Democratic and Popular Republic of Algeria
Ministry of Higher education and Scientific Research

University of Constantine
Faculty of exact science
Physics department
Theoretical Physics Laboratory

Order N°:
Series:

Thesis

Submitted to the department of physics in candidacy for the degree of doctorate in theoretical physics

Presented to obtain the diplomat of

Doctorate LMD

Specialty: Physics

Option: Theoretical physics

Theme

Pseudo-squeezed coherent states

By the candidate:

Elaihar Lamine

Presented on the: 04/03/2023

Jury:

Chairwoman:	B. Bentag	Prof.	UFM. Constantine 1
Supervisor:	M. Maamache	Prof.	UFA. Setif
Examiners:	N. Baadji	Prof.	Univ. M'sila
	A. Boumali	Prof.	Univ. Tebessa
	A. Bounames	Prof.	Univ. Jijel

بسم الله الرحمن الرحيم

شكر و تقدير

أشكر و أحمد الله العلي القدير الذي أنعم عليَّ بنعمة العقل والدين. القائل في محكم التنزيل" وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ "

وأيضا وفاءً وتقديراً واعترافاً مني بالجميل أتقدم بجزيل الشكر لأولئك الذين لم يألوا جهداً في مساعدتنا في مجال البحث العلمي، وأخص بالذكر الأستاذ الفاضل: أ. معماش مصطفى على إشرافه لهذه المذكرة وصاحب فضل في توجيهي ومساعدتي طوال سنوات إنجازها، فجزاه الله كل خير وحفظه هو و عائلته.

ولا أنسى أن أتقدم بجزيل الشكر للأساتذة الذين قبلوا أن يكونوا في لجنة مناقشة مذكرتي، الأستاذة بن طاق باية رئيسة والأساتذة الممتحنين بعجي نجيب، بومعالي عبد المالك و بونامس عبد الحفيظ.

أتقدم بجزيل شكري إلى كل من مدوا لي يد العون والمساعدة لإتمام هذه المذكرة على أكمل وجه وأخص بالذكر وليد كوسة، نعيمة مانع، بوقرة ياسين.

وأخيرا أشكر زملائي في الدكتوراه لقمان سعد عزام، عبد الرزاق لقويزي، بلال علوش، وإلى كل من تعذر علينا ذكره.

إهداء

أهدي هذا العمل المتواضع إلى والديّ العزيزين حفظهما الله اللذان وقفا بجانبي في السرّاء والضرّاء

إلى إخوتي

و إلى كل من جمعني بهم منبر العلم

لمين العيهار

Contents

Introduction

Classical and quantum mechanics are two theories that study the motion of physical objects, the first one is stricted to macroscopic scale whereas the second describes microscopic and macroscopic objects.

While describing the quantum mechanics of bound and unbound particles is successful through the wave mechanical formulation, some properties though can not be represented by a wave-like description i.e. an electron spin degree of freedom can't be expressed as an action of a gradient operator. Thus, the reformulation of quantum mechanics to a framework that involves only operators is useful. A state vector or a wave function ψ in the Dirac notation is represented by $|\psi\rangle$, also known as "ket", likewise, any wavefunction can be expanded as a superposition of basis state vectors

$$|\psi\rangle = \lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle + \lambda_3 |\psi_3\rangle + \cdots . \tag{1}$$

Consequently, we define the "bra" $\langle\psi|$ that defines together with the ket, a scalar product

$$\langle\phi|\,\psi\rangle \equiv \int_{-\infty}^{+\infty} \phi^* (x)\, \psi (x)\, dx, \tag{2}$$

correspondingly, we deduce the identity $\langle\phi|\,\psi\rangle = \langle\psi|\,\phi\rangle^*$. The space and momentum representation of the wave function is given as $\psi (x) = \langle x|\,\psi\rangle$ and $\psi (p) = \langle p|\,\psi\rangle$ respectively. Moreover, B is said to be an opoerator if it maps a state vector $|\alpha\rangle$ into another $|\beta\rangle$, i.e. $B|\alpha\rangle = |\beta\rangle$, nevertheless, if

$$B|\alpha\rangle = b|\alpha\rangle , \tag{3}$$

where b is real, accordingly, we can say that $|\alpha\rangle$ is an eigenfunction or an eigenstate of the operator B with the eigenvalue b, it is known that for any quantum observable O there is an operator O that acts on a wave function $|\phi\rangle$, in which if the system is in a state characterized by the wave function, then the expectation value is said to be

$$\langle O\rangle = \langle\phi|\, O\, |\phi\rangle = \int dx \phi^* (x)\, O \phi (x) . \tag{4}$$

A hermitian linear operator is an observable, i.e. $B (a_c |\alpha\rangle + b_c |\beta\rangle) = a_c (B|\alpha\rangle) + b_c (B|\beta\rangle)$, where a_c and b_c are complex numbers, hence, it is appropriate to define the adjoint or the

Hermitian conjugate, the adjoint of a linear operator O is defined as

$$\langle \alpha | \, O \, \beta \rangle = \int dx \alpha^* \left(O\beta \right) = \int dx \beta \left(O^\dagger \alpha \right)^* = \left\langle O^\dagger \alpha \right| \beta \rangle \, , \tag{5}$$

an operator O is called self-adjoint or Hermitian if $O^\dagger = O$, where the symbol "†" denotes the adjoint operation. The eigenfunctions of Hermitian operators form an orthonormal complete basis, for example $\langle i | \, j \rangle = \delta_{ij}$, in consequence, we obtain the resolution of identity if we sum over a complete set of states

$$\sum_i |i\rangle \langle i| = \mathbb{I}, \tag{6}$$

hence, any state function can be expanded if we use the resolution of identity

$$\psi\left(x\right) = \langle x | \, \psi \rangle = \sum_i \langle x | \, i \rangle \langle i | \, \psi \rangle = \sum_i \langle i | \, \psi \rangle \, \phi_i \left(x\right), \tag{7}$$

where $\phi_i\left(x\right) = \langle x | \, i \rangle$.

Since we are able to expand and develop an eigenfunction one can say that we have the means to inspect the time evolution, thus, the wave function can evolve in time by applying the time evolution operator, i.e. for a time-dependent Hamiltonian

$$|\psi\left(t\right)\rangle = U\left(t\right) |\psi\left(0\right)\rangle \, , \tag{8}$$

where $U\left(t\right) = e^{-i \int \partial t H/\hbar}$ and it is found by integrating the following time-dependent Schrödinger equation

$$H \, |\psi\rangle = i\hbar \partial_t \, |\psi\rangle \, , \tag{9}$$

also, the time evolution operator is unitary $UU^\dagger = \mathbb{I}$.

Correspondingly, the expectation values can also evolve through time, if we assume that the operator O is time dependent, then we have

$$\frac{d}{dt} \langle \phi | \, O \, |\phi\rangle = \partial_t \left(\langle \phi | \right) O \, |\phi\rangle + \langle \phi | \, \partial_t O \, |\phi\rangle + \langle \phi | \, O \left(\partial_t \, |\phi\rangle \right), \tag{10}$$

from the time-dependent Schrödinger equation and the fact that the Hamiltonian is Hermitian, we obtain

$$\begin{aligned} \frac{d}{dt} \langle \phi | \, O \, |\phi\rangle &= \frac{i}{\hbar} \left[\langle \phi | \, HO \, |\phi\rangle - \langle \phi | \, OH \, |\phi\rangle \right] + \langle \phi | \, \partial_t O \, |\phi\rangle \, , \\ &= \frac{i}{\hbar} \langle \phi | \left[H, O \right] |\phi\rangle + \langle \phi | \, \partial_t O \, |\phi\rangle \, , \end{aligned} \tag{11}$$

if the operator O is time-independent, then we have

$$\frac{d}{dt}\langle\phi|\,O\,|\phi\rangle = \frac{i}{\hbar}\langle\phi|\,[H,O]\,|\phi\rangle\,, \tag{12}$$

the preceding relation is also known as the Ehrenfest theorem, from which we notice that when $[H,O]=0$, the expectation value of O is a constant of motion and obeys the laws of classical mechanics. The expectation values of the position and momentum operators are respectively given as

$$\frac{d}{dt}\langle x\rangle = \left\langle\frac{\partial H}{\partial p}\right\rangle, \qquad\qquad \frac{d}{dt}\langle p\rangle = -\left\langle\frac{\partial H}{\partial x}\right\rangle, \tag{13}$$

these relations are the equivalent of Hamilton's classical equations of motion. Another description to obtain classical results is achieved with the coherent states. These states which were introduced for the first time in 1926 [1] are related to the harmonic oscillator which is one of the fairly small number of quantum mechanical problems that can be exactly solved, the problem provides a foundation for our understanding of many significantly important physical problems, including molecular vibrations, the vibrational excitations of solids i.e. phonons, and the quantization of the electromagnetic field for example the photons. Indeed, the one-dimensional harmonic oscillator is one of the most important systems in quantum field theory.

The expectation values of the coordinate and momentum in the coherent states are the same as the expectation values of the position and momentum in the classical theory of the harmonic oscillator. The coherent states describes a state in a system from which the ground state wave packet is displaced from the origin of the system, this state can be related to classical solutions by a particle oscillating with an amplitude equivalent to the displacement, these states are expressed as eigenvectors of the annihilation operator, they were familiarized by Klauder [2, 3], and were later on presented by the work of Glauber in 1963 [4].

Coherent states are remarkable quantum states that are important in many fields of physics [2, 5], such as quantum optics where they play a particular role in laser physics. These states were introduced for the hamonic oscillator by Schrödinger in 1926 [1]. In 1963 Glauber [4] introduced the coherent states of the radiation field as eigenstates of the annilation operator [6], whereas Klauder [7] used coherent states to verify the relation between the quantum system and the classical system. Squeezed states [8, 9] which are a special class of minimum uncertainty states, have received considerable attention due to their important applications in optical communication, photon detection techniques, gravitational wave detection, and noise-free amplification.

Squeezed states represent a generalization of coherent states and were introduced by different authors [10, 11, 12, 13, 14, 15, 16, 17]: Stoler [11, 18], Lu [19, 20], Yuen [21] and Hollenhorst [13] who originated them with the name "squeezed states".

Similarly, to the coherent states the squeezed states of the harmonic oscillator are the states that are attained by acting on the ground state with an exponential that consists of terms of the quadratic forms of the creation and annihilation operators. In addition, these states are common for which $\Delta x \Delta p = \hbar/2$, therefore reaching the saturation of the uncertainty bound.

In addition, the use of the squeezed states allow for continuous measurement improvements, and it is now becoming widely accepted i.e. in the gravitational wave detectors tests the squeezed states improved measurements sensitivity.

With the aim to define squeeezed coherent states for time-dependent non-Hermitian systems, we present in the first chapter, the definition of coherent and squeezed states for the Hermitian harmonic oscillator in the stationary case then we develop their time evolution. In the second chapter, we construct the time-dependent pseudo-squeezed coherent states by introducing pseudo-squeezed bosonic ladder operators defined as time-dependent non-Hermitian linear invariants and related to their adjoint operators via the bounded Hermitian invertible operator or metric operator. As an illustration, we study in the third chapter the time-dependent non-Hermitian displaced harmonic oscillator, in which we find interesting results that lead to this work.

Chapter 1

Coherent and squeezed states for the harmonic oscillator

Coherent states together with squeezed states constitute the foundation and cornerstone of the theoretical framework of modern optics. As shown in the literature, this framework starts from the harmonic oscillator creation and annihilation operators. In this part, we present the definition of these states, we distinguish the different methods to obtain the coherent states as well as the squeezed states, moreover, listing some of their properties.

1.1 Definition of the coherent states

Coherent states are quantum states that exhibit a classical behavior, i.e. the mean values of the position and momentum operators in the coherent states have properties close to the classical values of the position and momentum.

Furthermore, coherent states have been extensively studied by several physicists and different definitions have emerged as a result of that process. If we ought to summarize their work, we can keep in mind some of the distinct but equivalent methods of obtaining the mentioned coherent states [1, 4, 22, 23, 24, 25, 26, 27, 28, 29].

These are:

(1) The states that minimize this uncertainty relation

$$(\Delta x)^2 (\Delta p)^2 \geq \frac{1}{4}, \tag{1.1}$$

where $(\Delta x)^2$ and $(\Delta p)^2$ represent the dispersions of the position and of the momentum respectively

$$\Delta x = \left(\left\langle x^2\right\rangle - \langle x\rangle^2\right)^{\frac{1}{2}}, \qquad \Delta p = \left(\left\langle p^2\right\rangle - \langle p\rangle^2\right)^{\frac{1}{2}}, \tag{1.2}$$

and the operators of the position and momentum are given in case of the harmonic oscillator by

$$x = \sqrt{\hbar\,(2m\omega)^{-1}}\left(a^\dagger + a\right), \tag{1.3}$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger - a\right) = i\hbar\frac{\left(a^\dagger - a\right)}{\sqrt{2}d}\ , \tag{1.4}$$

where d is a quantum length scale from the harmonic oscillator that can be built from $\hbar$,m and ω

$$d \equiv \sqrt{\hbar/m\omega}. \tag{1.5}$$

Consequently, the position and the momentum expectation values are

$$\langle x\rangle = \langle\alpha|x|\alpha\rangle = \frac{d}{\sqrt{2}}\left\langle\alpha\right|\left(a^\dagger + a\right)\left|\alpha\right\rangle = \frac{d}{\sqrt{2}}\left(\alpha^* + \alpha\right) = d\sqrt{2}\,\mathrm{Re}\left(\alpha\right), \tag{1.6}$$

$$\langle p\rangle = i\hbar\left(d\sqrt{2}\right)^{-1}\left\langle\alpha\right|\left(a^\dagger - a\right)\left|\alpha\right\rangle = \frac{\hbar\sqrt{2}}{d}\,\mathrm{Im}\left(\alpha\right), \tag{1.7}$$

thus,

$$\alpha = \mathrm{Re}\left(\alpha\right) + i\,\mathrm{Im}\left(\alpha\right) = \frac{\langle x\rangle}{d\sqrt{2}} + i\frac{\langle p\rangle\, d}{\hbar\sqrt{2}}\ , \tag{1.8}$$

therefore, Δx and Δp are expressed as

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\ , \qquad \Delta p = \sqrt{\frac{\hbar m\omega}{2}}\ . \tag{1.9}$$

(2) The eigenfunctions of the annihilation operator, also called Glauber states, defined as

$$a\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \tag{1.10}$$

where we note that the parameter α is complex, and the action of a on the state $|\alpha\rangle$ can be computed as the following

$$a\left|\alpha\right\rangle = a\exp\left(\alpha a^\dagger - \alpha^* a\right)\left|0\right\rangle = \left[a, \exp\left(\alpha a^\dagger - \alpha^* a\right)\right]\left|0\right\rangle, \tag{1.11}$$

and if we look at the commutation relation between the non-Hermitian ladder operators

$$\left[a, a^{\dagger}\right] = 1, \tag{1.12}$$

it follows that

$$a\left|\alpha\right\rangle = \left[a, \alpha a^{\dagger} - \alpha^{*}a\right] \exp\left(\alpha a^{\dagger} - \alpha^{*}a\right)\left|0\right\rangle, \tag{1.13}$$

therefore, we get (1.10), also, the normalized coherent states are given as[30]

$$\left|\alpha\right\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left|n\right\rangle, \tag{1.14}$$

and their adjoint

$$\left\langle\alpha\right| = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha^{*}\right)^n}{\sqrt{n!}} \left\langle n\right|. \tag{1.15}$$

(3) Displacement-operator methods, i.e.: applying the displacement operator

$$D(\alpha) = \exp\left[\left(\alpha a^{\dagger} - \alpha^{*}a\right)\right], \tag{1.16}$$

on the ground state $\left|0\right\rangle$ as

$$\left|\alpha\right\rangle = D(\alpha)\left|0\right\rangle = \exp\left[\left(\alpha a^{\dagger} - \alpha^{*}a\right)\right]\left|0\right\rangle, \tag{1.17}$$

The displacement operator $D(\alpha)$ is unitary

$$D^{\dagger}\left(\alpha\right) = D^{-1}\left(\alpha\right) = D\left(-\alpha\right), \tag{1.18}$$

$$D^{\dagger}\left(\alpha\right) D\left(\alpha\right) = D\left(\alpha\right) D^{\dagger}\left(\alpha\right) = \mathbb{I}. \tag{1.19}$$

Introducing the Baker-Campbell-Hausdorff formula [31, 32, 33]

$$e^{A} B e^{-A} = B + [A, B] + \frac{1}{2!}\left[A, [A, B]\right] \ \cdots, \tag{1.20}$$

consequently, we recall the commutator identity

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B} = e^{\frac{1}{2}[A,B]} e^{B} e^{A}, \tag{1.21}$$

it is worth to note that A and B are two operators such that

$$\left[[A, B], A\right] = \left[[A, B], B\right] = 0, \tag{1.22}$$

then

$$e^{A}e^{B} = e^{[A,B]}e^{B}e^{A}. \tag{1.23}$$

The operator $D(\alpha)$ (1.16) can be expressed as

$$D\left(\alpha\right) = e^{\frac{-|\alpha|^2}{2}}e^{\alpha\, a^{\dagger}}e^{-\alpha^{*}a}, \tag{1.24}$$

or

$$D\left(\alpha\right) = e^{\frac{|\alpha|^2}{2}}e^{-\alpha^{*}a}e^{\alpha\, a^{\dagger}}, \tag{1.25}$$

and therefore its action on a and $a^{\dagger}$ yields

$$D^{\dagger}\left(\alpha\right)aD\left(\alpha\right) = a + \alpha, \tag{1.26}$$

$$D^{\dagger}\left(\alpha\right)a^{\dagger}D\left(\alpha\right) = a^{\dagger} + \alpha^{*}, \tag{1.27}$$

we can easily demonstrate that

$$D\left(\alpha+\beta\right) = D\left(\alpha\right)D\left(\beta\right)e^{\mathbf{i}\,\mathrm{Im}\,\alpha\beta^{*}}. \tag{1.28}$$

Additionally, the formula in (1.21) implies that the coherent state form in (1.17) can be written as the following expression

$$\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^{\dagger}}\left|0\right\rangle. \tag{1.29}$$

1.2 Properties and time evolution of the Coherent states

The coherent states has a number of properties that can be listed as

1. The coherent states are not orthogonal between them

$$\left\langle\beta\right|\alpha\rangle = \left\langle 0\right|D^{\dagger}\left(\beta\right)D\left(\alpha\right)\left|0\right\rangle, \tag{1.30}$$

by using (1.14), we obtain

$$\left\langle\beta\right|\alpha\rangle = e^{\frac{-|\beta|^2}{2}-\frac{|\alpha|^2}{2}+\beta^{*}\alpha}, \tag{1.31}$$

which shows that the coherent states $\left|\alpha\right\rangle$ and $\left|\beta\right\rangle$ are not mutually orthogonal, and that the squared modulus $\left\langle\beta\right|\alpha\rangle$ indicates the distance measure between the coherent states.

2. The coherent states are normalized, simply by puting $\beta = \alpha$, we obtain

$$\langle \alpha | \, \alpha \rangle = \mathbb{I}. \tag{1.32}$$

3. Also, the coherent states form an overcomplete basis

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| \, d^2\alpha = \mathbb{I}, \tag{1.33}$$

we can demonstrate this identity if we pose that $\alpha = ue^{i\theta}$, thus, $d^2\alpha = udud\theta$ and by using (1.14), we obtain

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| \, d^2\alpha = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{\infty} udu \int_0^{2\pi} \frac{d\theta}{\pi} \frac{u^{n+m}}{\sqrt{n!m!}} e^{i(n-m)\theta} e^{-u^2} |n\rangle \langle m| \,, \tag{1.34}$$

and thus,

$$\frac{1}{\pi} \int |\alpha\rangle \langle \alpha| \, d^2\alpha = \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \int_0^{\infty} dve^{-v} v^n, \tag{1.35}$$

in which we used $\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{nm}$, the variable change $u^2 = v$, and after we use the integration $\int_0^{\infty} dve^{-v}v^n = n!$, we obtain the previous relation.

4. The expectation values $\langle x\rangle$,$\langle p\rangle$ and $\langle H\rangle$ in $|\alpha\rangle$, remain constantly equal to their classical counterpart. We define the time evolution of coherent states by using the Heisenberg picture, from (1.17) we get

$$|\alpha, t\rangle \equiv e^{-\frac{i}{\hbar}Ht} |\alpha\rangle = e^{-\frac{i}{\hbar}Ht} e^{\left(\alpha a^\dagger - \alpha^* a\right)} e^{\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}Ht} |0\rangle \,, \tag{1.36}$$

then, we obtain the following form of the time evolution of coherent state

$$|\alpha, t\rangle = \exp\left[\alpha a^\dagger(t) - \alpha^* a(t)\right] \exp\left[\frac{-i\omega t}{2}\right] |0\rangle \,, \tag{1.37}$$

where we have generalized the idea of a Heisenberg operator as

$$O_H(t) = e^{iHt/\hbar} O_{Shr} e^{-iHt/\hbar}, \tag{1.38}$$

Furthermore, we now introduce the relations $a(t) = e^{i\omega t}a$, and $a^\dagger(t) = e^{-i\omega t}a^\dagger$ we can rewrite the time evolution of the coherent state (1.37) to be in the following form

$$|\alpha, t\rangle = \exp\left[\frac{-i\omega t}{2}\right] \exp\left[\alpha e^{-i\omega t} a^\dagger - \alpha^* e^{i\omega t} a\right] |0\rangle \,, \tag{1.39}$$

by comparing it to (1.17) and precisely the exponential, we notice that it is in fact the displacement operator with a slight change where $\alpha \to \alpha e^{-i\omega t}$, we can say that $\alpha(t) = \alpha e^{-i\omega t}$, thus, the time evolution of the coherent state can be expressed also in the following form

$$|\alpha, t\rangle = \exp\left[\frac{-i\omega t}{2}\right]\left|e^{-i\omega t}\alpha\right\rangle = \exp\left[\frac{-i\omega t}{2}\right]|\alpha(t)\rangle\,, \tag{1.40}$$

this interprets time evolution of a coherent states $|\alpha\rangle$, up to an irrelevant phase the state remains a coherent state with a time changing parameter $e^{-i\omega t}\alpha$, also the state is represented by a vector that rotates in the clockwise direction with angular velocity ω in the complex plane α that can be represented with two axis, the first one is a real axis that gives $\langle x\rangle$ up to a proportionality constant, and the second one is an imaginary axis that gives $\langle p\rangle$ up to a proportionality constant. This can be interpreted as a phase space and the evolution regarding any state is represented by a circle, as demonstrated in the next figure

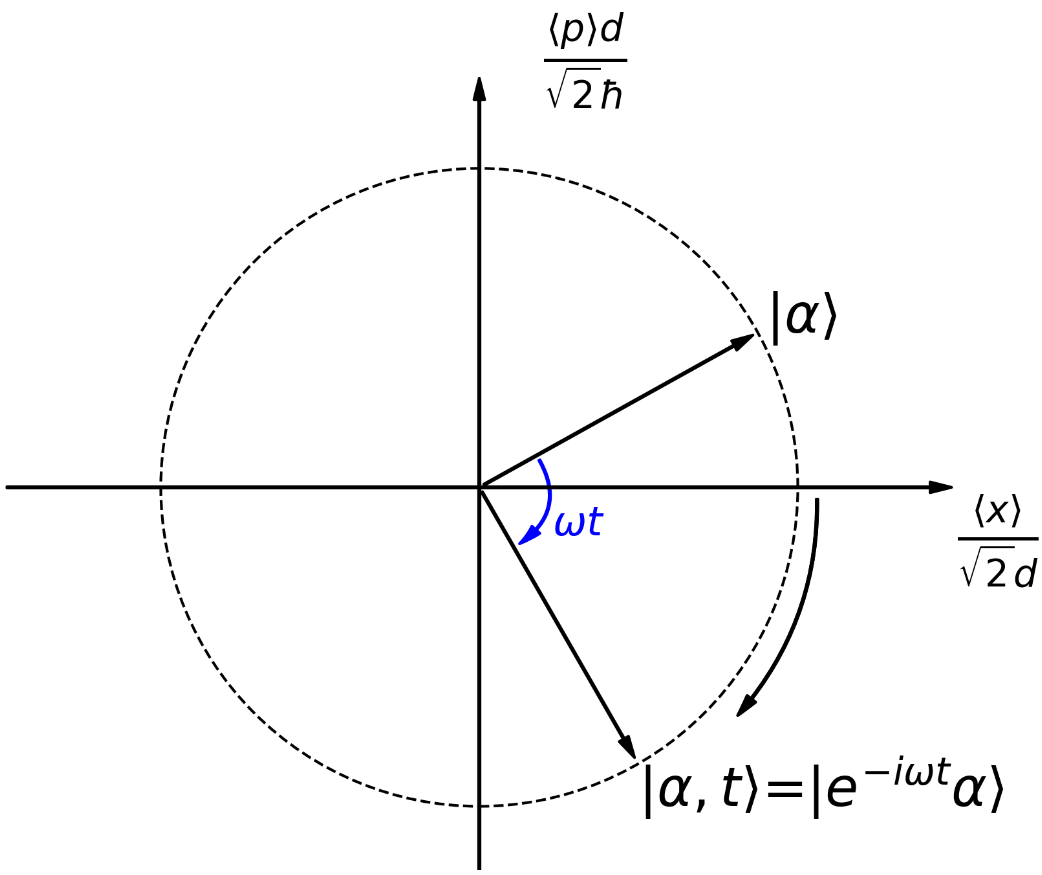

Figure 1.2.1 represents the time evolution of the coherent state $|\alpha\rangle$.

The figure (1.2.1) states that the real and the imaginary parts of α determine the expectation values $\langle x\rangle$ and $\langle p\rangle$ respectively. Noticeably, throughout the time evolution the parameter α of the coherent state rotates clockwise with an angular velocity ω.

In an alternative way, there is a conventional calculation of the time evolution by expanding the exponential in (1.29), where we know that the coherent states refer to a set of vectors in the Hilbert space and refer to a special kind of quantum mechanical state of light field that is defined in (1.14) and its definition is also known as the coherent state in the n-representation, which implies that there is a coherent state in the x-representation that will be mentioned later on. By using the action $\exp\left[-iHt/\hbar\right]$, we thus obtain

$$|\alpha,t\rangle \equiv \exp\left[-iHt/\hbar\right]|\alpha\rangle = \frac{\exp\left[-\frac{|\alpha|^2}{2}\right]\sum_{n=0}^{\infty}\alpha^n \exp\left[-\frac{i\hbar\omega\left[\left(n+\frac{1}{2}\right)t\right]}{\hbar}\right]}{\sqrt{n!}}|n\rangle\,, \tag{1.41}$$

where we find

$$|\alpha,t\rangle = \exp\left[\frac{-i\omega t}{2}\right]\exp\left[-\frac{|\alpha|^2}{2}\right]\frac{\sum_{n=0}^{\infty}\left(\exp\left[-i\omega t\right]\alpha\right)^n}{\sqrt{n!}}|n\rangle\,, \tag{1.42}$$

we note that

$$|e^{-i\omega t}\alpha|^2 = |\alpha|^2, \tag{1.43}$$

therefore, the time evolved coherent state can be written in the following form

$$|\alpha,t\rangle = \exp\left[-\frac{i\omega t}{2}\right]\left|e^{-i\omega t}\alpha\right\rangle, \tag{1.44}$$

this confirms our previous result in (1.40).

1.3 Generalized coherent states: linear invariants approach

We introduce the non-Hermitian invariant linear operator as

$$A(t) = f(t)\,q + ig(t)\,p, \tag{1.45}$$

$$A^{\dagger}(t) = f^{*}(t)\,q - ig^{*}(t)\,p. \tag{1.46}$$

where $f(t)$ and $g(t)$ are time-dependent complex functions. An operator $I(t)$ is said invariant if it satisfies the Von-Neumann equation

$$\frac{\partial I^{PH}(t)}{\partial t} = \frac{i}{\hbar}\left[I^{PH}(t), H(t)\right]. \tag{1.47}$$

The generalized coherent states can be defined with the invariant operators $A(t)$ and $A^{\dagger}(t)$, as they can be considered as the annihilation and creation operators, respectively, where $\left[A(t), A^{\dagger}(t)\right] = \mathbb{I}$, as

$$A(t)|\alpha, t\rangle = \alpha|\alpha, t\rangle, \tag{1.48}$$

the eigenstates of the operator $A(t)$ are the generalized coherent states, also the number operator can be written as $N(t) = A^{\dagger}(t)A(t)$. The generalized coherent states $|\alpha, t\rangle$ can be obtained from the action of the displacement operator $D(\alpha, t)$ on the vacuum state $|0, t\rangle$ defined by $A(t)|0, t\rangle = 0$ as

$$|\alpha, t\rangle = D(\alpha, t)|0, t\rangle, \tag{1.49}$$

where the displacement operator is given by

$$D(\alpha, t) = \exp\left[\alpha A^{\dagger}(t) - \alpha^{*}A(t)\right], \tag{1.50}$$

we emphasize that the generalized coherent states have the same properties as the ones mentioned before.

1.4 Definition of the squeezed states

Squeezed states [8, 9] are a special class of minimum-uncertainty states, have received considerable attention due to their important applications in optical communication, photon detection techniques, gravitational wave detection and noise-free amplification.

Squeezed states represent a generalization of coherent states were introduced by different authors [10, 11, 12, 13, 14, 15, 16, 17]: Stoler[11, 18], Lu[19, 20], Yuen[21] and Hollenhorst [13] whom originated them with the name "squeezed states".

Similar to the coherent states, the squeezed states can be defined by some distinct but equivalent ways:

1. To obtain squeezed states [34, 35, 36, 37], one applies both the squeeze and displacement operators on the ground state

$$D(\alpha)S(\xi)\left|0\right\rangle \equiv \left|\alpha,\xi\right\rangle . \tag{1.51}$$

We find in the literature different squeeze operators introduced for obtaining squeezed states that are given as [8, 11, 18, 38, 39, 40]

$$\begin{aligned} S_1\left(\xi\right) &= \exp(\frac{1}{2}\left[\xi a^2 - \xi^* a^{\dagger^2}\right]), & (1.52)\\ S_2\left(\xi\right) &= \exp(\frac{1}{2}\left[\xi^* a^{\dagger^2} - \xi a^2\right]), & (1.53)\\ S_3\left(\xi\right) &= \exp(\frac{1}{2}\left[\xi^* a^2 - \xi a^{\dagger^2}\right]), & (1.54)\\ S_4\left(\xi\right) &= \exp(\frac{1}{2}\left[\xi a^{\dagger^2} - \xi^* a^2\right]), & (1.55)\end{aligned}$$

where $\xi = r\exp\left[i\theta\right]$ is an arbitrary complex parameter and a and $a^\dagger$ are the lowering and raising operators, which satisfy the commuation relation (1.12), the numbers r and θ are real and known as the *squeezed factor* and the *squeeze angle*, respectivly, and they are defined in the intervals $0 \leq r < \infty$ and -$\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$, we notice that $S_1\left(\xi\right) = S_2^\dagger\left(\xi\right)$ and $S_3\left(\xi\right) = S_4^\dagger\left(\xi\right)$.

Let us consider the unitary squeeze operator defined by

$$S(\xi) = \exp\left[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)\right], \tag{1.56}$$

whose properties are summarized as

(a) Unitarity

$$S^\dagger\left(\xi\right) = S^{-1}\left(\xi\right) = S\left(-\xi\right). \tag{1.57}$$

(b) $S(\xi)$ acts upon the lowering and raising operators, a and its adjoint $a^\dagger$, i.e.

$$\begin{aligned} b = S(\xi)aS^{-1}(\xi) &= \cosh\left|\xi\right|\, a - \frac{\xi}{\left|\xi\right|}\sinh\left|\xi\right| a^\dagger,\\ b^\dagger = S(\xi)a^\dagger S^{-1}(\xi) &= \cosh\left|\xi\right|\, a^\dagger - \frac{\xi^*}{\left|\xi\right|}\sinh\left|\xi\right| a\ , \end{aligned} \tag{1.58}$$

which is known as the Bogolyubov transformation [41].

(c) If the squeeze parameter ξ is null, $\xi = 0$, we obtain the coherent states that we defined above

$$D(\alpha)S(0)\left|0\right\rangle = D(\alpha)\left|0\right\rangle = \left|\alpha, 0\right\rangle = \left|\alpha\right\rangle. \tag{1.59}$$

Noticeably, since $D\left(\alpha\right)$ and $S(\xi)$ are both unitary operators, then

$$\langle\alpha, \xi|\alpha, \xi\rangle = \mathbb{I}. \tag{1.60}$$

An equivalent form of defining the two-photon coherent state [34] reverses the order of the $D(\alpha)$ and $S(\xi)$ such that

$$D(\alpha)S(\xi) = S(\xi)D(\gamma), \tag{1.61}$$

$$\gamma(\alpha, \xi) = \alpha \cosh|\xi| - \alpha^* \frac{\xi}{|\xi|} \sinh|\xi| \,. \tag{1.62}$$

2. The annihilation- (or, more generally, ladder-) operator method. Using a Bogolyubov transformation (1.58), the operator $D(\alpha)S(\xi)aS^{-1}(\xi)D^{-1}(\alpha)$ can be expressed as a linear combination of a and $a^\dagger$, this transformation provides an eigenvalue relation for the squeezed coherent states,

$$\left(\cosh|\xi|\ a - \frac{\xi}{|\xi|}\sinh|\xi|\, a^\dagger\right)\left|\alpha, \xi\right\rangle = \gamma(\alpha, \xi)\left|\alpha, \xi\right\rangle, \tag{1.63}$$

with $\gamma(\alpha, \xi)$ given in Eq. (1.62).

3. Minimum-Uncertainty Method, in which the squeezed states can be obtained as states which satisfy, rather than the Heisenberg uncertainty relation, the Schrödinger-Robertson uncertainty relation

$$\Delta A \Delta B \geq \frac{|\langle C\rangle|}{2}, \qquad \text{where } [A, B] = iC. \tag{1.64}$$

A squeezed state is obtained if the variance in one of the latter observables met the condition

$$(\Delta A)^2 < \frac{|\langle C\rangle|}{2}, \text{ or } (\Delta B)^2 < \frac{|\langle C\rangle|}{2}, \tag{1.65}$$

if the condition (1.65) is verified and, in addition, the relation (1.64) is found to be an equality, i.e.

$$\Delta A \Delta B = \frac{|\langle C\rangle|}{2}, \tag{1.66}$$

then the state is called an ideal squeezed state.

As an illustration, A and B can be expressed in the dimensionless position and momentum operators form X and P of (1.3) and (1.4), respectively, where

$$X = \frac{1}{2}(a + a^\dagger), \qquad P = \frac{1}{2i}(a - a^\dagger), \tag{1.67}$$

thus, from (1.64), we obtain

$$\Delta X \Delta P \geq \frac{1}{4}, \quad \text{where } [X, P] = (-2i)^{-1}, \tag{1.68}$$

furthermore, the squeezed states are obtained if (1.65) is verified, also, an ideal squeezed state is obtained if in addition to (1.65) the relation in (1.68) is found to be an equality i.e.

$$\Delta X \Delta P = \frac{1}{4}. \tag{1.69}$$

1.5 Time evolution of the squeezed states

We define the time evolution of squeezed coherent states using the Heisenberg picture and from (1.51) we get

$$|\alpha, \xi, t\rangle = \exp\left[-\frac{i}{\hbar}Ht\right]|\alpha, \xi\rangle\,, \tag{1.70}$$

it follows that

$$|\alpha, \xi, t\rangle = e^{\left(-\frac{i}{\hbar}Ht\right)} D(\alpha)\ e^{\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}Ht} S(\xi)\, e^{\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}Ht}\,|0\rangle\,, \tag{1.71}$$

by replacing $D(\alpha)$ and $S(\xi)$, we obtain

$$|\alpha, \xi, t\rangle = e^{[-\frac{i}{\hbar}Ht]} e^{[\alpha a^\dagger - \alpha^* a]} e^{\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}Ht} e^{[\frac{1}{2}\left(\xi\left(a^\dagger\right)^2 - \xi^*(a)^2\right)]} e^{\frac{i}{\hbar}Ht} e^{-\frac{i}{\hbar}Ht}\,|0\rangle\,, \tag{1.72}$$

then by using the Baker-Campbell-Hausdorff formula (1.20), we now have the time dependent squeezed coherent states, defined as

$$|\alpha, \xi, t\rangle = \exp\left[\frac{-i\omega t}{2}\right] D_\alpha(t)\, S_\xi(t)\,|0\rangle\,, \tag{1.73}$$

where

$$D_\alpha(t) = \exp\left[\alpha a^\dagger(t) - \alpha^* a(t)\right], \tag{1.74}$$

$$S_\xi(t) = \exp\left[\frac{1}{2}\left(\xi a^\dagger(t)\, a^\dagger(t) - \xi^* a(t)\, a(t)\right)\right], \tag{1.75}$$

Chapter 2

Time-dependent pseudo-squeezed coherent states

2.1 Time-dependent non-Hermitian systems

$\mathcal{PT}$ -symmetry and pseudo-Hermiticity are two notions that have emerged widely in the literature showing that non-Hermitian systems may have real energy spectrum.

$\mathcal{PT}$ -symmetry that was developped in 1998 by Bender and Boettcher [42] signifies the symmetry parity-time, where $\mathcal{P}$ is a linear operator and it represents the parity, or an in-depth definition, it represents the space reflection, while the time reversal $\mathcal{T}$ is an anti linear operator. The two operators commute $[\mathcal{P}, \mathcal{T}] = 0$ but not known to be equal, their square is the identity $(\mathcal{PT})^2 = \mathbb{I}$, where $\mathcal{P}^2 = \mathcal{T}^2 = \mathbb{I}$. The action of $\mathcal{P}$ and $\mathcal{T}$ on the operators of position x, momentum p and the imaginary number i are given as

$$\mathcal{P}\left\{x \rightarrow -x \quad , \quad p \rightarrow -p \quad , \quad i \rightarrow i\right\}, \tag{2.1}$$

$$\mathcal{T}\left\{x \rightarrow x \quad , \quad p \rightarrow -p \quad , \quad i \rightarrow -i\right\}, \tag{2.2}$$

we find in some literature, the authors define the operator $\mathcal{T}$ as time changing also $t \rightarrow -t$ [43, 44, 45, 46, 47, 48]. A Hamiltonian H is $\mathcal{PT}$-symmetric if it satisfies

$$[H, \mathcal{PT}] = 0, \tag{2.3}$$

the notion of $\mathcal{PT}$ -symmetry was generalized by Mostafazadeh when he introduced the notion of pseudo-Hermiticity [49, 50, 51]. The Hamiltonian H is said to be pseudo-hermitian if it satisfies the following

$$H^{\dagger} = \eta H \eta^{-1} \tag{2.4}$$

where $H^{\dagger}$ is the adjoint of H, and η is a Hermitian bounded invertible operator. Writing the eigenvalues equations of H and $H^{\dagger}$ as

$$\begin{aligned} H\left|\psi_n\right\rangle &= E_n\left|\psi_n\right\rangle, & (2.5)\\ H^{\dagger}\left|\phi_n\right\rangle &= E_n^{*}\left|\phi_n\right\rangle. & (2.6)\end{aligned}$$

where the eigenvectors $\left|\psi_n\right\rangle$ and $\left|\phi_n\right\rangle$ form a biorthonormal basis

$$\left\langle\phi_m\right.\left|\psi_n\right\rangle = \delta_{mn}. \tag{2.7}$$

The closure relation reads

$$\sum_n \left|\psi_n\right\rangle\left\langle\phi_n\right| = \sum_n \left|\phi_n\right\rangle\left\langle\psi_n\right| = \mathbb{I}, \tag{2.8}$$

therefore, H and $H^{\dagger}$, are given as

$$H = \sum_n E_n\left|\psi_n\right\rangle\left\langle\phi_n\right|, \qquad H^{\dagger} = \sum_n E_n^{*}\left|\phi_n\right\rangle\left\langle\psi_n\right|. \tag{2.9}$$

The pseud-Hermicity connects also the Hamiltonian H to a Hermitian one h as

$$h = \rho H \rho^{-1}, \tag{2.10}$$

where the Dyson transformation operator ρ is linear, bounded and invertible. In fact, from the preceding relation we can obtain (2.4), therefore,

$$\rho H \rho^{-1} = \left(\rho^{-1}\right)^{\dagger} H^{\dagger}\rho^{\dagger}, \tag{2.11}$$

we multiply it from the left by $\rho^{\dagger}$, and from the right with $\left(\rho^{\dagger}\right)^{-1}$, noting that $\eta = \rho^{\dagger}\rho$

$$\rho^{\dagger}\rho H \rho^{-1}\left(\rho^{\dagger}\right)^{-1} = \eta H \eta^{-1} = H^{\dagger}, \tag{2.12}$$

now, we consider the eigenvalues of the hermitian Hamiltonian h

$$h\left|\chi_n\right\rangle = E_n\left|\chi_n\right\rangle. \tag{2.13}$$

the transformation ρ allows us to pass from the eigenfuctions of h to the eigenfunctions of H

$$|\chi_n\rangle = \rho\,|\psi_n\rangle\,. \tag{2.14}$$

The eigenfunctions of $|\chi_n\rangle$ form an orthonormal basis in which the inner-product is preserved, i.e.

$$\langle\chi_m\,|\chi_n\rangle = \delta_{mn}, \tag{2.15}$$

by using (2.14), we obtain

$$\langle\chi_m\,|\chi_n\rangle = \langle\psi_m|\,\rho^{\dagger}\rho\,|\psi_n\rangle = \langle\psi_m|\,\eta\,|\psi_n\rangle = \langle\psi_m|\,\psi_n\rangle_{\eta} = \delta_{mn}, \tag{2.16}$$

the latter is known as the pseudo-inner product or the η-inner product.

The study of time-dependent non-Hermitian Hamiltonian systems has led to a controversial between physicists: Mostafazadeh [52, 53, 54] said that the evolution of a pseudo-Hermitian Hamiltonian $H(t)$ is unitary only if the metric operator is time-independent. Znojil [55, 56, 57] demonstrated the unitary evolution of a time-dependent system does not necessitate a time-independent metric operator, it can be obtained with a time-dependent one. Whereas Fring and Moussa [58, 59] established a time-dependent quasi-Hermiticity relation.

We summarize the three different points of views in what follows:

1) Ali Mostafazadeh point of view: Let $U^{H}(t)$ be the time-evolution operator associated to the non-Hermitian Hamiltonian $H(t)$

$$H(t)\,U^{H}(t) = i\hbar\frac{\partial}{\partial t}U^{H}(t)\,, \tag{2.17}$$

where $U(0)=\mathbb{I}$, and $\psi(t)$,$\phi(t)$ are eigenstates with $U^{H}(t)$ defining their time evolution as

$$\psi(t) = U^{H}(t)\,\psi(0)\,, \qquad\qquad \phi(t) = U^{H}(t)\,\phi(0)\,, \tag{2.18}$$

the time independence is given to the scalar product $\langle\psi(t)\,,\phi(t)\rangle_{\eta(t)}$ by the unitary evolution. The pseudo scalar product $\langle.,.\rangle_{\eta(t)}$ is also valid for $\eta(t)$ i.e. (2.16)

$$\begin{aligned}\langle\psi(t)\,,\phi(t)\rangle_{\eta(t)} &= \langle\psi(t)\,|\eta(t)\,\phi(t)\rangle\\ &= \langle\psi(0)|\,U^{\dagger H}(t)\,\eta(t)\,U^{H}(t)\,|\phi(0)\rangle = \langle\psi(0)|\,\eta(0)\,|\phi(0)\rangle\,,\end{aligned} \tag{2.19}$$

it follows that

$$U^{\dagger H}(t)\,\eta(t)\,U^{H}(t)=\eta(0)\Longleftrightarrow\eta(t)=\left[U^{\dagger H}(t)\right]^{-1}\eta(0)\left[U^{H}(t)\right]^{-1},\tag{2.20}$$

thus, we obtain

$$\eta^{-1}(t)=U^{\dagger H}(t)\,\eta^{-1}(0)\,U^{H}(t),\tag{2.21}$$

by using the latter relation, the relation in (2.17) gives

$$H^{\dagger}(t)=\eta(t)\,H(t)\,\eta^{-1}(t)-i\hbar\eta(t)\,\frac{\partial}{\partial t}\eta^{-1}(t).\tag{2.22}$$

the preceding equation demonstrates that $H(t)$ is η-pseudo-Hermitian only if η is time independent (2.4).

2) Milozlav Znojil point of view: Znojil states that the time evoution of the quasi-Hermitian quantum systems is generated by the time evolution of non observable generator H_{gen} different than H. The associated Schrödinger time dependent equation of the hamiltonian $h(t)$ is

$$h(t)\left|\varphi(t)\right\rangle=i\hbar\frac{\partial}{\partial t}\left|\varphi(t)\right\rangle,\qquad\qquad \left|\varphi(t)\right\rangle=U^{h}(t)\left|\varphi(0)\right\rangle,\tag{2.23}$$

and can be written in terms of the unitary ($U^{h}(t)\left[U^{h}(t)\right]^{\dagger}=\mathbb{I}$) time evolution operator $U^{h}(t)$ as

$$i\hbar\frac{\partial}{\partial t}U^{h}(t)\left|\varphi(0)\right\rangle=h(t)\,U^{h}(t)\left|\varphi(0)\right\rangle,\tag{2.24}$$

where $h(t)=\rho(t)\,H(t)\,\rho^{-1}(t)$, therefore, the Schrödinger equation solution is given in the following form

$$\left|\varphi(t)\right\rangle=U^{h}(t)\left|\varphi(0)\right\rangle,\tag{2.25}$$

and gratifies

$$\left\langle\varphi(t)\,|\varphi(t)\right\rangle=\left\langle\varphi(0)\,|\varphi(0)\right\rangle,\tag{2.26}$$

which indicates that the norm is constant.

Znojil[57] is distinguishing two time evolution forms given as

$$\begin{aligned} \left|\phi\left(t\right)\right\rangle &= U_R\left(t\right)\left|\phi\left(0\right)\right\rangle, \quad (2.27)\\ \left\langle\phi'\left(t\right)\right| &= \left\langle\phi'\left(0\right)\right|U_L\left(t\right), \quad (2.28)\end{aligned}$$

where $U_R\left(t\right) = \rho^{-1}\left(t\right)U^h\left(t\right)\rho\left(0\right)$ and $U_L\left(t\right) = \rho^{-1}\left(0\right)U^{\dagger h}\left(t\right)\rho\left(t\right)$ and they act on $|\phi(t)\rangle = \rho^{-1}\left(t\right)\left|\varphi\left(t\right)\right\rangle$ and $\left\langle\phi'\left(t\right)\right| = \left\langle\varphi'\left(t\right)\right|\rho\left(t\right)$, respectively, this suggests that there is two different methods of representing the wave function in (2.25), simple computing leads to the time evolution rule of the action on the right as well as on the left. The differential equations of the right time evolution operators $U_R\left(t\right)$ and the left $U_L\left(t\right)$ are as follows

$$\begin{aligned} i\hbar\partial_t U_R\left(t\right) &= -i\hbar\rho^{-1}\left(t\right)\left[\partial_t\rho\left(t\right)\right]U_R\left(t\right) + H\left(t\right)U_R\left(t\right), \quad (2.29)\\ i\hbar\partial_t U_L^{\dagger}\left(t\right) &= \left[i\hbar\partial_t\rho^{\dagger}\left(t\right)\right]\left[\rho^{-1}\left(t\right)\right]^{\dagger}U_L^{\dagger}\left(t\right) + H^{\dagger}\left(t\right)U_L^{\dagger}\left(t\right). \quad (2.30)\end{aligned}$$

In consequence, the states $|\phi\rangle$ and $|\phi'\rangle$ satisfy the Schrödinger equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\phi\left(t\right)\right\rangle &= H_{gen}\left(t\right)\left|\phi\left(t\right)\right\rangle, \quad (2.31)\\ i\hbar\frac{\partial}{\partial t}\left|\phi'\left(t\right)\right\rangle &= H_{gen}^{\dagger}\left(t\right)\left|\phi'\left(t\right)\right\rangle, \quad (2.32)\end{aligned}$$

where

$$\begin{aligned} H_{gen}\left(t\right) &= H\left(t\right) - i\hbar\rho^{-1}\left(t\right)\partial_t\rho\left(t\right), \quad (2.33)\\ H_{gen}^{\dagger}\left(t\right) &= H^{\dagger}\left(t\right) + i\hbar\partial_t\rho^{\dagger}\left(t\right)\left(\rho^{-1}\left(t\right)\right)^{\dagger}. \quad (2.34)\end{aligned}$$

This point of view also demonstrates the time evolution to be unitary, if we take the normalization time differential, we obtain

$$\partial_t\left\langle\phi'\left(t\right)\left|\phi\left(t\right)\right.\right\rangle = 0. \quad (2.35)$$

Noticeably, two opposite point of views emerges while constructing the quantum time dependent quasi-hermitian systems.

3) Fring and Moussa point of view: Fring and Moussa confirm that when $\eta(t)$ is time dependent the relations of quasi-Hermicity (2.10) and (2.12) are not valid, thus, approving Mostafazadeh's point of view. We take two time dependent Shrödinger equations

$$h(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle, \qquad H(t)|\phi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle, \tag{2.36}$$

where $h(t)$ is hermitian $h(t) = h^{\dagger}(t)$ while $H(t)$ is not $H(t) \neq H^{\dagger}(t)$, Fring and Moussa claim that the operators are not called Hamiltonians unless they generate the time evolution of the system under consideration, In which they should satisfy the time dependent Schrödinger euqation. Afterwards, they claim the two solutions $|\psi(t)\rangle$ and $|\phi(t)\rangle$ to be connected by a time dependent invertible operator $\rho(t)$

$$|\psi(t)\rangle = \rho(t)|\phi(t)\rangle, \tag{2.37}$$

if we substitute the latter relation into (2.36), we find that the two Hamiltonians are connected to each other as

$$h(t) = \rho(t) H(t) \rho^{-1}(t) - i\hbar\rho^{-1}(t)\partial_t\rho(t), \tag{2.38}$$

we note that $h(t)$ and $H(t)$ are not related with a similarity transformation as in the time independent case, or as in the time dependent case where the metric operator is time independent. They refer to the preceding equation as the time dependent Dyson relation, thus generalizing its time independent counterpart. If we take the adjoint of (2.38) and use the Hermicity of $h(t)$ we obtain the relation between $H(t)$ and its adjoint

$$H^{\dagger}(t)\eta(t) - \eta(t) H(t) = i\hbar\partial_t\eta(t), \tag{2.39}$$

thus, defining $\eta(t) = \rho^{\dagger}(t)\rho(t)$ as a metric operator, the latter relation replaces the well known standard quasi-Hermicity relation in the context of the non-Hermitian time dependent quantum mechanincs [58, 59].

Besides the three approaches listed above, the pseudo-invariants theory is a useful theory for the study of time-dependent non-Hermitian systems that we will use for the construction of pseudo-squeezed coherent states.

2.2 Pseudo-squeezed coherent states:

To build a time-dependent pseudo-squeezed coherent state, we consider a non-Hermitian time-dependent case, therefore, we should find the integrals of motion, and we will choose the annihilation and creation operators which are very convenient for the study of pseudo-coherent states and pseudo-squeezed coherent states, to show that the pseudo-squeezed coherent states constitute a non-orthogonal overcomplete system which yields a resolution of the identity operator.

In addition, coherent states diagonalize the annihilation operator a involved in the harmonic oscillator algebra. We stress that the eigenstates of a and other non-Hermitian operators are not orthogonal. Consequently, we introduce the diagonalization [60, 61, 62, 63] of the complex time-dependent combinations of the annihilation and creation operators

$$A(t) = u(t)\,a + v(t)\,a^{\dagger}, \tag{2.40}$$

$A(t)$ is an operator in which is constructed [60, 61] as a non-Hermitian invariant for the quantum variable frequency oscillator with the Hamiltonian $h(t) = \frac{1}{2}\left[p^2 + \omega^2(t)\,q^2\right]$, where

$$\frac{\partial A(t)}{\partial t} - i\left[A(t), h(t)\right] = 0, \qquad \frac{\partial A^{\dagger}(t)}{\partial t} - i\left[A^{\dagger}(t), h(t)\right] = 0, \tag{2.41}$$

and satisfies the following commutation relation

$$\left[A(t), A^{\dagger}(t)\right] = 1. \tag{2.42}$$

Therefore, the operators $A(t)$ and $A^{\dagger}(t)$ can be considered respectively as the lowering and raising operators and can be used for the construction of the time-dependent coherent states for the system under consideration.

2.2.1 Pseudo-bosons and pseudo-linear invariants

Similarly, to the time-independent case, we now introduce the time-dependent pseudo-bosonic coherent states, where we emphasize that now we use a time-dependent creation, annihilation and metric operators, where a linear metric operator $\eta = \rho^{\dagger}\rho$ connects a non-Hermitian Hamiltonian to its Hermitian conjugate $H^{\dagger} = \eta H \eta^{-1}$, where now H is η-pseudo-Hermitian with respect to a positive-definite inner product defined by $\langle ., . \rangle_{\eta} = \langle . | \eta | . \rangle$.

More recently, considerable attention has been paid to an alternative formalism for the description of non-Hermitian systems, based on the concept of the so-called pseudo-bosons [64, 65, 66]. Pseudo-bosons are a pseudo-Hermitian extension of usual bosons. In fact, they are a very particular modification of the standard canonical bosonic commutation rule $\left[A, A^{\dagger}\right]$ = 1, obtained from

$$\left[A, \bar{A}\right] = [\bar{A}^{\dagger}, A^{\dagger}] = 1, \tag{2.43}$$

where the operators A and $\bar{A}$ are related to their adjoint operator $\bar{A}^{\dagger}$ and $A^{\dagger}$, respectively, via the bounded Hermitian invertible operator or metric operator η as

$$\begin{aligned} \bar{A}^{\dagger} &= \eta A \eta^{-1}, \\ A^{\dagger} &= \eta \bar{A} \eta^{-1}. \end{aligned} \tag{2.44}$$

Coherent states are generally studied in the Hilbert space $\mathcal{H}$ generated through a self adjoint Hamiltonian (i.e. $H = H^{\dagger}$). However, under the above assumptions, we can introduce different kinds of pseudo-coherent states in a non-Hermitian case. The notion of pseudo-coherent states, in connection with pseudo-bosons, originally introduced in [64] and then analysed from a more mathematically oriented perspective, in [65], has been considered as a non-Hermitian generalization of coherent states. The pseudo-coherent states for the pseudo-Hermitian boson systems are defined as eigenstates of the corresponding pseudo-boson annihilation operators A and $\bar{A}^{\dagger}$

$$A \left|\psi_{\alpha}\right\rangle = \alpha \left|\psi_{\alpha}\right\rangle, \qquad \bar{A}^{\dagger} \left|\phi_{\alpha}\right\rangle = \alpha \left|\phi_{\alpha}\right\rangle, \quad \alpha \in \mathbb{C}. \tag{2.45}$$

and satisfy the resolution of the identity

$$\frac{1}{\pi} \int_{\mathbb{C}} \left|\phi_{\alpha}\right\rangle \langle\psi_{\alpha}| \, d\alpha^{*} d\alpha = \frac{1}{\pi} \int_{\mathbb{C}} |\psi_{\alpha}\rangle \left\langle\phi_{\alpha}\right| d\alpha^{*} d\alpha = \mathbb{I}. \tag{2.46}$$

with $\left|\phi_{\alpha}\right\rangle = \eta \left|\psi_{\alpha}\right\rangle$. These pseudo-coherent states $\left|\psi_{\alpha}\right\rangle$ and $\left|\phi_{\alpha}\right\rangle$ can be generated respectively from the vacuum states $\left|\psi_{0}\right\rangle$ and $\left|\phi_{0}\right\rangle$ by the action of displacement operators $D(\alpha)$ and $\overline{D}^{\dagger}(\alpha)$, respectively,

$$D(\alpha) = \exp(\alpha \bar{A} - \alpha^{*} A), \qquad \overline{D}^{\dagger}(\alpha) = \exp(\alpha A^{\dagger} - \alpha^{*} \bar{A}^{\dagger}) \tag{2.47}$$

where $\overline{D}^{\dagger}(\alpha) = \eta D(\alpha) \eta^{-1}$ is the complementary pseudo-unitary displacement operator of $D(\alpha)$. Bagarello et al. [67] introduced the bi-squeezed states that can be considered as a suitable non-Hermitian extension of the squeezed states. Bi-squeezed states are defined by the action of the squeezing operator on the vacuum state $\left|\psi_{_0}\right\rangle$ of pseudo-boson annihilation operator A. Moreover, the bi-squeezed coherent states, are defined as the successive applications of the displacement and of the squeezing operators on the vacuum $\left|\psi_{_0}\right\rangle$. In our approach, the pseudo-bosonic squeezed coherent states are generated by displacing the squeezed state or the ground state.

If we identify $A\left(t\right),\ \bar{A}\left(t\right),\ \bar{A}^{\dagger}\left(t\right)$ and $A^{\dagger}\left(t\right)$ as integrals of motion, where $A\left(t\right),\ \bar{A}\left(t\right)$ are associated to the time-dependent non-Hermitian Hamiltonian $H\left(t\right)$ whereas $\bar{A}^{\dagger}\left(t\right),\ A^{\dagger}\left(t\right)$ are associated to the latter Hamiltonians adjoint $H^{\dagger}\left(t\right)$, thus they are time-dependent invariant operators verifying the following equations

$$\frac{\partial A\left(t\right)}{\partial t}-i\left[A\left(t\right),H\left(t\right)\right]=0,\qquad \frac{\partial \overline{A}\left(t\right)}{\partial t}-i\left[\overline{A}\left(t\right),H\left(t\right)\right]=0, \tag{2.48}$$

$$\frac{\partial \overline{A^{\dagger}}\left(t\right)}{\partial t}-i\left[\overline{A^{\dagger}}\left(t\right),H^{\dagger}\left(t\right)\right]=0,\qquad \frac{\partial A^{\dagger}\left(t\right)}{\partial t}-i\left[A^{\dagger}\left(t\right),H^{\dagger}\left(t\right)\right]=0, \tag{2.49}$$

where the Hamiltonian $H\left(t\right)$ governs the time-dependent Schrödinger equation

$$\left|\Psi^{H}\left(t\right)\right\rangle=e^{i\varphi(t)}\left|\psi\left(t\right)\right\rangle. \tag{2.50}$$

2.2.2 Pseudo-bosonic coherent states

Consequently, in order to construct pseudo-bosonic coherent states, we consider in analogy with the time-independent case reported above, that the invariant operators $A\left(t\right)$, $\overline{A}\left(t\right)$, $\overline{A^{\dagger}}\left(t\right)$ and $A^{\dagger}\left(t\right)$ as time-dependent pseudo-bosonic annihilation and creation operators associated to $H\left(t\right)$ and $H^{\dagger}\left(t\right)$ respectively, that verify the so-called Weyl-Heisenberg commutation relations

$$\left[A\left(t\right),\overline{A}\left(t\right)\right]=\left[\overline{A^{\dagger}}\left(t\right),A^{\dagger}\left(t\right)\right]=\mathbb{I}. \tag{2.51}$$

These operators act on a dense subspace $\mathcal{D}$ of $\mathcal{H}$. The operators $A\left(t\right)$ and $\overline{A}\left(t\right)$ associated to $H\left(t\right)$ are related to the operators $\overline{A^{\dagger}}\left(t\right)$ and $A^{\dagger}\left(t\right)$ associated to $H^{\dagger}\left(t\right)$ via the time-dependent bounded Hermitian invertible operator $\eta\left(t\right)$ as

$$A\left(t\right)=\eta^{-1}\left(t\right)\overline{A^{\dagger}}\left(t\right)\eta\left(t\right),\qquad \overline{A}\left(t\right)=\eta^{-1}\left(t\right)A^{\dagger}\left(t\right)\eta\left(t\right), \tag{2.52}$$

where the pseudo-bosonic coherent states are generated by the action on the vacuum states $\{\left|\psi_0\left(t\right)\right\rangle,\ \left|\phi_0\left(t\right)\right\rangle\}$ of the pseudo-displacement operators $\left\{D^{H}\left(\alpha,t\right),D^{H^{\dagger}}\left(\alpha,t\right)\right\}$, in which

$$\left|\psi_{\alpha}\left(t\right)\right\rangle=D^{H}\left(\alpha,t\right)\left|\psi_0\left(t\right)\right\rangle=\exp\left[\alpha\overline{A}\left(t\right)-\alpha^{*}A\left(t\right)\right]\left|\psi_0\left(t\right)\right\rangle, \tag{2.53}$$

and

$$\left|\phi_{\alpha}\left(t\right)\right\rangle = D^{H^{\dagger}}\left(\alpha,t\right)\left|\phi_{0}\left(t\right)\right\rangle, \tag{2.54}$$

we note that $D^{H^{\dagger}}\left(\alpha,t\right)$ is the pseudo-adjoint of $D^{H}\left(\alpha,t\right)$, i.e.

$$D^{H^{\dagger}}\left(\alpha,t\right) = \eta\left(t\right) D^{H}\left(\alpha,t\right)\eta^{-1}\left(t\right) = \exp\left[\alpha A^{\dagger}\left(t\right) - \alpha^{*}\overline{A^{\dagger}}\left(t\right)\right]. \tag{2.55}$$

Additionally, the vacuum states are defined by

$$A\left(t\right)\left|\psi_{0}\left(t\right)\right\rangle = 0, \qquad \overline{A^{\dagger}}\left(t\right)\left|\phi_{0}\left(t\right)\right\rangle = 0. \tag{2.56}$$

Consequently, the vacuum states $\left|\psi_{0}\left(t\right)\right\rangle$ and $\left|\phi_{0}\left(t\right)\right\rangle$ are related to each other as

$$\left|\phi_{0}\left(t\right)\right\rangle = \eta\left(t\right)\left|\psi_{0}\left(t\right)\right\rangle, \tag{2.57}$$

where the same expression for $\{\left|\psi_{\alpha}\left(t\right)\right\rangle,\ \left|\phi_{\alpha}\left(t\right)\right\rangle = \eta\left(t\right)\left|\psi_{\alpha}\left(t\right)\right\rangle\}$, and can be obtained by defining them as eigenstates of the annihilation operators $\left\{A\left(t\right),\ \overline{A^{\dagger}}\left(t\right)\right\}$ with a complex time-independent eigenvalue α, i.e.

$$A\left(t\right)\left|\psi_{\alpha}\left(t\right)\right\rangle = \alpha\left|\psi_{\alpha}\left(t\right)\right\rangle, \qquad \overline{A^{\dagger}}\left(t\right)\left|\phi_{\alpha}\left(t\right)\right\rangle = \alpha\left|\phi_{\alpha}\left(t\right)\right\rangle. \tag{2.58}$$

Particularly, the choice of the normalization condition as

$$\left\langle\psi_{0}\left(t\right)\right|\eta\left(t\right)\left|\psi_{0}\left(t\right)\right\rangle = 1, \tag{2.59}$$

leads to

$$\left\langle\psi_{\alpha}\left(t\right)\right|\eta\left(t\right)\left|\psi_{\alpha}\left(t\right)\right\rangle = 1, \tag{2.60}$$

and, then the integral

$$\frac{1}{\pi}\int_{\mathcal{C}}\eta(t)\left|\psi_{\alpha}(t)\right\rangle\left\langle\psi_{\alpha}(t)\right| d\alpha^{*}d\alpha = \frac{1}{\pi}\int_{\mathcal{C}}\left|\phi_{\alpha}(t)\right\rangle\left\langle\phi_{\alpha}(t)\right|\eta^{-1}(t)d\alpha^{*}d\alpha = \mathbb{I}, \tag{2.61}$$

in which the integral is an identity operator.

2.2.3 Pseudo-squeezed coherent states

Another important class of quantum states are the squeezed states which are generated by the action of the squeezing operator

$$S^{H}\left(\xi,t\right) = \exp\left[\frac{1}{2}\left(\xi\overline{A}^{2}\left(t\right) - \xi^{*}A^{2}\left(t\right)\right)\right], \tag{2.62}$$

on the vacuum state $|\psi_0(t)\rangle$ of $A(t)$, also the squeezed vacuum state is denoted as

$$|\xi, t\rangle = S^H(\xi, t)\,|\psi_0(t)\rangle\,, \tag{2.63}$$

where ξ is the complex squeeze time-independent parameter. The definition in the preceding equation resembles that of the coherent states in equation (2.53), but with the linear displacement operator $D^H(\alpha, t)$ replaced by the squeeze operator of the equation (2.62), whose exponent is quadratic in the mode creation and destruction operators.

$$B(t) = S^H(\xi, t)\,A(t)\,S^{-1H}(\xi, t) = \cosh|\xi|\,A(t) - \frac{\xi}{|\xi|}\sinh|\xi|\overline{A}(t)\,, \tag{2.64}$$

$$\overline{B}(t) = S^H(\xi, t)\,\overline{A}(t)\,S^{-1H}(\xi, t) = \cosh|\xi|\,\overline{A}(t) - \frac{\xi^*}{|\xi|}\sinh|\xi|A(t)\,. \tag{2.65}$$

Noticeably, the product of any two quantum invariants is also another quantum invariant, the same holds for the sum of quantum invariants. In this form, it is straightforward that the preceding $B(t)$ and $\overline{B}(t)$ are indeed quantum invariant operators verifying the following equation

$$\frac{\partial B(t)}{\partial t} - i\,[B(t), H(t)] = 0, \quad \frac{\partial \overline{B}(t)}{\partial t} - i\left[\overline{B}(t), H(t)\right] = 0, \tag{2.66}$$

the ladder operators $B(t)$ and $\overline{B}(t)$ associated to $H(t)$ are related to the operators $\overline{B^\dagger}(t)$ and $B^\dagger(t)$ associated to $H^\dagger(t)$ via the time-dependent bounded Hermitian invertible operator $\eta(t)$ as

$$B(t) = \eta^{-1}(t)\,\overline{B^\dagger}(t)\,\eta(t)\,, \qquad \overline{B}(t) = \eta^{-1}(t)\,B^\dagger(t)\,\eta(t)\,. \tag{2.67}$$

In order to construct the time-dependent pseudo-bosonic squeezed coherent states, we consider, in analogy with the pseudo-bosonic coherent states case reported above, the invariant operators $B(t)$ and $\overline{B}(t)$ and their related ones via $\eta(t)$, $\overline{B^\dagger}(t)$ and $B^\dagger(t)$, as time-dependent pseudo-bosonic squeezed ladder operators associated to $H(t)$ and $H^\dagger(t)$, respectively, that verify the commutation relations

$$\left[B(t), \overline{B}(t)\right] = \left[\overline{B^\dagger}(t), B^\dagger(t)\right] = \mathbb{I}. \tag{2.68}$$

Alternately, we may define the squeezed states in a different way where we start from the squeezed vacuum (2.63), where

$$B(t)\,|\xi, t\rangle = S^H(\xi, t)\,A(t)\,|\psi_0(t)\rangle = 0. \tag{2.69}$$

Since the pseudo-bosonic coherent states $\left|\psi_{\alpha}(t)\right\rangle$ are generated by using the pseudo-displacement operator $D^{H}(\alpha, t)$ applied on the vacuum $\left|\psi_{0}(t)\right\rangle$, we may generate the set of pseudo-bosonic squeezed coherent states $\left|\psi_{\alpha,\xi}(t)\right\rangle$ by displacing the squeezed vacuum state. A more general pseudo-bosonic squeezed coherent states $\left|\psi_{\alpha,\xi}(t)\right\rangle$ may be obtained by applying the pseudo-squeezed displacement operator $T(\gamma, t)$ to the equation (2.63), where we obtain

$$\left|\psi_{\alpha,\xi}(t)\right\rangle = T(\gamma, t)\left|\xi, t\right\rangle = \exp\left[\gamma \overline{B}(t) - \gamma^{*} B(t)\right]\left|\xi, t\right\rangle, \tag{2.70}$$

where γ will be defined later on. Obviously, for $\xi = 0$ we just obtain the pseudo-coherent states. The properties of the pseudo-squeezed coherent states $\left|\psi_{\alpha,\xi}(t)\right\rangle$ may be proved to parallel those of the pseudo-coherent states $\left|\psi_{\alpha}(t)\right\rangle$. Since our pseudo-squeezed coherent states are closely related to the ones of the pseudo-coherent states $\left|\psi_{\alpha}(t)\right\rangle$, other constructions of squeezed coherent states can be considered using the ladder operator $B(t)$, where we obtain

$$B(t)\left|\psi_{\alpha,\xi}(t)\right\rangle = \gamma(\alpha, \xi)\left|\psi_{\alpha,\xi}(t)\right\rangle, \tag{2.71}$$

where the equality in the latter equation is from

$$T(\gamma, t) B(t) T^{-1}(\gamma, t) = B(t) - \gamma, \qquad T(\gamma, t) B^{\dagger}(t) T^{-1}(\gamma, t) = B^{\dagger}(t) - \gamma^{*}. \tag{2.72}$$

The use of the properties of the squeezed operator given in equations (2.64) and (2.65) leads to

$$T(\gamma, t) = S^{H}(\xi, t) D^{H}(\gamma, t) S^{-1H}(\xi, t) = D^{H}(\alpha, t). \tag{2.73}$$

The pseudo-squeezed coherent states $\left|\psi_{\alpha,\xi}(t)\right\rangle$ are obtained by first acting with the pseudo-squeezed displacement operator $T(\gamma, t)$ on the pseudo-squeezed vacuum states $\left|\xi, t\right\rangle$ or with the displacement operator $D^{H}(\alpha, t)$ on the pseudo-squeezed vacuum states $\left|\xi, t\right\rangle$. This transformation provides an eigenvalue relation for the pseudo-squeezed coherent state where

$$\gamma(\alpha, \xi) = \cosh|\xi|\alpha - \frac{\xi}{|\xi|}\sinh|\xi|\alpha^{*}. \tag{2.74}$$

On the other hand, when acting with the pseudo-squeezed displacement operator $T(\gamma, t)$ on the pseudo vacuum $\left|\psi_{0}(t)\right\rangle$, we obtain the pseudo-bosonic coherent states $\left|\psi_{\alpha}(t)\right\rangle$. Knowing that, the pseudo-vacuum states $\{\left|\psi_{0}(t)\right\rangle, \left|\phi_{0}(t)\right\rangle\}$ of $\left\{A(t), \overline{A}^{\dagger}(t)\right\}$ respectively are related to each other as $|\phi_0(t)\rangle = \eta(t)|\psi_0(t)\rangle$, consequently the pseudo-squeezed vacuum states $|\xi, t\rangle$ and $|\widetilde{\xi, t}\rangle$ of $B(t)$ and $\overline{B}^{\dagger}(t)$ are linked to each other as

$$|\widetilde{\xi, t}\rangle = \eta(t)|\xi, t\rangle = \left[S^{H}(\xi, t)\right]^{-1\dagger}\eta(t)|\psi_0(t)\rangle = \left[S^{H}(\xi, t)\right]^{-1\dagger}|\phi_0(t)\rangle. \tag{2.75}$$

Pseudo-bosonic squeezed coherent state $\left|\phi_{\alpha,\xi}(t)\right\rangle$, associated to $H^{\dagger}(t)$, can be also obtained from the action of the displacement operator

$$T^{H^{\dagger}}(\gamma, t) = \left[\eta(t)T(\gamma, t)\eta^{-1}(t)\right] = \left[T(\gamma, t)\right]^{-1\dagger}, \tag{2.76}$$

on the pseudo-squeezed vacuum state $|\widetilde{\xi, t}\rangle$ of $\overline{B}^{\dagger}(t)$ as

$$\left|\phi_{\alpha,\xi}(t)\right\rangle = \left[T(\gamma, t)\right]^{-1\dagger}|\widetilde{\xi, t}\rangle = \exp\left[\gamma B^{\dagger}(t) - \gamma^{*}\overline{B}^{\dagger}(t)\right]|\widetilde{\xi, t}\rangle. \tag{2.77}$$

The pseudo-squeezed coherent states $\left|\phi_{\alpha,\xi}(t)\right\rangle$ are eigenstates of the operator $\overline{B}^{\dagger}(t)$ with the complex time-independent eigenvalue γ where

$$\overline{B}^{\dagger}(t)\left|\phi_{\alpha,\xi}(t)\right\rangle = \gamma\left|\phi_{\alpha,\xi}(t)\right\rangle. \tag{2.78}$$

Therefore, the normalization condition in a similar way to (2.59) $\langle\psi_0(t)|\eta(t)|\psi_0(t)\rangle = \mathbb{I}$, leads to

$$\left\langle\psi_{\alpha,\xi}(t)\right|\eta(t)\left|\psi_{\alpha,\xi}(t)\right\rangle = \left\langle\phi_{\alpha,\xi}(t)\right.\left|\psi_{\alpha,\xi}(t)\right\rangle = \mathbb{I}, \tag{2.79}$$

which show that the pseudo-bosonic squeezed coherent states form an overcomplete set in that the identity can be resolved as

$$\frac{1}{\pi}\int_{c}\eta(t)\left|\psi_{\alpha,\xi}(t)\right\rangle\left\langle\psi_{\alpha,\xi}(t)\right|d\gamma^{*}d\gamma = \frac{1}{\pi}\int_{c}\left|\phi_{\alpha,\xi}(t)\right\rangle\left\langle\phi_{\alpha,\xi}(t)\right|\eta^{-1}(t)\,d\gamma^{*}d\gamma = \mathbb{I}. \tag{2.80}$$

Chapter 3

Application: Non-Hermitian displaced harmonic oscillator

After introducing in detail how to construct the time-dependent pseudo-squeezed coherent states, we now manage an explicit example, namely the time-dependent non-Hermitian displaced harmonic oscillator.

Let us consider the non-Hermitian displaced harmonic oscillator described by the Hamiltonian

$$H(t) = \omega(t)\, a^{\dagger} a + \beta(t)\, a + \lambda(t)\, a^{\dagger}, \tag{3.1}$$

where a and $a^{\dagger}$ are bosonic annihilation and creation operators of a light field mode verifying $\left[a,\ a^{\dagger}\right] = 1$, and the coefficients $\omega(t)$, $\beta(t)$ and $\lambda(t)$ are time-dependent complex parameters defined as

$$\begin{aligned} \omega(t) &= |\omega(t)| \exp\left[i\varphi_{\omega}(t)\right], \\ \beta(t) &= |\beta(t)| \exp\left[i\varphi_{\beta}(t)\right], \\ \lambda(t) &= |\lambda(t)| \exp\left[i\varphi_{\lambda}(t)\right]. \end{aligned} \tag{3.2}$$

Let the linear non-Hermitian pseudo-bosonic invariant operator be in the following form

$$A(t) = \left[\delta_1(t)\, a + \delta_2(t)\, a^{\dagger}\right], \tag{3.3}$$

where

$$\left(\delta_1(t) \neq \delta_2(t)\right) \in \mathbb{R}, \tag{3.4}$$

the invariance condition in (2.48) leads to the following equations

$$\begin{aligned}\dot{\delta}_1(t) &= i\omega(t)\,\delta_1(t)\,,\\ \dot{\delta}_2(t) &= -i\omega(t)\,\delta_2(t)\,,\end{aligned} \tag{3.5}$$

$$\beta(t)\,\delta_2(t) = \lambda(t)\,\delta_1(t)\,, \tag{3.6}$$

if we insert (3.2) in (3.5), we find

$$\delta_1(t) = \exp\left[-\int_0^t |\omega(t)|\sin(\varphi_\omega)\,dt'\right], \qquad \delta_2(t) = \exp\left[\int_0^t |\omega(t)|\sin(\varphi_\omega)\,dt'\right], \tag{3.7}$$

and if inserting (3.2) in (3.6), we find

$$\delta_1(t)\,|\lambda(t)|\cos(\varphi_\lambda) = \delta_2\,|\beta(t)|\cos(\varphi_\beta)\,, \qquad \delta_1(t)\,|\lambda(t)|\sin(\varphi_\lambda) = \delta_2\,|\beta(t)|\sin(\varphi_\beta)\,, \tag{3.8}$$

Now, to determine the pseudo-operator $\overline{A}(t)$, defined from the pseudo-Hermicity relation (2.52), we define the ansatz for the time-dependent metric $\eta(t)$ [68, 69, 70, 71]

$$\begin{aligned}\eta(t) &= \exp\left[2\left(\epsilon(t)\left(a^\dagger a + \frac{1}{2}\right) + \mu(t)\,a^2 + \mu^*(t)\,a^{\dagger 2}\right)\right],\\ &= \exp\left[\frac{1}{2}\vartheta_+(t)\,a^{\dagger 2}\right]\exp\left[\frac{1}{2}\ln\vartheta_0(t)\left(a^\dagger a + \frac{1}{2}\right)\right]\exp\left[\frac{1}{2}\vartheta_-(t)\,a^2\right],\end{aligned} \tag{3.9}$$

with

$$\begin{aligned}\vartheta_+(t) &= \frac{2(2\mu^*)\sinh\theta}{\theta\cosh\theta - 2\epsilon\sinh\theta} = -\zeta(t)e^{-i\varphi(t)},\\ \vartheta_0(t) &= \left(\cosh\theta - \frac{2\epsilon}{\theta}\sinh\theta\right)^{-2} = \zeta^2(t) - \chi(t), \quad \theta = 2\sqrt{\epsilon^2 - 4|\mu|^2},\\ \vartheta_-(t) &= \frac{2(2\mu)\sinh\theta}{\theta\cosh\theta - 2\epsilon\sinh\theta} = -\zeta(t)e^{+i\varphi(t)},\\ \chi(t) &= -\frac{\cosh\theta + \frac{2\epsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{2\epsilon}{\theta}\sinh\theta},\end{aligned} \tag{3.10}$$

using the Baker-Campbell-Hausdorff formula (1.20) to obtain

$$\begin{aligned}\exp\left[\tfrac{1}{2}\vartheta_-(t)\,a^2\right]a\exp\left[-\tfrac{1}{2}\vartheta_-(t)\,a^2\right] &= a,\\ \exp\left[\tfrac{1}{2}\vartheta_+(t)\,a^{\dagger 2}\right]a\exp\left[-\tfrac{1}{2}\vartheta_+(t)\,a^{\dagger 2}\right] &= a - \vartheta_+(t)\,a^\dagger,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\exp\left[\tfrac{1}{2}\ln\vartheta_0(t)\left(a^\dagger a + \tfrac{1}{2}\right)\right]a\exp\left[-\tfrac{1}{2}\ln\vartheta_0(t)\left(a^\dagger a + \tfrac{1}{2}\right)\right] &= \frac{a}{\sqrt{\vartheta_0(t)}},\\ \exp\left[\tfrac{1}{2}\vartheta_+(t)\,a^{\dagger 2}\right]a^\dagger\exp\left[-\tfrac{1}{2}\vartheta_+(t)\,a^{\dagger 2}\right] &= a^\dagger,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\eta(t)\,a\eta^{-1}(t) &= \tfrac{1}{\sqrt{\vartheta_0}}(a-\vartheta_+(t)\,a^\dagger),\\ \eta(t)\,a^\dagger\eta^{-1}(t) &= \tfrac{1}{\sqrt{\vartheta_0}}\left(\vartheta_-(t)\,a-\chi(t)\,a^\dagger\right).\end{aligned} \tag{3.13}$$

Thus, by using (2.52) with (3.13) the pseudo-operator $\overline{A}(t)$, is expressed in the following form

$$\overline{A}(t)=\frac{1}{\sqrt{\vartheta_0(t)}}\left([\delta_1(t)+\vartheta_+(t)\,\delta_2(t)]\,a^\dagger-[\vartheta_-(t)\,\delta_1(t)+\chi(t)\,\delta_2(t)]\,a\right). \tag{3.14}$$

The operator $\overline{A}(t)$ verify the bosonic commutation relation (2.51)

$$\begin{aligned}&\left[A(t),\overline{A}(t)\right]=A(t)\,\overline{A}(t)-\overline{A}(t)\,A(t)\\ &=\left(\delta_1(t)\,a+\delta_2(t)\,a^\dagger\right)\left(\tfrac{1}{\sqrt{\vartheta_0}}\left([\delta_1+\vartheta_+\delta_2]\,a^\dagger-[\vartheta_-\delta_1+\chi\delta_2]\,a\right)\right)\\ &-\left(\tfrac{1}{\sqrt{\vartheta_0}}\left([\delta_1+\vartheta_+\delta_2]\,a^\dagger-[\vartheta_-\delta_1+\chi\delta_2]\,a\right)\right)\left(\delta_1(t)\,a+\delta_2(t)\,a^\dagger\right)\\ &=\mathbb{I},\end{aligned} \tag{3.15}$$

which imply the constraint

$$\delta_1^2(t)+\chi\delta_2^2(t)+(\vartheta_++\vartheta_-)\,\delta_1(t)\,\delta_2(t)=\sqrt{\vartheta_0}. \tag{3.16}$$

The time-dependent pseudo-bosonic squeezed ladder operators $B(t)$ and $\bar{B}(t)$ can be determined by using the equations (2.63), (2.64) and (2.65) as

$$\begin{aligned}&B(t)=\cosh|\xi|\left(\delta_1(t)\,a+\delta_2(t)\,a^\dagger\right)\\ &-\tfrac{\xi}{|\xi|}\sinh|\xi|\left(\tfrac{1}{\sqrt{\vartheta_0}}\left([\delta_1(t)+\vartheta_+(t)\,\delta_2(t)]\,a^\dagger-[\vartheta_-(t)\,\delta_1(t)+\chi(t)\,\delta_2(t)]\,a\right)\right),\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\overline{B}(t)=\cosh|\xi|\left(\tfrac{1}{\sqrt{\vartheta_0}}\left([\delta_1(t)+\vartheta_+(t)\,\delta_2(t)]\,a^\dagger-[\vartheta_-(t)\,\delta_1(t)+\chi(t)\,\delta_2(t)]\,a\right)\right)\\ &-\tfrac{\xi^*}{|\xi|}\sinh|\xi|\left(\delta_1(t)\,a+\delta_2(t)\,a^\dagger\right),\end{aligned} \tag{3.18}$$

hence, the operators $B(t)$ and $\bar{B}(t)$ have a linear combination of a and $a^\dagger$ and can be expressed as

$$B(t)=\begin{aligned}&\left(\delta_1(t)\cosh|\xi|+\tfrac{\xi}{|\xi|\sqrt{\vartheta_0}}\sinh|\xi|\,[\vartheta_-\delta_1+\chi\delta_2]\right)a\\ &+\left(\delta_2(t)\cosh|\xi|-\tfrac{\xi}{|\xi|\sqrt{\vartheta_0}}\sinh|\xi|\,[\delta_1+\vartheta_+\delta_2]\right)a^\dagger,\end{aligned} \tag{3.19}$$

$$\bar{B}(t)=\begin{aligned}&\left(\tfrac{1}{\sqrt{\vartheta_0}}\cosh|\xi|\,[\delta_1+\vartheta_+\delta_2]-\tfrac{\xi^*}{|\xi|}\sinh|\xi|\delta_2(t)\right)a^\dagger\\ &-\left(\tfrac{1}{\sqrt{\vartheta_0}}\cosh|\xi|\,[\vartheta_-\delta_1+\chi\delta_2]+\tfrac{\xi^*}{|\xi|}\sinh|\xi|\delta_1(t)\right)a.\end{aligned} \tag{3.20}$$

Let us express the equation (3.3), (3.14), (3.19) and (3.20) in position and momentum operators representation for the case where $a = \frac{1}{\sqrt{2}}(x + ip)$ and also $a^{\dagger} = \frac{1}{\sqrt{2}}(x - ip)$ which imply

$$A(t) = [fx + igp], \qquad \overline{A}(t) = \left[\widetilde{f}x - i\widetilde{g}p\right], \tag{3.21}$$

where

$$f = \frac{1}{\sqrt{2}}[\delta_1 + \delta_2], \qquad g = \frac{1}{\sqrt{2}}[\delta_1 - \delta_2], \tag{3.22}$$

also

$$\widetilde{f} = \frac{1}{\sqrt{2\vartheta_0}}[(1 - \vartheta_-)\delta_1(t) + (\vartheta_+ - \chi)\delta_2(t)], \tag{3.23}$$

$$\widetilde{g} = \frac{1}{\sqrt{2\vartheta_0}}[(1 + \vartheta_-)\delta_1(t) + (\vartheta_+ + \chi)\delta_2(t)]. \tag{3.24}$$

Additionally, the condition (2.51) gives

$$g(t)\widetilde{f}(t) + f(t)\widetilde{g}(t) = 1. \tag{3.25}$$

Furthermore, the operators $B(t)$ and $\bar{B}(t)$ can also be written as

$$B(t) = [f_\xi x + ig_\xi p], \qquad \bar{B}(t) = \left[\widetilde{f}_\xi x - i\widetilde{g}_\xi p\right], \tag{3.26}$$

where the condition in (2.68) and

$$f_\xi = \cosh|\xi| f - \frac{\xi}{|\xi|}\sinh|\xi|\widetilde{f}, \qquad g_\xi = \cosh|\xi| g + \frac{\xi}{|\xi|}\sinh|\xi|\widetilde{g}, \tag{3.27}$$

together with

$$\widetilde{f}_\xi = \cosh|\xi|\widetilde{f} - \frac{\xi^*}{|\xi|}\sinh|\xi| f, \qquad \tilde{g}_\xi = \cosh|\xi|\widetilde{g} + \frac{\xi^*}{|\xi|}\sinh|\xi| g, \tag{3.28}$$

imply that

$$g_\xi\widetilde{f}_\xi + f_\xi\widetilde{g}_\xi = 1. \tag{3.29}$$

3.1 Pseudo-squeezed coherent states in position representation

In order to construct the pseudo-squeezed coherent states in the position representation, the pseudo-squeezed vacuum state in the x-represetation is required. This will lead us to solve the eigenvalue equation $B(t)|\xi,t\rangle = 0$ and $\overline{B}^{\dagger}(t)|\widetilde{\xi,t}\rangle$ in the x-representation

$$B(t)\langle x|\xi,t\rangle = \left(f_{\xi}(t)\,x + g_{\xi}(t)\frac{\partial}{\partial x}\right)\xi(x,t) = 0, \tag{3.30}$$

$$\overline{B}^{\dagger}(t)\langle x|\widetilde{\xi,t}\rangle = \left(\widetilde{f}_{\xi}^{*}(t)\,x + \widetilde{g}_{\xi}^{*}(t)\frac{\partial}{\partial x}\right)\widetilde{\xi(x,t)} = 0. \tag{3.31}$$

Therefore, the solutions of the above equations are

$$\xi(x,t) = \left(\frac{1}{2\pi\widetilde{g}_{\xi}g_{\xi}}\right)^{\frac{1}{4}}\exp\left[-\frac{f_{\xi}}{2g_{\xi}}x^{2}\right], \tag{3.32}$$

and

$$\widetilde{\xi(x,t)} = \eta(t)\,\xi(x,t) = \left(\frac{1}{2\pi\widetilde{g}_{\xi}^{*}g_{\xi}^{*}}\right)^{\frac{1}{4}}\exp\left[-\frac{\widetilde{f}_{\xi}^{*}}{2\widetilde{g}_{\xi}^{*}}x^{2}\right], \tag{3.33}$$

where the coefficients $\left(\frac{1}{2\pi\widetilde{g}_{\xi}g_{\xi}}\right)^{\frac{1}{4}}$ and $\left(\frac{1}{2\pi\widetilde{g}_{\xi}^{*}g_{\xi}^{*}}\right)^{\frac{1}{4}}$ come from the binormalization relation between the pseudo-vacuum state $|\widetilde{\xi,t}\rangle$ and $|\xi,t\rangle$

$$\langle\xi,t|\,\eta(t)\,|\xi,t\rangle = \int\widetilde{\xi^{*}(x,t)}\,\xi(x,t)\,dx = 1. \tag{3.34}$$

As mentioned before, the pseudo-squeezed coherent state is obtained by the action of the displacement operator $T(\gamma,t)$ on the pseudo-squeezed vacuum (2.70). Accordingly, by expressing $T(\gamma,t)$ in terms of

$$\left\{\begin{array}{l} x = g_{\xi}\overline{B} + \widetilde{g}_{\xi}B \\ ip = \widetilde{f}_{\xi}B - f_{\xi}\overline{B} \end{array}\right., \tag{3.35}$$

we find

$$\begin{aligned} T(\gamma,t) &= \exp\left[\gamma\overline{B}(t) - \gamma^{*}B(t)\right] = \exp\left[\gamma\left(\widetilde{f}_{\xi}x - i\widetilde{g}_{\xi}p\right) - \gamma^{*}\left(f_{\xi}x + ig_{\xi}p\right)\right] \\ &= \exp\left[\left(\gamma\widetilde{f}_{\xi} - \gamma^{*}f_{\xi}\right)x - i\left(\gamma\widetilde{g}_{\xi} + \gamma^{*}g_{\xi}\right)p\right], \end{aligned} \tag{3.36}$$

and by using the relation (1.21) and the following commutation relation

$$[x,p] = i\hbar, \qquad \hbar = 1, \tag{3.37}$$

thus, we obtain

$$\begin{aligned} T(\gamma,t) &= \exp\left[-\tfrac{i}{2}\langle p\rangle_{\eta}\langle x\rangle_{\eta}\right]\exp\left[i\langle p\rangle_{\eta}x\right]\exp\left[-i\langle x\rangle_{\eta}p\right] \\ &= \exp\left[-\tfrac{i}{2}\langle p\rangle_{\eta}\langle x\rangle_{\eta}\right]\exp\left[i\langle p\rangle_{\eta}x\right]\exp\left[-i\left(\tfrac{\langle x\rangle_{\eta}-\langle x\rangle_{\eta}^{*}}{2}\right)p\right]\exp\left[-i\left(\tfrac{\langle x\rangle_{\eta}+\langle x\rangle_{\eta}^{*}}{2}\right)p\right], \end{aligned} \tag{3.38}$$

where

$$\langle x\rangle_\eta = \langle\psi_{\alpha,\xi}(t)|\,\eta x\,|\psi_{\alpha,\xi}(t)\rangle = \left\langle\left(g_\xi\overline{B}+\widetilde{g}_\xi B\right)\right\rangle_\eta = \gamma\widetilde{g}_\xi+\gamma^* g_\xi, \tag{3.39}$$

$$i\langle p\rangle_\eta = i\langle\psi_{\alpha,\xi}(t)|\,\eta p\,|\psi_{\alpha,\xi}(t)\rangle = \left\langle\left(\widetilde{f}_\xi B - f_\xi\overline{B}\right)\right\rangle_\eta = \gamma\widetilde{f}_\xi-\gamma^* f_\xi, \tag{3.40}$$

by the same method, we deduce

$$\begin{aligned}\left[T^{-1}(\gamma,t)\right]^\dagger &= \exp\left[-\tfrac{i}{2}\langle p\rangle_\eta^*\langle x\rangle_\eta^*\right]\exp\left[i\langle p\rangle_\eta^* x\right]\exp\left[-i\langle x\rangle_\eta^* p\right]\\ &= \exp\left[-\tfrac{i}{2}\langle p\rangle_\eta^*\langle x\rangle_\eta^*\right]\exp\left[i\langle p\rangle_\eta^* x\right]\exp\left[-i\left(\tfrac{\langle x\rangle_\eta^*-\langle x\rangle_\eta}{2}\right)p\right]\exp\left[-i\left(\tfrac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right)p\right],\end{aligned} \tag{3.41}$$

When the operators defined above in (3.38) and (3.41) act on the pseudo-squeezed vacuums given in (3.32) and (3.33), we obtain the pseudo-squeezed coherent states as

$$\psi_{\alpha,\xi}(x,t) = \left(\frac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{\frac14}\exp\left[-\frac{i}{2}\langle p\rangle_\eta\langle x\rangle_\eta\right]\exp\left[i\langle p\rangle_\eta x-\frac{f_\xi}{2g_\xi}\left(x-\langle x\rangle_\eta\right)^2\right], \tag{3.42}$$

$$\phi_{\alpha,\xi}(x,t) = \left(\frac{1}{2\pi\widetilde{g}_\xi^* g_\xi^*}\right)^{\frac14}\exp\left[-\frac{i}{2}\langle p\rangle_\eta^*\langle x\rangle_\eta^*\right]\exp\left[i\langle p\rangle_\eta^* x-\frac{\widetilde{f}_\xi^*}{2\widetilde{g}_\xi^*}\left(x-\langle x\rangle_\eta^*\right)^2\right], \tag{3.43}$$

with

$$\exp\left[-i\langle x\rangle_\eta p\right]\xi(x,t) = \xi\left(x-\langle x\rangle_\eta,t\right). \tag{3.44}$$

Noting that these last two equations (3.42), (3.43) can also be written in a more appropriate form, where

$$\begin{aligned}\psi_{\alpha,\xi}(x,t) &= \left(\tfrac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{\frac14}\exp\left[-\tfrac{i}{2}\langle p\rangle_\eta\langle x\rangle_\eta\right]\exp\left[i\langle p\rangle_\eta x\right]\\ &\times\exp\left[-i\left(\tfrac{\langle x\rangle_\eta-\langle x\rangle_\eta^*}{2}\right)p\right]\exp\left[-\tfrac{f_\xi}{2g_\xi}\left(x-\left[\tfrac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right]\right)^2\right],\end{aligned} \tag{3.45}$$

$$\begin{aligned}\phi_{\alpha,\xi}(x,t) &= \left(\tfrac{1}{2\pi\widetilde{g}_\xi^* g_\xi^*}\right)^{\frac14}\exp\left[-\tfrac{i}{2}\langle p\rangle_\eta^*\langle x\rangle_\eta^*\right]\exp\left[i\langle p\rangle_\eta^* x\right]\\ &\times\exp\left[-i\left(\tfrac{\langle x\rangle_\eta^*-\langle x\rangle_\eta}{2}\right)p\right]\exp\left[-\tfrac{\widetilde{f}_\xi^*}{2\widetilde{g}_\xi^*}\left(x-\left[\tfrac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right]\right)^2\right].\end{aligned} \tag{3.46}$$

Therefore, the density $\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2 = \langle\phi_{\alpha,\xi}(t)|\psi_{\alpha,\xi}(t)\rangle$ can be expressed as a function of $\left(\langle x\rangle_\eta+\langle x\rangle_\eta^*\right)$ as the following

$$\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2 = \left(\frac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{\frac12}\exp\left[-\frac{1}{2\widetilde{g}_\xi g_\xi}\left(x-\frac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right)^2\right], \tag{3.47}$$

and the latter represents a gaussian wave packet centred at $\left[x-\left(\frac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right)\right]$. We see from this equation that the width of this gaussian wave packet varies with the time and is identical to $\sigma=\widetilde{g}_\xi g_\xi$. This wave packet is represented in the figures 3.1.1-3. it also readily verified that the time-dependent pseudo-probability density is conserved

$$\begin{aligned}\int dx\left|\rho\left(t\right)\psi_{\alpha,\xi}\left(x,t\right)\right|^2 &= \int dx\left(\psi_{\alpha,\xi}\left(x,t\right)\eta\left(t\right)\right)\psi_{\alpha,\xi}\left(x,t\right)\\ &= \int dx\left(\phi_{\alpha,\xi}^*\left(x,t\right)\right)\psi_{\alpha,\xi}\left(x,t\right)\\ = \int dx\left(\frac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{\frac{1}{2}}\exp\left[-\frac{1}{2\widetilde{g}_\xi g_\xi}\left(x-\frac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right)^2\right] &= \left(\frac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{\frac{1}{2}}\left(\frac{1}{2\pi\widetilde{g}_\xi g_\xi}\right)^{-\frac{1}{2}}=1.\end{aligned} \tag{3.48}$$

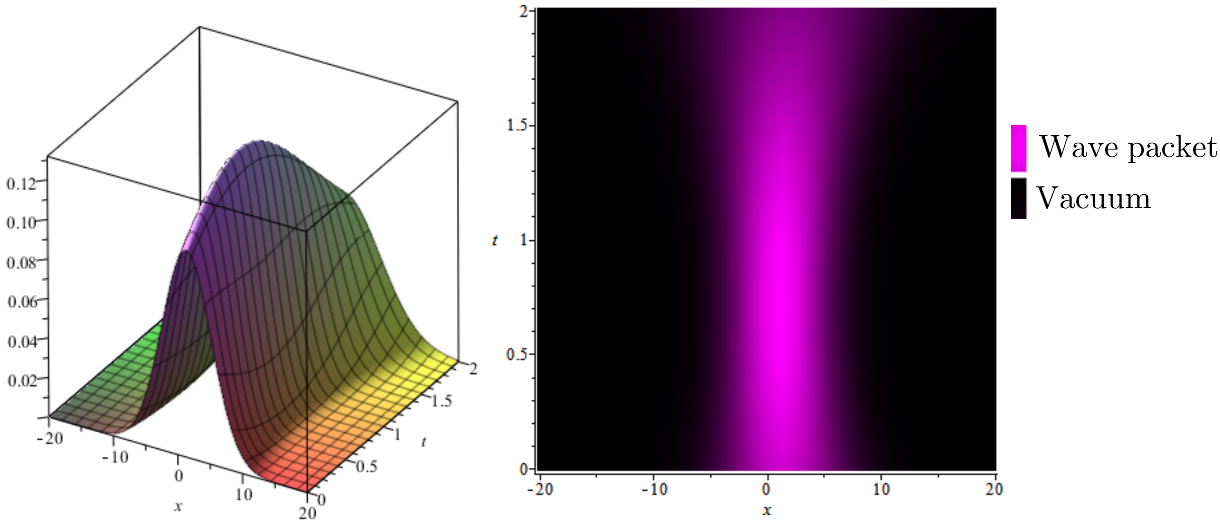

Figure 3.1.1 Time evolution of the wave packet $\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2$ with $\xi=1,\vartheta_+=\vartheta_-=0$, $\vartheta_0=-\chi=e^{2\epsilon}$, $\epsilon=2t-arc\sinh(2)$, $\omega=-i$ and $\alpha=1+i$.

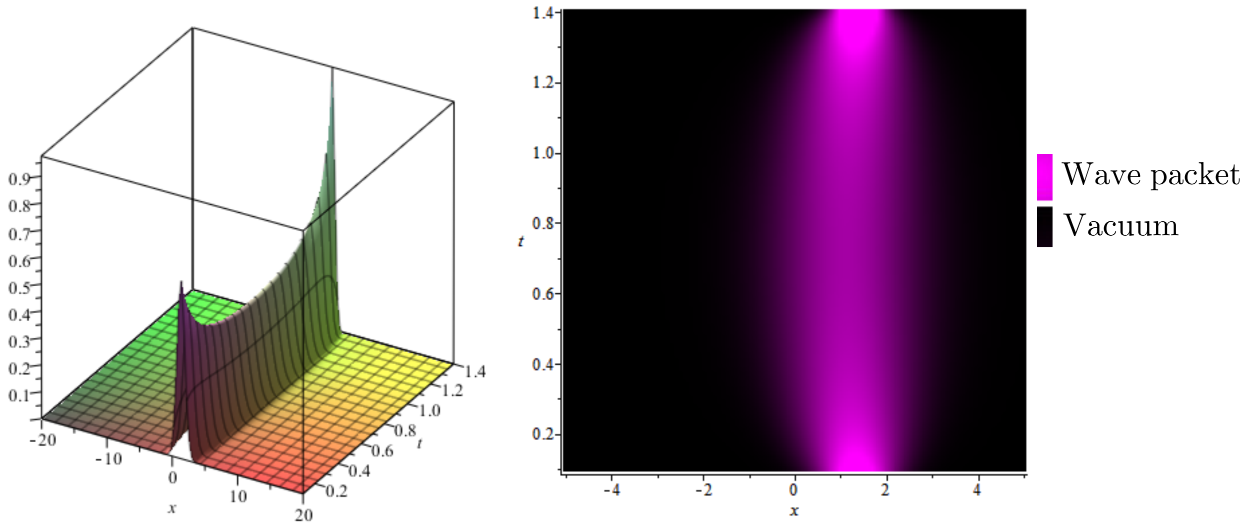

Figure 3.1.2 Time evolution of the wave packet $\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2$ for $\xi = 0, \vartheta_+ = \vartheta_- = 0$, $\vartheta_0 = -\chi = e^{2\epsilon}, \epsilon = 2t - arc\sinh(2)$, $\omega = -i$ and $\alpha = 1 + i$, $t \in \,]0, 1.44[$.

We have illustrated the pseudo-probability density, equation (3.47), in the figures 3.1.1-3 from various parameters (ξ, $\vartheta_+ = \vartheta_-$, ϑ_0, ϵ, ω, α and t). These figures correspond to the wave packet for a particle moving along the positive and negative x axis. Figure 3.1.1 shows that, although the shape of the wave packet is always kept to be gaussian. As a consequence, the width of the packet gradually becomes broader over time whereas its height, $1/\left[2\sigma\right]$, decreases. Figure 3.1.2 however, the wave packet in it that corresponds to pseudo-boson coherent states ($\xi = 0$) which is finite only when $t \in \,]0, \; 1.44[$ and it has an obvious pronounced peak which is situated at $x = \left(\langle x\rangle_\eta\right)_R = \frac{\langle x\rangle_\eta + \langle x\rangle_\eta^*}{2}$.

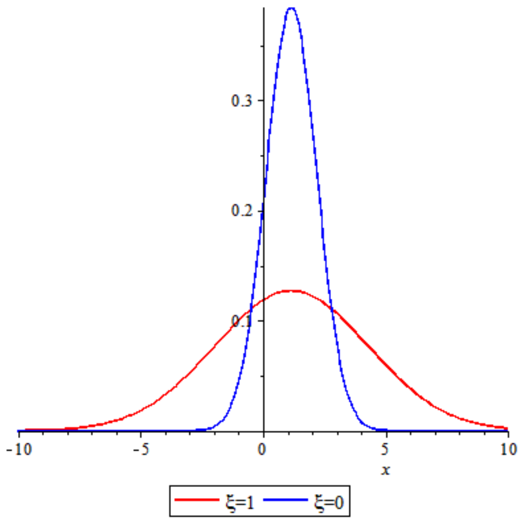

Figure 3.1.3 $\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2$ **for** $\xi = 1$ and $\xi = 0$, at time $t = 1$
$\vartheta_+ = \vartheta_- = 0, \vartheta_0 = -\chi = e^{2\epsilon},\ \epsilon = 2 - arc\sinh(2),\ \omega = -i$ and $\alpha = 1 + i$.

Figure 3.1.3 shows that $\left|\rho\left(t\right)\psi_{\alpha,\xi}\left(x,t\right)\right|^2$ is a function which has a peak at $x = \left(\langle x\rangle_\eta\right)_R$ with the width σ and an amplitude of $1/\sigma$, whose integral between $-\infty$ and $+\infty$ is equal to 1.

Noticeably, (3.39) and (3.40) yields a complex quantities, the equations (3.45), (3.46) and (3.47) prompt us to define the expectation value of an operator O in a given pseudo-squeezed coherent state as the real part of O, namely,

$$\left(\langle O\rangle_\eta\right)_R \equiv \frac{1}{2}\left[\langle O\rangle + \langle O\rangle^*\right]. \tag{3.49}$$

Since our aim is to compute the Heisenberg uncertainty relations, it is required to calculate the preceding equation for $O = x, x^2, p$ and p^2. Using the expression for the expectation value of an operator O given by the latter equation, the corresponding dispersion, defined in the usual way, is in the following form

$$(\Delta O)_R^2 = \frac{1}{2}\left[\left\langle O^2\right\rangle_\eta - \langle O\rangle_\eta^2\right] + \frac{1}{2}\left[\left\langle O^2\right\rangle_\eta^* - \left(\langle O\rangle_\eta^*\right)^2\right], \tag{3.50}$$

it follows, after evaluating the dispersion in the position

$$\begin{aligned}(\Delta x)_R^2 &= \tfrac{1}{2}\left[\langle x^2\rangle_\eta - \langle x\rangle_\eta^2\right] + \tfrac{1}{2}\left[\langle x^2\rangle_\eta^* - \left(\langle x\rangle_\eta^*\right)^2\right], \\ &= \tfrac{1}{2}\left[g_\xi\widetilde{g}_\xi + g_\xi^*\widetilde{g}_\xi^*\right],\end{aligned} \tag{3.51}$$

and momentum

$$
\begin{aligned}
(\Delta p)_R^2 &= \tfrac{1}{2}\left[\langle p^2\rangle_\eta - \langle p\rangle_\eta^2\right] + \tfrac{1}{2}\left[\langle p^2\rangle_\eta^* - \left(\langle p\rangle_\eta^*\right)^2\right], \\
&= \tfrac{1}{2}\left[f_\xi \widetilde{f}_\xi + f_\xi^* \widetilde{f}_\xi^*\right],
\end{aligned}
\tag{3.52}
$$

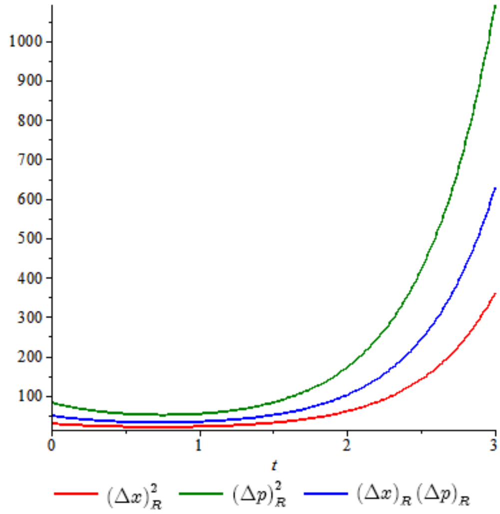

Figure 3.1.4 Time dependent dispersions in position, momentum and the uncertainty product for $\vartheta_+ = \vartheta_- = 0, \vartheta_0 = -\chi = e^{2\epsilon}, \epsilon = 2t - arc\sinh(2)$, $\omega = -i$ and $\alpha = 1 + i$.

Accordingly, the pseudo expectation value of x, x^2, p and p^2 in the gaussian state $\psi_{\alpha,\xi}(t)$ in (3.42) are given by

$$
\langle x\rangle_\eta = \left\langle \left(g_\xi \overline{B} + \widetilde{g}_\xi B\right)\right\rangle_\eta = \gamma \widetilde{g}_\xi + \gamma^* g_\xi, \tag{3.53}
$$

$$
\left\langle x^2\right\rangle_\eta = \left\langle \left(g_\xi \overline{B} + \widetilde{g}_\xi B\right)\left(g_\xi \overline{B} + \widetilde{g}_\xi B\right)\right\rangle_\eta = \left[\widetilde{g}_\xi^2 \gamma^2 + g_\xi^2 \gamma^{*^2} + g_\xi \widetilde{g}_\xi \left(2|\gamma|^2 + 1\right)\right], \tag{3.54}
$$

$$
i\langle p\rangle_\eta = \left\langle \left(\widetilde{f}_\xi B - f_\xi \overline{B}\right)\right\rangle_\eta = i\left(\gamma^* f_\xi - \gamma \widetilde{f}_\xi\right), \tag{3.55}
$$

$$
\left\langle p^2\right\rangle_\eta = -\left\langle \left(\widetilde{f}_\xi B - f_\xi \overline{B}\right)\left(\widetilde{f}_\xi B - f_\xi \overline{B}\right)\right\rangle_\eta = -\left[f_\xi^2 \gamma^* + \widetilde{f}_\xi^2 \gamma^2 - f_\xi \widetilde{f}_\xi \left(2|\gamma|^2 + 1\right)\right]. \tag{3.56}
$$

It follows from the equation (3.49) that the expression for the Heisenberg uncertainty relations $(\Delta x)_R^2 (\Delta p)_R^2$, is written as in the following form

$$
(\Delta x)_R (\Delta p)_R = \frac{1}{2}\left[\left(g_\xi \widetilde{g}_\xi + g_\xi^* \widetilde{g}_\xi^*\right)\left(f_\xi \widetilde{f}_\xi + f_\xi^* \widetilde{f}_\xi^*\right)\right]^{\frac{1}{2}}. \tag{3.57}
$$

While in figure 3.1.4 the uncertainty product and dispersions are represented for $\xi = 1$, in the following figure 3.1.5 they are represented for $\xi = 0$.

Figure 3.1.4 also illustrates that $(\Delta x)_R$, $(\Delta p)_R$ and $(\Delta x)_R\,(\Delta p)_R$ decrease when time $t \in\,]0,\ 0.72]$ while they increase when $t > 0.72$.

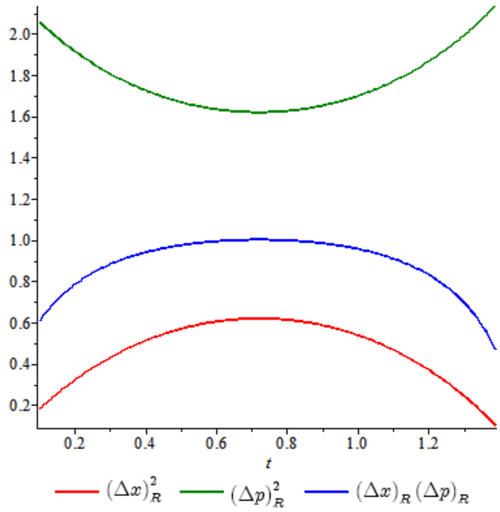

Figure 3.1.5 Time dependent dispersions in position, in momentum and the uncertainty product for $\vartheta_+ = \vartheta_- = 0, \vartheta_0 = -\chi = e^{2\epsilon}, \epsilon = 2t - arc\sinh(2)$, $\omega = -i$ and $\alpha = 1 + i$.

Figure 3.1.5 shows that $(\Delta x)_R$ and $(\Delta x)_R\,(\Delta p)_R$ increase with time while $(\Delta p)_R$ decreases. Hence, the corresponding uncertainty relation $(\Delta x)_R\,(\Delta p)_R$ increases with time. For $t \in\,]0.72,\ 1.44[$ the inverse occurs. It does not seem for figure 3.1.4, that the pseudo-squeezed coherent states produce any squeezing. Figure 3.1.5 however, it is clear that the quadrature squeezing is achieved.

3.2 Time evolution of the pseudo-squeezed coherent states $|\Psi_{\alpha,\xi}(t)\rangle$

The final step consists in determining the Schrödinger solution which is an eigenstate of the pseudo-invariant operator $B(t)$ multiplied by a time-dependent factor [72]

$$\Psi^H_{\alpha,\xi}(x,t) = \exp\left[i\varphi_{\alpha,\xi}\right]\psi_{\alpha,\xi}(x,t)\,, \tag{3.58}$$

this phase [69, 70] is given as

$$\begin{aligned}\dot{\varphi}_{\alpha,\xi} &= \int \psi^*_{\alpha,\xi}(x,t)\,\eta\,(i\partial_t - H)\,\psi_{\alpha,\xi}(x,t)\,dx\\ &= \int \phi^*_{\alpha,\xi}(x,t)\,(i\partial_t - H)\,\psi_{\alpha,\xi}(x,t)\,dx,\end{aligned} \tag{3.59}$$

where the Hamiltonian in (3.1) expressed in variable operators x and p is

$$H(t) = \frac{\omega}{2}\left(p^2 + x^2\right) + \frac{1}{\sqrt{2}}(\beta + \lambda)\, x + \frac{i}{\sqrt{2}}(\beta - \lambda)\, p - \frac{\omega}{2}\ . \tag{3.60}$$

The phase is calculated by substituting the expressions in (3.42) and (3.43) into the equation (3.59), we thus obtain

$$\begin{aligned}\left\langle \psi_{\alpha,\xi}(x,t)\right|\eta(t)\, H(t)\left|\psi_{\alpha,\xi}(x,t)\right\rangle = \tfrac{\omega}{2}\left(\left\langle p^2\right\rangle_\eta + \left\langle x^2\right\rangle_\eta\right) + \tfrac{1}{\sqrt{2}}(\beta+\lambda)\left\langle x\right\rangle_\eta \\ + \tfrac{i}{\sqrt{2}}(\beta-\lambda)\left\langle p\right\rangle_\eta - \tfrac{\omega}{2},\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}\left\langle \psi_{\alpha,\xi}(x,t)\right|\eta(t)\, i\partial_t\left|\psi_{\alpha,\xi}(x,t)\right\rangle = -\tfrac{i}{4}\left(\tfrac{\dot{\widetilde{g}}_\xi g_\xi + \widetilde{g}_\xi \dot{g}_\xi}{g_\xi \widetilde{g}_\xi}\right) + \tfrac{1}{2}\left(\left\langle \dot{x}\right\rangle_\eta \left\langle p\right\rangle_\eta - \left\langle x\right\rangle_\eta \left\langle \dot{p}\right\rangle_\eta\right) \\ + i\left(\tfrac{f_\xi \dot{g}_\xi - \dot{f}_\xi g_\xi}{2g_\xi^2}\right)(\Delta x)^2,\end{aligned} \tag{3.62}$$

therefore, the phase can be expressed as

$$\begin{aligned}\dot{\varphi}_{\alpha,\xi} = -\tfrac{i}{4}\left(\tfrac{\dot{\widetilde{g}}_\xi g_\xi + \widetilde{g}_\xi \dot{g}_\xi}{g_\xi \widetilde{g}_\xi}\right) + \tfrac{1}{2}\left(\left\langle \dot{x}\right\rangle_\eta \left\langle p\right\rangle_\eta - \left\langle x\right\rangle_\eta \left\langle \dot{p}\right\rangle_\eta\right) + i\left(\tfrac{f_\xi \dot{g}_\xi - \dot{f}_\xi g_\xi}{2g_\xi^2}\right)(\Delta x)^2 \\ - \left[\tfrac{\omega}{2}\left(\left\langle p^2\right\rangle_\eta + \left\langle x^2\right\rangle_\eta\right) + \tfrac{1}{\sqrt{2}}(\beta+\lambda)\left\langle x\right\rangle_\eta + \tfrac{i}{\sqrt{2}}(\beta-\lambda)\left\langle p\right\rangle_\eta - \tfrac{\omega}{2}\right],\end{aligned} \tag{3.63}$$

where

$$(\Delta x)^2 = \tilde{g}_\xi g_\xi, \tag{3.64}$$

and $\left\langle \dot{p}\right\rangle_\eta$ and $\left\langle \dot{x}\right\rangle_\eta$ are replaced by the classical equations,

$$\left\langle \dot{p}\right\rangle_\eta = -\frac{\partial H(t)}{\partial \left\langle x\right\rangle_\eta} = -\omega\left\langle x\right\rangle_\eta - \frac{1}{\sqrt{2}}(\beta+\lambda), \tag{3.65}$$

$$\left\langle \dot{x}\right\rangle_\eta = \frac{\partial H(t)}{\partial \left\langle p\right\rangle_\eta} = \omega\left\langle p\right\rangle_\eta + \frac{i}{\sqrt{2}}(\beta-\lambda). \tag{3.66}$$

Finally, the phase (3.59) can be simplified by using the equation (2.66) and (3.26), thus, we have

$$\dot{\varphi}_{\alpha,\xi} = \omega(t)\left(\frac{1}{2} - \frac{1}{4g_\xi(t)\,\widetilde{g}_\xi(t)}\right). \tag{3.67}$$

Consequently, the evolved pseudo-squeezed coherent states can be written as

$$\Psi_{\alpha,\xi}(x,t) = \exp\left[i\int_0^t \omega(t')\left(\frac{1}{2} - \frac{1}{4g_\xi(t')\,\widetilde{g}_\xi(t')}\right)dt'\right]\psi_{\alpha,\xi}(x,t)\,. \tag{3.68}$$

Moreover, after a straightforward calculation, we obtain the expectation value of x in the pseudo-squeezed coherent states $\Psi_{\alpha,\xi}(x,t)$

$$\begin{aligned}\langle x\rangle(t) &= \langle \Psi_{\alpha,\xi}(x,t)|\eta(t)x|\Psi_{\alpha,\xi}(x,t)\rangle \\ &= \left|\exp\left[i\int_0^t \omega(t')\left(\tfrac{1}{2}-\tfrac{1}{4g_\xi(t')\widetilde{g}_\xi(t')}\right)dt'\right]\right|^2 \left\langle \psi_{\alpha,\xi}(x,t)|\eta(t)x|\psi_{\alpha,\xi}(x,t)\right\rangle \\ &= \left|\exp\left[i\int_0^t \omega(t')\left(\tfrac{1}{2}-\tfrac{1}{4g_\xi(t')\widetilde{g}_\xi(t')}\right)dt'\right]\right|^2 \left(\gamma\widetilde{g}_\xi+\gamma^* g_\xi\right).\end{aligned} \tag{3.69}$$

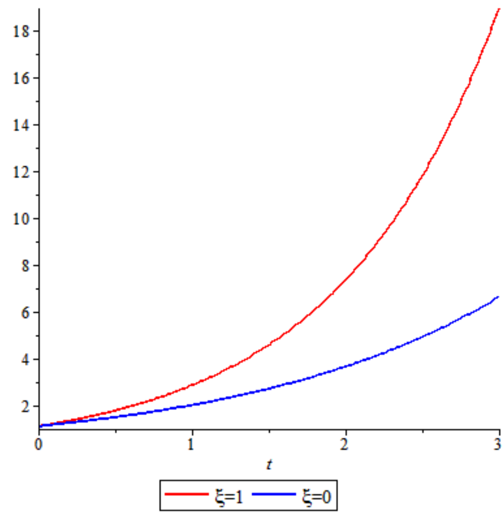

Figure 3.1.6 Represents variation of $\langle x\rangle(t)$

Figure 3.1.6 has been produced by the parameters $\vartheta_+ = \vartheta_- = 0,\ \vartheta_0 = -\chi = 1,\ \omega = -i$ and also $\alpha = 1+i$, which have been used in the latter equation. In which it shows that $\langle x\rangle(t)$ for the pseudo-squeezed coherent states $(\xi = 1)$ increases rapidly than $\langle x\rangle(t)$ of the pseudo-boson coherent states $(\xi = 0)$.

Conclusion

In this thesis, we have distinguished the different methods to obtain the coherent and squeezed states in the stationary case that are

- The minimum uncertainty method,
- The annihilation or more general the ladder operator method,
- The squeeze and displacement operator methods.

Also, we list some of their properties and we develop the time evolution of the coherent and squeezed states.

Consequently, we have constructed time-dependent pseudo-bosonic squeezed coherent states solutions of the Schrödinger equation for non-Hermitian Hamiltonians. Our construction of pseudo-bosonic squeezed coherent states is based on the introduction of time-dependent pseudo-bosonic squeezed ladder operators subjected to time-dependent metric such that the latter ones are integrals of motion. The pseudo-squeeze ladder operators (or invariant operators) obtained as a Bogolyubov transformation of pseudo-bosonic annihilation (or creation) operators, are used to define the pseudo-bosonic squeezed coherent states by the pseudo-displacement operator method acting on the ground state.

These pseudo-bosonic squeezed coherent states form a quasi-normalized and quasi-overcomplete set of states in the Hilbert space and are eigenfunctions of the introduced ladder operators. As discussed earlier, whereas a pseudo-coherent state is generated by linear terms in $A(t)$ and $\overline{A}(t)$ in the exponent, the pseudo-squeezed coherent state requires quadratic terms.

As an illustration, we have treated in detail the non-Hermitian time-dependent displaced harmonic oscillator. Thus, we have introduced a set of linear integrals of motion that are pseudo-squeeze ladder operators. With the help of these operators, we constructed in position representation, pseudo-bosonic squeezed coherent states as displaced squeezed ground state.

We have determined the associated phase of the evolved pseudo-bosonic squeezed coherent states solution of the Schrödinger equation, this solution is in a Gaussian form and preserves Schrödinger minimum uncertainty.

Bibliography

[1] E. Schrödinger, "Der stetige Übergang von der Mikro- zur Makromechanik," Naturwissenschaften, vol. 14, no. 28, pp. 664–666, Jul. 1926.

[2] J. R. Klauder and B. Skagerstam, Coherent States. Singapore: World Scientific, 1985.

[3] J. R. Klauder, "The action option and a Feynman quantization of spinor fields in terms of ordinary c-numbers"," Ann. Phys. Elsevier BV, vol. 11, no. 2, pp. 123–168, 1960.

[4] R. J. Glauber, "The Quantum Theory of Optical Coherence," Phys Rev, vol. 130, no. 6, pp. 2529–2539, Jun. 1963.

[5] W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev Mod Phys, vol. 62, p. 867, 1990.

[6] R. J. Glauber, "Coherent and Incoherent States of the Radiation Field," Phys Rev, vol. 131, no. 6, pp. 2766–2788, Sep. 1963.

[7] J. R. Klauder and E. C. G. Sudarshan, Fundamentals of Quantum Optics. Benjamin, NY, 1968.

[8] D. F. Walls, "Squeezed states of light," Nature, vol. 306, no 5939, Art. no 5939, nov. 1983.

[9] B. L. Schumaker, "Quantum mechanical pure states with Gaussian wave functions," Phys. Rep., vol. 135, no 6, p. 317-408, 1986.

[10] H. Takahasi, "Information theory of quantum-mechanical channels," in Advances in Communication Systems, vol. 1, Elsevier, 1965, pp. 227–310.

[11] D. Stoler, "Equivalence Classes of Minimum Uncertainty Packets," Phys. Rev. D, vol. 1, no. 12, pp. 3217–3219, Jun. 1970.

[12] H. P. Yuen, “Two-photon coherent states of the radiation field,” Phys. Rev. A, vol. 13, no. 6, p. 2226, 1976.

[13] J. N. Hollenhorst, “Quantum limits on resonant-mass gravitational-radiation detectors,” Phys. Rev. D, vol. 19, no. 6, p. 1669, 1979.

[14] I. Fujiwara and K. Miyoshi, “Pulsating States for Quantal Harmonic Oscillator,” Prog. Theor. Phys., vol. 64, no. 2, pp. 715–718, Aug. 1980.

[15] V. V. Dodonov, E. V. Kurmyshev, and V. I. Man’ko, “Generalized uncertainty relation and correlated coherent states,” Phys. Lett. A, vol. 79, no. 2, pp. 150–152, Sep. 1980.

[16] A. K. Rajagopal and J. T. Marshall, “New coherent states with applications to time-dependent systems,” Phys. Rev. A, vol. 26, no. 5, pp. 2977–2980, Nov. 1982.

[17] H. P. Yuen, “Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions,” Phys. Rev. Lett., vol. 51, no. 9, pp. 719–722, Aug. 1983.

[18] D. Stoler, “Equivalence Classes of Minimum-Uncertainty Packets. II,” Phys. Rev. D, vol. 4, no 6, p. 1925-1926, sept. 1971.

[19] E. Y. C. Lu, “New coherent states of the electromagnetic field,” Lett. Al Nuovo Cimento 1971-1985, vol. 2, no 24, p. 1241-1244, déc. 1971.

[20] E. Y. C. Lu, “Quantum correlations in two-photon amplification,” Lett. Al Nuovo Cimento 1971-1985, vol. 3, no 14, p. 585-589, avr. 1972.

[21] H. P. Yuen, “Two-photon coherent states of the radiation field,” Phys. Rev. A, vol. 13, no 6, p. 2226-2243, juin 1976.

[22] J. R. Klauder, “Continuous-Representation Theory. I. Postulates of Continuous-Representation Theory,” J. Math. Phys., vol. 4, no. 8, pp. 1055–1058, Aug. 1963.

[23] E. C. G. Sudarshan, “Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams,” Phys. Rev. Lett., vol. 10, no. 7, pp. 277–279, Apr. 1963.

[24] J. Klauder and B. Skagerstam, Coherent States: Applications in Physics and Mathematical Physics. WORLD SCIENTIFIC, 1985.

[25] W.-M. Zhang, D. H. Feng, and R. Gilmore, "Coherent states: Theory and some applications," Rev. Mod. Phys., vol. 62, no. 4, pp. 867–927, Oct. 1990.

[26] M. M. Nieto and L. M. Simmons, "Coherent States for General Potentials," Phys. Rev. Lett., vol. 41, no. 4, pp. 207–210, Jul. 1978.

[27] M. M. Nieto and L. M. Simmons, "Coherent states for general potentials. I. Formalism," Phys. Rev. D, vol. 20, no. 6, pp. 1321–1331, Sep. 1979.

[28] M. M. Nieto, L. M. Simmons, and V. P. Gutschick, "Coherent states for general potentials. VI. Conclusions about the classical motion and the WKB approximation," Phys. Rev. D, vol. 23, no. 4, pp. 927–933, Feb. 1981.

[29] D. F. Walls and G. J. Milburn, Quantum Optics, 2nd ed. Berlin Heidelberg: Springer-Verlag, 2008.

[30] R. J. Glauber, The Quantum Theory of Optical Coherence, Phys. Rev.130 (1963) 2529; Photon Correlations, Phys. Rev. Lett. 10 (1963) 84; Coherent and Incoherent States of the Radiation Field, Phys. Rev. 131 (1963) 2766 .

[31] H. F. Baker, "Alternants and Continuous Groups," Proc. Lond. Math. Soc., vol. s2-3, no. 1, pp. 24–47, Jan. 1905.

[32] J. E. Campbell, "On a Law of Combination of Operators (Second Paper)*," Proc. Lond. Math. Soc., vol. s1-29, no. 1, pp. 14–32, 1897.

[33] F. Hausdorff, "Die symbolische Exponentialformel in der Gruppentheorie, Ber," Verh Saechs Akad Wiss Leipz. Math Phys Kl, vol. 58, pp. 19–48, 1906.

[34] H. P. Yuen, Two-photon coherent states of the radiation field, Phys. Rev. A 13, 2226 (1976).

[35] J.N. Hollenhorst, "Quantum limits on resonant-mass gravitational-radiation detectors", Phys. Rev. D19, 1669 (1979).

[36] C.M. Caves, “Quantum-mechanical noise in an interferometer”, Phys. Rev. D23, 1693 (1981).

[37] D.F. Walls, “Squeezed states of light”, Nature 306, 141 (1983).

[38] J. N. Hollenhorst, “Quantum limits on resonant-mass gravitational-radiation detectors,” Phys. Rev. D, vol. 19, no 6, p. 1669-1679, mars 1979.

[39] C. M. Caves, “Quantum-mechanical noise in an interferometer,” Phys. Rev. D, vol. 23, no 8, p. 1693-1708, avr. 1981.

[40] R. A. Fisher, M. M. Nieto, et V. D. Sandberg, “Impossibility of naively generalizing squeezed coherent states,” Phys. Rev. D, vol. 29, no 6, p. 1107-1110, mars 1984.

[41] M. M. Nieto and D. R. Truax, Holstein-Prirnakof/Bogoliubov Transformations and the Multiboson System, Fortsch. Phys. 45, 145 (1997).

[42] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry, Phys. Rev. Lett. **80**, 5243 (1998).

[43] O. E. Alon, Dynamical symmetries of time-periodic Hamiltonians, Phys. Rev. A 66, 013414 (2002).

[44] A. de Sousa Dutra, M. B. Hott and V. G. C. S. dos Santos, Time-dependent non-Hermitian Hamiltonians with real energies, Europhys. Lett. 71, 166 (2005).

[45] C. Yuce, Time-dependent $\mathcal{PT}$ symmetric problems, Phys. Lett. A 336, 290 (2005).

[46] C. Yuce, Complex spectrum of a spontaneously unbroken $\mathcal{PT}$ symmetric Hamiltonian, arXiv:quant-ph/0703235v1.

[47] N. Moiseyev, Crossing rule for a $\mathcal{PT}$-symmetric two-level time-periodic system, Phys. Rev. A 83, 052125 (2011).

[48] X. Luo, J. Huang, H. Zhong, X. Qin, Q. Xie, Y. S. Kivshar and C. Lee, Pseudo-Parity-Time Symmetry in Optical Systems, Phys. Rev. Lett. 110, 243902 (2013).

[49] A. Mostafazadeh, Pseudo-Hermiticity versus $\mathcal{PT}$ symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian, J. Math. Phys. **43**, 205-214 (2002).

[50] A. Mostafazadeh, Pseudo-Hermiticity versus $\mathcal{PT}$ symmetry.II. A complete characterization of non-Hermitian Hamiltonians with real spectrum, J. Math. Phys. **43**, 2814-2816 (2002).

[51] A. Mostafazadeh, Pseudo-Hermiticity versus $\mathcal{PT}$ symmetry.III. Equivalence of Pseudo Hermiticity and the presence of antilinear symmetries, J. Math. Phys. **43**, 3944-3951 (2002).

[52] A. Mostafazadeh, "Time-dependent pseudo-Hermitian Hamiltonians defining a unitary quantum system and uniqueness of the metric operator," Phys. Lett. Sect. B, vol. 650, no. 2–3, pp. 208–212, 2007.

[53] A. Mostafazadeh, Comment on "Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution" 0711.0137v1 [quant-ph] (2007)A.

[54] Mostafazadeh, Comment on "Reply to Comment on Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution" 0711.1078v2 [quant-ph] (2007).

[55] M. Znojil, "Time-dependent quasi-Hermitian Hamiltonians and the unitarity of quantum evolution," ArXiv07105653 Quant-Ph, Oct. 2007.

[56] M. Znojil, "Reply to Comment on "Time-dependent quasi-Hermitian Hamiltonians and the unitary quantum evolution"," ArXiv07110514 Quant-Ph, Nov. 2007.

[57] M. Znojil, "Which operator generates time evolution in Quantum Mechanics?," Dec. 2007.

[58] A. Fring and M. H. Y. Moussa, "Unitary quantum evolution for time-dependent quasi-Hermitian systems with nonobservable Hamiltonians," Phys. Rev. A, vol. 93, no. 4, p. 042114, Apr. 2016.

[59] A. Fring and M. H. Y. Moussa, "Non-Hermitian Swanson model with a time-dependent metric," Phys. Rev. A, vol. 94, no. 4, p. 042128, Oct. 2016.

[60] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, "Coherent States and Transition Probabilities in a Time-Dependent Electromagnetic Field," Phys. Rev. D, vol. 2, no. 8, pp. 1371–1385, Oct. 1970.

[61] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, "Linear adiabatic invariants and coherent states," J. Math. Phys., vol. 14, no. 5, pp. 576–582, May 1973.

[62] V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, "Integrals of the motion, green functions, and coherent states of dynamical systems," Int. J. Theor. Phys., vol. 14, no. 1, pp. 37–54, Sep. 1975.

[63] V. V. Dodonov and V. I. Man'ko, "Coherent states and the resonance of a quantum damped oscillator," Phys. Rev. A, vol. 20, no. 2, pp. 550–560, Aug. 1979.

[64] D. A. Trifonov, "Pseudo-Boson Coherent and Fock States," ArXiv09023744 Quant-Ph, Feb. 2009.

[65] F. Bagarello, "Pseudobosons, Riesz bases, and coherent states," J. Math. Phys., vol. 51, no. 2, p. 023531, Feb. 2010.

[66] F. Bagarello, "Intertwining operators for non-self-adjoint Hamiltonians and bicoherent states," J. Math. Phys., vol. 57, no. 10, p. 103501, Oct. 2016.

[67] F. Bagarello, F. Gargano, and S. Spagnolo, "Bi-squeezed states arising from pseudobosons," J. Phys. Math. Theor., vol. 51, no. 45, p. 455204, Oct. 2018.

[68] A. Fring and M. H. Y. Moussa, "Non-Hermitian Swanson model with a time-dependent metric," Phys. Rev. A, vol. 94, no. 4, p. 042128, Oct. 2016.

[69] M. Maamache, O. Kaltoum Djeghiour, N. Mana, and W. Koussa, "Pseudo-invariants theory and real phases for systems with non-Hermitian time-dependent Hamiltonians," Eur. Phys. J. Plus, vol. 132, no. 9, p. 383, Sep. 2017.

[70] W. Koussa, N. Mana, O. K. Djeghiour, and M. Maamache, "The pseudo Hermitian invariant operator and time-dependent non-Hermitian Hamiltonian exhibiting a SU(1,1) and SU(2) dynamical symmetry," J. Math. Phys., vol. 59, no. 7, p. 072103, Jul. 2018.

[71] B. Khantoul, A. Bounames, and M. Maamache, "On the invariant method for the time-dependent non-Hermitian Hamiltonians," Eur. Phys. J. Plus, vol. 132, pp. 1–7, 2017.

[72] H. R. Lewis and W. B. Riesenfeld, "An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field," J. Math. Phys., vol. 10, no. 8, pp. 1458–1473, Aug. 1969.

Published articles

ملخص:

في هذه الأطروحة، نقوم ببناء حالات متماسكة شبه مضغوطة تعتمد على الوقت باستخدام مشغلات سلم بوزونية مضغوطة يتم تعريفها على أنها ثوابت خطية غير هيرميتية تعتمد على الوقت، والتي ترتبط بمشغلين مساعدين عبر مشغل متري يعتمد على الوقت. في حين يتم الحصول على مشغلي السلم من التحويل المضغوط لمشغلي الخفض والرفع. كتوضيح، ندرس المذبذب التوافقي غير الهرميتي المعتمد على الوقت، و تحليل خصائص هذه الحالات فيما يتعلق بالموضع ومبدأ عدم اليقين.

الكلمات المفتاحية: الحالات متماسكة، الحالات متماسكة المضغوطة، تناظر شبه التكافؤ، شبه هرميتية، الأنظمة المعتمدة على الوقت، العوامل اللامتغيرة، الحالات متماسكة شبه المضغوطة.

Abstract:

In this thesis we construct time-dependent pseudo-squeezed coherent states using pseudo-squeezed bosonic ladder operators defined as time dependent non-Hermitian linear invariants that are related to their adjoint operators via a time-dependent metric operator. These ladder operators are obtained from the squeezed transformation of the pseudo-bosonic annihilation and creation operators. As an illustration, we study the time-dependent non-Hermitian displaced harmonic oscillator and the properties of these states are analyzed with respect to the localization in position and to uncertainty principle.

Keywords: Coherent states, squeezed states, pt-symmetry, pseudo-hermicity, time-dependent systems, invariant operators, pseudo-squeezed coherent states.

Résumé :

Dans cette thèse, nous construisons des états cohérents pseudo-squeezed dépendant du temps en utilisant des opérateurs pseudo-squeezed bosoniques d'annihilation et de création définis comme des invariants non-hermitique linéaires dépendant du temps qui sont liés à leurs opérateurs adjoints via un opérateur métrique dépendant du temps. Ces opérateurs sont obtenus à partir de la transformation squeezed des opérateurs d'annihilation et de création pseudo-bosonique. En effet, comme une illustration, nous étudions l'oscillateur harmonique non-hermitique déplacé dépendant du temps, et les propriétés de ces états sont analysées par rapport à la localisation en position et au principe d'incertitude.

Mots clés: Les états cohérents, les états squeezed, pt-symétrie, pseudo-hermiticité, systèmes dépendant du temps, opérateurs invariant, les etats cohérent pseudo-squeezed.