easy to show that difference

$$\sum_{l=0}^{k_n-1}\frac{1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}-w_n^{'}(t)\,U_n(t)\,dt$$
$$-\sum_{l=0}^{k_n-1}\frac{-1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}U_n(t)\,\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n dt$$

converge to zero as $k_n\to\infty$ and $T=k_n\tau_n$; since

$$\frac{-1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}U_n(t)\left(w_n^{'}(t)-\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\tau_n\right)dt$$

is estimated by

$$C\frac{1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}U_n(t)\left|\left(w_n^{'}(t)-\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\right)\right|dt=O\left(\tau_n^2\right).$$

The second term from $\pm$ construction could be estimated as

$$\left\|\sum_{l=0}^{k_n-1}\frac{-1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}U_n(t)-(I_n-\tau_nA_n)^{-l-1}dt\frac{w_n(l\tau_n+\tau_n)-w_n(l\tau_n)}{\tau_n}\right\|$$

$$\le C\sum_{l=1}^{k_n-1}\frac{1}{w_n(0)}\int_{l\tau_n}^{(l+1)\tau_n}\|U_n(t)-U_n(l\tau_n+\tau_n)\|\,dt+C\tau_n$$
$$+C\sum_{l=0}^{k_n-1}\left\|U_n(l\tau_n+\tau_n)-(I_n-\tau_nA_n)^{-l-1}\right\|\tau_n$$

$$\le C\left(\sum_{l=1}^{k_n-1}\frac{\tau_n}{l}+\sum_{l=0}^{k_n-1}\frac{\tau_n}{l+1}\right)+\tau_n.$$

Where we used the fact that for any $t\in[j\tau_n,(j+1)\tau_n]\,,1\le j\le k_n-1$,

$$\|U_n(t)-U_n(j\tau_n+\tau_n)\|\le C\frac{\tau_n}{j\tau_n}.$$

The Theorem is proved. □

Abstract. In this work, we are interested to the study of boundary values problems with integral boundary conditions. We obtain the existence and uniqueness of solutions with a priori estimate, and prove the Fredholm solvability of the problem. Finally, we apply an iteration approximation method to approximate an initial condition of a boundary value problem for an abstract first order homogeneous linear differential equation with an integral boundary condition on a Banach space.

Key words. Abstract Differential Equation, Integral condition, Analytic semigroup, Semigroup with singularity Fredholm property, Ill-posed Problem.

Résumé. Dans ce travail, on s'intéresse a l'étude des problèmes aux limites avec des conditions aux limites intégrales. On obtient l'existence et l'unicité de la solution du problème avec une éstimation a priori, et on démontre la solvabilité de Frédholm du problème. Enfin, on applique une méthode d'approximation par itération pour approximer une condition initiale d'un problème aux limite pour une équation différentielle linéaire homogène abstraite du premier ordre avec une condition aux limites intégrale sur un espace de Banach.

Mot clés. Equation Différentielle Abstraite, Condition intégrale , Semigroupe analytique, Semi-groupe à singularities, Propriété de Fredholm, Problème mal posé.

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